

Chapter 2

An Overview on Steffensen-Type Methods

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Abstract In this chapter we present an extensive overview of Steffensen-type methods. We first present the real study of the methods and then we present the complex dynamics related this type of methods applied to different polynomials. We also provide an extension to Banach space settings and an application to a Boundary Value Problem. We finish this chapter with contributions to this matter made by other authors.

2.1 Introduction

One of the most studied problems in Numerical Analysis is the approximation of nonlinear equations. A powerful tool is the use of iterative methods. It is well-known that Newton's method,

$$x_0 \in \Omega, \quad x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n \in \mathbb{N},$$

is one of the most used iterative methods to approximate the solution x^* of $F(x) = 0$. The quadratic convergence and the low operational cost of Newton's method ensure that it has a good computational efficiency.

If we are interesting in methods without using derivatives, then Steffensen-type methods will be a good alternative. These methods only compute divided differences

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and can be used for nondifferentiable problems. Moreover, they have the same order of convergence than the Newton-type methods. For instance, if the evaluation of $F'(x)$ at each step of Newton's method is approximated by a divided difference of first order $[x, x + F(x); F]$, we will obtain the known method of Steffensen,

$$x_0 \in \Omega, \quad x_n = x_{n-1} - [x_{n-1}, x_{n-1} + F(x_{n-1}); F]^{-1} F(x_{n-1}), \quad n \in \mathbb{N},$$

which has quadratic convergence and the same computational efficiency as Newton's method. Recall that a bounded linear operator $[x, y; F]$ from X into X is called divided difference of first order for the operator F on the points x and y if $[x, y; F](x - y) = F(x) - F(y)$. Moreover, if F is Fréchet differentiable, then $F'(x) = [x, x; F]$.

The organization of the paper is as follows. We start in Sect. 2.2 with the study of scalar equations. We present in Sect. 2.2.1.1 some convergence analysis and some dynamical aspects of the methods. Some numerical experiments and the dynamics associated to the previous analysis is presented in Sect. 2.2.1.2. In Sect. 2.4, we study the extension of these schemes to a Banach space setting and give some semilocal convergence analysis. Finally, some numerical experiments, including differentiable and nondifferentiable operators, are presented in Sect. 2.5. Finally, other contributions are reported in Sect. 2.6.

2.2 The Real Case

Steffensen's method is a root-finding method [39], similar to Newton's method, named after Johan Frederik Steffensen. It is well known that Steffensen's method also achieves quadratic convergence for smooth equations, but without using derivatives as Newton's method does. In this section, we recall the convergence analysis for semismooth equations that is less popular.

2.2.1 Semismooth Equations

In [9, 31] the definition of semismooth functions is extended to nonlinear operators. We say that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is semismooth at x if F is locally Lipschitz at x and the following limit

$$\lim_{V \in \partial F(x + th') h' \rightarrow h, t \downarrow 0} Vh'$$

exists for any $h \in \mathbb{R}^n$, where ∂F is the generalized Jacobian defined,

$$\partial F(x) = \text{conv} \partial_B F(x)$$

Most nonsmooth equations involve semismooth operators at practice [32]. We say that F is strongly semismooth at x if F is semismooth at x and for any $V \in \partial F(x+h)$, $h \rightarrow 0$, $Vh - F'(x; h) = O(\|h\|^2)$.

For $n = 1$, we denote by $\delta F(x, y)$ the divided differences of the form:

$$\delta F(x, y) = \frac{F(x) - F(y)}{x - y}.$$

For the convergence analysis we will need the following result.

Lemma 1 Suppose that F is semismooth at x^* and denote the lateral derivatives of F at x^* by

$$d^- = -F'(x^* -), \quad d^+ = F'(x^* +)$$

then

$$\begin{aligned} d^- - \delta F(u, v) &= o(1) \quad u \uparrow x^*, v \uparrow x^*, \\ d^+ - \delta F(u, v) &= o(1) \quad u \downarrow x^*, v \downarrow x^*. \end{aligned}$$

Moreover if F is strongly semismooth at x^* , then

$$\begin{aligned} d^- - \delta F(u, v) &= O(|u - x^*| + |v - x^*|) \quad u, v < x^*, \\ d^- - \delta F(u, v) &= O(|u - x^*| + |v - x^*|) \quad u, v > x^*. \end{aligned}$$

2.2.1.1 A Modification of Steffensen's Method and Convergence Analysis

The classical Steffensen's method can be written as

$$x_{n+1} = x_n - \delta F(x_n, x_n + F(x_n))^{-1} F(x_n).$$

Our iterative procedure would be considered as a new approach based in a better approximation to the derivative $F'(x_n)$ from x_n and $x_n + F(x_n)$ in each iteration. It takes the following form

$$x_{n+1} = x_n - \delta F(x_n, \tilde{x}_n)^{-1} F(x_n) \quad (2.1)$$

where $\tilde{x}_n = x_n + \alpha_n |F(x_n)|$.

