Chapter 2
Mechanics

In this chapter, the reader will be introduced to a variety of non-Newtonian phenomena exhibited by real fluids, namely stress relaxation, nonlinear creep, shear-thinning and shear-thickening, thixotropy, development of normal stress differences in simple shear flows, yield, etc. This is followed by a section dedicated to the basic kinematical definitions. The next subsection deals with the definition of frame-indifference and the restrictions that are a consequence of its requirement. In the following subsection the balance laws, and the Clausius–Duhem inequality[1] (which is interpreted in this book as the second law of thermodynamics) are recorded and this is followed with a subsection on constitutive relations wherein the reader is introduced to the fluid of the differential type and to its special subclasses such as fluids of complexity n, and fluids of grade n. In this chapter, the reader is also introduced to the three categories of fluids, namely the differential, rate, and integral type fluids. The classical Navier–Stokes model is a fluid of grade 1 and can be obtained as a special subclass of fluids of grade 2 by setting appropriate terms that appear in the constitutive relation to be zero.

2.1 Introductory Remarks

It is hard to define precisely what one means by a fluid. David Goldstein [124] aptly remarks "Precisely what do we mean by the term liquid? Asking what is a liquid is like asking what is life; we usually know it when we see it, but the existence of some doubtful cases makes it hard to define precisely." In fact, the situation is far worse than that described by Goldstein. In the pitch drop experiment set up by Professor

[1] A thermodynamic framework has been put into place by Rajagopal and Srinivasa in [234] that appeals to the maximization of the rate of entropy production to provide a basis for fluids of grade 2 and the same framework can be used to develop the model of the grade 3 fluid and the Bingham fluid.

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Parnell in 1927 that yet continues to date, pitch was poured into a funnel with a sealed bottom and allowed to consolidate. The stem was cut in 1930. The first drop fell in December 1938, the second in February 1947, and the sixth in April 1979 (see Edgeworth et al. [93] for details). Thus, a person observing the pitch in the funnel for even an hour would conclude that the material did no flow and was a solid, while were one to wait 8 years one would conclude that the pitch was a fluid that flowed. The same pitch, when hammered shatters like a brittle solid. Thus, whether a body behaves in a fluid-like manner depends on the time and length scale of observation, as well as the magnitude of the forces a body is subject to.

Maxwell [184] captures the quintessential character of a fluid when he writes: “In the case of viscous fluid it is *time* which is required, and if enough time is given, the very smallest force will produce a sensible effect, such as would require a very large force if suddenly applied.” Maxwell’s comments bring into focus the most important factor in defining fluid-like and solid-like behavior: the notion of *time scales*, *length scales* and *force scales*. What are the time, length, and force scales of interest? Are we interested in motions discernible to the naked eye in nanoseconds? Are we interested in motions of the order of nanometers over a period of years? Based on which of the two above questions we are interested in, the response of a body would be considered fluid-like or solid-like. We shall not get into a lengthy discussion of this issue here. By a fluid-like body, we mean a body which moves in a manner discernible to the naked eye for a time scale of observation that is meaningful for the forces under consideration.

The Oxford English Dictionary [1] defines a fluid as “Having the property of flowing, …consisting of particles that move freely along themselves so as to give way before the slightest pressure, ….” The first part of the above definition does not convey much meaning as “flowing” is regarded as a property of a fluid, and with regard to the second part, it is now commonly recognized that it is the inability to withstand a shear stress rather than “pressure” that is characteristic of a fluid. Of course, one cannot get rid of the time and length of scales of observation or the magnitude of the shear stress in determining the inability of the body to withstand the shear stress. We shall not get into a more detailed discussion of the nature of fluids.

Details concerning the historical development of fluid mechanics can be found in the authoritative treatises by Tokaty [269] and Truesdell [270]. Here, we shall briefly discuss the main contributions of Newton and those that followed. In his immortal Principiae in 1687, Newton [196] discusses in detail the resistance occurring due to the motion of fluids. He states that “The resistance arising from the want of lubricity in parts of fluids is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.” Such a constitutive description which provides the genesis for the fluid model that is popularly referred to as a Newtonian fluid does not provide a clear mathematical definition of the model. Navier [193] in 1823, Poisson [213] in 1831 and Stokes [259] in 1845, greatly generalized the ideas of Newton to arrive at a model that bears the names of the first and last of these authors. It is interesting to note that very different paths were taken by these authors to arrive essentially at the same result. As Stokes [259] observes:

"I afterwards found that Poisson had written a memoir on the same subject, and on referring to it, I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society." In a footnote he adds: "The same equations have also been obtained by Navier in the case of an incompressible fluid (Mémoire de l'Académie, t.VI, p. 389), but his principles differ from mine still more than Poisson's." The model that Stokes derived in 1845, satisfied the requirements of frame-indifference and isotropy.

Stokes' work is all the more remarkable that he arrived at a model far more general than that which bears his name partially, in that he recognized that the viscosity could possibly depend on the pressure. He then provides a justification for a class of flows where one could view viscosity as a constant: "Let us consider in what cases it is allowable to suppose μ to be independent of pressure. It has been concluded by Du Buat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure… I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of pressure." On the basis of this additional assumption, Stokes arrives at the equations of motion for incompressible fluids like water, which in our current notation would read[2]

$$\mu\,\Delta \boldsymbol{v} + \varrho\,\boldsymbol{b} = \mathbf{grad}\,p + \varrho\frac{d\boldsymbol{v}}{dt}, \tag{2.1.1}$$

$$\operatorname{div}\boldsymbol{v} = 0. \tag{2.1.2}$$

The above equations of motion are a consequence of assuming that the fluid is incompressible and assuming furthermore a constitutive relation for the Cauchy stress that is linear in the symmetric part of the velocity gradient. They are referred to as the incompressible Navier–Stokes equations and they have been quite successful in describing the laminar flow of many liquids such as water. However, the classical linearly viscous fluid model (Newtonian fluid model, Navier–Stokes fluid model) cannot adequately describe the laminar response of many polymeric liquids, biological fluids, foams, and slurries. Their departure from the Newtonian fluid response is manifest in a variety of ways, namely the ability of the fluid to

1. shear-thin or shear-thicken,
2. exhibit thixotropy,
3. allow stress relaxation,
4. creep in a nonlinear manner,

[2]This is the vector version of equation (13) in Stokes' paper [259]. We have not yet introduced the notation that will be followed in this book. Here, μ is the viscosity, ϱ the density, $\boldsymbol{v}$ the velocity, p the pressure, $\boldsymbol{b}$ the specific body force, and Δ denotes the Laplacian, $\mathbf{grad}$ the gradient, div the divergence and $\frac{d}{dt}$ the material time derivative, see Section 2.3 for more details.

5. develop normal stress differences,
6. and exhibit a threshold for the shear stress before it starts to flow (commonly referred to as "yielding" behavior).

[3]The last of the above characteristics of non-Newtonian fluids contradicts our earlier motivation that a fluid cannot sustain a shear stress. Thus condition (6) should be interpreted as there being a threshold for the flow with regard to some reasonable time and length scale of observation, and some specified range of forces.

Many fluids exhibit one or more of these characteristics, and some all of them. In order to describe the response of such complex fluids a plethora of fluid models have been developed, all of them being referred to as non-Newtonian fluid models.

We shall discuss the different characteristics displayed by non-Newtonian fluids in Section 2.2. The delineation of a viscoelastic material as a solid or a fluid is not an easy matter. We can at best classify such materials as being fluid-like or solid-like. As we remarked earlier, such categorizations depend on inherent time scales associated with the material in comparison with the time scale of the observation. In this book, we are concerned with the fluid-like response of materials. It is quite possible for a viscoelastic liquid to display more "elastic response" than a viscoelastic solid, and for a viscoelastic solid to display more "dissipative" response than a viscoelastic liquid. Maxwell [184] discussed this issue with a clarity that is not to be found even in the most current texts. He was able to draw a clear distinction between the response of a soft solid and a viscoelastic fluid. He remarked: "What is required to alter the form of a soft solid is a sufficient force, and this, when applied produces its effect at once. In the case of a viscous fluid it is *time* which is required, and if enough time is given, the very smallest force will produce a sensible effect, such as would require a very large force if suddenly applied. Thus a block of pitch may be so hard that you cannot make a dent in it by striking it with your knuckles; and yet in course of time, it will flatten itself by its own weight, and glide down-hill like a stream of water." Maxwell is making the distinction between the instantaneous response of pitch (asphalt/bitumen) and its long-time behavior.

We shall discuss the notion of stress relaxation in some detail later. A measure of how the relaxation takes place is given by a characteristic time associated with the body, referred to as the relaxation time. An important nondimensional number that one comes across while studying the mechanics of viscoelastic fluids is the DEBORAH[4] number that is the ratio of a characteristic time associated with the body,

[3]There are several phenomena associated with the response of non-Newtonian fluids such as "elastic turbulence," "drag reduction due to turbulence," "secondary flows in turbulent flows of non-Newtonian fluids," etc. that are a consequence of some of the above-mentioned response characteristics of non-Newtonian fluids coupled with the effects of turbulence. As turbulence, to date, has defied proper understanding, even within the context of fluids that in the laminar range are described by the classical Navier–Stokes fluid, we shall not discuss such characteristics in the book.

[4]If one waits sufficiently long, then one could discern motion in nearly all bodies due to the effect of gravity. Deborah [80] remarks in the Old Testament that "Even mountains quaked in the presence of the Lord." Usually, Deborah's statement is translated to read "Even mountains flowed in the presence of the Lord," but this translation does not seem to be felicitous (see Rajagopal [228] for a detailed discussion concerning this issue).

to a time interval associated with the process. In considering problems involving non-Newtonian fluids, we will come across a variety of nondimensional numbers in addition to the usual Reynolds number that one comes across while considering isothermal flows of the Navier–Stokes fluid. Depending on which of these nondimensional numbers are dominant, we can have a variety of response characteristics.

Many non-Newtonian fluids have memory, that is the stress in the fluid at current time can depend on how the fluid has deformed in the past. The incompressible Newtonian fluid on the other hand has no memory in that the stress in the fluid is completely determined to within an indeterminate spherical stress by the current value of the symmetric part of the velocity gradient.

Models that have been developed to describe the non-Newtonian response of fluids can be broadly classified into three categories

1. fluids of the differential type,
2. fluids of the rate type,
3. fluids of the integral type.

Fluids in which the stress is determined by the velocity gradient and its various higher time derivatives are called fluids of the differential type. In incompressible fluids of the differential type, as soon as the motion ceases, the stress reduces to an indeterminate spherical stress. Unlike fluids of the differential type, in which the stress is given explicitly as a function of the velocity gradient and its higher time derivatives, in rate type fluid models there is an implicit relationship between the stress and its higher time derivatives.[5] Integral type models are those in which the Cauchy stress at the current time is given in terms of an integral over the past time of the history of the relative deformation gradient. Some rate type models can be expressed as integral type models, but not all rate type models can be expressed in such a manner. In general, rate type models define a class of models rather than just one model (cf. Truesdell and Noll [276], p. 95).

