

# Integrability in Action: Solitons, Instability and Rogue Waves

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**Abstract** Integrable nonlinear equations modeling wave phenomena play an important role in understanding and predicting experimental observations. Indeed, even if approximate, they can capture important nonlinear effects because they can be derived, as amplitude modulation equations, by multiscale perturbation methods from various kind of wave equations, not necessarily integrable, under the assumption of weak dispersion and nonlinearity. Thanks to the mathematical property of being integrable, a number of powerful computational techniques is available to analytically construct special interesting solutions, describing coherent structures such as solitons and rogue waves, or to investigate patterns as those due to shock waves or behaviors caused by instability. This chapter illustrates a selection of these techniques, using first the ubiquitous Nonlinear Schrödinger (NLS) equation as a prototype integrable model, and moving then to the Vector Nonlinear Schrödinger (VNLS) equation as a natural extension to wave coupling.

## 1 Introduction to Integrability and Solitons

Many nonlinear partial differential equations which model dispersive wave propagation possess *solitary wave* solutions. In most physical contexts these special solutions describe the motion in one, two or three-dimensional space of a *bump*, possibly modulating a carrier plane-wave, whose profile depends on the particular nonlinear terms which appear in the wave equation itself. Among the nonlinear wave equations which have been derived in many physical applications, we consider here those special ones which prove to be *integrable* and model wave motion in 1-dimensional space. The solitary wave solutions of integrable equations, because of their exceptional mathematical properties, have been termed *solitons* [1]. The first

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observation of a soliton dates back to 1834 (John Scott Russell's *wave of translation* [2]). Among the peculiar properties which are distinctive of integrability, we point out the following (see also Sect. 2):

- existence and explicit construction of infinitely many independent conservation laws;
- existence and explicit construction of  $N$ -soliton solutions, for any  $N$ ;
- existence of a nonlinear generalization of the Fourier transform, the *Spectral transform*, which provides a tool to investigate the solution of special initial-boundary value problems.

Because of these properties, integrable nonlinear wave equations may be understood as the limit to *infinitely many* degrees of freedom of classical Liouville-integrable dynamical systems. Some of these integrable wave equations are relevant as approximate models in various physical contexts. In these cases one may say that Nature and Mathematics go well hand in hand as the powerful methods of integrability allow for analytical description/prediction of wave behaviors.

The reader who is not familiar with the theory of integrable systems, and, in particular, with the theory of solitons, may find it of interest to have a preliminary look at the following rather long, and yet partial, list of model equations of physical interest which have been proven to be integrable ( $t$  is the evolution variable and  $x$  is the space coordinate, while partial differentiation is indicated by a subscript)

- Korteweg-de Vries (KdV) equation:

$$u_t - u_{xxx} = 6uu_x \quad (1)$$

- Benjamin-Ono (BO) equation:

$$u_t - Hu_{xx} = uu_x \quad (2)$$

where the Hilbert operator  $H$  is defined as  $Hf(x) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} dy \frac{f(y)}{y-x}$

- complex modified Korteweg de Vries (cmKdV) equation:

$$u_t - u_{xxx} = 6s|u|^2u_x, \quad s = \pm 1 \quad (3)$$

- Sine-Gordon (SG) equation:

$$u_{tt} - u_{xx} = -\sin u \quad (4)$$

- Nonlinear Schrödinger (NLS) equation:

$$iu_t + u_{xx} = 2s|u|^2u, \quad s = \pm 1 \quad (5)$$

(continued)

- Long Wave-Short Wave (LWSW) equation:

$$iu_t + u_{xx} + vu = 0, \quad v_t = (|u|^2)_x \quad (6)$$

- Derivative Nonlinear Schrödinger (DNLS) equation:

$$iu_t + u_{xx} = is(|u|^2 u)_x, \quad s = \pm 1 \quad (7)$$

- Massive Thirring Model (MTM):

$$\begin{aligned} iu_{1t} + iu_{1x} - u_2 &= |u_2|^2 u_1 \\ iu_{2t} - iu_{2x} - u_1 &= |u_1|^2 u_2 \end{aligned} \quad (8)$$

- Vector Nonlinear Schrödinger (VNLS) equation:

$$iu_{jt} + u_{jxx} = 2 \left( \sum_{n=1}^N s_n |u_n|^2 \right) u_j, \quad j = 1, \dots, N \quad (9)$$

where  $s_n = \pm 1$ . For  $N = 2$ ,  $s_1 = s_2 = -1$  this is the Manakov model.

- Three Wave Resonant Interaction (3WRI) equations:

$$\begin{aligned} u_{1t} + V_1 u_{1x} &= u_2^* u_3^* \\ u_{2t} + V_2 u_{2x} &= -u_1^* u_3^* \\ u_{3t} + V_3 u_{3x} &= u_1^* u_2^* \end{aligned} \quad (10)$$

where the asterisk denotes complex conjugation and where  $V_1$ ,  $V_2$  and  $V_3$  are real constants.

All these nonlinear integrable equations have the common property of being the condition that two *linear* first order homogeneous ordinary differential equations, one with respect to the variable  $x$  and the other with respect to the variable  $t$ , be compatible with each other. This pair of linear equations is commonly referred to as *Lax pair*. In order to clarify and detail this scheme, we give here few examples, which may serve as well as guidelines for further computational exercises. In general a linear homogeneous ordinary differential equation with respect to the variable  $y$  takes the form

$$\psi_y = M(y)\psi, \quad \psi = \psi(y), \quad (11)$$

where  $M$  is a  $N \times N$  matrix whose entries are functions of the independent variable  $y$ , and  $\psi(y)$  is a  $N$ -dimensional vector solution. Thus in this notation a Lax pair reads as

$$\psi_x = X\psi, \quad \psi_t = T\psi, \quad \psi = \psi(x, t), \quad (12)$$

where  $X$  and  $T$  are  $N \times N$  matrices and the vector  $\psi$  is required to solve both equations. It is plain that, for a generic choice of  $X(x, t)$  and  $T(x, t)$ , only the trivial solution  $\psi = 0$  solves the pair of Eqs. (12), whereas a non vanishing solution  $\psi$  exists if the compatibility condition  $\psi_{xt} = \psi_{tx}$  holds true, namely, as implied by (12), if the matrices  $X, T$  satisfy the equation

$$X_t + XT = T_x + TX, \quad \text{or} \quad X_t - T_x + [X, T] = O, \quad (13)$$

where  $[A, B] = AB - BA$  and  $O$  stands for the zero matrix. It is moreover crucial for the integrability that the matrices  $X, T$  depend also on a complex variable  $\lambda$ , the so-called *spectral variable*, say  $X = X(x, t, \lambda)$ ,  $T = T(x, t, \lambda)$ , with the additional strong requirement that the compatibility condition (13) holds for *any* value of  $\lambda$ . The following few explicit examples illustrate how indeed some of the integrable equations in the list above follow from compatibility conditions of the form (13). To this aim let  $X$  be the  $2 \times 2$  traceless matrix

$$X = i\lambda\sigma_3 + Q(x, t), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x, t) = \begin{pmatrix} 0 & v(x, t) \\ u(x, t) & 0 \end{pmatrix}. \quad (14)$$

This is the simplest choice since  $X(x, t, \lambda)$  is polynomial of first degree in the spectral variable  $\lambda$  while its dependence on  $x$  and  $t$  comes through the functions  $u(x, t)$  and  $v(x, t)$  which will eventually play the role of wave fields. As for the second equation of the Lax pair (12), the matrix  $T(x, t, \lambda)$  may be taken as a third degree polynomial

$$T = \lambda^3 T_3 + \lambda^2 T_2 + \lambda T_1 + T_0. \quad (15)$$

In this case both sides of the compatibility equation (13) are fourth degree polynomials in  $\lambda$  so that Eq. (13) yields five matrix algebraic/differential equations which can be easily solved (this step being left to the diligent reader). The solution of these equations can be conveniently given as the following expression of the four

matrix coefficients  $T_j$  [see (15)]:

$$\begin{aligned} T_3 &= -4ic_3\sigma_3 \\ T_2 &= -4c_3Q + 2ic_2\sigma_3 \\ T_1 &= 2ic_3\sigma_3(Q_x - Q^2) + 2c_2Q \\ T_0 &= c_3([Q, Q_x] + Q_{xx} - 2Q^3) - ic_2\sigma_3(Q_x - Q^2) \end{aligned} \quad (16)$$

while the additional fifth equation

$$Q_t = c_3(Q_{xxx} - 6Q^2Q_x) - ic_2\sigma_3(Q_{xx} - 2Q^3) \quad (17)$$

yields the dispersive nonlinear wave equation for the matrix  $Q(x, t)$ , namely for its two entries  $u(x, t)$ ,  $v(x, t)$ . The two parameters  $c_3$  and  $c_2$  are arbitrary. It is now left to the reader to verify that the KdV equation, the cmKdV equation and the NLS equation which appear in the list of integrable wave equations given above are just special cases of the evolution equation (17). Precisely, the choice  $c_3 = 1$ ,  $c_2 = 0$  and the reduction condition  $v = -1$  yields the KdV equation (1), while setting again  $c_3 = 1$ ,  $c_2 = 0$  but with the condition  $v = -su^*$ ,  $s = \pm 1$ , leads to the cmKdV equation (3), and finally the NLS equation (5) corresponds to the choice  $c_3 = 0$ ,  $c_2 = 1$  together with the reduction  $v = su^*$ ,  $s = \pm 1$ .