These parameters $\alpha_n \in \mathbb{R}$ will be a control of the good approximation to the derivative. Theoretically, if $\alpha_n \rightarrow 0$, then

$$\delta F(x_n, \tilde{x}_n) \rightarrow F'(x_n).$$

In order to control the stability in practice, but having a good resolution at every iteration, the parameters α_n can be computed such that

$$tol_c \ll |\alpha_n|F(x_n)|F(x_n)| \leq tol_u,$$

where tol_c is related with the computer precision and tol_u is a user's free parameter.

As the classical Steffensen's method the modification (2.1) needs two evaluations of the function in each iteration and it is quadratically convergent in the smooth case. In the next theorem, we prove that the iterative method (2.1) is quadratically convergent for strongly semismooth equations as well.

Theorem 1 *Suppose that F is semismooth at a solution x^* of $F(x) = 0$. If d^- and d^+ are nonzero, then the algorithm (2.1) is well defined in a neighborhood of x^* and converges to x^* Q -superlinearly. Furthermore, if F is strongly semismooth at x^* , the converge to x^* is Q -quadratic.*

Proof We may choose x_0 sufficiently close to x^* (and/or α_n sufficiently small) such that we have either $x_0, \tilde{x}_0 > x^*$ or $x_0, \tilde{x}_0 < x^*$. According to Lemma 1 is well defined for $k = 0$. It is easy to check that

$$|\tilde{x}_n - x_n| = O(|F(x_n)|^2) = O(|x_n - x^*|^2).$$

Then from Lemma 1,

$$\delta F(x_n, \tilde{x}_n) = \delta F(x_n, x^*) + o(1) = d^+ + o(1) \text{ (or } d^- + o(1)).$$

Thus,

$$\begin{aligned} |x_{n+1} - x^*| &= |x_n - x^* - \delta F(x_n, \tilde{x}_n)^{-1} F(x_n)| \\ &\leq |\delta F(x_n, \tilde{x}_n)^{-1}| |F(x_n) - F(x^*) - \delta F(x_n, \tilde{x}_n)(x_n - x^*)| \\ &\leq |\delta F(x_n, \tilde{x}_n)^{-1}| |\delta F(x_n, x^*) - \delta F(x_n, \tilde{x}_n)| |x_n - x^*| \\ &= o(|x_n - x^*|). \end{aligned}$$

And we obtain superlinear convergence of $\{x_n\}$. If F is strongly semismooth at x^* , we may prove similarly the Q -quadratic convergence of $\{x_n\}$. \square

At practice, this modified Steffensen's method will present some advantages. Firstly, since in general $\delta F(x_n, \tilde{x}_n)$ is a better approximation to the derivative $F'(x_n)$ than $\delta F(x_n, x_n + \epsilon|F(x_n)|F(x_n))$ the convergence will be faster (the first iterations will be better). Secondly, the size of the neighborhood can be higher, that is, we can consider worse starting points x_0 (taking α_0 sufficiently small), as we will see at the numerical experiments. Finally, if we consider ϵ sufficiently small in order to obtain similar results at the first iterations and solving the above mentioned disadvantages, then some numerical stability problems will appear at the next iterations.

See [32] and its references for more details on this topic.

2.2.1.2 Numerical Experiments and Conclusions

In order to show the performance of the modified Steffensen's method, we have compared it with the classical Steffensen's method and the modified secant's type method proposed in [34]. We consider $tol_u = 10^{-8} \gg tol_c$. We have tested on several semismooth equations. Now, we present one.

We consider

$$F(x) = \begin{cases} k(x^4 + x) & \text{if } x < 0 \\ -k(x^3 + x) & \text{if } x \geq 0 \end{cases} \quad (2.2)$$

where k is a real constant.