As we remarked earlier, Stokes recognized that the viscosity of a fluid could depend upon the pressure. There is considerable experimental evidence indicating that this is indeed the case and a thorough discussion of the experimental results prior to 1930 can be found in the authoritative treatise by Bridgman [52] on the physics of high pressure. As early as 1893, Barus [21] suggested that the viscosity is related to the pressure through

$$\mu(p) = \mu_0 \exp(\alpha\, p), \;\; \alpha > 0.$$

Later, according to Bridgman [52], Andrade proposed the following relationship between the viscosity, pressure, density, and temperature:

$$\mu(p, \varrho, \theta) = A\, \varrho^{\frac{1}{2}} \exp\left[\frac{B}{\theta}(p + D\, \varrho^2)\right],$$

[5]Recently, it has been shown that the Cauchy stress in many rate type models can be expressed as an elastic response from an evolving natural configuration, i.e., the stress can be expressed in terms of a Cauchy–Green stretch tensor which obeys an evolution equation (see Rajagopal and Srinivasa [235]).

where A, B, and D are constants, θ denotes the temperature and ϱ the density (see also the paper by Andrade [11] wherein the pressure dependence of the viscosity is discussed in detail).

Various other relationships between the viscosity and pressure have been proposed. At very high pressures, experiments suggest that the fluid is close to undergoing glass transition and equation such as those suggested by Barus and Andrade seem to be inappropriate. For a discussion of the relevant issues, we refer the readers to some of the experimental work in the area (Cutler et al. [72], Griest et al. [126], Johnson and Cameron [143], Johnson and Greenwood [144], Johnson and Tevaarwerk [145], Bair and Winer [19], Roelands [248], Paluch et al. [209], Irving and Barlow [140], Bender et al. [25] and Bair and Kottle [18]).

In liquids such as water, the change in density due to pressure seems to correlate reasonably well with the relation (see Dowson and Higginson [84]),

$$\varrho = \varrho_0 \left[1 + \frac{0.6\, p}{1 + 1.4\, p} \right],$$

where ϱ_0 is the reference density (density where the pressure is zero).

While the density changes by about 3 to 4 percent as the pressure changes from 2 to 3 GPa, the viscosity changes by approximately 10^8 percent. Thus, it would be reasonable that fluids such as water and organic liquids be modeled as incompressible fluids with pressure dependent viscosities when the pressure varies over a reasonably wide range. However, under operating conditions as those encountered in flows in pipes and channels, the viscosity varies ever so slightly that it can be considered a constant.

When the viscosity of the fluid depends on the pressure, the balance of linear momentum reduces to

$$\mu(p) \Delta \boldsymbol{v} + \varrho\, \boldsymbol{b} + \left[\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \right) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \right)^T \right] \mathbf{grad}\, \mu(p) = \mathbf{grad}\, p + \varrho \frac{d\boldsymbol{v}}{dt},$$

$$\operatorname{div} \boldsymbol{v} = 0.$$

The variation of the viscosity with pressure is of great significance in a variety of important technological applications such as elastohydrodynamics (see Szeri [261]). Recently, Rajagopal and Szeri [238] have shown that the additional term involving the gradient of the viscosity in the above equation cannot be neglected while considering problems such as elastohydrodynamics. It is possible that the viscosity of an incompressible fluid can depend on both the pressure and the symmetric part of the velocity gradient. These fluids cannot be characterized by an explicit relationship between the stress and the symmetric part of the velocity gradient; the relationship is implicit. Such fluid models are also non-Newtonian models and they could be of the differential, rate, or integral type. We shall not consider these models here, the interested reader can find a detailed discussion of the mathematical properties of the equations governing the flows of such fluids in the review article by Málek and Rajagopal [170].

In addition to the above types of models, other forms of models based on the notion of fractional derivatives, the notion of the conformation tensor, etc. have been proposed. Here, we shall be concerned with incompressible fluids of the differential and rate type, wherein the stress is expressed in terms of the kinematical quantities.

In recent years, there has been considerable amount of interest in the study of non-Newtonian fluids of the differential type (the Navier–Stokes fluid is also a fluid of the differential type). Despite such extensive studies, a considerable amount of confusion and lack of clarity seems to prevail in the interpretation of the results concerning the flows of fluids of the differential type. Though repeated attempts at clarifying the issues have not seemed to have put paid to erroneous and inaccurate conclusions, we shall endeavor to set the records straight. As there has been much progress in the past decade, our task might be easier in understanding the response of these fluids.

There has been a great deal of controversy surrounding the thermodynamics of such fluids. In the two decades that followed the work of Coleman and Noll [69], in which they used the Clausius–Duhem inequality to obtain restrictions on the forms of constitutive relations that are allowable, it became common practice in continuum mechanics to use such a procedure. However, for a variety of reasons, serious reservations were expressed against using the procedure, the most compelling objection being the appeal to the body being subject to arbitrary processes to obtain these restrictions. As no single constitutive relation is expected to hold for all arbitrary processes that any body is subject to, any specific constitutive relation that is used will be valid for only a certain class of processes beyond which the constitutive relation is not expected to hold. We believe that such an objection is well taken. The main problem stems from trying to obtain necessary and sufficient conditions that will ensure that the Clausius–Duhem inequality should hold. If on the other hand, one is only interested in sufficient conditions that will guarantee that the Clausius–Duhem inequality holds, then one can overcome the objections that are raised. In special cases, the procedure used by Coleman and Noll [69] led to sensible results and in the absence of an alternate procedure to obtain restrictions on constitutive relations, it gained credence. For instance, in the case of a Newtonian fluid, such a procedure would lead to the viscosity being nonnegative, a perfectly reasonable result, reasonable in that such a fluid exhibited reasonable agreement with experiments, its stability characteristics, and its mathematical properties.

Another serious drawback in using the Clausius–Duhem inequality, as advocated by Coleman and Noll [69], stems from the fact that the radiation is eliminated from the reduced dissipation equation by substituting for it from the energy equation. Thus, the second law, as interpreted in the above approach, places no restriction whatsoever on the radiation. While, this may not be of much consequence in many problems involving the flow of fluids, it is of great significance in other problems. From a philosophical standpoint, ignoring radiation altogether, or for that matter being incapable of recognizing the restrictions that need to be placed on the radiation, is an unacceptable situation. We shall not get into a detailed discussion of this issue here. The interested readers can find details in Rajagopal and Tao [239].

However, controversy surfaced concerning the consequences of using Clausius–Duhem inequality and the apparent predictions of experiments for a special subclass of fluids of the differential type, namely fluids of grade two. We shall not get into a discussion of these issues but refer the reader to Dunn and Rajagopal [88], where the relevant issues are discussed at length. Suffice it to say that fluids of grade two that satisfy the Clausius–Duhem inequality and whose specific Helmholtz potential is a minimum in equilibrium exhibit reasonable response. Of course, it is perfectly acceptable for one to yet object to the use of the Clausius–Duhem inequality on the grounds that one cannot appeal to arbitrary processes, in which case we have to make appropriate restrictions to the constitutive relation that guarantees that the second law is met (i.e., we can a priori assume constitutive relations that guarantee that the entropy production is nonnegative. Our constitutive assumptions are "sufficient" to assume that the second law is met).

We shall not be wedded to the consequences of the restrictions due to the Clausius–Duhem inequality. However, we shall show that noncompliance with thermodynamic considerations leads to fluids that exhibit unacceptable physical characteristics: instability of the rest state of the fluid. Thus, while from a general philosophical point of view, the thermodynamic procedure can be rightfully called into question, in the case of some special models of fluids of the differential type that are considered, they do lead to models that exhibit reasonable physical characteristics.

This book is devoted to mechanics issues, and in some cases mathematical issues, concerning the flows of incompressible fluids of differential and rate type. Some numerical issues that arise during the solution of the equations that govern the flows of these fluids are addressed in the reference by Girault and Hecht [112]. Among others, we shall examine two important questions: the boundary conditions and the development of boundary layers.

An important mathematical issue that surfaces when we consider the flows of a Navier–Stokes fluid instead of an Euler fluid is the necessity for an additional boundary condition. In the case of an Euler fluid, we need to only require that the normal component of the velocity be zero at an impervious boundary while we need the adherence condition in the case of a Navier–Stokes fluid. A similar situation, i.e., a need to augment boundary conditions, presents itself when we consider fluids of higher grade.

Unlike the flows of a classical Newtonian fluid which lead to second order partial differential equations for which the "no-slip" boundary condition is sufficient for well-posed problems, flows of fluids of grades two and three lead to partial differential equations of third order and in general the "no-slip" boundary condition is inadequate to guarantee uniqueness of solutions. In fact, an infinite number of solutions have been exhibited for several simple boundary value problems (cf. Rajagopal and Gupta [230], Rajagopal and Kaloni [232]). Not only is the "no-slip" condition insufficient for certain flows (those for which $\boldsymbol{v} \cdot \boldsymbol{n} \neq 0$), but also there are many flows of fluids wherein the fluid "stick-slips" at the boundary. It has been experimentally observed (cf. Hatzikiriakos and Dealy [131], Ramamurthy [240], Kraynik and Schowalter [152]) in the flow of many polymer melts that the fluid adheres to the boundary, provided that the pressure gradient (the corresponding shear stress at the wall) is below a certain critical value. However, when the pressure gradient is increased beyond this critical value, the fluid starts to slip at the boundary. This sticking and slipping at the boundary leads to surface instabilities in an extruded melt producing a phenomenon popularly referred to as the "shark-skin effect" (cf. [131]).

One of the cornerstones of classical fluid mechanics is the boundary layer approximation developed by Prandtl [216, 217] for the Navier–Stokes equation. The physical basis for the boundary layer approximation is the experimentally observed fact that the vorticity, in the flow of fluids such as water along a solid boundary, is confined in a narrow region adjacent to the boundary, this region being referred to as the boundary layer. This confinement of vorticity adjacent to the solid wall leads to a great simplification, namely the flow outside the boundary layer is approximated as the flow due to an Euler fluid, while the flow inside the boundary layer is that of a Newtonian fluid (the equations being further simplified). In the case of a Newtonian fluid, boundary layers arise at sufficiently high Reynolds numbers, the layer becoming more pronounced (narrower) as the Reynolds number increases. In non-Newtonian fluids, confinement of vorticity adjacent to solid boundaries is not necessarily a consequence of the Reynolds number being sufficiently large. In certain non-Newtonian fluids, it is possible to have confinement of vorticity adjacent to the boundary in the limit of Reynolds number tending to zero. In other non-Newtonian fluids, boundary layers can develop much in the same manner as in the case of Newtonian fluids, at high Reynolds number. However, a much richer "class" of boundary layers are possible in non-Newtonian fluids: there could be confinement (concentration) of various kinematical quantities due to elastic effects, shear-thinning, etc. We shall illustrate these issues via specific examples.

2.2 Non-Newtonian Behavior

Let us discuss the departure from Newtonian response that we mentioned in the introduction (see also Huilgol [138] and Málek and Rajagopal [169]).