Different choices of the two matrices  $X$ ,  $T$  which appear in the Lax pair (12) generate, by the same scheme based on the compatibility equation (13), different integrable nonlinear (systems of) partial differential equations. For instance, if  $X$  and  $T$  are still  $2 \times 2$  but their dependence on the spectral variable  $\lambda$  is rational rather than polynomial, one may obtain the SG equation (4) and the MTM (8). If instead  $X$  and  $T$  are higher rank  $R > 2$ , the 3WRI system (10), the LWSW (6) equations as well as the Manakov model [(9) for  $N = 2$ ], are obtained for  $R = 3$ , while the rank  $R = N + 1$  is required to derive the system of  $N$  coupled NLS equations (9).

Once a nonlinear wave equation has been associated to a Lax pair according to the method we have sketched here, what is this association good for?

The answer to this question is contained in the huge collection of research results which accumulated during the last half-century. Thus we conclude this section with a list of works on the subject which is neither exhaustive nor complete, but which may guide the interested reader in the vast land of integrability and its applications. A starting point for a reader with no previous exposure to integrability are the books [3, 4] which present an overview of solitons in applications. These are complemented by more classical (and at times more detailed) textbooks such as [5–13]. An overview on the origin of soliton theory and a fairly complete set of references of its origins can be found in [14]. A more mathematical introduction to the theory of integrable systems is presented in [15]. This text complements collections such as [16, 17] and classical textbooks such as [18, 19]. In [20] the *universality* of integrable systems is well explained, whereas the link with multiple scale analysis is given in [21] and reference therein. An introduction to the theory of

nonlinear waves can be found in the monographs [22–25]. As the NLS equation plays a central role in our exposition, we draw the reader’s attention to a few monographs on this fundamental model [19, 26–28].

## 2 Integrability in Action: The NLS Equation as Study Case

The Nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} = 2s|u|^2u, \quad s = \pm 1, \quad (18)$$

has been first derived [29] in optics in the self-focusing case  $s = -1$ . However it arose again and again in different physical contexts, and it has been then recognized as a *universal* equation that models amplitude modulation of a quasi-monochromatic wave due to weak nonlinearities. Its universality stems from its derivation by multiscale perturbation theory [20, 21, 30, 31] from very large families of nonlinear dispersive wave equations (the nonlinear terms being treated as perturbation of the linear ones) which includes for instance Maxwell equations in Kerr and  $\chi_2$  media, Euler equations in ocean physics and Einstein gravitational field equations, among many others. In particular it can be derived by multiscale perturbation also from integrable equations, e.g. from the SG, KdV and cmKdV equations (see Sect. 1). Though its integrability has been discovered independently [32], from this very last fact one can predict that indeed the NLS equation should be integrable itself [20]. The aim of this section is to shortly illustrate a number of important consequences of the Lax pair associated to the NLS equation. In particular, we first show how to derive *infinitely many* local conservation laws. It is also shown here that the technique of transforming the Lax pair by a *Darboux transformation* leads to the algebraic construction, from a known solution, of a novel solution of the NLS equation.

### 2.1 Conservation Laws from the Lax Pair

Let us first consider how to obtain from the Lax pair an infinite sequence of local conservation laws,

$$\rho_t^{(n)} + f_x^{(n)} = 0, \quad n = 1, 2, \dots, \quad (19)$$

where the functions  $\rho^{(n)}(x, t)$  and  $f^{(n)}(x, t)$  are the conserved densities and, respectively, the corresponding currents. The method we follow here is applicable to solutions of the NLS equation (18) which vanish sufficiently fast as the variable  $|x|$  goes to infinity, namely  $u(x, t) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . The extension to different boundary conditions is possible with some extra technical efforts. The Lax pair

associated to the NLS equation is given by (12) where the matrices  $X$  and  $T$  are respectively given by (14), where  $v = su^*$ , and by (15) and (16) with  $c_3 = 0$ ,  $c_2 = 1$ . Since it is convenient to proceed by performing our computation in the algebra of matrices, we consider now the  $2 \times 2$  matrix  $\Psi(x, t, \lambda)$  whose column vectors are two linearly independent solutions of the Lax pair, namely

$$\Psi_x = X\Psi, \quad \Psi_t = T\Psi. \quad (20)$$

When  $x \rightarrow \pm\infty$  the matrix  $Q$  vanishes and the matrix solution  $\Psi$  goes to a solution  $\Psi^{(\pm)}$  of the Lax pair with  $Q = 0$ . Just for mere sake of convenience, we choose the solution  $\Psi$  which satisfies the boundary condition

$$\Psi \rightarrow \Psi^{(-)} = \exp[i\lambda\sigma_3(x + 2\lambda t)], \quad x \rightarrow -\infty. \quad (21)$$

It is also convenient to introduce the matrix  $\Phi(x, t, \lambda) = \Psi(\Psi^{(-)})^{-1}$  which satisfies the pair of equations

$$\Phi_x = i\lambda[\sigma_3, \Phi] + Q\Phi, \quad \Phi_t = 2i\lambda^2[\sigma_3, \Phi] + (2\lambda Q + i\sigma_3 Q^2 - i\sigma_3 Q_x)\Phi. \quad (22)$$

More conveniently to our purposes, we rewrite these equations in the following form

$$\begin{cases} (\sigma_3\Phi)_x = i\lambda(\Phi - \sigma_3\Phi\sigma_3) + \sigma_3 Q\Phi \\ (\sigma_3\Phi)_t = 2i\lambda^2(\Phi - \sigma_3\Phi\sigma_3) + (2\lambda\sigma_3 Q + iQ^2 - iQ_x)\Phi \end{cases} \quad (23)$$

which shows that the two functions

$$R(x, t, \lambda) = \text{tr}(\sigma_3 Q\Phi), \quad F(x, t, \lambda) = -\text{tr}[(2\lambda\sigma_3 Q + iQ^2 - iQ_x)\Phi], \quad (24)$$

where  $\text{tr}(M)$  stands for the trace of the matrix  $M$ , satisfy, by cross-differentiating the two Eqs. (23), the conservation equation

$$R_t + F_x = 0. \quad (25)$$

Note that this continuity equation is direct consequence of the Lax pair (20) and that it depends on the spectral variable  $\lambda$  through the matrix  $\Phi$  [see the definition (24)]. It now remains to extract from this last Eq. (25) conserved densities and currents whose expression contains only the solution  $u(x, t)$  of the NLS equation (18). This step is done via the following theorem:

**Theorem 1** *The matrix  $\Phi(x, t, \lambda)$  which solves the system (22), with the boundary value  $\Phi \rightarrow \mathbf{I}$  as  $x \rightarrow -\infty$ , has the following asymptotic expansion as  $|\lambda|$  becomes very large, say around the point at infinity of the  $\lambda$ -plane,*

$$\Phi = \mathbf{I} + \frac{1}{\lambda}\Phi_1 + \frac{1}{\lambda^2}\Phi_2 + \frac{1}{\lambda^3}\Phi_3 + \dots \quad (26)$$

where the matrix coefficients  $\Phi_n$  depend only on  $x$  and  $t$ .

The computation of these coefficients  $\Phi_n$  can be done recursively by inserting the expansion (26) into the first of the differential equations (22), and by splitting the matrix  $\Phi$  into its diagonal part  $\Phi^{(d)}$  and its off-diagonal part  $\Phi^{(o)}$ , namely  $\Phi = \Phi^{(d)} + \Phi^{(o)}$ . By taking into account the boundary condition  $\Phi \rightarrow \mathbf{1}$  as  $x \rightarrow -\infty$  [see (21)], the upshot of these computations is summarized by the following formulae (in self evident notation)

$$\begin{aligned} \Phi_0^{(d)} &= \mathbf{1}, \quad \Phi_0^{(o)} = 0, \quad \Phi_n^{(d)} = \int_{-\infty}^x dy Q(y, t) \Phi_n^{(o)}(y, t), \quad n \geq 1 \\ \Phi_{n+1}^{(o)} &= \frac{1}{2i} \sigma_3 (\Phi_{nx}^{(o)} - Q \Phi_n^{(d)}), \quad n \geq 0. \end{aligned} \quad (27)$$