For $x_0 = 0.1$ and $k = 1$, all the iterative method are Q-quadratically convergent, see Table 2.1. Nevertheless, for ϵ small the method proposed in [34] has problems with the last iterations. If we consider a stop criterium in order to avoid this problems then we would not be arrived to the convergence. However, our scheme converges without stability pathologies.

If we consider now $x_0 = 1$ and $k = 10$, the classical Steffensen's method and the modified secant method with $\epsilon = 1$ have problems of convergence, in fact they need 258 and 87,174 iterations to converge respectively, see Table 2.2.

The other schemes obtain similar results as before, see Table 2.3.

Finally, in Tables 2.4 and 2.5 we take different initials guesses and different values of k . In these tables, we do not write the results for Steffensen's and for $\epsilon = 1$ because in all cases the method do not converge after 10^6 iterations. On the other hand, if ϵ is not small enough the convergence is slow, but if it is too small stability

Table 2.1 Error, Eq. (2.2) $k = 1, x_0 = 0.1$

Iter.	Steff.	$\epsilon = 1$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-8}$	$tol_u = 10^{-8}$
1	$1.38e-03$	$3.62e-04$	$2.99e-04$	$2.99e-04$	$2.99e-04$
2	$5.09e-11$	$5.19e-14$	$1.72e-13$	$5.21e-09$	$2.26e-14$
3	$0.00e+00$	$0.00e+00$	<i>NaN</i>	<i>NaN</i>	$0.00e+00$

Table 2.2 Iterations and error, Eq. (2.2) $k = 10, x_0 = 1$

Steff.	$\epsilon = 1$
256 $1.00e-02$	87172 $3.22e-02$
257 $1.41e-06$	87173 $3.42e-05$
258 $0.00e+00$	87174 $0.00e+00$

Table 2.3 Error, Eq. (2.2) $k = 10, x_0 = 1$

Iter.	$\epsilon = 10^{-4}$	$\epsilon = 10^{-8}$	$tol_u = 10^{-8}$
5	$1.26e-02$	$1.13e-02$	$1.13e-02$
6	$7.49e-07$	$4.79e-07$	$4.92e-07$
7	$9.08e-14$	$7.84e-09$	$0.00e+00$
8	$8.32e-15$	<i>NaN</i>	
9	<i>NaN</i>		

Table 2.4 Final iteration and error, Eq. (2.2), $x_0 = 1$

k	$\epsilon = 10^{-4}$	$\epsilon = 10^{-8}$	$tol_u = 10^{-8}$
10^3	$50870\ 3.88e - 13$	$14\ 3.68e - 15$	$12\ 0.00e + 00$
10^4	$> 10^6$	$17\ 0.00e + 00$	$15\ 0.00e + 00$
10^6	$> 10^6$	$> 10^6$	$20\ 0.00e + 00$
10^8	$> 10^6$	$> 10^6$	$26\ 0.00e + 00$
10^{16}	$> 10^6$	$> 10^6$	$47\ 0.00e + 00$

Table 2.5 Final iteration and error, Eq. (2.2), $k = 1$

x_0	$\epsilon = 10^{-4}$	$\epsilon = 10^{-8}$	$tol_u = 10^{-8}$
4	$13\ 4.20e - 13$	$9\ 6.14e - 09$	$9\ 0.00e + 00$
8	$457095\ 1.27e - 13$	$11\ 2.42e - 09$	$12\ 0.00e + 00$
16	$> 10^6$	$20\ 3.46e - 09$	$14\ 0.00e + 00$
32	$> 10^6$	$> 10^6$	$16\ 0.00e + 00$

problems appear, as we said before. Our iterative method gives goods results in all the cases.

2.3 Dynamics

In the last years many authors has been studied the dynamics of iterative methods [7, 8, 13, 14, 27]. This classical methods require the computation of the inverse of derivatives which is well known that it can involves a very high computational cost, so other authors have worked in developing tools in order to study nondifferentiable methods [28] and studying the dynamics them [10, 15, 26].

We begin the study with the modification of the following classical iterative methods:

1. Newton

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2. Two-steps

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}.$$

3. Chebyshev

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n)\right) \frac{f(x_n)}{f'(x_n)}.$$

4. *Halley*

$$x_{n+1} = x_n - \left(\frac{1}{1 - \frac{1}{2}L_f(x_n)} \right) \frac{f(x_n)}{f'(x_n)},$$

where

$$L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}.$$

We denote by $[\cdot, \cdot; f]$ and $[\cdot, \cdot, \cdot; f]$ the first and the second divided difference of the function f .