2.2.1 Shear-Thinning and Shear-Thickening

Consider a simple shear flow, i.e., the velocity field $\boldsymbol{v}$ has the form

$$\boldsymbol{v} = u(y)\boldsymbol{i}, \tag{2.2.1}$$

in a cartesian coordinate system. In the case of a Newtonian fluid, the shear stress T_{xy} is directly proportional to the rate of shear $u'(y)$, the constant of proportionality being the viscosity μ

$$T_{xy} = \mu\,\kappa,$$

where $\kappa = u'(y)$.

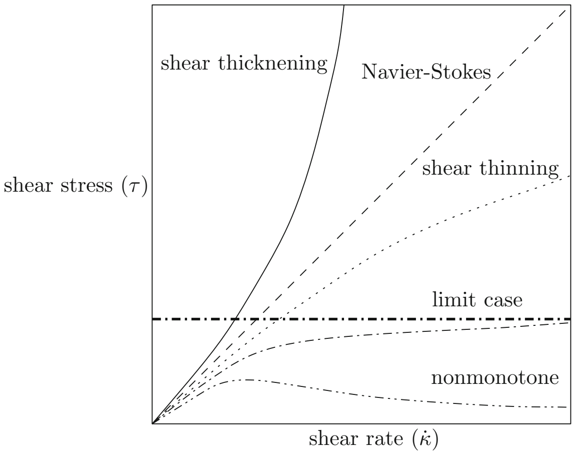


Fig. 2.1 Shear dependent behavior of non-Newtonian fluids

In a general simple fluid (cf. Noll [197]), it can be shown that

$$T_{xy} = \tau(\kappa) := [\tilde{\mu}(\kappa)]\kappa, \tag{2.2.2}$$

where $\tilde{\mu}(\kappa)$ is the generalized viscosity and we shall see later that $\tau(\kappa)$ is one of the three viscometric functions. It is possible that the variation of the shear stress with the shear rate is such that the derivative of the generalized viscosity $\tilde{\mu}(\kappa)$ with respect to κ is negative. Such fluids are called shear-thinning fluids. On the other hand, if the derivative of the generalized viscosity with respect to κ is positive, it is called a shear-thickening fluid. Thus, the variation of the shear stress with respect to the shear rate in a simple shear flow has one of the forms depicted in Figure 2.1.

Fluids with shear rate dependent viscosity are used widely in the chemical engineering field, ice mechanics, geology, and hemodynamics (see Bird et al. [39]). Issues concerning the existence, regularity of solution and stability for fluids with shear rate dependent viscosity can be found in Málek et al. [171].

2.2.2 *Thixotropy*

Thixotropy is not a very well-understood phenomenon and is usually confused with shear-thinning. The phenomenon of thixotropy is exhibited by suspensions of colloidal and non-colloidal particles that form a flocculated system, and it is also observed in cross-linked gels. The main characteristic of thixotropic fluids is the change in material properties, such as viscosity and elasticity, with time. Whereas in a shear-thinning fluid, the viscosity changes with the shear rate, in a thixotropic fluid the viscosity can decrease with time while the shear rate remains constant. Such changes in a thixotropic material are a consequence of structural changes in the underlying microstructure of the fluid. In view of this, usually a scalar parameter referred to as the "structure parameter" is introduced. The evolution of the "structure parameter" with time is given by a kinetic equation that is derived from microstructural considerations, but is in most instances ad hoc. Many thixotropic fluids also exhibit “yield” and the properties of such bodies are sensitive to the effect of temperature. In some thixotropic fluids changes in temperature can lead to crystallization which in turn affects the flow characteristics of the fluid. Another important aspect that needs to be taken into account while describing thixotropic fluids is the effect of Brownian motion, which in turn is affected by the temperature. Hence, in order to describe the response of thixotropic fluids, one needs to consider a fully thermodynamic framework.

The detailed reviews by Mewis and Wagner [190] and Barnes [20] provide a description of the main features of thixotropic fluids and also copious references to the subject. De Souza Mendes, Thomson and coworkers have developed models to describe the response of thixotropic fluids (see de Souza Mendes [75, 76], de Souza Mendes and Thomson [78, 79]). Many of these models, while they are very useful in that they have been able to explain reasonably well several of the experimental observations in one-dimensional shear flows, are ad hoc and mostly restricted to one-dimension, and none of them have a rigorous thermodynamic underpinning. Recently, de Souza Mendes et al. [77] developed a thermodynamic framework to generate properly invariant three-dimensional models, which when restricted to one dimension can capture the main features of the one-dimensional models developed by de Souza Mendes [75, 76] and de Souza Mendes and Thomson [78, 79]. The thermodynamic analysis by de Souza Mendes et al. [77] does not explain all the features exhibited by thixotropic fluids, however, it can serve as the starting point for the development of a more complete model. Moreover, most of the experiments that have been carried out thus far are one-dimensional shear flow experiments, thus before a complete model can be put into place, it is necessary to carry out experiments in more complex domains as well as a more thorough experimental study of the dependence of the response characteristic of thixotropic fluids on temperature.

We shall not discuss thixotropy in further detail as the fluids that will be considered in this book do not exhibit thixotropy.

2.2.3 Stress Relaxation

Next, consider a fluid that is subject to a step shear strain of the following form:

$$\varepsilon_{xy} = \begin{cases} 0 \text{ for all} & t < 0, \\ \varepsilon_0 \text{ for all} & t \geq 0. \end{cases}$$

In certain fluids, the shear stress that is required to maintain the stresses would decrease and tend to zero with time. As the shear stresses required to maintain the strain decrease, the fluid is said to stress relax. In the case of certain solids, a similar type of response is elicited; however, the shear stress decreases to a nonzero asymptotic value (Figure 2.2).

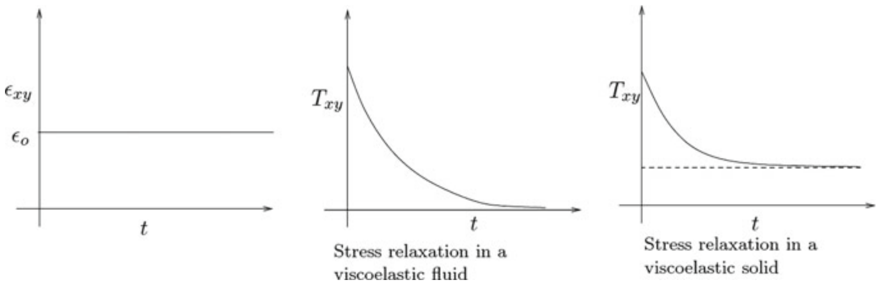


Fig. 2.2 Stress relaxation in viscoelastic bodies

Many materials which stress-relax also exhibit instantaneous "elastic" response. A Newtonian fluid neither has an instantaneous "elastic" response nor does it have the ability to stress relax. In a Newtonian fluid, in response to the step strain there would be a Dirac measure for the shear stress. We shall see later that fluids of the differential type share this feature in common with the Newtonian fluid; they are incapable of stress relaxation and they are incapable of instantaneous elastic response.

2.2.4 Creep

Consider a fluid that is subject to a constant shear stress. If the fluid were Newtonian, the shear strain would increase linearly with respect to time. However, there are many materials in which the variation of the strain with respect to time is nonlinear. When certain materials are subject to a shear stress of the form

$$T_{xy} = \begin{cases} 0 \ \text{ if } t < 0, \\ T_0 \text{ if } 0 \leq t < \alpha, \\ 0 \ \text{ if } t \geq \alpha, \end{cases}$$

they typically respond in the manner shown in Figure 2.3.

In a viscoelastic fluid capable of an instantaneous elastic response, the strain will asymptotically tend to a constant value while in a viscoelastic solid capable of an instantaneous elastic response, the strain will asymptotically tend to zero. This phenomenon is called "creep".

2.2.5 Normal Stress Differences

Once again, let us consider a flow of the form (2.2.2). In the case of incompressible Newtonian fluids where the viscosity μ is constant, we have

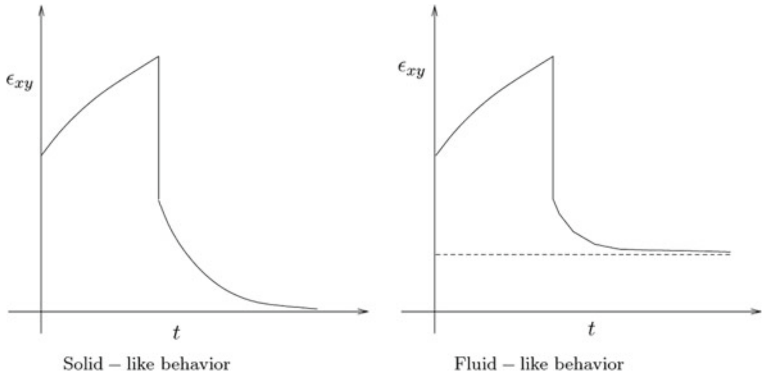


Fig. 2.3 Nonlinear creep

$$T_{xx} = T_{yy} = T_{zz} = -p,$$
$$T_{xy} = \mu\kappa, \ T_{xz} = T_{yz} = 0.$$

Thus

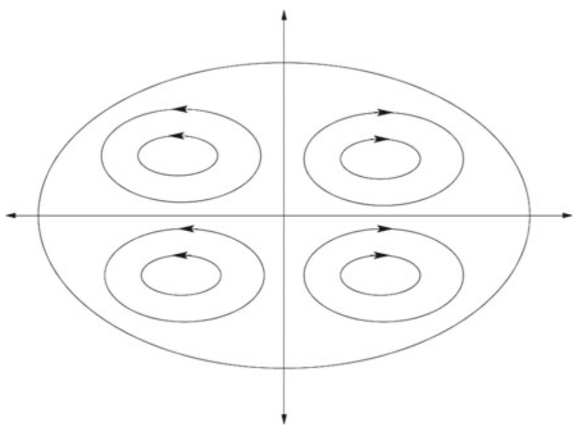
$$T_{xx} - T_{yy} = 0, \ T_{xx} - T_{zz} = 0.$$

We will see later that in many fluids of differential type, we have

$$T_{xx} - T_{yy} \neq 0, \ T_{xx} - T_{zz} \neq 0,$$

and thus normal stress differences develop in a simple shear flow. These normal stress differences are the cause for many interesting phenomena such as "die-swell," "rod-climbing" (see Figure 2.5), and secondary flows in pipes of noncircular cross-sections (see Figure 2.4).

