Fuzzy Probability Theory I: Discrete Case

I. Burak Parlak and A. Cağrı Tolga

Abstract This chapter introduces the underlying theory of Fuzzy Probability and Statistics related to the differences and similarities between discrete probability and possibility spaces. Fuzzy Probability Theory for Discrete Case starts with the fundamental tools to implement an immigration of crisp probability theory into fuzzy probability theory. Fuzzy random variables are the initial steps to develop this theory. Different models for fuzzy random variables are designated regarding the fuzzy expectation and fuzzy variance. In order to derive the observation related to fuzzy discrete random variables, a brief summary of alpha-cuts is introduced. Furthermore, essential properties of fuzzy probability are derived to present the measurement of fuzzy conditional probability, fuzzy independency and fuzzy Bayes theorem. The fuzzy expectation theory is studied in order to characterize fuzzy probability distributions. Fuzzy discrete distributions; Fuzzy Binomial and Fuzzy Poisson are introduced with different examples. The chapter is concluded with further steps in the discrete case.

Keywords Fuzzy random variable · Fuzzy conditional probability · Fuzzy independency · Fuzzy Bayes theorem · Fuzzy Hypergeometric distribution · Fuzzy Binomial distribution · Fuzzy Poisson distribution

1 Introduction

When certainties occur people typically look back to gathered data and try to estimate future events. Traditionally, one of the methodologies; well known as probability theory has met these requirements in dealing with uncertainty and

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imprecision. However in full-uncertainty cases, probability theory may not be considered sufficient and it should be integrated with fuzzy logic to enhance its robustness. Full-uncertainty can be described as no one has data on the occurrence of possible cases moreover in some cases nobody know anything about these becoming true possible events. For example, think about space missions: likewise landing of the Rosetta spacecraft to the comet 67P. Additionally think about Mars relocation mission. Of course scientists can compute all the probabilities however despite all the observations made by *Phoenix* and *Curiosity* at Mars the events what will happen in the near future in the Mars mission contain deep vaguenesses. By these contingent events, people try to estimate the various events and additionally their probabilities of course.

Probability measures in fuzzy sets were first revealed by Zadeh [1]. In his work, Zadeh stated that an extension by fuzzy sets might eventually broaden the domain of practicability of probability theory, notably in those fields in which fuzziness is an expansive phenomenon. Then, in his another paper published after 10 years from the previous one, he claimed that the imprecision which is intrinsic in natural languages is, in the main, possibilistic rather than probabilistic in nature [2]. Fuzzy random variables (FRVs) were defined by Kwakernaak and he put forward several theories about independent fuzzy variables for the first time in the literature [3]. Then, he added algorithms about fuzzy random variable after 1 year, and also he gave examples for the discrete case [4].

Liu and Liu offered a new concept of fuzzy expected values related with Choquet integral occurring by chance with random variables (RVs) [5]. They contemplated a fuzzy simulation technique in order to calculate the expected value of general fuzzy variable. Also, a new description of scalar expected value operator for fuzzy random variables was given initially in their paper [6]. Buckley developed fuzzy probabilistic definitions and theorems about fuzzy probability in his pioneer books [7–9]. Nguyen and Wu presented some mathematical background of probability theory for linguistic fuzzy data and introduced several practical examples in their book [10]. Recently, Shapiro reviewed the fuzzy probability theory and summarized the application fields in order to represent fuzzy random variables and the variations between probability and possibility spaces [11–13].

In a nutshell, a fuzzy random variable is a random variable (RV) which is defined using a membership function related to a fuzzy set. However, this definition could be interpreted within the *Probability Space* and the *Possibility Space: Ps*. Let Ω be the sample space, $\mathcal{F}$ be the σ-algebra of subsets of Ω and P be the probability on Ω. $(\Omega, \mathcal{F}, P)$ is the 3-tuple which is called the *Probability Space*. On the other hand, let Θ be the sample space, $P(\Theta)$ be power set of Θ and Ps be the possibility on Θ. Then, $(\Theta, P(\Theta), Ps)$ is the 3-tuple which is called the *Possibility Space*.

In order to compare these triples, let us consider the following example; in a football match of a national league, total number of goals is determined between 1 and 8. The question is to illustrate the meaning between the probable number of goals and the possible number of goals. This difference is represented in Fig. 1. The left chart describes the probability values for the number of goals. On the other hand, the right chart shows the possibility of the number of goals. It is remarkable

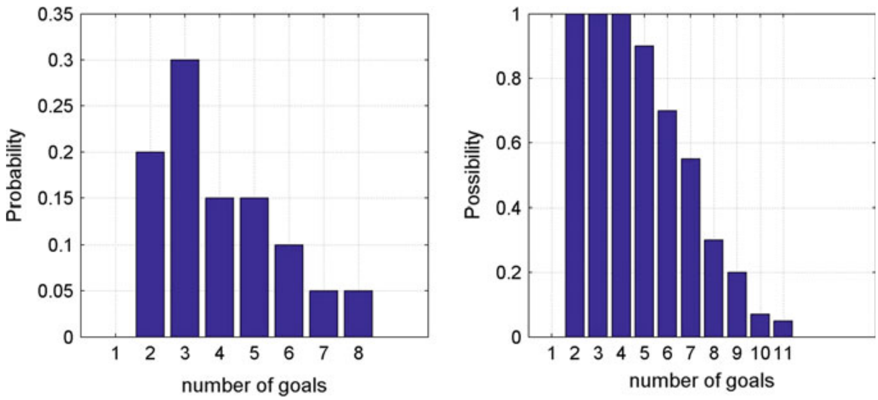

Fig. 1 The probability and possibility of having a certain number of goals in a football match

Table 1 Properties of probability and possibility spaces

Probability space	Possibility space
$(\Omega, \mathcal{F}, P)$	$(\Theta, P(\Theta), Ps)$
Ω: Sample space	Θ: Sample space
$\mathcal{F}$: σ-algebra of subsets	$P(\Theta)$: power set of Θ
P: Probability of Ω	Ps: Possibility on Θ
$P(\Omega) = 1$	$Ps(\Theta) = 1$
$P(A) \geq 0$	$Ps(\emptyset) = 0$
$P\{\bigcup_{i=1}^{\infty} A_i\}$ A_i : *disjoint events*	$Ps\{\bigcup_i A_i\} = sup_i\, Ps\{A_i\}$

that having one or two goals have different probabilities, but they are equally possible in a football match.

The differences between probability space $(\Omega, \mathcal{F}, P)$ and possibility space $(\Theta, P(\Theta), Ps)$ are detailed within the studies of Shapiro [11–13]. As a summary, Table 1 condenses the properties for both spaces as follows;

Discrete possibility space is plotted in Fig. 2 as a graph of membership functions for each discrete event.