Our modify Steffensen-type methods associated to the above schemes write:

1. *Modify Steffensen*

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, x_n + \alpha_n f(x_n); f]}.$$

2. *Modify Steffensen-Two-steps*

$$y_n = x_n - \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]}.$$

3. *Modify Steffensen-Chebyshev*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n) \right) \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]}.$$

4. *Modify Steffensen-Halley*

$$x_{n+1} = x_n - \left(\frac{1}{1 - \frac{1}{2}L_f(x_n)} \right) \frac{f(x_n)}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]},$$

where

$$L_f(x) = \frac{f(x)[x_n - \alpha_n f(x_n), x_n, x_n + \alpha_n f(x_n); f]}{[x_n - \alpha_n f(x_n), x_n + \alpha_n f(x_n); f]^2}.$$

These methods depend, in each iteration, of some parameters α_n . These parameters are a control of the good approximation to the derivatives. In order to control

the accuracy and stability in practice, the α_n can be computed such that

$$tol_c < \frac{tol_u}{2} \leq \|\alpha_n f(x_n)\| \leq tol_u,$$

where tol_c is related with the computer precision and tol_u is a free parameter for the user.

The classical Steffensen-type methods use $\alpha_n = 1$.

In this section we compare the dynamics of the above methods to introduce the benefits of using the parameters α_n . In the experiments we have taken $tol_u = 10^{-6}$.

We approximate the roots of polynomials. We use different colored painting regions of convergence of each root and dark violet is used for no convergence.

We include only the examples for $p(z) = z^3 - 1$ but similar conclusions are obtained for other examples.

The clear conclusion is that the good approximation of the derivatives (for instance using the parameters α_n) is crucial to remain the characteristic of the basins of attraction. The classical Steffensen-type methods ($\alpha_n = 1$) have smaller basins of attraction and great regions of no convergence (Figs. 2.1, 2.2, 2.3 and 2.4).

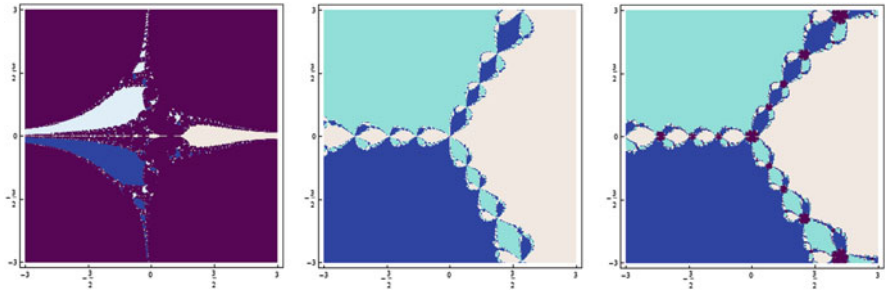


Fig. 2.1 Basins of attraction for $p(z) = z^3 - 1$. *Left*: Steffensen's method, *Middle*: Newton's method, *Right*: modified Steffensen's method

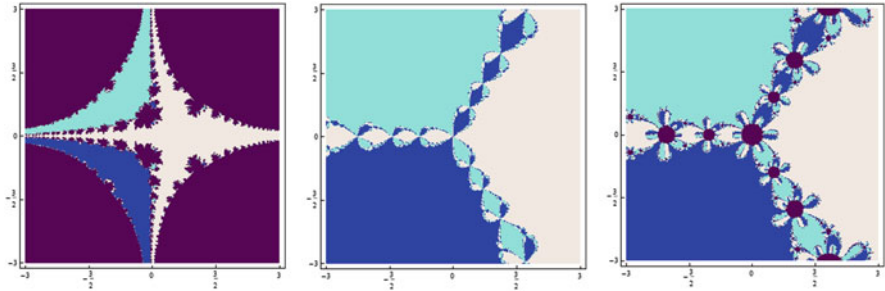


Fig. 2.2 Basins of attraction for $p(z) = z^3 - 1$. *Left*: two-steps Steffensen's method, *Middle*: two-steps Newton's method, *Right*: modified two-step Steffensen's method