Fig. 2.4 Flow in a pipe of noncircular cross-section: vortices due to normal stress differences



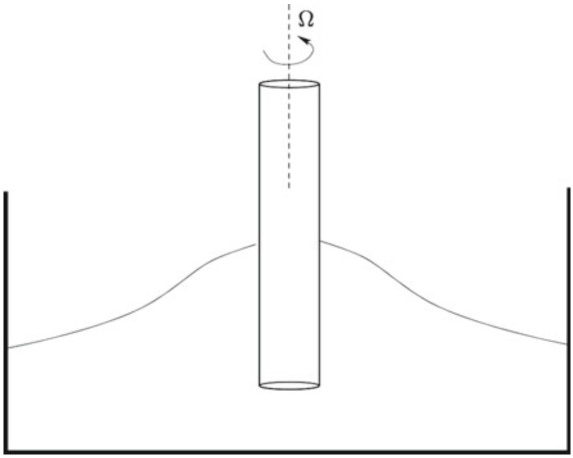


Fig. 2.5 Rod-climbing due to normal stress differences

The development of normal stresses are common to many types of nonlinear response of materials. In nonlinear solids, the Poynting effect, first studied by Wertheim (see Bell [24]), and later by Poynting [214] and Thomson (Lord Kelvin) and Tait [149], is a consequence of unequal normal stresses developing due to shear. A similar phenomenon in non-Newtonian fluids is called the Weissenberg effect (cf. Weissenberg [282]), and in granular materials such a phenomenon was observed and characterized by Reynolds [243]. Most interestingly, simple shear in nonlinear materials is characterized by the normal stresses as well as the shear stresses.

2.2.6 Yield

There are certain materials which flow only after a certain threshold for the shear stress is reached; such materials are called "Bingham plastic fluids" and the threshold value for the stress is called the yield stress. However, as a fluid is a material that cannot resist a shear stress given sufficient amount of time, the existence of such a threshold is merely due to the time scale of the experiment not being sufficiently large. Since we are at times forced to work within certain time scales, it is found convenient to describe the material as a fluid, but with a "yield" condition (see Figure 2.6). But we must take great care to recognize that existence of a yield stress is only a convenient approximation and the graph in this figure is that of a set-valued function.

At this juncture, it is worthwhile recalling the distinction between a fluid and a solid that was offered by Maxwell [184], as it is clearer than any of the distinctions made in the works that followed. Maxwell states: "Thus, a tallow candle is much softer than a sealing wax; but if the candle and the stick of sealing wax are laid horizontally between two supports, the sealing wax will in a few weeks in summer

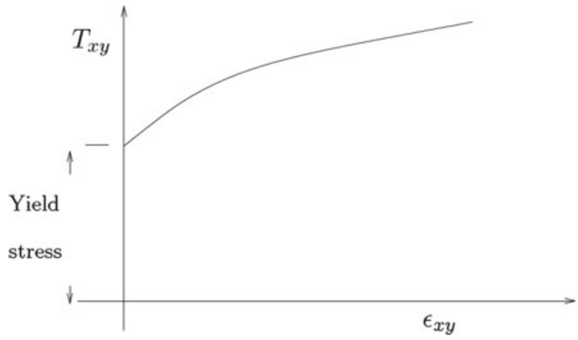


Fig. 2.6 A material with a yield-stress

bend with its own weight, while the candle remains straight. The candle is therefore a soft solid and the sealing wax a very viscous fluid."

2.3 Preliminaries

2.3.1 Kinematics

Let $\kappa_R(\mathcal{B})$ and $\kappa_t(\mathcal{B})$ denote a reference and current configuration of the abstract body $\mathcal{B}$, in a three-dimensional euclidean space. A motion $\boldsymbol{\chi}_{\kappa_R}$ of the body is a one-to-one mapping that assigns to each point $\boldsymbol{X} \in \kappa_R(\mathcal{B})$ a point $\boldsymbol{x} \in \kappa_t(\mathcal{B})$ at time t, i.e.,

$$\boldsymbol{x} = \boldsymbol{\chi}_{\kappa_R}(\boldsymbol{X}, t). \tag{2.3.1}$$

If the choice of the reference configuration changes, the form of the motion will also change. The velocity $\boldsymbol{v}$ is defined by

$$\boldsymbol{v} := \frac{\partial}{\partial t}\boldsymbol{\chi}_{\kappa_R}, \tag{2.3.2}$$

and the deformation gradient $\boldsymbol{F}_{\kappa_R}$ is defined through

$$\boldsymbol{F}_{\kappa_R} := \frac{\partial}{\partial \boldsymbol{X}}\boldsymbol{\chi}_{\kappa_R} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}. \tag{2.3.3}$$

Now, for any function φ defined on $\mathcal{B} \times R$, we can define a function $\tilde{\varphi}$ on $\kappa_R(\mathcal{B}) \times R$ and a function $\hat{\varphi}$ on $\kappa_t(\mathcal{B}) \times R$ such that

$$\varphi(\boldsymbol{P}, t) = \tilde{\varphi}(\boldsymbol{X}, t) = \hat{\varphi}(\boldsymbol{x}, t). \tag{2.3.4}$$

Suppose $\tilde{\varphi} : \kappa_R(\mathcal{B}) \times R \mapsto F_1$ and $\hat{\varphi} : \kappa_t(\mathcal{B}) \times R \mapsto F_2$, where F_1 and F_2 are inner product spaces. We use the following notation:

$$\frac{\partial \varphi}{\partial t} := \frac{\partial \tilde{\varphi}}{\partial t}, \ \mathbf{Grad}\,\varphi := \frac{\partial \tilde{\varphi}}{\partial \boldsymbol{X}}, \tag{2.3.5}$$

$$\dot{\varphi} = \frac{d\varphi}{dt} := \frac{\partial \hat{\varphi}}{\partial t}, \ \mathbf{grad}\,\varphi := \frac{\partial \hat{\varphi}}{\partial \boldsymbol{x}}. \tag{2.3.6}$$

The specification in terms of $(\boldsymbol{X}, t)$ is usually referred to as Lagrangian specification (used earlier by Euler) and the specification in terms of $(\boldsymbol{x}, t)$ is usually referred to as Eulerian specification (used earlier by d'Alembert). Thus, with the above notation:

$$\boldsymbol{v} = \frac{d}{dt}\boldsymbol{\chi}_{\kappa_R}, \ \boldsymbol{F}_{\kappa_R} = \mathbf{Grad}\,\boldsymbol{\chi}_{\kappa_R}. \tag{2.3.7}$$

The derivative $\frac{d\varphi}{dt}$ is called the material time derivative or the substantial time derivative of the quantity φ. The expression $\frac{\partial \varphi}{\partial t}$ is called the local time derivative and denotes the variation with time in the quantity at a fixed location $\boldsymbol{x}$. The material time derivative denotes the variation in the quantity as observed by a person moving with the particle. A flow is said to be steady if $\frac{\partial \varphi}{\partial t} = 0$, for all quantities φ associated with the flow.

The velocity gradient $\boldsymbol{L}$ is defined by

$$\boldsymbol{L} = \mathbf{grad}\,\boldsymbol{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j}. \tag{2.3.8}$$

We also denote by

$$\operatorname{div} \boldsymbol{v} = \operatorname{tr}(\mathbf{grad}\,\boldsymbol{v}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}, \tag{2.3.9}$$

$$\operatorname{Div} \boldsymbol{v} = \operatorname{tr}(\mathbf{Grad}\,\boldsymbol{v}). \tag{2.3.10}$$

We shall assume that $\boldsymbol{F}_{\kappa_R}$ is non-singular. For convenience, we shall suppress the suffix κ_R from all the quantities henceforth. It follows that

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\,\boldsymbol{F}^{-1}. \tag{2.3.11}$$

Let $\boldsymbol{\xi}$ denote the position occupied by $\boldsymbol{X}$ at time τ. Then

$$\boldsymbol{\xi} = \boldsymbol{\chi}(\boldsymbol{X}, \tau) = \boldsymbol{\chi}(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), \tau) := \boldsymbol{\chi}_t(\boldsymbol{x}, \tau). \tag{2.3.12}$$

The mapping $\boldsymbol{\chi}_t$ is referred to as the relative deformation, namely the motion of the body with respect to the current configuration as the reference. Thus,

$$\boldsymbol{v} = \frac{\partial \boldsymbol{\chi}_t}{\partial \tau}\Big|_{\tau=t}. \tag{2.3.13}$$

The relative deformation gradient $\boldsymbol{F}_t(\boldsymbol{x}, \tau)$ is defined through

$$\boldsymbol{F}_t(\boldsymbol{x}, \tau) = \frac{\partial \boldsymbol{\chi}_t}{\partial \boldsymbol{x}} = \mathbf{grad}\, \boldsymbol{\chi}_t = \boldsymbol{F}(\boldsymbol{X}, \tau)\boldsymbol{F}^{-1}(\boldsymbol{x}, t), \tag{2.3.14}$$

and the gradient of the velocity $\boldsymbol{L}$ is related to the relative deformation gradient $\boldsymbol{F}_t(\tau)$ through

$$\boldsymbol{L}(\boldsymbol{x}, t) = \dot{\boldsymbol{F}}_t(\boldsymbol{x}, t).$$

It follows from the Polar Decomposition Theorem that the deformation gradient $\boldsymbol{F}$ can be expressed as

$$\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}, \tag{2.3.15}$$

where $\boldsymbol{R} \in \mathcal{O}$, the group of orthogonal transformations[6] and $\boldsymbol{U}$, $\boldsymbol{V}$ are positive definite and symmetric linear transformations; $\boldsymbol{R}$ is called the rotation and $\boldsymbol{U}$ and $\boldsymbol{V}$ the stretch tensors. Similarly, the polar decomposition theorem of the relative deformation gradient $\boldsymbol{F}_t(\boldsymbol{x}, \tau)$ leads to

$$\boldsymbol{F}_t(\boldsymbol{x}, \tau) = \boldsymbol{R}_t(\boldsymbol{x}, \tau)\boldsymbol{U}_t(\boldsymbol{x}, \tau) = \boldsymbol{V}_t(\boldsymbol{x}, \tau)\boldsymbol{R}_t(\boldsymbol{x}, \tau),$$

$\boldsymbol{U}_t(\boldsymbol{x}, \tau)$ and $\boldsymbol{V}_t(\boldsymbol{x}, \tau)$ are the right and left relative stretch tensors and $\boldsymbol{R}_t(\boldsymbol{x}, \tau)$ is the relative rotation tensor.