Note that this recursion equations generate the expression of all coefficients  $\Phi_n^{(d)}$  and  $\Phi_n^{(o)}$  in a way that is well suitable to symbolic computation. Equation (25) clearly yields infinitely many local conservation laws via the expansions

$$R = \frac{1}{\lambda} R_1 + \frac{1}{\lambda^2} R_2 + \frac{1}{\lambda^3} R_3 + \dots, \quad F = \frac{1}{\lambda} F_1 + \frac{1}{\lambda^2} F_2 + \frac{1}{\lambda^3} F_3 + \dots, \quad (28)$$

namely

$$R_{nt} + F_{nx} = 0, \quad n \geq 1. \quad (29)$$

Here we give the first three conserved quantities and leave the computation of the currents to the interested reader. It is convenient to define the conserved densities as

$$\rho_n = i s 2^{n-1} R_n = i s 2^{n-1} \text{tr}(\sigma_3 Q \Phi_n^{(o)}), \quad n \geq 1, \quad (30)$$

and the time-independent quantities, i.e. the constants of the motion, as

$$C_n = \int_{-\infty}^{+\infty} dx \rho_n(x, t), \quad n \geq 1, \quad (31)$$

to arrive at the well known expressions

$$C_1 = \int_{-\infty}^{+\infty} dx |u|^2, \quad C_2 = \int_{-\infty}^{+\infty} dx \text{Im}(u u_x^*), \quad C_3 = H - \frac{1}{6} C_1^3, \quad (32)$$

where the functional

$$H = \int_{-\infty}^{+\infty} dx (|u_x|^2 + s |u|^4), \quad (33)$$



is the standard Hamiltonian which yields the NLS equation (18) via the Hamilton equation

$$u_t = -i \frac{\delta H}{\delta u^*} . \quad (34)$$

## 2.2 The Initial Value Problem and Particular Solutions

Before proceeding further we shortly comment on the way the Lax pair (20) gives the method of investigating the initial value problem  $u(x, 0) \rightarrow u(x, t)$  for the NLS equation (18). This method generalizes the well known Fourier analysis as applied to a linear equation with constant coefficients, for instance the linear Schrödinger equation  $iu_t + u_{xx} = 0$ . In the linear case, one introduces the Fourier Transform (*FT*) of the solution

$$\hat{u}(k, t) = \int_{-\infty}^{+\infty} dx e^{ikx} u(x, t) \quad (35)$$

together with its simple time dependence  $\hat{u}(k, t) = \hat{u}(k, 0)e^{-ik^2 t}$  and then one performs the chain of steps

$$u(x, 0) \rightarrow \hat{u}(k, 0) \rightarrow \hat{u}(k, t) \rightarrow u(x, t).$$

Note that in the last step it is essential that the Fourier map  $u(x, t) \rightarrow \hat{u}(k, t)$  (35) be invertible. Note also that we are dealing here with the class of solutions  $u(x, t)$  which vanish as  $|x| \rightarrow \infty$  sufficiently fast so that their Fourier integral is well defined. The initial value problem for the NLS equation (18) can be investigated in a similar way. In this case the Fourier transform has to be replaced by a new transform known as the *spectral transform* (or scattering transform) (*ST*). The map  $u(x, t) \rightarrow ST$  which takes the solution  $u(x, t)$  into its spectral transform *ST* is computed by considering the first equation of the Lax pair (20) as the eigenvalue problem [see (14)]

$$L\Psi = \lambda\Psi , \quad L \equiv -i\sigma_3\partial_x + i\sigma_3Q , \quad (36)$$

for the differential operator  $L$ . While we refer the reader to textbooks, e.g. [6, 12], for details, we limit ourself to pinpoint here differences and similarities between the *ST* and the *FT*. In the *FT* (35) the spectral variable  $k$  takes its values on the real line, which is also the continuum spectrum of the differential operator  $L$ . In contrast, the *ST* is defined not only on the continuum spectrum of  $L$  but also, if it exists, on its discrete spectrum which consists of a finite number of complex values of  $k$ . The solutions of the NLS equation which correspond to these discrete eigenvalues are the soliton solutions. Like the *FT*, also the *ST* has the similarly simple and explicit exponential dependence on the time variable  $t$ , namely the nonlinear mapping

$u(x, t) \rightarrow ST(t)$  transforms the nonlinear dynamics of the NLS equation into a trivial linear dynamics. It is moreover easy to show that the approximate expression of the  $ST$  obtained by linearizing the nonlinear transformation  $u(x, t) \rightarrow ST(t)$  reduces, on the continuum spectrum, to the  $FT$ . Similarly to the linear case, also for the NLS equation the solution of the initial value problem goes via the steps

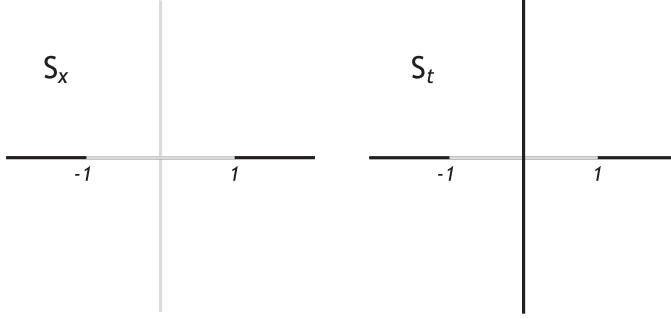
$$u(x, 0) \rightarrow ST(0) \rightarrow ST(t) \rightarrow u(x, t),$$

the last one requiring the (hard) task of reconstructing  $u$  from its  $ST$ . The problem of inverting the nonlinear map  $u(x) \rightarrow ST$  is very important in many applications, e.g. in medical imaging techniques and earth's crust geophysics, and it is a research field on its own known as *inverse problem*. The application of the mathematical methods of the inverse problem to the solution of nonlinear integrable wave equations is a well established technique known as *IST*, namely Inverse Spectral Transform, see e.g. [12], or Inverse Scattering Transform, see e.g. [6, 13].

The physical significance of the solutions corresponding to the continuum spectrum is of dispersive wave packets in contrast with the solutions corresponding to the discrete spectrum which are instead multi-soliton solutions. In this respect we notice that in the defocusing case  $s = 1$ , the operator  $L$  [see (36)] is formally Hermitian,  $L^\dagger = L$ , with the implication that all eigenvalues are real; in this case therefore no discrete spectrum is possible and the defocusing NLS equation has no soliton solutions. This is not the case for the focusing NLS with  $s = -1$  which possesses *bright* soliton solutions since  $L^\dagger \neq L$ . These conclusions drastically change if the solutions of the NLS equation do *not* vanish as  $|x|$  goes to infinity. In the class of solutions which are required to behave as plane waves (see e.g. (47)) with  $|u(x, t)| \rightarrow a = \text{constant}$  amplitude as  $|x| \rightarrow \infty$ , the  $ST$  of  $u$  has been as well defined and the method of solving the initial value problem has been extended to cover this class of solutions in both the defocusing case  $s = 1$  [19, 33–35] and in the focusing case  $s = -1$  [36]. In the defocusing case the NLS equation possesses soliton solutions which behave as plane waves at the boundary. These solutions are known as *dark* (grey or black) solitons. This occurs because the continuum spectrum is the real axis with the finite gap  $-a < \lambda < a$  within which real discrete eigenvalues may exist (see e.g. Fig. 1, where  $a = 1$ ). On the other hand, in the focusing case, in addition to the continuum spectrum, a complex discrete spectrum exists (see Sect. 3).

The spectral transform approach is able not only to deal with the initial value problem but also to explicitly construct those solutions which correspond to a purely discrete spectrum, namely the  $N$ -soliton solutions for any  $N$ . These special, yet important, solutions can be obtained also by a simpler direct algebraic technique which is known as the *Darboux transformation* (e.g. [37–40]), or *Dressing method* (e.g. [10, 41, 42]).

We conclude this section by shortly illustrating this method, and by providing the expression of the one soliton solution for solitons traveling in vacuum as well as over a continuous wave background.



**Fig. 1** Defocusing NLS:  $S_x = x$ -part spectrum/ $S_t = t$ -part spectrum of  $u^{(0)} = \exp(-2it)$

Before doing this we point out that, as a general rule which applies to many other wave models, the analytic expression of any explicitly known wave solution depends not only on the variables  $x, t$  but also on a number of parameters which may be related to physically relevant properties and features. In the following however, we omit to show in the expression of the soliton solutions those arbitrary parameters which can be inserted by taking into account the symmetries of the wave equation. In the present case the following five symmetries, or transformations  $u(x, t) \rightarrow u'(x, t)$  that leave the NLS equation unchanged, can serve this purpose:

1.  $x$ -translation  $u'(x, t) = u(x + \xi, t)$
2.  $t$ -translation  $u'(x, t) = u(x, t + \tau)$
3. phase factor  $u'(x, t) = e^{i\theta} u(x, t)$
4. rescaling  $u'(x, t) = pu(px, p^2t)$
5. Galilei change of frame  $u'(x, t) = e^{i\frac{v}{2}(x - \frac{v}{2}t)} u(x - vt, t)$