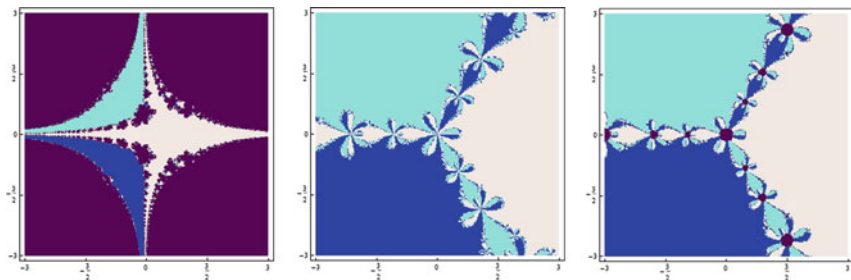


Fig. 2.3 Basins of attraction for $p(z) = z^3 - 1$. *Left*: Chebyshev-Steffensen's method, *Middle*: Chebyshev's method, *Right*: modified Chebyshev-Steffensen's method

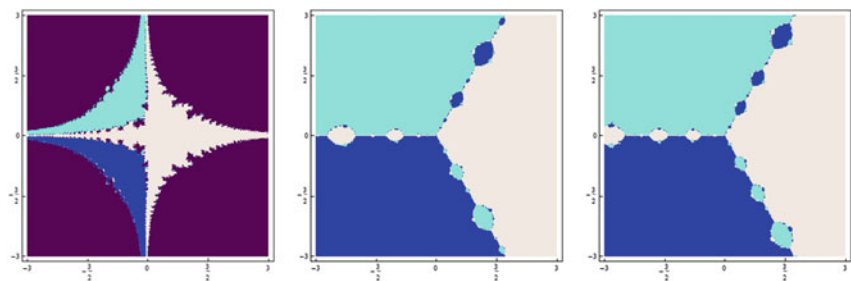


Fig. 2.4 Basins of attraction for $p(z) = z^3 - 1$. *Left*: Halley-Steffensen's method, *Middle*: Halley's method, *Right*: modified Halley-Steffensen's method

2.4 Extension to Banach Space Setting

We only consider the case of second order methods, but similar results can be found for higher order methods.

2.4.1 Convergence Analysis

We consider both type of equations: $F(x) = x$ and the usual $F(x) = 0$.

First of all, we must recall the expression of the method for fixed point type equations:

$$x_{n+1} = x_n + (I - [F(x_n), x_n; F])^{-1}(F(x_n) - x_n). \quad (2.3)$$

Theorem 2 *Let B be an open convex set of a Banach space X . Let $F : B \subset X \rightarrow X$ be a nonlinear operator, with divided difference in $B \subset X$. Let x_0 be such that*

$$\|F(x_0) - x_0\| \leq a_0 \quad (2.4)$$

$$\|(I - [\alpha_0(F(x_0) - x_0) + x_0, x_0; F])^{-1}\| \leq b_0 \quad (2.5)$$

$$\|[x', x''; F] - [y', y''; F]\| \leq k \cdot (\|x' - y'\| + \|x'' - y''\|) \quad (2.6)$$

para todo x', x'', y', y'' en $S_0 = \{x : \|x - x_0\| \leq \max(a_0, 2a_0b_0)\}$. Si $S_0 \subset B$, $\alpha_n < 2\alpha_{n-1}$ ($\alpha_n \in (0, 1], \forall n$) y $h = 2ka_0b_0(\alpha_0 + b_0) \leq \frac{1}{2}$ then, the sequence $\{x_n\}$ given by (2.3) is well defined and converges to a fixed point of $F(x)$. Moreover, x^* belong to the ball

$$\|x - x_0\| \leq \frac{a_0b_0(1 - \sqrt{1 - 2h_0})}{h_0}, \quad (2.7)$$

and the convergence radius is give by

$$\|x_n - x^*\| \leq \frac{a_0b_0(2h_0)^{2^n}}{2^n h_0}. \quad (2.8)$$

Finally, if condition (2.6) is held in $\|x - x_0\| \leq a_0 + \frac{a_0b_0(1 + \sqrt{1 - 2h_0})}{h_0} = a_0 + M_0$ the fixed point x^* in unique in the ball $\|x - x_0\| < M_0$.

The basic hypothesis given in the previous theorem is that the divided difference of F was Lipchitz in any ball in a neighbourhood of the initial iteration, in particular the Fréchet derivative of F exists. In some recent works [20–22] (for secant methods), Hernández and Rubio relax these hypotheses and they only suppose that the divided difference satisfy that

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|), \quad x, y, v, w \in B$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing continuous function in both components.

In the next theorem, we will extend that theory to our method

$$x_{n+1} = x_n - ([x_n, x_n + \alpha_n F(x_n); F])^{-1} F(x_n) \quad (2.9)$$

in order to solve the equation $F(x) = 0$.