In this chapter, fuzzy random variables will be defined and related results will be presented to link them with discrete fuzzy random variables. The discrete fuzzy probability function and its related expectation will be given also. While deepening in fuzzy random variables α-cuts need to be investigated. The consequent section will contain this alpha-cuts topic. Fuzzy probability will be discussed in Sect. 4. In Sect. 5, we will penetrate to discrete fuzzy expectation topic. Section 6 will make mention of fuzzy conditional probability. Fuzzy independency and fuzzy Bayes formula will be investigated in Sects. 7 and 8 successively. Then Fuzzy Hypergeometric distribution will be investigated in Sect. 9. After that, in Sect. 10 Fuzzy Binomial distribution will be expressed. And Fuzzy Poisson distribution,

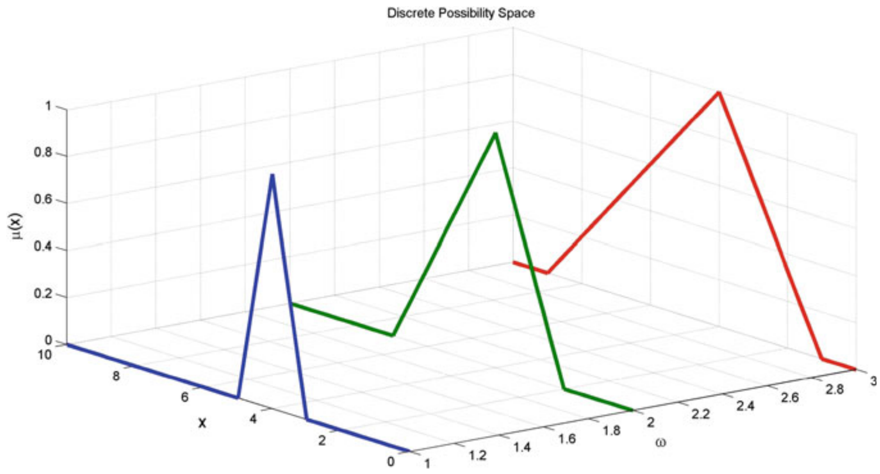

Fig. 2 The illustration of discrete possibility space

ending of the discrete probability distributions, will be explained in Sect. 11. All these Fuzzy discrete distributions will be intensified by illustrative examples. Finally, we will complete this chapter by an inclusive section.

2 Fuzzy Random Variables

The statement of fuzzy random variable (FRV) was firstly developed by Kwakernaak [3], as '*random variables whose values are not real, but fuzzy numbers*'. He defined a FRV as an ambiguous cognition of a crisp but unobservable random variable. For example, let's think about assigning an age to people in a conference. Consider X as their existing age that is an unexceptional random variable on the real line, of course at the positive side. But, someone can simply conceive a random variable x through a set of values as follows: "young", "middle age" and "old". Which means; someone conceive fuzzy sets as noteable results since the genuine X is not remarkable.

As Shapiro [12] remarked; Kwakernaak [4] introduced the fundamentals for a fuzzy random variable model. However before Kwakernaak's study there were essential studies made by various scientist on random sets [1, 2, 14–17]. Puri and Ralescu [18] developed new idea to generate the fuzziness and they stated that the expected value could be fuzzy but the variance should be scalar. Liu and Liu [6] asserted the law of truth conservation and they insisted on that possibility measure was inconsistent with the law of excluded middle and the law of contradiction. They maintained that the expected value and variance of any FRV should be scalar both. However, in contradiction to all these ideas in his books Buckley [7–9] developed another aspect that is generating the fuzziness and the model where the

Table 2 Summary of milestones on fuzzy probabilistic studies

Researchers	State of art
Zadeh 1968 [1]	Fuzzy probability measures
Kendall 1974 [14]	Random sets
Matheron 1975 [15]	Random sets
Fron 1976 [16]	Random fuzzy sets
Zadeh 1978 [2]	Possibility theory
Nguyen 1978 [17]	Random sets and belief functions
Kwakernaak 1978 [3], 1979 [4]	Fuzzy random variable
Puri and Ralescu 1986 [18]	Fuzzy random variables
Buckley 2004 [7], 2005 [8], 2006 [7]	Fuzzy discrete and continuous probability distributions
Couso et al. 2014 [24]	Ill perceived random sets

expected value and the variance are both fuzzy values. Table 2 summarizes fundamental studies on fuzzy probability theory and applications.

A fuzzy membership function would have different values in an interval. Therefore, a fuzzy random variable will be a random variable whose value would be a set using the fuzzy membership function.

Let x_i, $i \in \mathbb{N}$ be a discrete fuzzy random variable. If the values of x_i are denumarable, x_i is called a discrete fuzzy random variable. The probability function of a discrete fuzzy random variable is the representation of the discrete values and their respective probabilities. Furthermore, the fuzzy values of x_i are denoted as $\mu(x_i)$.

In discrete case, this membership function will be represented using fuzzy probability mass function whereas it will be the fuzzy probability density function in continuous case.

In order to generalize the FRV in discrete case, let A be a fuzzy subset of Ω. If $A(x) \neq 0$ for n times of x values in Ω, this subset A could be identified as a discrete fuzzy set. Let us suppose $A(x) \neq 0$ for $1 \leq i \leq n$ in Ω. Therefore, we may write the fuzzy set as follows

$$\tilde{A} = \mu_1, \ldots, \mu_i, \ldots, \mu_n \tag{1}$$

Here μ_i are called the membership values of x_i. In this chapter we will adapt the following expression $\tilde{A}(x_i) = \tilde{\mu}_i$, $1 \leq i \leq n$. In the generalized form, discrete fuzzy subsets could be any space in Ω. Therefore, we may note that α-cuts of discrete fuzzy sets of $\mathbb{R}$, the set of real numbers, do not produce closed, bounded intervals.

Let $X = x_1, \ldots, x_n$ be a finite set and suppose a probability function denominated by P should depicted on all subsets of X with $P(x_i) = a_i$, $1 \leq i \leq n$, $0 \leq a_i \leq 1$ for all i. As we recall from probability theory the summation of a_i $(1 \leq i \leq n)$ values should be equal to 1. The relation between X and P is identified as discrete probability distribution.

Even the probabilistic a_i values should be already known, they are generally estimated, or are observed by experiments. In order to immigrate the fuzzy case, let us start with the assumption of the uncertainty for some a_i and let us model them using fuzzy numbers as described recently. In practice, we may write some a_i as fuzzy numbers and the others as crisp number. However, we should apply fuzzy notation for both a_i numbers in order to facilitate the nomenclature and the calculations.

Moreover, we may write the uncertain a_i values as $\tilde{a}_i$; fuzzy values, and we may apply the probability theory in a similar way; $\forall\ a_i$ and nominate that $0 < \tilde{a}_i < 1$, $1 \leq i \leq n$. Throughout the rest of this chapter, this fuzzy nomenclature is used for given or estimated probabilities.

The probability value a_i is expressed as $\tilde{a}_i = a_i$. However, $\tilde{a}_i$ might be omitted and $\tilde{A}$ might be preferred in order to express the whole distribution in some cases. Therefore, FRV values $X = x_i$ coupled with fuzzy probabilistic values; $\tilde{A} = a_i$ are called a discrete fuzzy probability distribution. Finally, fuzzy P is $\tilde{P}$ and intrinsically $\tilde{P}(x_i) = \tilde{a}_i$, $1 \leq i \leq n$, $0 \leq \tilde{a}_i \leq 1$ could be written.

Consequently, in order to satisfy the summation of probability values, the following restriction should be taken into account when a missed observation is estimated regarding the fuzzy case. We can estimate a_i in $\tilde{a}_i[\alpha]$, all α, and $a_i \in \tilde{a}_i[1]$ should satisfy $\sum_{i=1}^{n} a_i = 1$.