The Cauchy–Green tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ are defined through

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2, \tag{2.3.16}$$

$$\boldsymbol{B} = \boldsymbol{F}\,\boldsymbol{F}^T = \boldsymbol{V}^2. \tag{2.3.17}$$

The Green–Saint Venant and the Almansi–Hamel strains, $\boldsymbol{E}$ and $\boldsymbol{e}$ are introduced through

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}), \tag{2.3.18}$$

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{B}^{-1}), \tag{2.3.19}$$

where $\boldsymbol{I}$ is the identity tensor. We can now proceed to define the relative Cauchy-Green tensors in a manner analogous to (2.3.16) and (2.3.17)

$$\boldsymbol{C}_t(\boldsymbol{x}, \tau) := [\boldsymbol{F}_t(\boldsymbol{x}, \tau)]^T[\boldsymbol{F}_t(\boldsymbol{x}, \tau)], \tag{2.3.20}$$

[6]More precisely, $\boldsymbol{R} \in \mathcal{O}^+$, i.e., $\det \boldsymbol{R} = 1$. This follows from the fact that, as a consequence of the conservation of mass, $\det \boldsymbol{F} > 0$ at all times. However, some authors relax this constraint by allowing $\boldsymbol{F}$ to become singular at some points in the presence of shocks.

and

$$\boldsymbol{B}_t(\boldsymbol{x},\tau) := [\boldsymbol{F}_t(\boldsymbol{x},\tau)][\boldsymbol{F}_t(\boldsymbol{x},\tau)]^T. \tag{2.3.21}$$

The Rivlin–Ericksen tensors $\boldsymbol{A}_n$ are defined through (cf. Rivlin and Ericksen [246])

$$\boldsymbol{A}_n = \left[\frac{\partial^n}{\partial\lambda^n}\boldsymbol{C}_t(\boldsymbol{x},\lambda)\right]|_{\lambda=t}. \tag{2.3.22}$$

It immediately follows that

$$\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T.$$

Usually, in the Navier–Stokes theory, the symmetric part of the velocity gradient is denoted by $\boldsymbol{D}$, i.e.,

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{A}_1 = \frac{1}{2}\left[\mathbf{grad}\,\boldsymbol{v} + (\mathbf{grad}\,\boldsymbol{v})^T\right] = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \tag{2.3.23}$$

and its skew part, called the spin tensor, is denoted by $\boldsymbol{W}$, i.e.,

$$\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T). \tag{2.3.24}$$

Thus, the symmetric part of the velocity gradient $\boldsymbol{D}$ and the skew part of the velocity gradient $\boldsymbol{W}$ are related to the relative stretch and the relative rotation through

$$\boldsymbol{D}(\boldsymbol{x},t) = \dot{\boldsymbol{U}}_t(\boldsymbol{x},t) = \dot{\boldsymbol{V}}_t(\boldsymbol{x},t),$$

and

$$\boldsymbol{W}(\boldsymbol{x},t) = \dot{\boldsymbol{R}}_t(\boldsymbol{x},t).$$

Let us define $\boldsymbol{L}_n$ through

$$\boldsymbol{L}_n(\boldsymbol{x},t) := \overset{(n)}{\boldsymbol{F}_t}(\boldsymbol{x},t),$$

where the superscript n denotes n material time derivatives. It immediately follows that

$$\boldsymbol{L}_n(\boldsymbol{x},t) = \mathbf{grad}\ \overset{(n)}{\boldsymbol{\chi}}.$$

It is possible to express the Rivlin–Ericksen tensors $\boldsymbol{A}_n$ by

$$\boldsymbol{A}_n = \boldsymbol{L}_n + \boldsymbol{L}_n^T + \sum_{j=1}^{n-1}\binom{n}{j}\boldsymbol{L}_j^T\boldsymbol{L}_{n-j} + \left[\frac{\partial^n}{\partial\lambda^n}\boldsymbol{C}_t(\boldsymbol{x},\lambda)\right]_{|\lambda=t}. \tag{2.3.25}$$

It follows from the definition of the Rivlin–Ericksen tensors (2.3.22) that the following recursive relation holds:

$$\boldsymbol{A}_n = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}_{n-1} + \boldsymbol{A}_{n-1}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_{n-1}, \quad \text{for } n = 2, \ldots \tag{2.3.26}$$

We shall see in the next subsection that the material time derivative of a frame-indifferent quantity is not necessarily frame-indifferent. However, if $\boldsymbol{A}$ is a second-order tensor, the quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} + \boldsymbol{A}\,\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A},$$

is frame-indifferent. Several other frame-indifferent derivatives are used in fluid mechanics; another that is particularly popular is the upper-convected Oldroyd derivative defined by

$$\overset{\triangledown}{\boldsymbol{A}} := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} - \boldsymbol{L}\,\boldsymbol{A} - \boldsymbol{A}\,\boldsymbol{L}^T, \tag{2.3.27}$$

where $\boldsymbol{A}$ is any second-order tensor. For instance,

$$\overset{\triangledown}{\boldsymbol{I}} = -2\boldsymbol{D} = -\boldsymbol{A}_1, \quad \overset{\triangledown\triangledown}{\boldsymbol{I}} = -2\,\overset{\triangledown}{\boldsymbol{D}}\,. \tag{2.3.28}$$

The vorticity $\boldsymbol{\omega}$ is the axial vector associated with the spin tensor $\boldsymbol{W}$

$$\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)^T, \tag{2.3.29}$$

and thus for all vectors $\boldsymbol{a}$

$$\boldsymbol{W}\,\boldsymbol{a} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{a}. \tag{2.3.30}$$

A source of confusion in fluid mechanics concerns the relationship between the time rate of the rotation tensor $\dot{\boldsymbol{R}}(t)$ and the spin tensor $\boldsymbol{W}$: the spin tensor is not the rate of rotation. Another interesting confusion that stems from imprecise use of language is that of "irrotational flow." Since $\boldsymbol{R}(t)$ is the rotation tensor, one would be led to believe that in an "irrotational flow," $\dot{\boldsymbol{R}}(t) = \boldsymbol{0}$. But this is not the case, for in an "irrotational flow," the vorticity $\boldsymbol{\omega}$ (and thus the spin tensor $\boldsymbol{W}$) is zero. It would be more appropriate to call such flows "vorticity-less flows." This concept is important to grasp, as we shall later develop boundary layer approximations based on the concept that, in some flows, the vorticity is confined close to boundaries. We now establish the relationship between the time rate of the rotation tensor and the spin tensor. A straightforward computation leads to

$$\boldsymbol{W} = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \frac{1}{2}\boldsymbol{R}\left[\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - \boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\right]\boldsymbol{R}^T,$$

because $\dot{\boldsymbol{R}}\,\boldsymbol{R}^T + \boldsymbol{R}\,\dot{\boldsymbol{R}}^T = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \boldsymbol{0}$ since $\boldsymbol{R}(t) \in \mathcal{O}$. Thus, the spin tensor $\boldsymbol{W}$ is clearly not the rate of rotation $\dot{\boldsymbol{R}}$. Also, $\dot{\boldsymbol{R}} = \boldsymbol{0}$ does not mean that $\boldsymbol{W} = \boldsymbol{0}$.

The terminology "stretching tensor" that is usually used to describe $\boldsymbol{D}$ can also be a source of confusion, as $\boldsymbol{D}$ is not the same as the rate of stretch $\dot{\boldsymbol{U}}$. Indeed, a simple calculation yields

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{R}\left[\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} + \boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\right]\boldsymbol{R}^T.$$

2.3.2 Frames, Frame-Indifference and Restrictions Due to Frame-Indifference

Two frames $\{\boldsymbol{x}, t\}$ and $\{\boldsymbol{x}^*, t^*\}$ are said to be related by a change of frame, if the two observers can agree on the measurement of length in the two frames, the measurement of time intervals, and the sense of time. It then follows that these two frames are related by

$$\boldsymbol{x}^* = \boldsymbol{c}(t) + \boldsymbol{Q}(t)[\boldsymbol{x} - \boldsymbol{x}^0], \tag{2.3.31}$$

$$t^* = t - a, \tag{2.3.32}$$

where $\boldsymbol{Q}(t) \in \mathcal{O}$, $\boldsymbol{x}^0$ is some fixed point in the frame $\{\boldsymbol{x}, t\}$, and a is a scalar constant. The term $\boldsymbol{c}(t)$ stands for a translation and $\boldsymbol{Q}(t)[\boldsymbol{x} - \boldsymbol{x}^0]$ stands for a rotation about the fixed point $\boldsymbol{x}^0$ if $\boldsymbol{Q}(t) \in \mathcal{O}^+$, i.e., det $\boldsymbol{Q}(t) = 1$, and it stands for a reflection if $\boldsymbol{Q}(t) \in \mathcal{O}^-$, i.e., det $\boldsymbol{Q}(t) = -1$. There is a considerable amount of controversy if we should require restrictions due to $\boldsymbol{Q}(t) \in \mathcal{O}$ or only to rigid body rotation, i.e., $\boldsymbol{Q}(t) \in \mathcal{O}^+$. Here, we shall require restrictions to merely rigid body rotations.

It follows from (2.3.31), that a directed line segment $\boldsymbol{\ell}$ transforms according to

$$\boldsymbol{\ell}^* = \boldsymbol{Q}(t)\,\boldsymbol{\ell}. \tag{2.3.33}$$

We say that vectors that transform according to the above rule are Frame-Indifferent or Objective. A scalar ψ that is such that

$$\psi^* = \psi, \tag{2.3.34}$$

is said to be Frame-Indifferent, and a second-order tensor $\boldsymbol{T}$ that transforms according to

$$\boldsymbol{T}^* = \boldsymbol{Q}(t)\,\boldsymbol{T}\,\boldsymbol{Q}(t)^T, \tag{2.3.35}$$

is said to be Frame-Indifferent.

Equations (2.3.31) and (2.3.32) imply that the velocity $\boldsymbol{v}$ transforms according to

$$\boldsymbol{v}^* = \dot{\boldsymbol{c}} + \dot{\boldsymbol{Q}}(t)[\boldsymbol{x} - \boldsymbol{x}^0] + \boldsymbol{Q}(t)\boldsymbol{v}, \tag{2.3.36}$$

and thus, we immediately recognize that velocity is not frame-indifferent. Equation (2.3.36) can be expressed as

$$\boldsymbol{v}^* - \boldsymbol{Q}(t)\boldsymbol{v} = \dot{\boldsymbol{c}} + \boldsymbol{A}[\boldsymbol{x}^* - \boldsymbol{c}], \tag{2.3.37}$$

where

$$\boldsymbol{A} = \dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}(t)^T = -\boldsymbol{A}^T.$$

The tensor $\boldsymbol{A}$ is called the angular velocity of the starred frame with respect to the unstarred frame. It follows from (2.3.37) that

$$\boldsymbol{a}^* - \boldsymbol{Q}(t)\boldsymbol{a} = \ddot{\boldsymbol{c}} + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{v} + \ddot{\boldsymbol{Q}}(t)[\boldsymbol{x} - \boldsymbol{x}^0],$$

where $\boldsymbol{a} = \dot{\boldsymbol{v}}$ is the acceleration, and the above can be rewritten as

$$\boldsymbol{a}^* - \boldsymbol{Q}(t)\boldsymbol{a} = \ddot{\boldsymbol{c}} + 2\boldsymbol{A}(\boldsymbol{v}^* - \dot{\boldsymbol{c}}) + (\dot{\boldsymbol{A}} - \boldsymbol{A}^2)(\boldsymbol{x}^* - \boldsymbol{c}).$$

Next, we find that under a change of frame, the deformation gradient transforms according to

$$\boldsymbol{F}^* = \boldsymbol{Q}(t)\,\boldsymbol{F}(t); \tag{2.3.38}$$

but since the deformation gradient is a second-order tensor, it follows from (2.3.35) that the deformation gradient is not frame-indifferent. Furthermore, it follows from (2.3.38) and (2.3.11) that

$$\begin{aligned} \boldsymbol{L}^* = \dot{\boldsymbol{F}}^*(\boldsymbol{F}^*)^{-1} &= \left(\dot{\boldsymbol{Q}}(t)\,\boldsymbol{F} + \boldsymbol{Q}(t)\,\dot{\boldsymbol{F}}\right)\boldsymbol{F}^{-1}\boldsymbol{Q}(t)^{-1} \\ &= \dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}(t)^T + \boldsymbol{Q}(t)\boldsymbol{L}\,\boldsymbol{Q}(t)^T; \end{aligned} \tag{2.3.39}$$

here, it is understood that $\boldsymbol{F}$ depends also on t, but we have suppressed this dependence to simplify the notation. Thus once again, the gradient of the velocity is not frame-indifferent. Now, let us consider how the symmetric part of the velocity gradient transforms. We have

$$\begin{aligned} \boldsymbol{D}^* &= \frac{1}{2}\left(\boldsymbol{L}^* + (\boldsymbol{L}^*)^T\right) \\ &= \frac{1}{2}\left(\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}(t)^T + \boldsymbol{Q}(t)\boldsymbol{L}\,\boldsymbol{Q}(t)^T + \boldsymbol{Q}(t)\,\dot{\boldsymbol{Q}}(t)^T + \boldsymbol{Q}(t)\boldsymbol{L}^T\boldsymbol{Q}(t)^T\right) \\ &= \frac{1}{2}\,\boldsymbol{Q}(t)\boldsymbol{D}\,\boldsymbol{Q}(t)^T, \end{aligned}$$

because $\boldsymbol{Q}(t) \in \mathcal{O}$. Thus, the symmetric part of the velocity gradient is frame-indifferent, and hence, the first Rivlin–Ericksen tensor is frame-indifferent. However, the material time derivative of the symmetric part of the velocity gradient and the material time derivative of the first Rivlin–Ericksen tensor are not frame-indifferent.