Theorem 3 *Let X be a Banach space. Let B be an open convex subset of X and let suppose that there exists a divided difference of first order of G such that*

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|), \quad x, y, v, w \in B$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing continuous function in both components. Let α_n be such that $\|\alpha_n G(x_n)\| \leq \text{tol}_u$.

Let $x_0 \in B$ and let suppose that

- 1) $\|\Gamma_0^{-1} := [x_0, x_0 + \alpha_0 G(x_0); F]^{-1}\| \leq \beta$.
- 2) $\|\Gamma_0^{-1} G(x_0)\| \leq \eta$.
- 3) Let $m = \beta\omega(\eta, \text{tol}_u)$. Let us suppose that

$$t(1 - \frac{m}{1 - \beta\omega(t, t + 2\text{tol}_u)}) - \eta = 0 \quad (2.10)$$

has a minimum positive root which we call R .

If $\beta\omega(R, R + 2\text{tol}_u) < 1$, $M := \frac{m}{1 - \beta\omega(R, R + 2\text{tol}_u)} < 1$ y $\overline{B(x_0, R)} \subset B$ then, the sequence given by (2.9) is well defined, belongs to $\overline{B(x_0, R)}$ and converges to the unique solution of $F(x) = 0$ in $\overline{B(x_0, R)}$.

2.5 Application to Boundary Value Problems

We consider the following boundary problem

$$y''(t) = f(t, y(t), y'(t)), \quad y(a) = \alpha, \quad y(b) = \beta, \quad (2.11)$$

choose a discretization of $[a, b]$ with N subintervals,

$$t_j = a + \frac{T}{N}j, \quad T = b - a, \quad j = 0, 1, \dots, N,$$

and propose the use of the multiple shooting method for solving it. First, in each interval $[t_j, t_{j+1}]$, we compute the function $y(t; s_0, s_1, \dots, s_{j-1})$ recursively, by solving the initial value problems

$$y''(t) = f(t, y(t), y'(t)), \quad y(t_j) = y(t_j; s_0, s_1, \dots, s_{j-1}), \quad y'(t_j) = s_j,$$

whose solution is denoted by $y(t; s_0, s_1, \dots, s_j)$.

To approximate a solution of problem (2.11), we approximate a solution of the nonlinear system of equations $F(s) = 0$, where $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and

$$\begin{cases} F_1(s_0, s_1, \dots, s_{N-1}) = s_1 - y'(t_1; s_0) \\ F_2(s_0, s_1, \dots, s_{N-1}) = s_2 - y'(t_2; s_0, s_1) \\ \vdots \\ F_{N-1}(s_0, s_1, \dots, s_{N-1}) = s_{N-1} - y'(t_{N-1}; s_0, s_1, \dots, s_{N-2}) \\ F_N(s_0, s_1, \dots, s_{N-1}) = \beta - y(t_N; s_0, s_1, s_{N-2}, s_{N-1}). \end{cases}$$

For this, we consider Steffensen's method and method (2.9) and compare their numerical performance. In our study, we consider the usual divided difference of first order. So, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, such that $\mathbf{u} \neq \mathbf{v}$, we consider $[\mathbf{u}, \mathbf{v}; F] = ([\mathbf{u}, \mathbf{v}; F]_{ij})_{i,j=1}^N \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$, where

$$[\mathbf{u}, \mathbf{v}; F]_{ij} = \frac{1}{u_j - v_j} (F_i(u_1, \dots, u_j, v_{j+1}, \dots, v_N) - F_i(u_1, \dots, u_{j-1}, v_j, \dots, v_N)).$$

For the initial slope $\mathbf{s}_0 = (s_0^0, s_1^0, \dots, s_{N-1}^0)$, to apply Steffensen's method and method (2.9), we consider

$$\left\{ \begin{array}{l} s_0^0 = \frac{\beta - \alpha}{b - a} = \frac{y(t_N) - y(t_0)}{t_N - t_0}, \\ s_1^0 = \frac{y(t_N) - y(t_1; s_0)}{t_N - t_1}, \\ s_2^0 = \frac{y(t_N) - y(t_2; s_0, s_1)}{t_N - t_2}, \\ \vdots \\ s_{N-1}^0 = \frac{y(t_N) - y(t_{N-1}; s_0, s_1, \dots, s_{N-2})}{t_N - t_{N-1}}. \end{array} \right.$$

In particular, to show the performance of method (2.9), we consider the following boundary value problem:

$$y''(t) = y(t) (y'(t)^2 + \cos^2 t), \quad y(0) = -1, \quad y(1) = 1.$$

In this case, we have $T = 1$ and consider three iterations of the schemes for $N = 2, 3$ and four subintervals in the multiple shooting method. The exact solution is obtained with ND-Solve of MATHEMATICA taking $y'(0) = 0.6500356840546128$ in order to have a trustworthy error for values near to 10^{-15} (tolerance in double precision).