2.1 The FRV Model by Puri and Ralescu

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $A(\mathbb{R}^n)$ emphasize the set of fuzzy subsets, $x : \mathbb{R}^n \to [0, 1], X : \Omega \to A(\mathbb{R}^n)$ be defined by $X_\alpha(\omega) = (x \in \mathbb{R}^n : X(\omega)(x) \geq \alpha)$, and $\mathcal{B}$ denote the Borel subsets of $\mathbb{R}^n$. An FRV by Puri and Ralescu is a function $X : \Omega \to A(\mathbb{R}^n)$ such that, for every $\alpha \in [0, 1]$:

$$\{(\omega, x) : x \in X_\alpha(\omega)\} \in \mathcal{F} \times \mathcal{B}$$

The most wonted measure is the Aumann-type mean for digitizing the central tendency of the distribution of an FRV model by Puri and Ralescu. In his work, Aumann [19] widened the real-valued variable's mean and maintained its principal essential characters and attitude. Before giving the statement of expected value and variance a definition of integrably boundedness has to be made:

A ϑ is an FRV as can be mentioned an integrably bounded FRV related with the probability space $(\Omega, \mathcal{F}, P)$ iff $\|\vartheta_0\| \in L^1(\Omega, \mathcal{F}, P)$, where, for the function f, $L^1(\Omega, \mathcal{F}, P) = \{f \,|\, f : \Omega \to \mathbb{R}, \mathcal{F}\text{-measurable}, \int |f|^1 dP < \infty\}$.

Let ϑ be an integrably bounded FRV related with $(\Omega, \mathcal{F}, P)$, and $S(A)$ be a nonempty bounded set as regard to the L^1(P)-norm, the expected value of ϑ is the unpaired fuzzy set $\tilde{E}(\vartheta \,|\, P)$ of $\mathbb{R}^n$ such that

$$\left(\tilde{E}(\vartheta \,|\, P)\right)_{\alpha} = \int_{\Omega} \vartheta_{\alpha} dP \quad \text{for all } \alpha \in [0, 1], \tag{2}$$

where $\int_{\Omega} \vartheta_{\alpha} dP = \left\{ \int_{\Omega} f dP \,|\, f \in S(\vartheta_{\alpha}) \right\}$ is the Aumann integral of ϑ_{α} as regard to P.

Variance of a Puri and Ralescu type FRV is argued by Feng et al. [20]. They claimed that, the variance should be used to observe the spread or deployment of the FRV around its expected value (EV) just like under the circumstances of real random variables. Ultimately, they illustrated the Puri and Ralescu type variance as a scalar shown below:

$$Var(\tilde{X}) = \frac{1}{2} \int_{0}^{1} [V(\underline{X}_{\alpha}) + V(\bar{X}_{\alpha})] d\alpha \tag{3}$$

Also in the literature, there are some other offers for scalar variance. Premier one is considering a numerical element of every fuzzy realization of the FRV as the midpoint of the support and then computing the deployment of these representative values.

2.2 The FRV Model by Buckley Based on Kwakernaak

As given in the previous subsection, let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A(\mathbb{R})$ emphasize the all fuzzy numbers' set in the real numbers set, $\mathbb{R}$. Particularly, $A(\mathbb{R})$ depicts the class of normal convex fuzzy subsets of $\mathbb{R}$ which has the severe α-levels for $\alpha \in [0, 1]$. Assignment class could be defined as U, and $U : \mathbb{R} \to [0, 1]$, i.e., $U(u) \in [0, 1]$, for all $u \in \mathbb{R}$, such that U_{α} is a non-empty severe interval, where

$$U_{\alpha} = \begin{cases} \{x \in \mathbb{R} \,|\, U(x) \leq \alpha & \text{if } \alpha \in (0, 1] \\ cl(suppU) & \text{if } \alpha = 0 \end{cases}$$

An FRV is an assignment $\vartheta : \Omega \to A(\mathbb{R})$ thus for each $\alpha \in [0, 1]$ and all $\omega \in \Omega$ the real-valued assignment:

$\inf \vartheta_{\alpha} : \Omega \to \mathbb{R}$, ensuring $\inf \vartheta_{\alpha}(\omega) = \inf \left(\vartheta(\omega)\right)_{\alpha}$, and

$\sup \vartheta_{\alpha} : \Omega \to \mathbb{R}$, ensuring $\inf \vartheta_{\alpha}(\omega) = \inf \left(\vartheta(\omega)\right)_{\alpha}$, are real valued RVs.

The central tendency of the distribution of an FRV model by Buckley based on Kwakernaak can be widened to the real-valued variable's mean and can be calculated as below:

$$\mu_{E(U)}(\vartheta) = \sup \{\mu_{\vartheta}(U) \,|\, U \in \mathcal{U}_{\mathcal{A}}, E(U) = \vartheta\}, \vartheta \in \mathbb{R} \tag{4}$$

where E indicates the usual expectation. Similarly, the fuzzy variance of ϑ is a fuzzy set on $[0, \infty)$ with

$$\mu_{V(U)}(\vartheta) = \sup \ \{\mu_{\vartheta}(U) \mid U \in \mathcal{U}_{\mathcal{A}}, V^2U = \sigma^2\}, \sigma^2 \in [0, \infty). \tag{5}$$

where, ϑ is an FRV, $\mathcal{U}_A$ is the collection of all $\mathcal{A}$-measurable RVs of Ω and V indicates the usual variance.

As stated in the previous sections, mean and variance of a fuzzy random variable in Liu and Liu's model are both scalar. One can calculate these values more easily than the cited FRV models above.

3 Alpha-Cuts

In a more universal perception, the random set could be produced by the α-cuts of A as stated in Zadeh's article [21]. To be more en detail, an α-cut, A_α, of A is a non-fuzzy set described by $A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, 0 \leq \alpha \leq 1$. The α-cuts are employed to be the main components of a random set; with α it is assumed to be uniformly distributed over the interval $[0, 1]$.

A fuzzy set A can be produced from a random set. Essentially, the identical result can be obtained without appearing of randomness. It is clear that a fuzzy set may be reproduced from its α-cuts both discriminating and additively. In order to explain this case, suppose that $\mu_{A_\alpha}(x)$ express the membership function of A_α. As A_α is non-fuzzy, it might be confused with the specific function of A_α. Then, the membership function of A can be displayed in terms of the membership functions of the A_α (a) discriminating as $\mu_A(x) = sup_\alpha(\alpha \wedge \mu_{A_\alpha}(x)), 0 \leq \alpha \leq 1$, where $\wedge$ expresses min, and (b) additively as

$$\mu_A(x) = \int_0^1 \mu_{A_\alpha}(x) d\alpha. \tag{6}$$

The illustrative representation is depicted in Fig. 3.

4 Fuzzy Probability

Let us start to define two (crisp) subsets A and B be of X. Moreover, we want to compute fuzzy probabilities which are denoted $\tilde{P}(A)$ and $\tilde{P}(B)$, respectively. We have to implement fuzzy algebra to calculate these values. In some a_i values there would be an uncertainty, however in discrete probability distribution there is no uncertainty. Therefore, the probability summation rule $a_1 + \cdots + a_n = 1$ for each a_i values in $\tilde{a}_i[\alpha]$ should be satisfied. This constraint will be served as the basis of our fuzzy algebra.