The initial step of the Darboux approach consists in linearly transforming the matrix solution of the Lax pair (20). Precisely, if  $\Psi^{(0)}$  is a solution of the Lax pair (20) with  $X, T$  replaced by  $X^{(0)}, T^{(0)}$ , and consequently  $Q$  replaced by  $Q^{(0)}$ , the Darboux transformation  $\Psi^{(0)} \rightarrow \Psi$ , reads

$$\Psi(x, t, \lambda) = D(x, t, \lambda) \Psi^{(0)}(x, t, \lambda), \quad (37)$$

where  $D$  is a  $2 \times 2$  matrix. We first observe that this linear transformation implies that also the new matrix  $\Psi$  satisfies a compatible Lax pair of equations, namely (20), where  $X = DX^{(0)}D^{-1} + D_x D^{-1}$  and  $T = DT^{(0)}D^{-1} + D_t D^{-1}$ . Thus the new matrices  $X, T$  satisfy themselves the compatibility equation (13) as a consequence of the compatibility equation  $X_t^{(0)} + X^{(0)}T^{(0)} = T_x^{(0)} + T^{(0)}X^{(0)}$  of the original Lax equations. Next one looks for a Darboux matrix  $D(x, t, \lambda)$  which satisfies the following (strong) conditions: (1) the new matrices  $X, T$  and the original ones  $X^{(0)}, T^{(0)}$  have the same structure (14) and (15) with (16); (2) the Darboux matrix has a polynomial dependence on the spectral variable  $\lambda$ . To our present purpose we

assume that this polynomial be first degree,

$$D(x, t, \lambda) = \lambda \mathbf{1} - M(x, t), \quad (38)$$

where the matrix  $M(x, t)$  has still to be found. To find it we use the symmetry condition

$$\Sigma D^\dagger(x, t, \lambda^*) \Sigma D(x, t, \lambda) = d_s(\lambda) \mathbf{1}, \quad (39)$$

where  $\Sigma = \mathbf{1}$  if  $s = -1$  and  $\Sigma = \sigma_3$  if  $s = 1$ , and  $d_s(\lambda)$  is a scalar  $x, t$ -independent second degree polynomial. This condition follows from the property  $Q^\dagger = sQ$  [or, equivalently,  $v = su^*$ , see (14)], and the pair of differential equations

$$D_x + DX^{(0)} = XD, \quad D_t + DT^{(0)} = TD, \quad (40)$$

which are implied by the Lax equations (20) together with the Darboux transformation (37). Once the matrix  $M$  has been computed, inserting the Darboux matrix expression (38) into the first of the two Eqs. (40) yields the expression  $Q = Q^{(0)} + i[\sigma_3, M]$ , or more explicitly and in self-evident notation,

$$u = u^{(0)} - 2iM_{21}, \quad (41)$$

of the new solution  $u(x, t)$  of the NLS equation. This expression can be given an alternative form of more practical use by involving the constant eigenvalues  $\alpha$  and  $\alpha^*$  of  $M(x, t)$ , together with their corresponding eigenvectors. While we skip detailing this computation, we limit our consideration here only to the focusing case  $s = -1$ . Thus, for the focusing NLS equation the Darboux matrix is

$$D(x, t, \lambda) = \lambda \mathbf{1} - \alpha^* \mathbf{1} - (\alpha - \alpha^*)P = \lambda \mathbf{1} - \alpha P - \alpha^*(\mathbf{1} - P), \quad (42)$$

where  $P(x, t)$  is the projection matrix

$$P(x, t) = \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} |z_1|^2 & z_1 z_2^* \\ z_1^* z_2 & |z_2|^2 \end{pmatrix}, \quad (43)$$

which projects on the eigenvector  $z(x, t)$  of the matrix  $M$  (corresponding to the eigenvalue  $\alpha$ ) with components  $z_1(x, t)$  and  $z_2(x, t)$ . It turns out that this eigenvector  $z$  is a vector solution of the original Lax pair with  $\lambda = \alpha$ , namely

$$z_x = X^{(0)}(x, t, \alpha)z, \quad z_t = T^{(0)}(x, t, \alpha)z. \quad (44)$$

The expression (41) takes now the more explicit and standard form

$$u = u^{(0)} - 2i(\alpha - \alpha^*) \frac{z_2 z_1^*}{|z_1|^2 + |z_2|^2}. \quad (45)$$

Note that here the complex number  $\alpha$  has to be strictly complex ( $\text{Im } \alpha \neq 0$ ) and that it is going to be an arbitrary parameter in the new solution  $u(x, t)$ . It should be also noticed that the applicability of the Darboux method obviously requires that the solution  $u^{(0)}$  of the NLS equation, as well as the solution  $\Psi^{(0)}$  of the corresponding Lax pair, be known. Before going into applications of the Darboux technique, let us summarize the computational scheme in the following steps: (1) fix the known solution  $u^{(0)}(x, t)$  and  $\Psi^{(0)}(x, t, \lambda)$ , (2) give an arbitrary complex value  $\alpha$  to the spectral variable, and fix an arbitrary constant vector  $\gamma$ , (3) compute the vector  $z(x, t) = \Psi^{(0)}(x, t, \alpha)\gamma$ , (4) apply the explicit formula (45). We also note that the constant vector  $\gamma$  in step (3) introduces an arbitrary complex parameter.

The simplest exercise now is the construction of the bright soliton solution. The starting known solution is the vacuum  $u^{(0)} = 0$  and the solution obtained via the Darboux method is

$$u = e^{it} \text{sech}(x) \quad (46)$$

for  $\alpha = i/2$  and  $\gamma = (1, 1)$ . Moreover, the corresponding operator  $L$  (36) possesses two discrete eigenvalues,  $\lambda_1 = \alpha = i/2$ ,  $\lambda_2 = \alpha^* = -i/2$  that are the roots of the polynomial  $\det D = (\lambda - \alpha)(\lambda - \alpha^*)$ , see (42). Consider now the Darboux construction of soliton solutions obtained when  $u^{(0)}$  is the continuous wave solution of the focusing NLS equation:

$$u^{(0)}(x, t) = e^{2it}. \quad (47)$$

In this case the general formula (45) leads to the expression

$$u = e^{2it} \left[ 1 + \cosh(\eta) \frac{2 \cosh(px) - e^{(\eta+iqt)} - e^{-(\eta+iqt)}}{\cos(qt) - \cosh(\eta) \cosh(px)} \right] \quad (48)$$

where  $p = 2 \sinh(\eta)$ ,  $q = 2 \sinh(2\eta)$ . This is a one-parameter family of solutions, the parameter  $\eta$  taking both real values  $0 \leq \eta < +\infty$  and imaginary values  $\eta = i\mu$ ,  $0 < \mu < +\infty$ . As in the previous case the discrete eigenvalues of the Lax equation (36) are on the imaginary axis of the  $\lambda$ -plane,  $\lambda_1 = i \cosh(\eta)$ ,  $\lambda_2 = -i \cosh(\eta)$ , and are again the roots of  $\det D$ . Further observations on this solution will be reported in the next section. We finally note that the Darboux technique can be applied in a similar way (with some extra care [43–45]) to the defocusing case. As already mentioned above, in the defocusing regime no bounded solutions are found if  $u^{(0)} = 0$ , while if  $u^{(0)} = \exp(-2it)$  the zeros of  $\det D$  are required to be real. The Darboux method can be extended to polynomial Darboux matrices  $D(x, t, \lambda)$  of higher degree in  $\lambda$  so as to construct solutions of the NLS

equation which describe the interaction of  $N$  solitons (e.g. [46, 47]). Moreover, this method applies as well to other Lax pairs and therefore to other integrable equations, f.i. to the VNLS equation (see Sect. 4) (e.g. [37, 40, 43–45, 48]).

### 3 NLS Equation: Linear Instability and Rogue Waves

Investigating the linear stability of a given solution  $u^{(0)}(x, t)$  of a nonlinear partial differential equation goes via the following standard computational steps: (1) linearizing the given nonlinear equation in the neighborhood of the given solution, (2) finding a complete set of solutions of this linear equation which are everywhere bounded in the space variables, (3) checking the boundedness of all these solutions over the entire time evolution. It is sufficient that some of this complete set of solutions of the linearized equation grow in time with no bound to declare that given solution  $u^{(0)}(x, t)$  of the nonlinear equation is linearly unstable. In particular, if the linearized equation is, or may be mapped into, an equation with constant coefficients, then the complete set of its solutions is the set of Fourier (continuous wave) exponentials of the type  $\exp[i(kx - \omega t)]$ . In such simple case the solution of the nonlinear equation is unstable if there exist real values of  $k$  such that the corresponding frequency  $\omega(k)$  has a non vanishing and positive imaginary part.

Here we show how the linear stability analysis can be alternatively handled if the nonlinear partial differential equation is integrable. Consistently with the previous section, we consider the integrability properties of the NLS equation (18), in both the defocusing ( $s = 1$ ) and focusing ( $s = -1$ ) regimes. Again the basic tool is the Lax pair (20) with (14). Let  $u^{(0)}(x, t)$  be the given solution of the NLS equation (18) whose stability we aim to investigate. Then the linearized NLS equation around this solution reads

$$i(\delta u)_t + (\delta u)_{xx} - 2su^{(0)2}(\delta u)^* - 4s|u^{(0)}|^2(\delta u) = 0, \quad (49)$$

where the function  $\delta u(x, t)$  is the small deviation from the given solution  $u^{(0)}$ , namely  $u = u^{(0)} + \delta u$ . Assume now that not only the solution  $u^{(0)}(x, t)$  is known, but that it is also known the explicit expression of an invertible matrix solution  $\Psi^{(0)}(x, t, \lambda)$ ,

$$\Psi^{(0)} = \begin{pmatrix} \psi_{11}^{(0)} & \psi_{12}^{(0)} \\ \psi_{21}^{(0)} & \psi_{22}^{(0)} \end{pmatrix}, \quad (50)$$

of the Lax pair (20) (with  $u$  replaced by  $u^{(0)}$ ). Then the following result provides the link between the linearized equation (49) and the Lax pair.