Similarly, it can be shown that

$$\boldsymbol{W}^* = \boldsymbol{Q}(t)\boldsymbol{W}\,\boldsymbol{Q}(t)^T + \boldsymbol{A},$$

and hence the spin tensor is not frame-indifferent.

It is straightforward to verify that all the Rivlin–Ericksen tensors of order n, defined by the recursive relation (2.3.26) are frame-indifferent. It is possible to introduce a variety of frame-indifferent time derivatives, and one that has proved particularly useful is called the upper-convected Oldroyd derivative, which was introduced in (2.3.27).

As far as the Cauchy–Green stretch tensors are concerned, we have:

$$\boldsymbol{B}^* = \boldsymbol{F}^*(\boldsymbol{F}^*)^T = (\boldsymbol{Q}(t)\,\boldsymbol{F})(\boldsymbol{F}^T\,\boldsymbol{Q}(t)^T) = \boldsymbol{Q}(t)\,\boldsymbol{B}\,\boldsymbol{Q}(t)^T,$$

while

$$\boldsymbol{C}^* = (\boldsymbol{F}^*)^T\boldsymbol{F}^* = (\boldsymbol{F}^T\,\boldsymbol{Q}(t)^T)(\boldsymbol{Q}(t)\,\boldsymbol{F}) = \boldsymbol{C}.$$

Thus, $\boldsymbol{B}$ is frame-indifferent while $\boldsymbol{C}$ is not. On the other hand, according to (2.3.14) and (2.3.20), the relative-stretch tensor $\boldsymbol{C}_t(\tau)$ can be expressed as (we have suppressed the dependence on $\boldsymbol{x}$)

$$\begin{aligned}\boldsymbol{C}_t(\tau) = \boldsymbol{F}_t(\tau)^T\boldsymbol{F}_t(\tau) &= \left[\boldsymbol{F}(\tau)\boldsymbol{F}(t)^{-1}\right]^T\left[\boldsymbol{F}(\tau)\boldsymbol{F}(t)^{-1}\right]\\ &= \boldsymbol{F}(t)^{-T}\boldsymbol{F}(\tau)^T\boldsymbol{F}(\tau)\boldsymbol{F}(t)^{-1},\end{aligned}$$

where we have used the notation superscript $-T$ to denote the transpose of the inverse. Thus,

$$\begin{aligned}\boldsymbol{C}_t^*(\tau) &= (\boldsymbol{Q}(t)\boldsymbol{F}(t))^{-T}\,(\boldsymbol{Q}(\tau)\boldsymbol{F}(\tau))^T\,\boldsymbol{Q}(\tau)\boldsymbol{F}(\tau)\,(\boldsymbol{Q}(t)\boldsymbol{F}(t))^{-1}\\ &= \boldsymbol{Q}(t)^{-T}\boldsymbol{F}(t)^{-T}\boldsymbol{F}(\tau)^T\boldsymbol{Q}(\tau)^T\boldsymbol{Q}(\tau)\boldsymbol{F}(\tau)\boldsymbol{F}(t)^{-1}\boldsymbol{Q}(t)^T\\ &= \boldsymbol{Q}(t)\left[\boldsymbol{F}(t)^{-T}\boldsymbol{F}(\tau)^T\right]\left[\boldsymbol{F}(\tau)\boldsymbol{F}(t)^{-1}\right]\boldsymbol{Q}(t)^T = \boldsymbol{Q}(t)\boldsymbol{C}_t(\tau)\boldsymbol{Q}(t)^T,\end{aligned}$$

that is $\boldsymbol{C}_t(\tau)$ is frame-indifferent. However, the Cauchy–Green stretch tensor $\boldsymbol{B}_t(\tau)$ is not frame-indifferent.

A common mistake made in mechanics is to assume that all scalars are frame-indifferent. This is not true. Consider, for instance, the euclidean norm of the velocity

$$|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}.$$

A glance at (2.3.36) shows that in general,

$$|\boldsymbol{v}^*|^2 = \boldsymbol{v}^*\cdot\boldsymbol{v}^* \neq |\boldsymbol{v}|^2.$$

This point cannot be overemphasized as viscosities that depend on the norms of the velocity have been used to describe the flows of fluids.

2.3.3 Balance Laws

In this section, we shall document the basic balance laws that a continuum has to satisfy. Our discussion will be brief and a reader interested in details concerning the same can find them in Truesdell [275], Jaunzemis [141], Malvern [174], or any other standard text in continuum mechanics.

Let ϱ_{κ_R} denote the density in the configuration $\kappa_R(\mathcal{B})$ and ϱ denote the current density in the current configuration $\kappa_t(\mathcal{B})$. The local form of the balance of mass is

$$\varrho \det \boldsymbol{F}_{\kappa_R} = \varrho_{\kappa_R}, \tag{2.3.40}$$

which, if $\boldsymbol{F}_{\kappa_R}$ is continuously differentiable with respect to time, takes the form

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}[\varrho \boldsymbol{v}] = 0. \tag{2.3.41}$$

The local form of the balance of linear momentum is

$$\varrho \frac{d\boldsymbol{v}}{dt} = \operatorname{div} \boldsymbol{T} + \varrho\, \boldsymbol{b}, \tag{2.3.42}$$

where $\boldsymbol{T}$ is the Cauchy stress and $\boldsymbol{b}$ the specific body force.

We shall assume that there are no body couples, and in the absence of body couples, the balance of angular momentum implies that the stress is symmetric

We do not concern ourselves with thermodynamic issues in this book. But since we consider models that are thermodynamically compatible in the sense that the fluids meet the second law of thermodynamics expressed in the form of the Clausius–Duhem inequality, in the motions that they undergo, and the requirement that their specific Helmholtz potential be a minimum at equilibrium, for the sake of completeness we shall record the balance of energy and the Clausius–Duhem inequality. The local form of the balance of energy states that

$$\varrho \frac{d\varepsilon}{dt} = \boldsymbol{T} \cdot \boldsymbol{L} - \operatorname{div} \boldsymbol{q} + \varrho\, r, \tag{2.3.43}$$

where ε is the specific internal energy, $\boldsymbol{q}$ is the heat flux vector and r the specific radiant heating. Finally, the Clausius–Duhem inequality reads, with η the specific entropy

$$\varrho \frac{d\eta}{dt} \geq -\operatorname{div}\left(\frac{\boldsymbol{q}}{\theta}\right) + \frac{\varrho\, r}{\theta}. \tag{2.3.44}$$

Requiring that the motions of a fluid meet the Clausius–Duhem inequality leads to restrictions on the signs of the coefficients. We shall see that when the material coefficients satisfy the restrictions imposed by the Clausius–Duhem inequality and the requirement that the specific Helmholtz free energy be a minimum when the fluid is locally at rest, we have models that are well behaved in that they exhibit physically reasonable characteristics. For instance, the state of rest of the fluid is stable to disturbances and basic flows, provided that they are sufficiently "slow," are asymptotically stable. From the mathematical point of view, for such fluids, one is able to prove existence of solutions. On the other hand, when the material moduli do not satisfy the requirements of the Clausius–Duhem inequality, the fluids exhibit unacceptable stability characteristics and anomalous behavior.

2.4 Constitutive Relations for Fluids of the Differential Type

In this section, we introduce the constitutive relations for fluids of the differential type, but to stress the difference with other fluids, we shall provide a brief discussion of fluid models of the rate and integral types and postpone details to Section 2.8.

Let us first define the notion of history. The history of a scalar, vector or tensor-valued function of time, φ, denoted by $\varphi^t(s)$, is defined through

$$\varphi^t(s) = \varphi(t-s), \quad \text{for } s \geq 0.$$

A fluid is said to be an incompressible simple fluid (cf. Noll [198], Truesdell and Noll [276]) if the Cauchy stress is determined to within a spherical part by the history of the deformation gradient, i.e.,

$$\boldsymbol{T} = -p\boldsymbol{I} + \mathfrak{F}[\boldsymbol{F}_t(t-s)]\,. \tag{2.4.1}$$

The constitutively determined part of the stress is referred to as the extra stress tensor. While many of the popular models fall into the class of simple fluids, there are many that do not. In fact, rate type materials do not fall into the class of simple materials. A material is said to be a material of the rate type (in the sense of Truesdell and Noll [276]) if

$$\overset{(n)}{\boldsymbol{T}} = \boldsymbol{f}(\boldsymbol{T}, \dot{\boldsymbol{T}}, \ldots, \overset{(n-1)}{\boldsymbol{T}}, \boldsymbol{F}, \dot{\boldsymbol{F}}, \ldots, \overset{(n)}{\boldsymbol{F}}), \tag{2.4.2}$$

recall that the superscript (n) denotes n material time derivatives. Of course, frame indifference will place restrictions on the possible forms that are allowable. As Truesdell and Noll [276] p. 95 remark, models of the class (2.4.2) do not model a simple material, but a class of simple materials. It is also important to bear in mind that relations such as (2.4.2) provide an implicit relation between the stress and the kinematical quantities, and it might not be possible to express the stress explicitly in terms of the kinematical quantities. When this is possible, we get in fact a fluid of complexity n as defined below.

The rate type models (2.4.2) are special subclasses of implicit constitutive relations that relate the histories of the stress and the deformation gradient of the form

$$\mathfrak{G}\Big[\varrho(t-s)\boldsymbol{T}(t-s), \boldsymbol{F}(t-s)\Big] = \boldsymbol{0}. \tag{2.4.3}$$

Prusa and Rajagopal [219] studied fluids described through implicit constitutive relations of the form

$$\mathfrak{L}\Big[\boldsymbol{T}(t-s),\, \boldsymbol{F}_t(t-s)\Big] = \boldsymbol{0}. \tag{2.4.4}$$

In the case of incompressible fluids, using the assumption of fading memory and restricting themselves to retarded motions, they are able to obtain approximations of (2.4.4) that have the same form as the Maxwell, Oldroyd-B, Rivlin–Ericksen, and numerous other popular models that are used in non-Newtonian fluid mechanics. The approximations used by Prusa and Rajagopal [219], just as those used earlier by Coleman and Noll [68] in the case of a simple fluid, do not lead to models, as these representations only hold in the special flows being considered (see Dunn and Rajagopal [88] for a discussion as to why such approximations are not models in their own right).