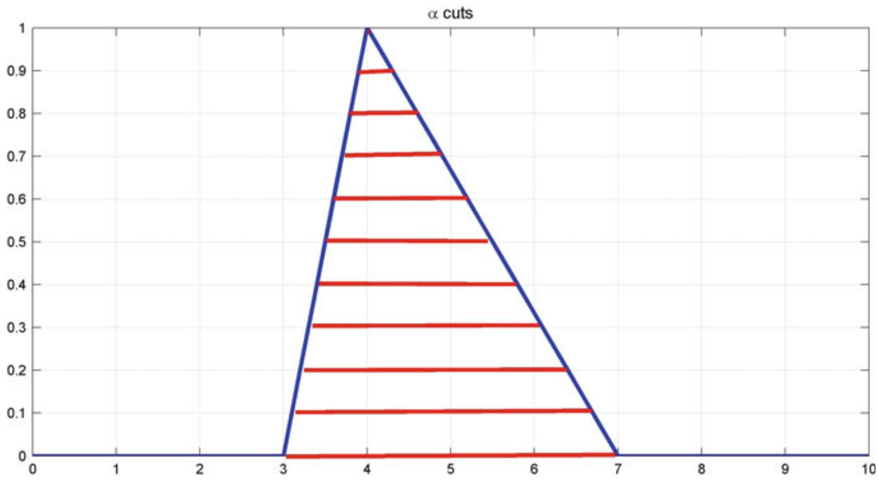

Fig. 3 The representation of alpha cuts for a triangular fuzzy number

We suppose that $A = x_1, \ldots, x_k,\ 1 \le k < n$, then define

$$\tilde{P}(A)[\alpha] = \left\{ \sum_{i=1}^{k} a_i \right\} \tag{7}$$

for $0 \le \alpha \le 1$, where $a_i \in \tilde{a}_i[\alpha], 1 \le i \le n, \sum_{i=1}^{n} a_i = 1$ which is related to the fuzzy algebra. First of all, a complete discrete probability distribution using the α-cuts should be determined. Secondly, a probability in Eq. (7) should be calculated.

Let us denote that $\tilde{P}(A)[\alpha]$ is not equal to the sum on fuzzy the intervals $\tilde{a}_i[\alpha]$ with the fuzzy algebra in $1 \le i \le k$. Let us try to complete the definition of $\tilde{P}(A)[\alpha]$ by introducing the α-cuts of a fuzzy number $P(A)$.

Initially, we note that $x_l, \ldots, x_m,\ x_i \ge 0,\ \sum_{i=1}^{n} x_i = 1$ and $f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{k} a_i$. Using these definitions, we might write that

1. If $A \cap B = \emptyset$, then $P(\tilde{A}) + P(\tilde{B}) \ge \tilde{P}(A \cup B)$.
2. If $A \subseteq B$, $\tilde{P}(A)[\alpha] = [p_{a1}(\alpha), p_{a2}(\alpha)]$ and $\tilde{P}(B)[\alpha] = [p_{b1}(\alpha), p_{b2}(\alpha)]$ then $p_{a_i}(\alpha) \le p_{b_i}(\alpha)$ for $i = 1, 2$ and $0 \le \alpha \le 1$
3. $0 \le \tilde{P}[A] \le 1$ all A with $\tilde{P}(\emptyset) = 0$, $\tilde{P}(X) = 1$
4. $\tilde{P}[A] + \tilde{P}[A'] \ge 1$, A' is the complement of A.
5. For $A \cap B \ne \emptyset$, $\tilde{P}(A \cup B) \le \tilde{P}(A) + \tilde{P}(B) - \tilde{P}(A \cup B)$

Example 1 Let n be 5 and the sets are; $X = \{a_1, a_2, a_3\}$ $Y = \{a_4, a_5\}$ and the fuzzy probabilities for each random variables are $\tilde{x}_i = (0.1, 0.2, 0.3)$ $1 \le i \le 5$.

Therefore, we may compute that; $\tilde{P}(X)[0] = [0.3, 0.9]$, $\tilde{P}(X)[1] = [0.6, 0.6]$ and $\tilde{P}(Y)[0] = [0.2, 0.6]$, $\tilde{P}(Y)[1] = [0.4, 0.4]$.

Example 2 Let n be 6 and the sets are; $X = \{a_1, a_2, a_3\}$ $Y = \{a_3, a_4\}$ and $Z = \{a_4, a_5, a_6\}$ the fuzzy probabilities for random variables are as follows; $\tilde{x}_i = (0.1, 0.15, 0.2)$ for $1 \le i \le 4$ $\tilde{x}_j = (0.15, 0.2, 0.22)$ for $5 \le j \le 6$.

Therefore, we may compute that; $\tilde{P}(X)[0] = [0.3, 0.6]$, $\tilde{P}(X)[1] = [0.45, 0.45]$, $\tilde{P}(Y)[0] = [0.2, 0.4]$, $\tilde{P}(Y)[1] = [0.3, 0.3]$ and $\tilde{P}(Z)[0] = [0.4, 0.64]$, $\tilde{P}(Z)[1] = [0.55, 0.55]$.

Furthermore, we may deduce the fuzzy probabilities of the intersection; $\tilde{P}(X \cap Y)[0] = [0.1, 0.2]$, $\tilde{P}(X \cap Y)[1] = [0.15, 0.15]$, $\tilde{P}(Y \cap Z)[0] = [0.1, 0.2]$, $\tilde{P}(Y \cap Z)[1] = [0.15, 0.15]$.

Finally the fuzzy union probabilities are as follows; $\tilde{P}(X \cup Y)[0] = [0.4, 0.8]$, $\tilde{P}(X \cup Y)[1] = [0.6, 0.6]$, $\tilde{P}(Y \cup Z)[0] = [0.5, 0.84]$, $\tilde{P}(Y \cup Z)[1] = [0.7, 0.7]$.

We may note that; $[0.4, 0.8] \neq [0.3, 0.6] + [0.2, 0.4] - [0.1, 0.2]$ and $[0.5, 0.84] \neq [0.2, 0.4] + [0.4, 0.64] - [0.1, 0.2]$.

Therefore, we should remark that $\tilde{P}(X \cup Y)$ and $\tilde{P}(Y \cup Z)$ can be expressed as a subset of $\tilde{P}(X) + \tilde{P}(Y) - \tilde{P}(X \cap Y)$, $\tilde{P}(Y) + \tilde{P}(Z) - \tilde{P}(Y \cap Z)$, respectively.

5 Fuzzy Discrete Expectation

Suppose X and Y are two random variables with joint probability density $f(x, y; \theta)$. where $x \in \mathbb{R}$ and the joint density function's vector of parameters should be $\theta = (\theta_1, \ldots \theta_n)$. Mostly cited parameters are anticipated employing a random sample from the population. These anticipations can be a point estimate or a confidence interval. In lieu of anticipation of a point, a confidence interval for each θ_i could be replaced by the probability density function to obtain an interval joint probability density function. Whereas more general form should be constructed and this necessitates formulation of the indefiniteness in the θ_i by replacing a fuzzy number for θ_i and acquire a joint fuzzy probability density function. For α-cuts of the fuzzy number utilized for θ_i, vide supra Sect. 3. If one wishes to implement the interval probability density functions utilizing the fuzzy numbers with α-cuts of course could be better. The joint fuzzy density functions are fulfilled by substituting fuzzy numbers for the vague parameters. For to clarify the nubiluous definitions fuzzy marginals should be discussed now.