**Theorem 2** *For any value of the variable  $\lambda$ , the function  $(\psi_{22}^{(0)}(x, t, \lambda))^2$  satisfies the linearized equation (49).*

Note that, for the vanishing solution  $u^{(0)} = 0$ ,  $(\psi_{22}^{(0)})^2 = \exp[-2i\lambda(x + 2\lambda t)]$  coincides with the Fourier mode  $\exp[i(kx - \omega t)]$ , with  $\omega(k) = k^2$  and  $k = -2\lambda$ , of the linear Schrödinger equation [see (49)]. If  $u^{(0)} \neq 0$ , in analogy with the previous case, we learn that the solutions  $(\psi_{22}^{(0)}(x, t, \lambda))^2$  of the linearized equation (49) play the role of generalized Fourier modes. In this generic case the spectral variable  $\lambda$  runs over the entire spectrum, both continuum and discrete (if any), of the  $x$ -part differential equation of the Lax pair [see (36)]. If the solution  $u^{(0)}$  is such that its corresponding functions  $(\psi_{22}^{(0)}(x, t, \lambda))^2$ , for some value of  $\lambda$  in the spectrum, grows with time with no bound, then this solution  $u^{(0)}$  is linearly unstable. In general the computations required by this procedure may not be explicitly doable. However for  $u^{(0)} = 0$  and  $u^{(0)} = \exp(-2ist)$ , i.e. for the vacuum and the continuous wave solution, the method can be carried out in explicit form. The linear stability of the vanishing solution is easily established for  $s = \pm 1$ . As for the continuous wave solution, in both the defocusing  $s = 1$  and focusing  $s = -1$  regimes, the discrete spectrum is empty and so it remains to compute the continuum spectrum. For the purpose of establishing the stability, it is convenient to compute the spectrum associated to both equations of the Lax pair (20). The  $x$ -part spectrum  $S_x$  is defined as the set of values of the spectral variable  $\lambda$  such that the corresponding solution  $\Psi^{(0)}$  of the Lax equation  $\Psi_x^{(0)} = X^{(0)}\Psi^{(0)}$  is bounded on the entire  $x$ -axis at any fixed time, and the  $t$ -part spectrum  $S_t$  is defined via the Lax equation  $\Psi_t^{(0)} = T^{(0)}\Psi^{(0)}$  in just the similar way. More explicitly, the Lax pair of equations for the matrix solution (50) reads

$$\begin{cases} \Psi_x^{(0)} = \begin{pmatrix} i\lambda & su^{(0)*} \\ u^{(0)} & -i\lambda \end{pmatrix} \Psi^{(0)} \\ \Psi_t^{(0)} = \begin{pmatrix} 2i\lambda^2 + is|u^{(0)}|^2 & 2\lambda su^{(0)*} - isu_x^{(0)*} \\ 2\lambda u^{(0)} + iu_x^{(0)} & -2i\lambda^2 - is|u^{(0)}|^2 \end{pmatrix} \Psi^{(0)}. \end{cases} \quad (51)$$

The solution of these equations with  $u^{(0)} = \exp(-2ist)$  can be conveniently written as

$$\Psi^{(0)}(x, t, \lambda) = \begin{pmatrix} e^{ist} & 0 \\ 0 & e^{-ist} \end{pmatrix} e^{i(xW - tF)}, \quad (52)$$

where the two matrices

$$W = \begin{pmatrix} \lambda & -is \\ -i & -\lambda \end{pmatrix}, \quad F = \begin{pmatrix} -2\lambda^2 & 2is\lambda \\ 2i\lambda & 2\lambda^2 \end{pmatrix} = -2\lambda W \quad (53)$$

depend only on  $\lambda$ . Next we compute the eigenvalues  $\pm w$  and  $\pm f$  of the matrices  $W$  and  $F$ , respectively. If  $g_{\pm}$  are their corresponding common eigenvectors, it follows

that  $Wg_{\pm} = \pm wg_{\pm}$ ,  $Fg_{\pm} = \pm fg_{\pm}$  with

$$w = \sqrt{\lambda^2 - s}, \quad f = -2\lambda\sqrt{\lambda^2 - s}. \quad (54)$$

The implication is that the Fourier modes take the vector expression

$$\Psi^{(0)}(x, t, \lambda)g_{\pm} = e^{\pm i(wx - ft)} \begin{pmatrix} e^{ist} & 0 \\ 0 & e^{-ist} \end{pmatrix} g_{\pm}, \quad (55)$$

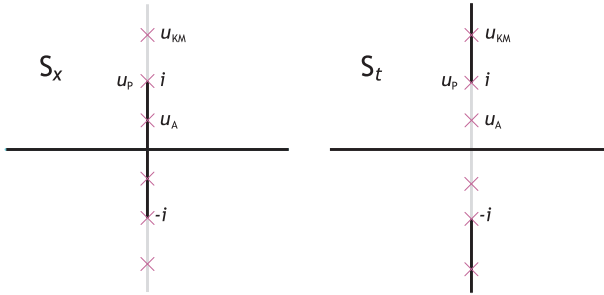
which clearly show the spectra  $S_x$  and  $S_t$ . Indeed, in the defocusing case  $s = 1$ , the “wave number”  $w$  is real if and only if  $\lambda$  is real but off the forbidden gap  $-1 < \lambda < 1$ .

So the spectrum  $S_x = \{-\infty < \lambda \leq -1\} \oplus \{1 \leq \lambda < +\infty\}$ . On the  $t$ -side,  $f$  is real if and only if  $\lambda \in S_t$  where  $S_t = S_x \oplus \{\lambda = i\nu, -\infty < \nu < +\infty\}$ , see Fig. 1. Since whenever  $w$  is real also  $f$  is real, we conclude that in this case the continuum wave solution  $u^{(0)} = \exp(-2it)$  is linearly stable.

Considering now the focusing case  $s = -1$ ; by reasoning in a similar way one derives from the expression (55), with (54), the  $x$ -spectrum,  $S_x = \{-\infty < \lambda < +\infty\} \oplus \{\lambda = i\nu, -1 \leq \nu \leq +1\}$  while the  $t$ -spectrum is  $S_t = \{-\infty < \lambda < +\infty\} \oplus \{\lambda = i\nu, -\infty < \nu \leq -1\} \oplus \{\lambda = i\nu, +1 \leq \nu < +\infty\}$ , see Fig. 2.

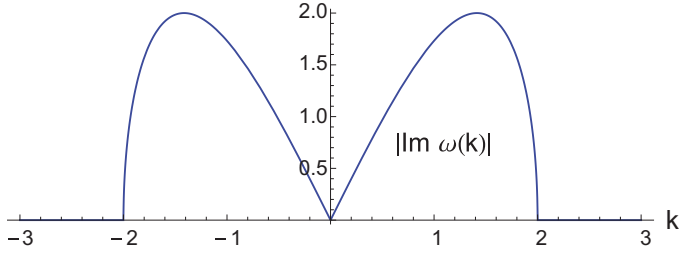
In this case the imaginary values of  $\lambda$  in the interval  $-1 < \text{Im}\lambda < +1$  belong to  $S_x$  but not to  $S_t$ , and therefore the solution  $u^{(0)} = \exp(2it)$  is linearly unstable. This result is known for water waves as Benjamin-Feir instability [49], and as modulational instability in optics [50] (see also [51]). It is common and convenient to characterize these wave phenomena by plotting the imaginary part of the wave frequency versus the wave number. To this purpose we consider again the Fourier-like mode (see above)  $(\psi_{22}^{(0)})^2 = e^{2it} \exp[i(kx - \omega t)]$  with

$$k = -2w = -2\sqrt{\lambda^2 + 1}, \quad \omega(k) = -k\sqrt{k^2 - 4}. \quad (56)$$



**Fig. 2** Focusing NLS:  $S_x = x$ -part spectrum/ $S_t = t$ -part spectrum of  $u^{(0)} = \exp(2it)$ ; the crosses indicate examples of solutions in the one parameter family (48)





**Fig. 3** NLS instability of  $u^{(0)} = \exp(2it)$ : imaginary part of frequency  $\omega(k)$  versus wave number  $k$ , see (56)

The resulting instability plot is shown in Fig. 3.