In Tables 2.6, 2.7, 2.8, 2.9 and 2.10, we observe that Steffensen's method obtains poor results. Notice that when N decreases (or the interval increases), the initial guess is less closer to the solution. This is the reason of the improvements of method (2.9) proposed in this work. For the worst case, $N = 2$, Steffensen's method diverges. And, for $N = 3, 4$, we observe clearly the second order of the methods, as well as the best performance of method (2.9).

Table 2.6 Method (2.9),
 $a = 0, b = 10^{-3}; N = 2$

n	$\ F(\mathbf{s}_n)\ _\infty$	$\ y(t) - y_n\ _\infty$	$\ y'(t) - y'_n\ _\infty$
1	$1.190 \dots \times 10^{-1}$	$1.190 \dots \times 10^{-1}$	$9.634 \dots \times 10^{-2}$
2	$6.292 \dots \times 10^{-3}$	$6.292 \dots \times 10^{-3}$	$6.297 \dots \times 10^{-3}$
3	$1.680 \dots \times 10^{-5}$	$1.680 \dots \times 10^{-5}$	$1.772 \dots \times 10^{-5}$

Table 2.7 Method (2.9),
 $a = 0, b = 10^{-3}; N = 3$

n	$\ F(s_n)\ _\infty$	$\ y(t) - y_n\ _\infty$	$\ y'(t) - y'_n\ _\infty$
1	$9.839 \dots \times 10^{-2}$	$2.041 \dots \times 10^{-2}$	$8.140 \dots \times 10^{-2}$
2	$7.274 \dots \times 10^{-4}$	$1.342 \dots \times 10^{-4}$	$5.189 \dots \times 10^{-4}$
3	$1.445 \dots \times 10^{-8}$	$1.169 \dots \times 10^{-8}$	$1.808 \dots \times 10^{-8}$

Table 2.8 Steffensen's
method; $N = 3$

n	$\ F(s_n)\ _\infty$	$\ y(t) - y_n\ _\infty$	$\ y'(t) - y'_n\ _\infty$
1	$2.665 \dots \times 10^{-1}$	$2.527 \dots \times 10^{-1}$	$3.865 \dots \times 10^{-1}$
2	$1.893 \dots \times 10^{-2}$	$1.893 \dots \times 10^{-2}$	$1.965 \dots \times 10^{-2}$
3	$6.407 \dots \times 10^{-4}$	$1.669 \dots \times 10^{-4}$	$4.999 \dots \times 10^{-4}$

Table 2.9 Method (2.9),
 $a = 0, b = 10^{-3}; N = 4$

n	$\ F(s_n)\ _\infty$	$\ y(t) - y_n\ _\infty$	$\ y'(t) - y'_n\ _\infty$
1	$4.680 \dots \times 10^{-2}$	$2.484 \dots \times 10^{-2}$	$6.815 \dots \times 10^{-2}$
2	$2.331 \dots \times 10^{-5}$	$8.762 \dots \times 10^{-5}$	$2.334 \dots \times 10^{-5}$
3	$3.636 \dots \times 10^{-9}$	$3.636 \dots \times 10^{-9}$	$4.208 \dots \times 10^{-9}$

Table 2.10 Steffensen's
method; $N = 4$

n	$\ F(s_n)\ _\infty$	$\ y(t) - y_n\ _\infty$	$\ y'(t) - y'_n\ _\infty$
1	$1.215 \dots \times 10^{-1}$	$5.081 \dots \times 10^{-2}$	$1.593 \dots \times 10^{-1}$
2	$6.728 \dots \times 10^{-3}$	$2.532 \dots \times 10^{-3}$	$5.457 \dots \times 10^{-3}$
3	$1.052 \dots \times 10^{-5}$	$3.891 \dots \times 10^{-6}$	$6.043 \dots \times 10^{-6}$

2.6 Other Contributions

Finally, we introduce briefly some recent contributions.