Since the θ_i in θ are uncertain fuzzy numbers $\tilde{\theta}_i$ are replaced for the θ_i, $1 \le i \le n$, that provides joint fuzzy density $f(x, y; \tilde{\theta})$. The fuzzy marginal for X is

$$f(x; \tilde{\theta}) = \sum_{k=1}^{\infty} f(x_k, y_k; \tilde{\theta}). \tag{8}$$

The fuzzy marginal for Y could be written by the same way. The α-cuts of the fuzzy marginals $f(x;\tilde{\theta})$ could be calculated by the following equation

$$f(x;\tilde{\theta})[\alpha] = \left\{ \sum_{k=1}^{\infty} f(x_k, y_k; \theta) \mid \theta_i \in \tilde{\theta}_i[\alpha], 1 \le i \le n \right\}, \tag{9}$$

for $0 \le \alpha \le 1$, and also we can derive an analogous equation for $f(y;\tilde{\theta})[\alpha]$. Equation (9) gives the α-cuts of a fuzzy set for each value of x.

At this juncture let $f(x;\theta)$ be the crisp marginal of x which means non-fuzzy marginal of x. Let's utilize $f(x;\theta)$ to procure the mean $\mu_x(\theta)$ and variance $Var_x(\theta)$ of X. The mean and variance of X are written as functions of θ as they are related with the values of the parameters. Suppose that $\mu_x(\theta)$ and $Var_x(\theta)$ are discrete functions of θ. Deriving the fuzzy mean and variance of the fuzzy marginal could be made by fuzzification of the crisp mean and variance. Subjacent theorem can be deduced from the previous explanations for X and also can be utilized for Y.

Theorem *The fuzzy mean and variance of the fuzzy marginal* $f(x;\tilde{\theta})$ *are* $\mu_x(\tilde{\theta})$ *and* $Var_x(\tilde{\theta})$ [8].

Proof An α-cut of the fuzzy mean of the fuzzy marginal for X is

$$\mathcal{M}_x(\tilde{\theta})[\alpha] = \left\{ \sum_{k=1}^{\infty} x_k f(x_k; \theta) \mid \theta_i \in \tilde{\theta}_i[\alpha], 1 \le i \le n \right\}, \tag{10}$$

for $0 \le \alpha \le 1$. Now the sum in Eq. (10) equals $\mathcal{M}_x(\theta)$ for each $\theta_i \in \tilde{\theta}_i$, $1 \le i \le n$. So

$$\mathcal{M}_x(\tilde{\theta})[\alpha] = \{\mu_x(\theta) \mid \theta_i \in \tilde{\theta}_i[\alpha], 1 \le i \le n\}. \tag{11}$$

Because of this, the fuzzy mean is $\mathcal{M}_x(\tilde{\theta})$. See the studies explained at the cited references in the various fuzzy distributions parts of this chapter.

The fuzzy variance with α-cuts could be written as follows:

$$Var_x(\tilde{\theta})[\alpha] = \{\sum_{k=1}^{\infty} (x_k - \mu_{x_k}(\theta))^2 f(x_k; \theta) \mid \theta_i \in \tilde{\theta}[\alpha], 1 \le i \le n\}, \tag{12}$$

for $0 \le \alpha \le 1$. But the sum in the above equation equals $Var_x(\theta)$ for each $\theta_i \in \tilde{\theta}_i$, $1 \le i \le n$. Due to this

$$Var_x(\tilde{\theta})[\alpha] = \{Var_x(\theta) \mid \theta_i \in \tilde{\theta}_i[\alpha], 1 \le i \le n\}. \tag{13}$$

So, the fuzzy variance is just $Var_x(\tilde{\theta})$. □

6 Fuzzy Conditional Probability

In probability theory, conditional probability serves us to introduce and calculate joint probabilities. In fuzzy probability theory, we will apply the same approach by using fuzzy random variables.

Let $M = x_l, \ldots, x_r$, $N = x_s, \ldots, x_t$ for $1 \le r \le s \le t \le n$ so that M and N are not disjoint. The fuzzy conditional probability of M given N should be defined naturally. The fuzzy conditional probability might be written as $P(M \mid N)$. Furthermore, the following definitions for fuzzy conditional probability could be presented. At first,

$$\tilde{P}(M \mid N) = \frac{\sum_{i=r}^{s} a_i}{\sum_{i=r}^{t} a_i} \quad a_i \in \tilde{a}_i[\alpha] \quad 1 \le i \le n \tag{14}$$

The numerator of the division is the sum of the a_i; in the intersection of M and N, while the denominator is the sum of the a_i in N. Then we may write;

$$\tilde{P}(M \mid N) = \frac{\tilde{P}(M \cap N)}{\tilde{P}(N)} \tag{15}$$

These definitions for fuzzy conditional probability would be considered as the fuzzy version of conditional probability theory. Therefore, we might interpret the fundamental characteristics of fuzzy conditional probability which are:

1. $0 \le \tilde{P}(M \mid N) \le 1$
2. $\tilde{P}(N \mid N) = 1$ crisp one
3. $\tilde{P}(M \mid N) = 1$ crisp if $N \subseteq M$
4. $\tilde{P}(M \mid N) = 0$ crisp if $N \cap M = \emptyset$
5. $\tilde{P}(M_1 \cup M_2 \mid N) \le \tilde{P}(M_1 \mid N) + \tilde{P}(M_2 \mid N)$ if $M_1 \cap M_2 = \emptyset$

Firstly, we may note that the first three properties will be directly related to the initial definition of fuzzy conditional probability. Let us assume an empty sum which will be equal to zero. As the numerator in Eq. 14 is an set empty, the fourth property will be true while these events M and N would be disjoint. For the last property in fuzzy conditional case, let us define that;

$$\tilde{P}(M_1 \cup M_2 \mid N)[\alpha] \subseteq \tilde{P}(M_1 \mid N)[\alpha] + \tilde{P}(M_2 \mid N)[\alpha] \quad 0 < \alpha < 1 \tag{16}$$

We evaluate the expression through α. Let us set $x = \frac{\beta + \gamma}{\theta}$ which belongs to $(M_1 \cup M_2 \mid N)[\alpha]$. For x we may write;

- β is equal to the sum of the a_i for $x_i \in M_1 \cap N$.
- γ is equal to the sum of the a_i for; $x_i \in M_2 \cap N$.
- θ is equal to the sum of the a_i for $x_i \in N$.

As a simplification, the sum of a_i will be equal to one as $a_i \in \tilde{a}_i[\alpha]$. Therefore, we may write that β/δ belongs to $\tilde{P}(M_1 \mid N)[\alpha]$ and γ/δ belongs to $\tilde{P}(M_2 \mid N)[\alpha]$.

7 Fuzzy Independency

In probability theory, the dependency or the independency of the events are crucial to observe the joints probabilities. However, the properties related to the moments are generally related to the independency to simplify the calculations. In the case of fuzzy probability, a similar reasoning would be applied to define the case for two events M and N.

As it is represented in fuzzy conditional probability, we may adapt two definitions; strong and weak independency for the events M and N.