It is worth observing that (2.4.2) provides an explicit expression for the nth material time derivative of the stress in terms of the $n-1$ material time derivatives of the deformation gradient. A more general implicit rate type relationship has the form

$$\boldsymbol{g}(\boldsymbol{T}, \ldots, \overset{(n)}{\boldsymbol{T}}, \boldsymbol{F}, \ldots, \overset{(m)}{\boldsymbol{F}}) = \boldsymbol{0}, \quad \text{where } n \text{ and } m \text{ are integers.} \tag{2.4.5}$$

A special case of such a relationship is the implicit relation

$$\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{F}) = \boldsymbol{0}, \tag{2.4.6}$$

for an elastic solid. Another special case is the relation

$$\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{A}_1) = \boldsymbol{0}. \tag{2.4.7}$$

The general representation for the implicit relationship between the stress and the first Rivlin–Ericksen tensor can be found in [227]. This representation includes as a special subclass the classical Stokesian fluid (see equation (2.4.12) that follows). The implicit relation also includes as a special subclass fluids modeled by

$$\boldsymbol{D} = \boldsymbol{f}(\varrho, \boldsymbol{T}).$$

Recently, Perlacova and Prusa [211] have used such implicit models to describe the response of colloids and slurries which cannot be described within the context of classical Stokesian fluids, as the relationship between the stress and the shear rate is multiple-valued.

In virtue of the constraint of incompressibility, the model

$$\boldsymbol{T} = -p\boldsymbol{I} + [\mu(p, \boldsymbol{A}_1)]\, \boldsymbol{A}_1 \tag{2.4.8}$$

for an incompressible fluid, with a viscosity that depends on the pressure p and the kinematical tensor $\boldsymbol{A}_1$, can be expressed as

$$\boldsymbol{T} = -\left(\frac{1}{3}\text{tr}(\boldsymbol{T})\right)\boldsymbol{I} + \left[\mu(\frac{1}{3}\text{tr}(\boldsymbol{T}), \boldsymbol{A}_1)\right]\boldsymbol{A}_1. \tag{2.4.9}$$

We see that the above constitutive relation (2.4.9) falls into the class of models (2.4.7) and does not belong to the class of materials defined through (2.4.2).

As we shall restrict our studies to purely mechanical issues in later chapters, that is to issues concerning the existence, uniqueness and stability of flows, exclusively with regard to the conservation of mass and the balance of linear momentum, we shall not provide constitutive relations for such fluids within the context of a fully thermodynamic framework. Therefore, the results that we shall establish cannot be impugned on the basis of the thermodynamic considerations being faulty. The interested reader can find a detailed and comprehensive treatment of the relevant issues in a critique of the thermodynamics of the fluids of the differential type by Dunn and Fosdick [87]. Results concerning the boundedness and stability of mechanical and thermal quantities for fluids of the differential type of grade n can be found in Rajagopal [222, 223].

An incompressible homogeneous isotropic[7] fluid of the differential type of complexity n is described by

$$\boldsymbol{T} = -p\boldsymbol{I} + \hat{\boldsymbol{f}}(\boldsymbol{L}, \dot{\boldsymbol{L}}, \dots, \overset{(n-1)}{\boldsymbol{L}}), \tag{2.4.10}$$

where $-p\boldsymbol{I}$ is the constraint response stress due to the constraint of incompressibility. The model (2.4.10) is not frame-indifferent. It follows from frame-indifference that the Cauchy stress in fluids of complexity n is related to the fluid motion through

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{f}(\boldsymbol{A}_1, \dots, \boldsymbol{A}_n), \tag{2.4.11}$$

where $\boldsymbol{A}_k$, $k = 1, \dots, n$, are the Rivlin–Ericksen tensors defined in (2.3.22). Such fluids were first studied by Rivlin and Ericksen in [246].

As mentioned earlier, results concerning the boundedness of both thermal and mechanical quantities for such fluids can be established (cf. Rajagopal [222]). We just cite a rather weak result concerning the boundedness of the kinetic energy associated with the flows of such fluids.

Theorem 2.4.1 *Suppose an incompressible fluid of complexity n is undergoing a process that is mechanically isolated*[8] *and meets the second law; then*

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[7] There is a popular misconception that by definition, a fluid is isotropic. If by fluid we mean a simple fluid in the sense of Noll and furthermore a body whose symmetry group is the unimodular group, then such fluids are isotropic. From such a narrow perspective one cannot have an anisotropic liquid, say a fluid model for a liquid crystal. On the other hand, the common definition that a fluid cannot support a shear stress indefinitely allows one to describe anisotropic fluids. For such fluids, the symmetry group is not the unimodular group (cf. Rajagopal [225] and Rajagopal and Srinivasa [235] for a discussion of the relevant issues).

[8] A body is said to be mechanically isolated if there is no working due to either the tractions on the boundary or due to the body forces.

$$0 \leq \int_{\Omega_t} \frac{\varrho}{2} |\boldsymbol{v}(t)|^2 \, dv \leq \int_{\Omega_{t'}} \frac{\varrho}{2} |\boldsymbol{v}(t')|^2 \, dv,$$

where $t' \leq t$. (That is, the kinetic energy is not increasing.)

The class of models described by (2.4.11) is too large to obtain sharper results. We can say a lot more for special subclasses of fluids of the differential type, namely fluids of grade n.

In this section, we shall be concerned with fluids of complexity one, i.e.,

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{f}(\boldsymbol{A}_1),$$

and fluids of complexity two, i.e.,

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{g}(\boldsymbol{A}_1, \boldsymbol{A}_2),$$

where the symbols $\boldsymbol{f}$ and $\boldsymbol{g}$ stand for functions in general. Let us consider first the more general case of a compressible fluid of complexity one. Such fluids are usually referred to as Stokesian fluids, as they were first studied in full detail from phenomenological point of view by Stokes [259].[9] The Cauchy stress in such fluids is given by

$$\boldsymbol{T} = \boldsymbol{f}(\varrho, \boldsymbol{A}_1). \tag{2.4.12}$$

Isotropy demands that

$$\forall \boldsymbol{Q} \in \mathcal{O}, \; \boldsymbol{f}(\varrho, \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{f}(\varrho, \boldsymbol{A}_1)\boldsymbol{Q}^T. \tag{2.4.13}$$

It then follows from standard representation theorems (cf. Spencer [256]) that

$$\boldsymbol{f}(\varrho, \boldsymbol{A}_1) = \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{A}_1 + \alpha_2\boldsymbol{A}_1^2, \tag{2.4.14}$$

where the α_i, $i = 0, 1, 2$, are functions of the density ϱ and the principal invariants of $\boldsymbol{A}_1$

$$I_1 = \operatorname{tr} \boldsymbol{A}_1, \; I_2 = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{A}_1)^2 - \operatorname{tr}(\boldsymbol{A}_1^2)\right], \; I_3 = \det \boldsymbol{A}_1.$$

Reiner [242] developed a model of the class (2.4.14) in which α_1 and α_2 are polynomials in the invariants, to describe the phenomenon of "dilatancy" (that property of a material by which it increases volume when stirred), a term coined by Reynolds [243] to describe the response of granular solids and granular solids infused with fluids. This behavior exhibited by granular materials is very similar to the swelling that takes place in non-Newtonian fluids due to shearing and the phenomenon of rod-climbing due to stirring. Normal stress differences are at the root of

[9] Reiner [242] derived and discussed special flows of a subclass of such fluids by using a power-series expansion. We shall discuss mathematical issues concerning these fluids as well as a more general subclass of those fluids that are referred to as Reiner–Rivlin fluids.

these phenomena in different types of materials. It is possible that α_1 and α_2 could be general functions of the invariants, and such fluids are called Reiner–Rivlin fluids.

The Reiner–Rivlin model (2.4.14) does not predict distinct normal stress differences in simple shear flow. While distinct normal stress differences have been observed in flows of certain polymeric liquids, one should bear in mind that there are many fluids that exhibit no normal stress differences, while exhibiting pronounced shear-thinning or shear-thickening. Many geological, biological materials, as well as food products and chemicals, are adequately described by the Reiner–Rivlin model (2.4.14). This fact has been largely overlooked by the mathematical community in view of the indictment of the model by Truesdell and Noll [276]: "Data was collected; they seem to indicate that for polyisobutylene solutions, $T^{<11>}$ and $T^{<22>}$ are not equal when κ becomes appreciably large. This experimental evidence was motive for the rejection of the Reiner–Rivlin equation (119.4) as an adequate basis of a physical theory and for the search of greater generality, in the last few years…." In a similar vein, Coleman et al. [67] observe: "Note that the two normal stress functions (30.4) resulting from the Reiner–Rivlin equation (30.2) are the same, which implies for example that the normal stresses in simple shear flow must coincide (see E.11). A relation of this kind can be compared with experiments. Data reported by Padden and De Witt show that they are not the same in some polymer solutions. This observation was a motive for the rejection of the Reiner–Rivlin theory. There is a large literature on the Reiner–Rivlin theory which must be regarded as obsolete today."

Unfortunately, these statements have been taken more seriously than they ought to have been. One cannot indict the whole class of models on the basis of experiments on polyisobutylene or for that matter on a few dilute polymeric liquids. While the Reiner–Rivlin model might be unsuitable for fluids that have distinct normal stress differences in simple shear flows, it is a perfectly reasonable model for shear-thinning and shear-thickening fluids. In fact the most popular models in chemical engineering, food rheology, glaciology and other areas are generalized power-law models or generalizations of the Navier–Stokes model that belong to the class of Reiner–Rivlin fluids. We notice that all the popular power-law fluid models and generalized Newtonian fluid models fall into the class of Stokesian fluids. Mathematical issues concerning such fluids have been studied in great detail (cf. the book by Málek et al. [168] for some of the recent results). We shall discuss these fluids later.

If we require the fluid to be linear in $\boldsymbol{A}_1$, then (2.4.14) simplifies to

$$\boldsymbol{f}(\varrho, \boldsymbol{A}_1) = \big[-p(\varrho) + \lambda(\varrho)(\operatorname{tr}\boldsymbol{A}_1)\big]\boldsymbol{I} + \mu(\varrho)\boldsymbol{A}_1, \tag{2.4.15}$$

where λ and μ are called the bulk and shear viscosities, and the negative sign in front of $p(\varrho)$ is due to the sign convention that compressive stresses are negative. A few remarks concerning the material moduli λ and μ are in order. While the material modulus μ can be determined directly through an experiment, the material modulus λ cannot be measured directly. It has to be inferred from measurements of μ and $(3\,\lambda + 2\,\mu)$, the latter being the bulk modulus of the fluid.