- In [17] the authors study the convergence of a Newton-Steffensen type method for solving nonlinear equations introduced by Sharma [37]. Under simplified assumptions regarding the smoothness of the nonlinear function, they show that the q -convergence order of the iterations is 3. Moreover, they show that if the nonlinear function maintains the same monotony and convexity on an interval containing the solution, and the initial approximation satisfies the Fourier condition, then the iterations converge monotonically to the solution. They also obtain a posteriori formulas for controlling the errors.
- Based on Steffensen's method, the paper [23] derives a one-parameter class of fourth-order methods for solving nonlinear equations. In the proposed methods, an interpolating polynomial is used to get a better approximation to the derivative of the given function. Each member of the class requires three evaluations of the given function per iteration. Therefore, this class of methods has efficiency index which equals 1.587.
- For solving nonlinear equations, the paper [33] suggests a second-order parametric Steffensen-like method, which is derivative free and only uses two evaluations of the function in one step. A variant of the Steffensen-like method which is still derivative free and uses four evaluations of the function to achieve cubic

convergence is also presented. Moreover, a fast Steffensen-like method with super quadratic convergence and a fast variant of the Steffensen-like method with super cubic convergence are proposed by using a parameter estimation. The error equations and asymptotic convergence constants are obtained for the discussed methods.

- In [34], a parametric variant of Steffensen-secant method and three fast variants of Steffensen-secant method for solving nonlinear equations are suggested. They achieve cubic convergence or super cubic convergence for finding simple roots by only using three evaluations of the function per step. Their error equations and asymptotic convergence constants are deduced. Modified Steffensen's method and modified parametric variant of Steffensen-secant method for finding multiple roots are also discussed.
- In [36], a family of fourth-order Steffensen-type two-step methods is constructed to make progress in including Ren-Wu-Bi's methods [23] and Liu-Zheng-Zhao's method [Z. Liu, Q. Zheng, P. Zhao, A variant of Steffensen's method of fourth-order convergence and its applications, *Appl. Math. Comput.* 216 (2010) 1978–1983.] as its special cases. Its error equation and asymptotic convergence constant are deduced. The family provides the opportunity to obtain derivative-free iterative methods varying in different rates and ranges of convergence.
- In [11], a family of Steffensen-type methods of fourth-order convergence for solving nonlinear smooth equations is suggested. In the proposed methods, a linear combination of divided differences is used to get a better approximation to the derivative of the given function. Each derivative-free member of the family requires only three evaluations of the given function per iteration. Therefore, this class of methods has efficiency index equal to 1.587. The new class of methods agrees with this conjecture.
- A new derivative-free iterative method for solving nonlinear equations with efficiency index equal to 1.5651 is presented in [18].
- In the paper [12], by approximating the derivatives in the well known fourth-order Ostrowski's method and in a sixth-order improved Ostrowski's method by central difference quotients, we obtain new modifications of these methods free from derivatives. The authors prove the important fact that the methods obtained preserve their convergence orders 4 and 6, respectively, without calculating any derivatives.
- The authors of [19] present a modification of Steffensen's method as a predictor-corrector iterative method, so that they can use Steffensen's method to approximate a solution of a nonlinear equation in Banach spaces from the same starting points from which Newton's method converges. They study the semilocal convergence of the predictor-corrector method by using the majorant principle.
- A derivative free method for solving nonlinear equations of Steffensen's type is presented in [17]. Using a self-correcting parameter, calculated by using Newton's interpolatory polynomial of second degree, the R -order of convergence is increased from 2 to 3. This acceleration of the convergence rate is attained without any additional function calculations, which provides a very high computational efficiency of the proposed method.

- The paper [38] proposes two classes of three-step without memory iterations based on the well known second-order method of Steffensen. Per computing step, the methods from the developed classes reach the order of convergence eight using only four evaluations, while they are totally free from derivative evaluation. Hence, they agree with the optimality conjecture of Kung-Traub for providing multi-point iterations without memory.
- In [40], based on some known fourth-order Steffensen-type methods, we present a family of three-step seventh-order Steffensen-type iterative methods for solving nonlinear equations and nonlinear systems. For nonlinear systems, a development of the inverse first-order divided difference operator for multivariable function is applied to prove the order of convergence of the new methods.

Other related works can be found in [1–6, 16, 24, 25, 29, 30, 35, 41–43].

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