The first expression to define the independency is based on the fuzzy conditional probability. M and N are characterized as strongly independent if

$$\tilde{P}(M \mid N) = \tilde{P}(M) \tag{17}$$

and

$$\tilde{P}(N \mid M) = \tilde{P}(N) \tag{18}$$

These expressions are not always obvious to define the independency. Therefore, we need to introduce a new term which is the weak independency in fuzzy probability theory. We may write that the events M and N are weakly independent if

$$\tilde{P}(M \mid N)[1] = \tilde{P}(M)[1] \tag{19}$$

and

$$\tilde{P}(N \mid M)[1] = \tilde{P}(N)[1] \tag{20}$$

In the second formulation where the events are characterized as weakly independent, we use the $\alpha = 1$ cuts to satisfy the equality. Therefore, events which are strongly independent are obviously weakly independent.

In order to conclude the fuzzy independency, we may start to use the conventional expression of independency based on the crisp way. Initially, the events M and N are said independent when

$$\tilde{P}(M \cap N) = \tilde{P}(M)\tilde{P}(N) \tag{21}$$

8 Fuzzy Bayes Formula

In the probability theory, Bayes formula formulates the relationship between the current and the prior information. It is supported by the theory of conditional probability. In this section, we will show the fuzzy interpretation of Bayes formula.

Let $X = x_1, \ldots, x_n$ be a random variable and let β_i, $1 \le i \le k$, be a partition of X. We assume that β_i are not empty sets, and they are mutually disjoint. The union of β_i is X. In a case where the probability of β_i is not known, we can develop a conditional probability to calculate β_i.

If θ_i are some priors that we may know and

$$p_{ij} = P\left(\beta_i \mid \theta_j\right) \tag{22}$$

where p_{ij} will generate the probability of β_i.

In order to calculate p_{ij}, we need the estimates $p_j = P(\theta_j)$. The probability p_j is defined as the prior probability distribution. The probability that the partition β_i is given when the priors θ_j has accomplished as follows;

$$P\left(\theta_j \mid \beta_i\right) = \frac{P\left(\beta_i \mid \theta_j\right)P\left(\theta_j\right)}{\sum_{j=1}^{J} P\left(\beta_i \mid \theta_j\right)P\left(\theta_j\right)} \qquad 1 \le j \le J \tag{23}$$

Moreover, $p_{kj} = P\left(\theta_j \mid \beta_i\right)$ is called the posterior probability distribution. The probability $P(\beta_i)$ might be calculated by integrating p_{ij} and p_j as follows;

$$P(\beta_i) = \sum_{j=1}^{J} P\left(\beta_i \mid \theta_j\right)P\left(\theta_j\right) \qquad 1 \le i \le k \tag{24}$$

For a specific event β_k which has occured, we might develop the prior for the posterior and calculate the probabilities of β_i as

$$P(\beta_i) = \sum_{j=1}^{J} P\left(\beta_i \mid \theta_j\right)P\left(\theta_j \mid \beta_k\right) \qquad 1 \le i \le k \tag{25}$$

Finally, Fuzzy Bayes formulation could be written using the α-cuts of fuzzy posterior distribution as follows;

$$\tilde{P}\left(\theta_j \mid \beta_k\right)[\alpha] = \frac{p_{ij}p_j}{\sum_{j=1}^{J} p_{ij}p_j} \tag{26}$$

9 Fuzzy Hypergeometric Distribution

In the discrete probability theory, the hypergeometric distribution is considered among the fundamental probability distributions where lot acceptance area is modeled using the probabilistic information. This formulation is developed for fuzzy case in the inspection of geospatial data by Tong and Wang [22].

It is assumed that there is a finite population concerning N units in hypergeometric distribution. Let's say some number D of these units contribute a class of interest which can be a success or a failure ($D \leq N$). This type of probability distribution describes the probability of x interests in n pulls without replacement. Then, x can be depicted as a hypergeometric random variable with the probability distribution as below:

$$P(x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad x = 0, 1, 2, \ldots, min(n, D) \tag{27}$$

In order to calculate the fuzzy probability, we need to use the fuzzy algebra for $\tilde{D} = N\tilde{r}$ and $\tilde{n} = N\tilde{l}$ and in addition we should have derive the minimum and the maximum of $\tilde{P}(x)$ using the α-cuts. The fuzzy hypergeometric distribution is characterized as the probability model for a fuzzy random sample selection of $\tilde{n}$ items without replacement from a lot of N items of which $\tilde{D}$ are non-conforming or defective. Therefore, fuzzy hypergeometric probability mass function is derived by using fuzzy numbers for the conforming items based on the approach of Tong and Wang [22] as follows;

$$\tilde{P}(x)[\alpha] = min\left\{\frac{\binom{\tilde{D}}{x}\binom{N-\tilde{D}}{\tilde{n}-x}}{\binom{N}{\tilde{n}}}\right\} \qquad 0 \leq \alpha \leq 1 \tag{28}$$

$$\tilde{P}(x)[\alpha] = max\left\{\frac{\binom{\tilde{D}}{x}\binom{N-\tilde{D}}{\tilde{n}-x}}{\binom{N}{\tilde{n}}}\right\} \qquad 0 \leq \alpha \leq 1 \tag{29}$$

Moreover, the fuzzy probability $\tilde{P}[\alpha]$ is obtained within $[P(r_1), P(r_2)]$. Therefore,

$$\tilde{P}(x)[\alpha] = \frac{\binom{Nr}{x}\binom{N-Nr}{Nl-x}}{\binom{N}{Nl}} \quad r \in \tilde{r}[\alpha] \;\; l \in \tilde{l}[\alpha] \;\; 0 \leq \alpha \leq 1 \tag{30}$$

Example 3 Suppose that a lot contains 2000 items, a fraction of $\tilde{r} = (0.25, 0.3, 0.4)$ which do not conform requirements. If a fraction of $\tilde{l} = (0.04, 0.05, 0.06)$ items is selected at random without replacement, then the fuzzy probability of finding one or fewer nonconforming items in the sample is as follows;

$$\tilde{P}\{x \leq 1\} = \tilde{P}\{x = 0\} + \tilde{P}\{x = 1\}$$

$$min\{\tilde{P}\{x \leq 1\}\} = \frac{\left(\binom{2000(0.25 + (0.3 - 0.25)\alpha)}{0}\right)\left(\binom{2000 - 2000(0.25 + (0.3 - 0.25)\alpha)}{2000(0.04 + (0.05 - 0.04)\alpha)}\right)}{\left(\binom{2000}{2000(0.04 + (0.05 - 0.04)\alpha)}\right)} + \frac{\left(\binom{2000(0.25 + (0.3 - 0.25)\alpha)}{0}\right)\left(\binom{2000 - 2000(0.25 + (0.3 - 0.25)\alpha)}{(2000(0.04 + (0.05 - 0.04)\alpha) - 1)}\right)}{\left(\binom{2000}{2000(0.04 + (0.05 - 0.04)\alpha)}\right)} \tag{31}$$

$$max\{\tilde{P}\{x \leq 1\}\} = \frac{\left(\binom{2000(0.4 + (0.3 - 0.4)\alpha)}{0}\right)\left(\binom{2000 - 2000(0.4 + (0.3 - 0.4)\alpha)}{2000(0.06 + (0.05 - 0.06)\alpha)}\right)}{\left(\binom{2000}{2000(0.06 + (0.05 - 0.06)\alpha)}\right)} + \frac{\left(\binom{2000(0.4 + (0.3 - 0.4)\alpha)}{0}\right)\left(\binom{2000 - 2000(0.4 + (0.3 - 0.4)\alpha)}{(2000(0.06 + (0.05 - 0.06)\alpha) - 1)}\right)}{\left(\binom{2000}{2000(0.06 + (0.05 - 0.06)\alpha)}\right)} \tag{32}$$