The linear stability analysis can tell that a solution is unstable by showing an exponential (or, in marginal cases, polynomial, see below) growth of that solution as time goes by. However, it is not able to tell the long time evolution. Depending on the perturbing deviation  $\delta u(x, t)$ , it may well happen that the growing perturbation eventually develop into a finite, likely soliton, solution of the NLS equation. Candidates to playing such role are indeed solutions with a simple spectral characterization such as those corresponding to discrete eigenvalues. One example of this type of solutions of the focusing NLS equation can be computed by means of one of the tool provided by its integrability, namely by the Darboux transformation. This construction has been done in the previous section, the outcome being the family of solutions (48). This is a one-parameter family, the parameter  $\eta$  being real if the corresponding pair of discrete eigenvalues,  $\lambda_1 = i \cosh(\eta)$ ,  $\lambda_2 = \lambda_1^* = -i \cosh(\eta)$  lie off the spectrum  $S_x$ , while  $\eta$  has to be imaginary, i.e.  $\eta = i\theta$ ,  $0 \leq \theta \leq \pi/2$ , if on the contrary the discrete eigenvalues  $\lambda_1 = i \cos(\theta)$ ,  $\lambda_2 = \lambda_1^* = -i \cos(\theta)$ , are required to be in the spectrum  $S_x$ , see Fig. 2. The corresponding solutions of the NLS equation,  $u_{KM}(x, t)$  if  $\eta$  is real, and  $u_A(x, t)$  if instead  $\eta$  is imaginary, have been separately found in [52, 53] and, respectively, in [54]. Here the subscript  $_{KM}$  indicates the Kuznetsov-Ma solution  $u_{KM}$  [52, 53] while the subscript  $_A$  indicates the Akhmediev solution  $u_A$  [54]. The  $x$  and  $t$  dependence of these two different types of solutions may be understood by looking at the position of their corresponding eigenvalues in the  $\lambda$ -plane. Indeed,  $u_{KM}(x, t)$  is localized in  $x$  and periodic in  $t$  since its corresponding eigenvalues are in  $S_t$  and off  $S_x$  while the opposite occurs for  $u_A(x, t)$  which is instead periodic in  $x$ . In order to show that indeed this last solution  $u_A(x, t)$  describes the fate of a small perturbation of the unstable continuum wave  $u^{(0)} = \exp(2it)$ , we first give this solution the more convenient expression

$$u = e^{2i(t-\theta)} [1 + i \sin(\theta) \frac{e^{t(\theta+iqt)} - e^{-i(\theta+iqt)}}{\cosh(qt) - \cos(\theta) \cos(px)}], \quad p = 2 \sin(\theta), \quad q = 2 \sin(2\theta) \quad (57)$$

and then we note that its asymptotic behavior

$$u(x, t) = e^{2it} [1 + \delta u(x, t) + O(e^{2qt})], \quad (58)$$

as  $t \rightarrow -\infty$ , is that of the background continuum wave perturbed by the small exponential tail

$$\delta u(x, t) = i \sin(2\theta) e^{-i\theta} e^{2it} \cos(px) e^{qt}, \quad (59)$$

which satisfies the linearized equation (49).

Let us finally consider the marginal case of the solution family (48) which corresponds to the parameter value  $\eta = 0$ , or equivalently, to the border value  $\lambda = \pm i$  of the corresponding eigenvalue, see Fig. 2. By performing this limit, the final expression has no more exponential functions in it since it features only a rational dependence on the variables  $x, t$ , which reads

$$u_P(x, t) = e^{2it} \left[ 1 - \frac{4(1 + 4it)}{1 + 4x^2 + 16t^2} \right]. \quad (60)$$

In this solution the subscript  $P$  stands for Peregrine and this is indeed the well known Peregrine soliton [55]; it gained relevance as model of water rogue waves [56–61], and later in other physical contexts [62–66]. Its peculiarity is that of appearing and disappearing over an unstable background while its amplitude reaches a maximum value which is three times that of the surrounding periodic wave. The suggestion that rogue waves, as they appear in nature, may be described by rational solutions has given a strong impulse in this direction, particularly to the mathematical side of this subject. Various extensions [45, 67–78] of the Peregrine soliton to other integrable wave models have soon been available and investigated, and still are to a considerable extent. Some of these developments are discussed in the next section.

## 4 Wave Coupling: Integrability and Rogue Waves

The dynamics of waves frequently requires that more than just one field propagates. For instance dealing with polarized light beams in a nonlinear (Kerr) medium naturally leads to consider both self interaction, as in the NLS equation, and cross interaction among the two different polarization fields. As a different mechanism causing similar wave-wave coupling, one may consider two different quasi-monochromatic waves with wave-numbers  $k_1, k_2$  propagating in the same medium with cubic nonlinearity. Then a multiscale analysis shows that, if the weak resonance condition  $v_g(k_1) = v_g(k_2)$  is satisfied,  $v_g(k)$  being the group velocity, the two waves interact with each other. In both these examples the resulting system of equations reads

$$\begin{cases} iu_{1t} + \gamma_1 u_{1xx} + (g_1 |u_1|^2 + g_{12} |u_2|^2) u_1 = 0 \\ iu_{2t} + \gamma_2 u_{2xx} + (g_2 |u_2|^2 + g_{21} |u_1|^2) u_2 = 0 \end{cases} \quad (61)$$

where the constant coefficients  $\gamma$ 's and  $g$ 's depend on the particular physical process. Integrable methods apply also to this system of equations provided the coefficients satisfy the conditions  $\gamma_1 = \gamma_2$ ,  $g_1 = g_{21}$ ,  $g_2 = g_{12}$ . Indeed, if these relations do not hold true, the system (61) is not integrable, and no Lax pair is associated to it. Thus there exist only three different integrable cases which, by appropriate cosmetic rescaling, see Sect. 2, take the form

$$\begin{cases} iu_{1t} + u_{1xx} - 2(s_1|u_1|^2 + s_2|u_2|^2)u_1 = 0 \\ iu_{2t} + u_{2xx} - 2(s_1|u_1|^2 + s_2|u_2|^2)u_2 = 0 \end{cases}, \quad s_1^2 = s_2^2 = 1. \quad (62)$$

Depending on the two signs  $s_1, s_2$ , we have the focusing case (Manakov model [79]),  $s_1 = s_2 = -1$ , the defocusing case  $s_1 = s_2 = 1$  and the mixed case  $s_1 = -s_2 = 1$  which models self-defocusing for the wave amplitude  $u_1$  and self-focusing for  $u_2$ . All of these cases have applications in fluid dynamics [80–82], optics [83–87] and Bose-Einstein condensates [88]. The system (62) clearly generalizes the NLS equation (18) and it is known as Vector Nonlinear Schrödinger (VNLS) equation since the two-component vector  $(u_1, u_2)$  can be easily generalized to a  $N$ -component vector for any  $N$ , see (9). As expected, it shares with the NLS equation several properties but it also differs under various aspects. Its Lax pair formally looks like (20) but now the matrices  $X$  and  $T$  are  $3 \times 3$  with

$$X = i\lambda \Sigma + Q(x, t), \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & s_1 u_1^* & s_2 u_2^* \\ u_1 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix}, \quad (63)$$

and

$$T = 2i\lambda^2 \Sigma + 2\lambda Q + i\Sigma(Q^2 - Q_x). \quad (64)$$

Starting from these expressions, and similarly to what has been done for the NLS equation, one can find an infinite number of local conservation laws of the form (29) [89], and can construct soliton solutions by means of the Darboux transformation (e.g. [43, 44]). Also the inverse spectral method can be extended to solve the initial value problem when the boundary values, as  $x \rightarrow \pm\infty$ , vanish, say  $u_1(x, t)$  and  $u_2(x, t) \rightarrow 0$  (see e.g. [79]), and also when  $u_1(x, t)$  and  $u_2(x, t)$  go, in the same limit of  $x$ , to a continuous wave solution (see e.g. [90]).

Here our discussion focuses on the construction of bounded rational solutions of (62) whose interest is well motivated by their application as rogue wave models. In analogy with the NLS equation, and according to a general common understanding of this phenomenon, the existence of rogue waves requires that they are superimposed to an unstable continuous background. However we do not address here the problem of determining the stability of the background solution, as done in the previous section for the NLS equation. We rather limit ourselves to point out that, quite differently from the NLS equation, instability occurs not only

in the focusing regime  $s_1 = s_2 = -1$ , but also in the defocusing and mixed cases (e.g. [45, 91, 92]). In the previous section we obtained the rational (alias Peregrine) solution (60) by taking the limit  $\eta \rightarrow 0$  of the  $\eta$ -dependent family (48). The analytic construction of rational solutions of multicomponent wave equations, as the VNLS system (62), by taking such limit is no longer convenient, if at all doable. In the following we show a direct way to solve this problem with no need to take this limit operation. We begin by observing that the solution (48) of the NLS equation is made out of exponential functions which come in, through the Darboux formula (45), from the exponential solution (52) of the Lax pair (with  $s = -1$ ). Thus, to our purpose, the main task is to change the exponential matrix function  $\exp[i(xW - tF)]$ , see (52), into a polynomial function. This is not possible if the matrix  $W(\lambda)$  (and therefore  $F(\lambda)$ ) is diagonalizable. It is instead possible if, for a special value of the spectral variable  $\lambda$ , the matrices  $W, F$  are not diagonalizable. If such value of  $\lambda$  exists, it is called *critical* and denoted  $\lambda_c$ . The expressions (53) and (54) clearly show that only if the eigenvalues coincide with each other, say if  $w(\lambda) = \sqrt{\lambda^2 + 1} = 0$ , the matrices  $W, F$  are not diagonalizable. Indeed, in this case there are two critical values  $\lambda_c = \pm i$ , and in fact, f. i. for  $\lambda_c = i$ , the matrix  $W$  takes the value

$$W(i) = i \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad (65)$$

which is nilpotent, namely  $W(i)^2 = 0$ . This property similarly holds for  $F(i)$  and for  $W(-i)$  and  $F(-i)$ . The implication is that the exponential  $\exp[i(xW(\lambda_c) - tF(\lambda_c))]$  is in fact the polynomial  $\mathbf{1} + i[xW(\lambda_c) - tF(\lambda_c)]$ , as implied by the Taylor expansion of the exponential function. It is now a simple exercise to obtain again the Peregrine solution (60) by this method.

