Preface

This book brings together several recent developments on the regularity theory for mean-field game systems. We detail several classes of methods and present a concise overview of the main techniques developed in the last few years. Most of the forthcoming material deals with simple and computation-friendly examples; this is intended to unveil the main ideas behind the methods rather than focus on the technicalities of particular cases.

The choice of topics presented here reflects the authors' perspective on this fast-growing field of research; it is by no means exhaustive or intended as a complete account of the theory. Rather—and in the best scenario—it serves as an introduction to the material available in scientific papers.

Book Outline

Mean-field games comprise a wide range of models with distinct properties. Accordingly, no single method addresses existence or regularity issues in all cases. In a restricted number of problems, existence questions on MFGs can be settled through explicit solutions or special transformations. Some of these explicit methods are presented in Chap. 2. Explicit solutions are also essential for the continuation arguments in Chap. 11.

When explicit solutions cannot be found, fixed-point methods, regularization techniques, and continuation arguments provide systematic tools to study the existence of solutions. Usually, a priori bounds are a key ingredient in existence proofs. These bounds are estimates for the size of solutions that are derived before the solution is known to exist. Then, it is often possible to show the existence of the solution. Unless otherwise stated, we work with classical (i.e., C^∞ or at least regular-enough solutions).

We begin our study of a priori bounds for MFGs in Chap. 3, where we examine the Hamilton–Jacobi equation. There, some of the estimates rely only on the optimal control interpretation (see Sect. 3.2) or parabolic regularization effects (see Sect. 3.5). In contrast, other results (see Sect. 3.3 or 3.4) illustrate a subtle interplay between these two mechanisms.

In Chap. 4, we consider transport and Fokker–Planck equations. Both equations preserve mass and positivity. However, the Fokker–Planck equation enjoys strong regularizing properties that we investigate in detail. The chapter ends with a brief discussion of relative entropy inequalities and weak solutions.

A recent development in the theory of solutions of Hamilton–Jacobi equations is the nonlinear adjoint method introduced by L.C. Evans. This method relies on coupling a Hamilton–Jacobi equation with a Fokker–Planck equation. This system resembles (1.1) with $F = 0$. In Chap. 5, we develop the main techniques of this method. The nonlinear adjoint method gives bounds for Hamilton–Jacobi equations that go beyond maximum principle methods. These bounds are obtained by careful integration techniques. In addition to bounds relevant to MFGs, to illustrate the method, we prove semiconcavity estimates and consider the vanishing viscosity problem.

Next, in Chap. 6, we develop techniques that are specific to mean-field games and that combine both equations. These bounds together with the estimates for the Hamilton–Jacobi equation or the Fokker–Planck equation improve earlier results.

Chapter 7 is devoted to stationary models. There, we develop a priori estimates for three different problems. First, we consider MFGs with polynomial dependence on m. To get Sobolev regularity, we combine the integral Bernstein estimate in Chap. 3 with the first-order estimates in Chap. 6. Next, we investigate two MFGs with singularities: the congestion problem and the logarithmic nonlinearity.

In Chaps. 8 and 9, we explore time-dependent MFGs. In the first of these two chapters, we consider models without singularities and illustrate two regularity regimes. The first regime corresponds to subquadratic Hamiltonians. In this case, the main tool is the Gagliardo–Nirenberg estimate discussed in Chap. 3. The second regime corresponds to quadratic and superquadratic Hamiltonians. For these, we get the regularity using the nonlinear adjoint method from Chap. 5. Time-dependent MFGs with singularities present substantial challenges and are examined in Chap. 9. There, we investigate logarithmic nonlinearities in the subquadratic setting and the short-time congestion problem.

Chapters 10 and 11 examine MFGs in the nonlocal and local cases, respectively. We use fixed-point methods to get the existence of solutions for nonlocal problems in both first-order and second-order cases. Besides their independent interest, nonlocal MFGs are used later to study local problems through a regularization procedure. Next, in Chap. 11, we present two techniques to prove the existence of solutions to MFGs. First, we discuss the regularization method. Then, we examine continuation arguments for both stationary and time-dependent problems.