$$\tilde{P}\{x \leq 1\} = \begin{bmatrix} \frac{\left(\binom{600\alpha)}{0}\right)\left(\binom{1400\alpha)}{100\alpha)}\right)}{\left(\binom{2000}{100\alpha)}\right)} + \frac{\left(\binom{600\alpha)}{1}\right)\left(\binom{1400\alpha)}{(100\alpha) - 1)}\right)}{\left(\binom{2000}{100\alpha)}\right)}, \\ \frac{\left(\binom{600\alpha)}{0}\right)\left(\binom{1400\alpha)}{100\alpha)}\right)}{\left(\binom{2000}{100\alpha)}\right)} + \frac{\left(\binom{600\alpha)}{1}\right)\left(\binom{1400\alpha)}{(100\alpha) - 1)}\right)}{\left(\binom{2000}{140\alpha)}\right)} \end{bmatrix} \tag{33}$$

Fuzzy Mean and Variance of Hypergeometric Distribution

The mean and variance of the hypergeometric distribution could be calculated

$$\tilde{\mu}[\alpha] = \frac{N\tilde{l} * N\tilde{r}}{N} \tag{34}$$

and

$$\tilde{\sigma}^2[\alpha] = \frac{N\tilde{l} * N\tilde{r}}{N}\left(1 - \frac{N\tilde{r}}{N}\right)\left(\frac{N - N\tilde{l}}{N-1}\right) \tag{35}$$

10 Fuzzy Binomial Distribution

Binomial distribution could be considered as the generalized form of Bernouilli distribution. Fuzzy Binomial distribution is defined with a fuzzy random variable.

In this section, the studies about binomial distribution in fuzzy form were firstly developed by Buckley using the α-cuts [8]. Kahraman and Kaya applied this model into fuzzy sampling by numerous examples [23].

Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete random variable. We start to apply the initial definition of Binomial distribution. A number of experiments n is considered independent, the probability of success is p and the probability of failure is $1 - p$ for a single experiment. Therefore X could be defined as a binomial random variable.

In order to generalize this expression, independent experiments should be repeated n times to gather the probability of x_i successes for $i \in [1, n]$. Thus;

$$P(x) = \binom{n}{x} p^x q^{nx} \qquad x = 0, 1, 2, ..., n \tag{36}$$

Fuzzy Binomial probability mass function is derived by using fuzzy numbers for the success: $\tilde{p}$ and the failure: $\tilde{q}$. In order to calculate the fuzzy probability, we need to use the fuzzy algebra for $\tilde{p}$ and $\tilde{q}$ and to derive the minimum and the maximum of $\tilde{P}(x)$ using the α-cuts as follows;

$$\tilde{P}(k_1)[\alpha] = min\left\{\binom{n}{x}\tilde{p}^x\tilde{q}^{nx}\right\} \qquad 0 \le \alpha \le 1 \tag{37}$$

$$\tilde{P}(k_2)[\alpha] = max\left\{\binom{n}{x}\tilde{p}^x\tilde{q}^{nx}\right\} \qquad 0 \le \alpha \le 1 \tag{38}$$

where $p \in \tilde{p}[\alpha]$, $q \in \tilde{q}[\alpha]$, $p + q = 1$.

Furthermore, the fuzzy probability $\tilde{P}[\alpha]$ is obtained within $[P(k_1), P(k_2)]$. Thus,

$$\tilde{P}(x) = \binom{n}{x} p^x q^{nx} \qquad 0 \le \alpha \le 1 \tag{39}$$

Example 4 For a determined time period, a shipyard company calculates that their yachts develop squeaks of indoor equipments in a measured percentage interval $(8, 12, 20)$ within the guarantee period. In a randomly delivery, 5 yachts reach the end of the guarantee period without any squeaks. Find the fuzzy probability in this case?

In order to solve this problem, we start to define p and q. Then, $\tilde{q} = 1 - \tilde{p} = 1 - (0.08, 0.12, 0.2) = (0.8, 0.88, 0.92)$. We need to calculate the fuzzy number $\tilde{P}[5]$. Therefore, we can rewrite the Eqs. 37, 38 as follows:

$$P(k_1)[\alpha] = min\left\{ \binom{5}{0} \tilde{p}^0 (1 - \tilde{p})^{(5-0)} \right\} \qquad 0 \le \alpha \le 1 \tag{40}$$

$$P(k_2)[\alpha] = max\left\{ \binom{5}{0} \tilde{p}^0 (1 - \tilde{p})^{(5-0)} \right\} \qquad 0 \le \alpha \le 1 \tag{41}$$

We obtain $P(k_1)[\alpha] = min\left\{ (1-p)^5 \right\}$, $P(k_2)[\alpha] = max\left\{ (1-p)^5 \right\}$. As the derivation $\frac{d(1-p)^5}{dp} \ge 0$, for $\tilde{p}[\alpha]$, where $\alpha = 0$, we may write the probability in the case of 5 yachts as follows;

$$\tilde{P}(5)[\alpha] = \left[(1 - p_1(\alpha))^5, (1 - p_2(\alpha))^5 \right] \tag{42}$$

Therefore, $\tilde{p}[\alpha] = [p_1(\alpha), p_2(\alpha)] = [0.08 + 0.04\alpha, 0.2 - 0.08\alpha]$ for $0 \le \alpha \le 1$.

Fuzzy Mean and Variance of Binomial Distribution

By using the α-cuts the fuzzy mean of Fuzzy Binomial distribution could be calculated as follows;

$$\tilde{\mu}[\alpha] = \sum_{i=1}^{n} x_i k_i \tag{43}$$

for $k_i \in \tilde{k}_i[\alpha]$, $1 \le i \le n$ and $\sum_{i=1}^{n} k_i = 1$.

Therefore, we may write;

$$\tilde{\mu}[\alpha] = \sum_{i=1}^{n} i \binom{n}{i} p^i q^{n-i} \tag{44}$$

which is equal to $\tilde{\mu}[\alpha] = n\tilde{p}$.

The variance is also calculated using the same principle;

$$\tilde{\sigma}^2[\alpha] = \sum_{i=1}^{n} (x_i - \mu_i)^2 k_i \tag{45}$$

Finally, we may write the fuzzy variance of the fuzzy binomial distribution as follows; $\tilde{\sigma}^2[\alpha] = n\tilde{p}\tilde{q}$.