The extension of this technique to the VNLS equation first requires the computation of the critical values  $\lambda_c$ . This can be done in a systematic way so as to find *all* such critical values which eventually lead to the construction of bounded rational solutions. The starting point is the expression

$$\begin{pmatrix} u_1^{(0)}(x, t) \\ u_2^{(0)}(x, t) \end{pmatrix} = \begin{pmatrix} a_1 e^{i(qx - vt)} \\ a_2 e^{-i(qx + vt)} \end{pmatrix}, \quad v = q^2 + 2(s_1 a_1^2 + s_2 a_2^2) \quad (66)$$

of the continuous wave solution of the VNLS equation (62) and of the corresponding matrix solution

$$\Psi^{(0)}(x, t, \lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(qx - vt)} & 0 \\ 0 & 0 & e^{-i(qx + vt)} \end{pmatrix} e^{i(xW(\lambda) - tF(\lambda))} \quad (67)$$

of the Lax pair (20). The  $3 \times 3$  constant matrices  $W, F$  depend only on  $\lambda$  and on the background solution parameters  $a_1, a_2, q$ . In this respect we note that the parameter  $q$  has no counterpart in the single wave NLS equation (18). Indeed it

represents the wave number mismatch of the two background components  $u_1^{(0)}$  and  $u_2^{(0)}$  as  $2q$  equals the difference of these wave numbers. Its novel important feature is to have a crucial effect on the stability of the solution (66). Again the Darboux transformation method, as applied to the seed solution (66), is the convenient tool to obtain the explicit expression of new solutions of the VNLS equation (see Sect. 2). As we are interested here in obtaining rational solution, our main concern is to find the critical values of  $\lambda$ , namely those values for which the matrix exponential  $e^{i(xW(\lambda)-tF(\lambda))}$  yields a polynomial dependence on  $x, t$ . In analogy with what we have shown above for the NLS equation, the main task is finding the eigenvalues of the matrices  $W(\lambda)$  and  $F(\lambda)$  together with their  $\lambda$ -dependence. Equivalently, and by dealing for instance with  $W$ , which is

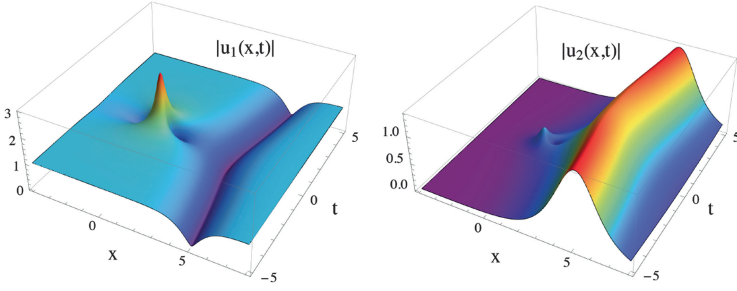
$$W(\lambda) = \begin{pmatrix} \lambda & -is_1a_1 & -is_2a_2 \\ -ia_1 & -\lambda - q & 0 \\ -ia_2 & 0 & -\lambda + q \end{pmatrix}, \quad (68)$$

one has to investigate the  $\lambda$ -dependence of the three roots  $w_1(\lambda)$ ,  $w_2(\lambda)$ ,  $w_3(\lambda)$  of the characteristic polynomial  $P(w) = \det[w\mathbf{1} - W(\lambda)]$  in the entire  $\lambda$ -plane. This task requires numerical computations in order to find in addition the dependence of the critical values  $\lambda_c$  on the continuous wave parameters  $q, a_1, a_2$ . We refer the reader to [45] for the way of classifying all critical values  $\lambda_c$  in the parameter space and we limit ourselves to make few comments and to show few plots. Because of the complicate Cardano expression for the roots of a third degree polynomial, only the solutions with  $q = 0$  can be found in simple explicit form. This particular case ( $q = 0$ ) yields the expression of the vector analog of the NLS Peregrine solution, and, as expected, this solution exists only in the focusing case  $s_1 = s_2 = -1$ . In fact this solution corresponds to two critical values,  $\lambda_c = \pm i\sqrt{a_1^2 + a_2^2}$ . However its expression, which reads

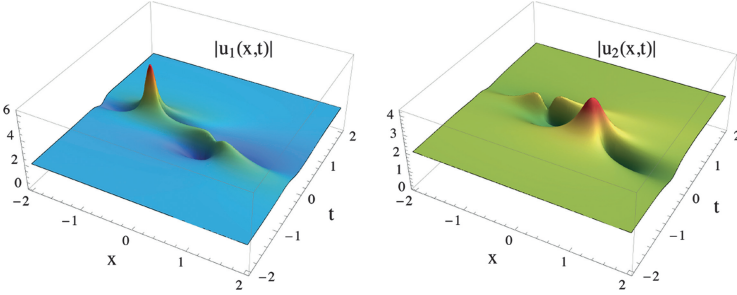
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{2ip^2t} \left[ \frac{(P_2 + |h|^2e^{2px})}{(M_2 + |h|^2e^{2px})} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{hP_1e^{(px+ip^2t)}}{(M_2 + |h|^2e^{2px})} \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} \right] \quad (69)$$

has the novel feature of showing a mixture of exponential and polynomial dependence on  $x, t$  since  $P_2, M_2$  are polynomials of degree 2, while  $P_1$  is a polynomial of first degree. Here  $p = \sqrt{a_1^2 + a_2^2}$  while  $h$  is an additional arbitrary complex parameter. Thus, if say  $a_2 = 0$ , this solution describes a dark soliton in the first component and a bright one in the second component which at the time of their interaction generate a Peregrine-type bump, see Fig. 4. Only if  $h = 0$  this solution features a Peregrine rogue wave in both components.

If the mismatch parameter  $q$  is different from zero, rogue wave type solutions exist in all regimes but not for all values of the parameters  $a_1, a_2, q$ . To the purpose of classifying all these solutions, it is convenient to separately consider the multiplicity of the three eigenvalues of the matrix  $W(\lambda)$  (68). It is plain that no critical values  $\lambda_c$



**Fig. 4**  $s_1 = s_2 = -1$ ,  $q = 0$ ,  $a_1 = 1$ ,  $a_2 = 0$ ,  $\lambda_c = i$ ,  $f = 0.1$



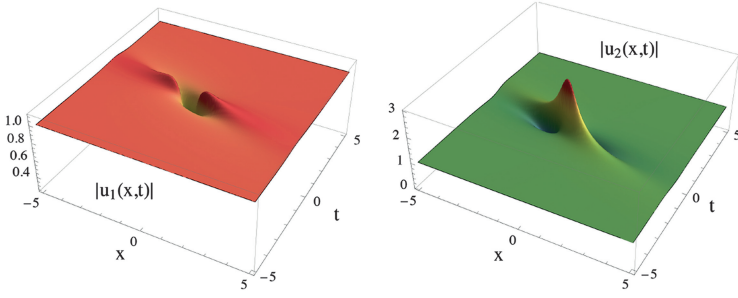
**Fig. 5**  $s_1 = s_2 = -1$ ,  $q = 1$ ,  $a_1 = a_2 = 2$ ,  $\lambda_c = i\frac{\sqrt{27}}{2}$

exist if all the eigenvalues are simple since in this case  $W$  and  $F$  are diagonalizable and the exponential  $\exp[i(xW(\lambda) - tF(\lambda))]$  cannot be a polynomial. In the case in which there is just one eigenvalue with multiplicity 3, only two critical values of  $\lambda$ ,  $\lambda_c = \pm i\frac{\sqrt{27}}{2}$ , exist and only in the focusing case  $s_1 = s_2 = -1$  with the restriction to the subset  $a_1 = a_2 = 2q$ . Figure 5 shows such a rogue wave.

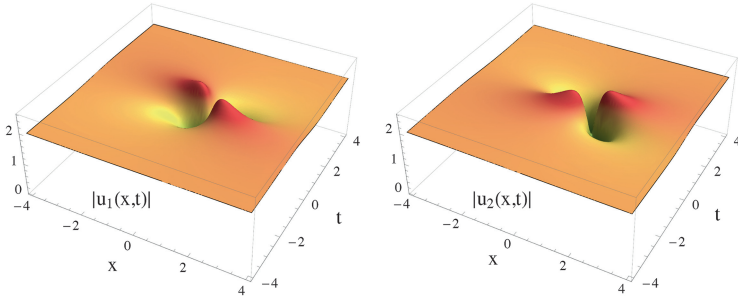
If instead the matrix  $W(\lambda)$  has an eigenvalue with multiplicity 2 there may exist several critical values  $\lambda_c$ . In this respect it is convenient to consider first the parameter subset  $s_1 = s_2$ ,  $a_1 = a_2$ ,  $q \neq 0$  because in this particular case the critical value  $\lambda_c$  can be explicitly computed. It turns out that in the focusing case  $s_1 = s_2 = -1$  four critical values exist for any value of  $q$  and  $a_1 = a_2$ , for one such solution see Fig. 6.

In the defocusing case  $s_1 = s_2 = 1$  threshold phenomena appear as no critical value  $\lambda_c$  (alias no rogue wave) exists if  $q^2 \geq 2a_1^2$  while (only) two critical values exist if  $q^2 < 2a_1^2$ , see Fig. 7.