Thanks

This book benefited immensely from the input from various colleagues and students. In particular, the authors are especially grateful to David Evangelista, Gustavo Madeira, José Ruidival dos Santos, Levon Nurbekyan, Lucas Fabiano Lima, Marc Sedjro, Mariana Prazeres, Renato Moura, Roberto Velho, and Teruo Tada, who read parts of the original manuscript and gave invaluable feedback. D. Gomes and V. Voskanyan were supported by KAUST baseline and start-up funds and KAUST SRI, Center for Uncertainty Quantification in Computational Science and Engineering. E. Pimentel was supported by FAPESP (Grant 2015/13011-6) and baseline funds of UFSCar Graduate Program in Mathematics (PPGM-DM-UFSCar).

Bibliographical Notes

Mean-field games were introduced independently and around the same time in the engineering community in [142, 143] and in the mathematics community in [164–167]. Many mathematical aspects of the theory were developed in [174], a course taught by Lions, and several of the techniques in this book can be traced to ideas outlined there.

Before the introduction of MFGs, systems combining a Hamilton–Jacobi equation with a Fokker–Planck or transport equation that resemble MFGs were considered in various settings. For example, the PDE approach to the Aubry–Mather theory [93–95], the problems in [89, 90], and the Benamou–Brenier formulation of the optimal transport problem [29] are forerunners of MFGs. The entropy-penalized scheme in [122] can be reinterpreted as a discrete-time mean-field game.

The goal of this book is to develop the regularity theory for MFGs. These problems have been investigated intensively in the last few years, and we give detailed references at the end of each chapter on the different models and problems. Due to space and time constraints, we cannot discuss the numerous applications of MFGs in engineering and in economics and the many recent results on stochastic methods, numerical analysis, and other MFG models. To make up for these omissions, next, we give a brief bibliography and refer the reader to the books and surveys [30, 61, 121, 133] for more material and references. Also, here, we do not develop the theory of weak solutions to MFGs and instead refer the reader to the following papers [62–64, 68, 98, 195, 196]. Furthermore, we do not discuss numerical methods for MFGs here; for that, see, for example, [1–4, 54, 70, 138–140, 160].

In the engineering community, emerging research includes power grids and energy management [14, 14, 148–150, 179], adaptive control [147, 184] and risk-sensitive or robust control [85, 86, 208, 210], robust MFGs [26, 209], learning [214], and networks [141], among several others [144, 206, 207]. Traffic and crowd models are an important and natural area of application of MFGs [42, 44, 45, 76, 83], as well as related problems on networks and graphs [27, 56, 57].

Some of the first MFG models were motivated by economic growth [167–169]. Subsequently, various problems in economics and finance have been considered in the literature, including socioeconomic models [37, 145, 185], inspection and corruption [153, 156] systemic risk [103], price formation [41, 46, 48, 49, 178], social dynamics [24], consensus [39, 185, 186, 198], and opinion dynamics [23, 36, 201, 202]. In the context of heterogeneous agent models (see [159]) with rational expectations (see [175]), MFGs became a popular modeling tool [5, 6, 176, 187]. An earlier model that predates the emergence of MFGs is the Aiyagari–Bewley–Huggett model [7, 38, 146]. A recent book [133] describes several MFG models in mathematical economics.

MFGs where the agents are subjected to correlated random forces were studied by stochastic methods in [71, 72, 74, 75, 154, 161–163]. The master equation was used to study problems with correlations in [33, 73] and deterministic problems in [105, 124]. An important tool in the study of the N player limit with or without correlations is the theory of nonlinear Markov processes [152]. Some applications of these methods were developed in [155, 157, 158]. Finally, minimax methods were considered in [11–13].

Several authors considered extensions of the original MFG framework. These include finite state mean-field games [21, 99, 113, 126, 128, 131, 132], problems with major and minor agents [183], multi-population models [77, 79, 97], extended MFGs [124, 129, 213], logistic population effects [120], problems with density constraints [180, 199], and obstacle-type problems [116].

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http://www.springer.com/978-3-319-38932-5

Regularity Theory for Mean-Field Game Systems
Gomes, D.A.; Pimentel, E.A.; Voskanyan, V.
2016, XIV, 156 p. 4 illus. in color., Softcover
ISBN: 978-3-319-38932-5