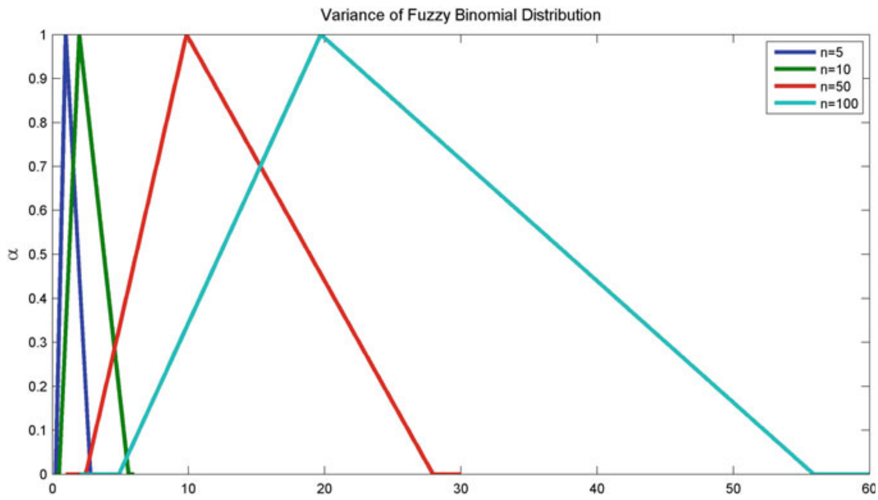

Fig. 4 The representation of variance analysis for fuzzy binomial distribution

Example 5 Let $\tilde{p}$ have a linear triangular fuzzy membership function $(0.14, 0.27, 0.65)$. Calculate the variance of fuzzy binomial distribution.

We know that;

$$\tilde{\sigma}^2[\alpha] = n\tilde{p}\tilde{q} \tag{46}$$

Furthermore, we may write $\tilde{\sigma}^2[\alpha] = n\tilde{p}(1 - \tilde{p})$ and we obtain; $(1 - \tilde{p})$ = $(0.35, 0.73, 0.86)$. In order to interpret the variance, MATLAB 2014a is used to sketch the variance for the different n values. We obtained the following Fig. 4.

11 Fuzzy Poisson Distribution

Poisson distribution is characterized by the experiments whose outputs are discrete in continuous space. A regular time observation could be considered as discrete values whereas time is continuous. On the other hand, Fuzzy Poisson distribution is represented by a fuzzy random variable.

The studies about Fuzzy Poisson distribution were firstly developed by Buckley using the α-cuts [8]. Kahraman and Kaya applied this model into fuzzy sampling by several examples [23].

In order to generate the fuzzy Poisson distribution we define $X = \{x_1, x_2, \ldots, x_n\}$ which is a discrete random variable. X has also the Poisson probability mass function. When the probability $P(x)$ is defined for the probability that $X = x$, we may write

$$P(x) = \frac{\lambda^x exp(-\lambda)}{x!} \tag{47}$$

where, $x = 0, 1, 2, \ldots, n$, and $\lambda > 0$.

Fuzzy Poisson probability mass function is derived by using fuzzy number $\tilde{\lambda} > 0$. Let us denote $\tilde{P}(x)$ to be the fuzzy probability that $X = x$. Therefore, we can calculate the fuzzy probability using α-cuts algebra,

$$\tilde{P}(x)[\alpha] = \frac{\lambda^x exp(-\lambda)}{x!}, \qquad \lambda \in \tilde{\lambda}[\alpha], \qquad 0 \le \alpha \le 1 \tag{48}$$

Furthermore, the fuzzy expression will depend on x where $\tilde{\lambda}[0]$ is observed. For a fixed x, let us calculate $m(\lambda) = \lambda e^{-\lambda}/x!$. When we observe the monotonicity of $m(\lambda)$, we remark that it is an increasing function for $\lambda < x$, the maximum value of $m(\lambda)$ is for $\lambda = x$ and finally, $m(\lambda)$ is a decreasing function for $\lambda > x$.

In order to summarize the way to analyze the Fuzzy Poisson distribution, let $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$ and $0 \le \alpha \le 1$. We note that;

1. For $x > \lambda_2(0)$, $\tilde{P}(x)[\alpha] = [m(\lambda_1), m(\lambda_2)]$
2. For $\lambda_1(0) > x$, $\tilde{P}(x)[\alpha] = [m(\lambda_2), m(\lambda_1)]$
3. For $x \in \tilde{\lambda}[0]$, $\beta, \alpha \in [0, 1]$

- $\tilde{P}(x)[\alpha] = [m(\lambda_1), m(x)]\ 0 \le \alpha \le \beta$
- $\tilde{P}(x)[\alpha] = [m(x), m(\lambda_2)]\ \beta \le \alpha \le 1$

Example 6 In order to illustrate an example of Fuzzy Poisson distributionlet x be the measurement of the defective percentage in a lot: $x = 0.1$ and $\tilde{\lambda} = (0.08, 0.12, 0.18)$. Determine the fuzzy probability $\tilde{P}(0.1)$.

Since $x = 0.1 \in [0.08, 0.18]$, let us start to evaluate the interval for $\tilde{\lambda}[0]$. $\tilde{\lambda}[\alpha] = [0.08 + 0.04\alpha, 0.18 - 0.06\alpha]$. In order to calculate $\tilde{P}(0.1)$, we must interpret the fuzzy intervals;

$$p_1(\alpha) = min\{m(\lambda)\}, \quad p_2(\alpha) = max\{m(\lambda)\} \quad \lambda \in \tilde{\lambda}[\alpha] \tag{49}$$

When the set of Eqs. (49) are examined, we may write;

$$\tilde{P}(0.1)[\alpha] = [m(0.08 + 0.04\alpha), m(0.1)] \quad 0 \le \alpha \le 0.5 \tag{50}$$

$$\tilde{P}(0.1)[\alpha] = [m(0.1), m(0.18 - 0.06\alpha)] \quad 0.5 \le \alpha \le 1 \tag{51}$$

Fuzzy Mean and Variance of Poisson Distribution

Furthermore, we need to calculate the fuzzy mean and the fuzzy variance of fuzzy Poisson probability distribution. Using the same principle in the fuzzy binomial distribution, we may write;

$$\tilde{\mu}[\alpha] = \left\{ \sum_{k=0}^{\infty} kh(\lambda) \right\} \tag{52}$$

This expression could be simplied into $\tilde{\mu} = \tilde{\lambda}$ which is similar to the crisp case. Let us calculate the variance with the similar way.

$$\tilde{\sigma}^2[\alpha] = \left\{ \sum_{k=0}^{\infty} (k - \mu)^2 h(\lambda) \right\} \tag{53}$$

which gives us the similar representation of crisp case; $\tilde{\sigma}^2 = \tilde{\lambda}$.

12 Conclusion

In an uncertain environment, like placing on the market of a new product, acceptance of defective lot sizes or interplanetary missions the occurrence of some events can not be anticipated through imprecise linguistic data. However, those cases necessitate formulation of probability distributions with fuzzy random variables. Fuzzy expectation in discrete case was provided in this chapter to find means for fuzzy distributions those are Fuzzy Hypergeometric, Fuzzy Binomial and Fuzzy Poisson distributions. And also fuzzy variances of those distributions were provided. We also dealt with fuzzy conditional probability. Independency with fuzzy random variables was offered before Fuzzy Bayes formula which are the basic of fuzzy probability theory. Additionally explanatory examples are employed for more paraphrasing. We tried to state the fuzzy probability theory by discrete form more clearly in a well-organized frame in this chapter. Fuzzy Hypergeometric distribution reinforced with an example was the additional contribution of this chapter to the literature among the books related with this topic.

For further research, general or interval type-2 fuzzy numbers can be integrated in the fuzzy discrete probability theory. Moreover, Fuzzy Hypergeometric, Fuzzy Binomial and Fuzzy Poisson distributions would be expressed within this framework. Finally, hidden Markov models might be extended by considering intuitionistic fuzzy probabilities.

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