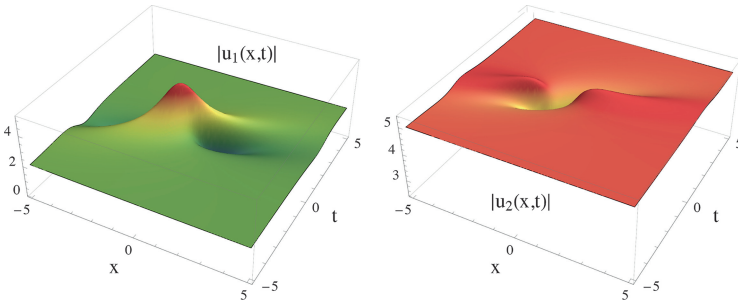
In order to explore the generic case  $a_1 \neq a_2$  and  $q \neq 0$  one may conveniently proceed by numerically computing the critical value  $\lambda_c$ , see the examples of rogue wave solutions shown in Figs. 8, 9, and 10, in, respectively, focusing, defocusing and mixed cases. Still it is explicitly found that the existence of rogue waves in the defocusing regime is conditioned by the inequality  $(a_1^2 + a_2^2)^3 - 12(a_1^4 - 7a_1^2a_2^2 +$



**Fig. 6**  $s_1 = s_2 = -1$ ,  $q = 1$ ,  $a_1 = a_2 = 1$ ,  $\lambda_c = \sqrt{\frac{3}{8}}\sqrt{-3 + i\sqrt{3}}$



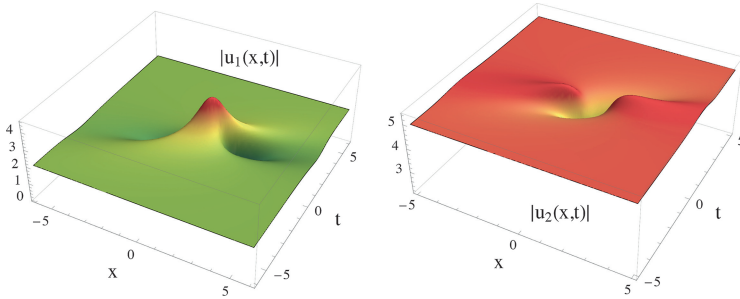
**Fig. 7**  $s_1 = s_2 = 1$ ,  $q = 1$ ,  $a_1 = a_2 = 2$ ,  $\lambda_c = \frac{i}{2}\sqrt{-13 + 16\sqrt{2}}$



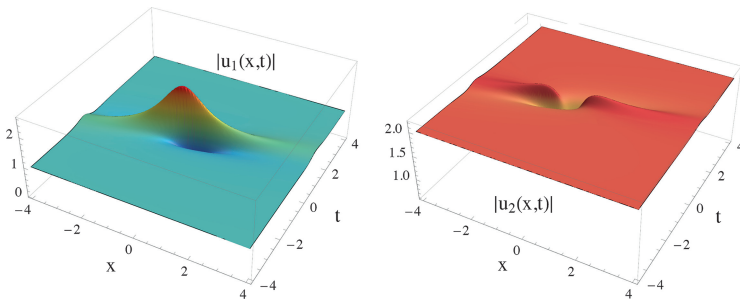
**Fig. 8**  $s_1 = s_2 = -1$ ,  $q = 1$ ,  $a_1 = 2$ ,  $a_2 = 5$ ,  $\lambda_c = 4.876 + 5.343i$

$a_2^4 q^2 + 48(a_1^2 + a_2^2)q^4 - 64q^6 > 0$  for the amplitudes  $a_1, a_2$  and the mismatch parameter  $q$ .

A detailed discussion of the existence of rogue waves as related to base-band modulational instability of the continuous wave background in the defocusing regime is reported in [93, 94], (see also [92]).



**Fig. 9**  $s_1 = s_2 = 1$ ,  $q = 1$ ,  $a_1 = 2$ ,  $a_2 = 5$ ,  $\lambda_c = -5.600 + 4.655i$



**Fig. 10**  $s_1 = -1$ ,  $s_2 = 1$ ,  $q = 1$ ,  $a_1 = 1$ ,  $a_2 = 2$ ,  $\lambda_c = -1.242 + 0.636i$

## 5 Integrability in Action: Beyond the NLS Model

Integrable nonlinear equations modeling wave phenomena, even if approximate, as they generally are, yet play an important role in understanding and predicting experimental observations. Thanks to the mathematical property of being integrable, a number of powerful computational techniques are available to investigate patterns as those due to shock waves, and even to analytically construct special interesting solutions such as multi-soliton and multi-rogue waves. To the purpose of illustrating how some of these methods work, in the previous sections we have considered the ubiquitous NLS equation as prototype integrable model, together with its extension to a system of two coupled NLS equations. All problems raised have been solved by starting from the Lax pair. Indeed, because of their dependence on the spectral variable, these two equations contain all the valuable information. Since the Lax pair plays such an essential role, it should be pointed out that finding which integrable partial differential equation is associated to a given Lax pair (as its compatibility condition) is rather easy. However the other way around is far from being a simple task, as no general method exists to prove, or disprove, the existence of a Lax pair associated with a *given* nonlinear partial differential equation. Attempts in this direction make use of a weaker definition of integrability, which can be tested by multi-scale expansion [21], or by recursively constructing symmetries [95, 96] of



the given nonlinear wave equation. Attempts to solve this problem by classifying Lax pairs also exist (see [97] and references therein).

Searching for new integrable wave equations naturally leads to change the matrices  $X(\lambda)$  and  $T(\lambda)$  of the Lax pair (20). This has been done in many ways, according to various purposes, during the last 45 years, and it is still common practice. In this respect we believe that the following examples may well give a good perspective of applications of integrability not only in the general context of nonlinear science but also in the more specific one of modeling wave phenomena. Let us consider first the two main features of the Lax pair of Eqs. (20), namely (1) the  $\lambda$ -dependence of the matrices  $X(\lambda)$  and  $T(\lambda)$ , and (2) their dimension. In fact searching for the Lax pair associated with the KdV and the cmKdV equations, (1) and (3), requires that the matrix  $T(\lambda)$  be a third degree polynomial of  $\lambda$  (e.g. [6]), while the matrix dimension is  $2 \times 2$  as for the NLS equation. Keeping this same matrix dimension but asking that the  $\lambda$ -dependence be rational rather than polynomial is required to obtain the SG equation (4) [98–100] and the MTM equation (8) [101, 102]. Increasing the matrix dimension is standard strategy to extend one scalar wave equation to a system of equations to model wave-wave interactions. This is the case for instance of the VNLS (62) which requires matrices of dimension  $3 \times 3$ . This same dimension is required to arrive at the system (10) which models the resonant interaction of three waves [103] (see also [104] and references therein), the  $\lambda$ -dependence of both  $X(\lambda)$  and  $T(\lambda)$  being polynomial of first degree. This system can be generalized to the so-called  $N$ -wave interaction equations (e.g. [105, 106] and references therein) which again model resonant interaction of  $N$  wave fields with quadratic nonlinearity. In this case the matrix dimension has to be  $n \times n$  with  $N = n(n - 1)/2$ . Also the Lax pair matrices associated with the LWSW equation (6), which models the resonant interaction of long waves with short waves [107, 108], are  $3 \times 3$  [107, 109, 110]. A further, and more substantial, way of changing the Lax pair is asking that this pair of equations be partial, rather than ordinary, differential equations by introducing more space variables. Examples of integrable equations in 2-space and 1-time dimensions are the Kadomtsev-Petviashvili [111] and Davey-Stewartson [112] equations which find their application in fluid dynamics. Other nonlinear wave equations can be added to those we have mentioned here which are integrable and also valuable in some applicative context.

As part of our discussion has been devoted to those solutions which model rogue waves, we conclude with the following collection of integrable equations which share the property of having rogue wave solutions. We observe that in fact not all integrable wave equations possess such type of solutions. In addition to the focusing NLS equation (5) with  $s = -1$  [55, 113] and VNLS equation (62) [45, 68, 93], whose rogue wave solutions have been discussed above, rogue wave solutions have been identified also for the cmKdV equation (3) [114], the DNLS equation (7) [115], the MTM (8) [116], the LWSW (6) [73, 117] and the 3WRI equation (10) [45, 71, 78]. Also the following integrable equations, among others, have been recently

reported to possess rogue wave solutions

- Hirota-Maxwell-Bloch (H-MB) equation [118]:

$$\begin{aligned} iu_t + a[u_{xx} + 2|u|^2u] + ib[u_{xxx} + 6|u|^2u_x] &= 2p \\ p_x &= 2i\omega p + 2\eta u \\ \eta_x &= -(up^* + u^*p) \end{aligned} \quad (70)$$

where  $a, b$  are arbitrary real constants.

- Sasa-Satsuma (SS) equation [119–121]:

$$iu_t + u_{xx} + 2|u|^2u + i\alpha[u_{xxx} + 3(|u|^2)_xu + 6|u|^2u_x] = 0 \quad (71)$$

the real constant coefficient  $\alpha$  being arbitrary.

- Kadomtsev-Petviashvili I (KP-I) equation [122, 123]:

$$(u_t - u_{xxx} - 6uu_x)_x + u_{yy} = 0 \quad (72)$$

- Davey-Stewartson II (DS-II) equation [124]:

$$iu_t + u_{xx} - u_{yy} + 2s|u|^2u = 2\phi u, \quad \phi_{xx} + \phi_{yy} = 2s(|u|^2)_{xx}, \quad s = \pm 1 \quad (73)$$

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