In a seminal paper that has had tremendous influence on the development of fluid dynamics, Stokes [259] suggested that there were several flows wherein the density of the fluid remains nearly constant and wherein one could make the assumption

$$3\,\lambda + 2\,\mu = 0.$$

This assumption, referred to as the "Stokes Assumption," has become a central part of fluid dynamics. Maxwell [182–184] claimed to have derived the "Stokes Assumption" for monatomic gases within the context of the kinetic theory of gases, and this gave further credibility to the "Stokes Assumption." However, as Truesdell [271] points out, Maxwell's results are a direct consequence of the assumptions he makes for the pressure and the temperature and thus the results are actually built into the assumptions. Despite Maxwell's work, Stokes [259] was far from convinced by his own assumption. He remarks (Stokes [260]): "Although I have shown (Volume 1, p. 119) that on the admission of a supposition which Poisson would have allowed, the two constants in his equation are reduced to one, and the equations take the form (1), and although Maxwell obtained the same equations from his kinetic theory of gases (Philosophical Transactions for 1867, p. 81) I have always felt that the correctness of the value $\mu/3$ for the coefficient in the last term of (1) does not rest on as firm a basis as the correctness of the equations of motion of an incompressible fluid, for which the last term does not come in at all." In fact, the "Stokes Assumption" is clearly wrong. There is overwhelming experimental evidence that contradicts the assumption despite which the assumption continues to be used. A detailed discussion as to why the "Stokes Assumption" is inapt can be found in the recent paper by Rajagopal [229] wherein a new interpretation and development of the Navier–Stokes constitutive relation is provided.

If we now require in addition that the fluid be incompressible, then as

$$\frac{1}{2}\mathrm{tr}\,\boldsymbol{A}_1 = \mathrm{div}\,\boldsymbol{v} = 0, \tag{2.4.16}$$

and since ϱ is a constant, because the fluid is incompressible and homogeneous,[10] we find that (2.4.15) reduces to

$$\boldsymbol{T} = \boldsymbol{f}(\boldsymbol{A}_1) = -p\boldsymbol{I} + \mu\boldsymbol{A}_1, \tag{2.4.17}$$

where p is indeterminate due to the constraint of incompressibility. This is the popular incompressible Navier–Stokes fluid model. This model approximates exceptionally well the response of liquids like water, in certain flows. However, such a model fails to describe the response of water through a wide range of pressures. As mentioned at the beginning of the introduction, Stokes [259] recognized that the model (2.4.17)

[10] If the fluid is incompressible but inhomogeneous then the density is not a constant everywhere, the density of a specific material point remains the same, i.e., $\varrho = \varrho(\boldsymbol{x})$.

is obtained under certain assumptions that do not apply to all flows and this matter warrants discussion.

In classical mechanics, one encounters a variety of constraints such as incompressibility, inextensibility, etc. Of course no material is truly incompressible; it is merely that the variations in the density are small enough to be negligible even under large spherical stresses. Such materials are approximated as bodies that are incompressible, and the stresses in such bodies are determined to within an indeterminate constraint stress. In the case of (2.4.17), the indeterminate part is $-p\boldsymbol{I}$. In classical mechanics, it is assumed that the constraint force does no work, but it is important to bear in mind that this is merely an assumption and a variety of other assumptions could be made concerning the constraint (cf. Liu [166]). In this context, it is worthwhile to recognize an important result due to Gauss [110] where he makes the point that the constraint response merely enforces the constraint and it may or may not do work. As written in the introduction, Stokes [259] recognized that it is possible for the material moduli to depend on the constraint stress; for instance, the viscosity μ could depend on the pressure. This assumption of Stokes is valid for incompressible lubricants used for elastohydrodynamic lubrication where the fluid is subject to a wide range of pressures. Under such severe pressures, it is found that the viscosity varies exponentially with the pressure! (cf. Andrade [11] and Bridgman [51]).

In the case of incompressible fluids, we saw that the material moduli can depend on the constraint response. Thus, in the case of an incompressible fluid of complexity one, the Cauchy stress could take the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1(p, I_1, I_2)\boldsymbol{A}_1 + \alpha_2(p, I_1, I_2)\boldsymbol{A}_1^2, \tag{2.4.18}$$

where

$$p = -\frac{1}{3}(\operatorname{tr}\boldsymbol{T}).$$

This model is fundamentally different from (2.4.14) in that (2.4.18) provides an implicit constitutive relation, i.e., equation (2.4.18) is of the form

$$\boldsymbol{f}(\boldsymbol{T}, \boldsymbol{A}_1) = \boldsymbol{0}.$$

If one requires that the fluid of complexity one be incompressible and also that the stress depend linearly on $\boldsymbol{A}_1$, then (2.4.18) reduces to

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu(p)\boldsymbol{A}_1.$$

There is a large body of experimental work that attests to the fact that, while the density changes are very small (i.e., of the order of a percent or less), the viscosity may change by a factor of 10^8 when the pressure changes considerably. In fact, if the pressures are sufficiently high, as it is the case in elastohydrodynamic lubrication, it is even possible that "glass transition" might take place (cf. Szeri [261]). A thorough discussion of the early experimental work on the variation of viscosity with pressure can be found in the authoritative text on the physics of high pressure by Bridgman [52].

It is however possible that in constrained materials, the material moduli can depend on the constraint response. In this case, the isotropy condition (2.4.13) and the constraint will lead to a representation different from (2.4.14). This leads to a class of models that have not been studied with the intensity that they deserve.

In the case of a compressible fluid of complexity two, isotropy requires that

$$\forall \boldsymbol{Q} \in \mathcal{O},\ \boldsymbol{g}(\varrho,\ \boldsymbol{Q}\,\boldsymbol{A}_1\boldsymbol{Q}^T,\ \boldsymbol{Q}\,\boldsymbol{A}_2\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{g}(\varrho,\ \boldsymbol{A}_1,\ \boldsymbol{A}_2)\,\boldsymbol{Q}^T.$$

Once again, using results from representation theory, we find that

$$\begin{aligned}\boldsymbol{g}(\varrho, \boldsymbol{A}_1, \boldsymbol{A}_2) = {} & \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{A}_1 + \alpha_2\boldsymbol{A}_1^2 + \alpha_3\boldsymbol{A}_2 + \alpha_4\boldsymbol{A}_2^2 + \alpha_5[\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1] \\ & + \alpha_6[\boldsymbol{A}_1^2\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1^2] + \alpha_7[\boldsymbol{A}_2^2\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{A}_2^2],\end{aligned}$$

where the coefficients $\alpha_0, \ldots, \alpha_7$ depend on the density and on the invariants

$$I_1 = \mathrm{tr}\,\boldsymbol{A}_1,\ I_2 = \mathrm{tr}\,\boldsymbol{A}_1^2,\ I_3 = \mathrm{tr}\,\boldsymbol{A}_1^3,\ I_4 = \mathrm{tr}(\boldsymbol{A}_1\boldsymbol{A}_2),$$

$$I_5 = \mathrm{tr}(\boldsymbol{A}_1^2\boldsymbol{A}_2),\ I_6 = \mathrm{tr}\,\boldsymbol{A}_2,\ I_7 = \mathrm{tr}\,\boldsymbol{A}_2^2,\ I_8 = \mathrm{tr}\,\boldsymbol{A}_2^3,\ I_9 = \mathrm{tr}(\boldsymbol{A}_2^2\boldsymbol{A}_1).$$

We shall be concerned primarily with fluids of grade two, i.e., fluids for which the Cauchy stress is

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2, \tag{2.4.19}$$

and a special subclass of fluids of grade three. In a general fluid of grade three, the Cauchy stress has the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_1\boldsymbol{A}_3 + \beta_2[\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1] + \beta_3(\mathrm{tr}\,\boldsymbol{A}_1^2)\boldsymbol{A}_1.$$

If the above fluid of grade three is to be thermodynamically compatible (cf. Fosdick and Rajagopal [101]), then it follows that the above constitutive relation reduces to

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(\mathrm{tr}\,\boldsymbol{A}_1^2)\boldsymbol{A}_1, \tag{2.4.20}$$

where

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \beta_3 \geq 0,\ \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}. \tag{2.4.21}$$

When $\beta_3 = 0$, the above restrictions reduce to

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \alpha_1 + \alpha_2 = 0, \tag{2.4.22}$$

which are the restrictions obtained by Dunn and Fosdick [87] for fluids of grade two.[11] We immediately notice that thermodynamically compatible fluids of grades two and three are fluids of complexity two.

In a much cited paper that has also been much misunderstood, Coleman and Noll [68] showed that a model of the form (2.4.19) could be obtained as a truncation at second order from the expansion for the stress in a simple fluid with a very special fading memory within the context of what they termed as retarded (slowed down) motions. The documentation of the mathematical procedure that they adopted and the result that they obtained is referred to as the "Retardation Theorem." Coleman and Noll [68] did not obtain models for describing the response of fluids, nor did they claim that they obtained models. They essentially showed that for very special simple fluids undergoing very special motions, when the expression for the stress is truncated using a special expansion procedure, at second order the truncation led to an expression of the form (2.4.19). However, outside of all the special circumstances that were assumed, their procedure remains mute with regard to the form the stress takes on in the simplified structure. This notwithstanding, a great deal of misunderstanding has resulted leading to overblown statements that the restrictions (2.4.22) that are a consequence of the Clausius–Duhem inequality are incorrect as they are contradicted by experiments, thereby impugning the Clausius–Duhem inequality. A detailed discussion of all the relevant issues will take us far afield and so we refer the reader to a critical review of the issues by Dunn and Rajagopal [88]. Here, we quote a clear and succinct discussion of the issues that has unfortunately, by and large, gone mostly unheeded (see Truesdell [274]):

"In the experimental literature the result of Coleman and Noll is sometimes given a vastly exaggerated statement such as: "For sufficiently slow flows the second-order fluid is a valid approximation to any simple fluid." Coleman and Noll neither stated nor proved any such thing. In particular, they never claimed that all simple fluids have fading memory in the sense of Coleman and Noll, for it is easy to provide examples of fluids that do not. Second, while their general theory concerns many different kinds of approximation, their only application to "slow" flows refers to those obtained by retardation of a given flow. Other definitions of "slow" lead to somewhat different results. Finally, they never claimed nor proved any relation at all between the solutions of differential equations of motion for the fluid of grade 2 and the solutions of the equations of motion of the general simple fluid that the fluid of grade 2 approximates. An example known from the kinetic theory of gases shows that it is risky to jump to conclusions in matters of this kind."

Recently, a thermodynamic framework has been put into place to describe bodies undergoing entropy producing processes. It appeals to the following central idea: the rate of entropy production (rate of dissipation for purely isothermal processes) must be maximized, and this among a set of admissible constitutive relations that automatically meet that the entropy production be nonnegative. It can be shown easily within that framework that the classical Navier–Stokes model corresponds to a fluid

[11] Ting [268] studied several flows of fluids of grade two, that is fluids modeled by (2.4.19). None of the problems he studied had bounded solutions when α_1 was negative.

whose specific Helmholtz potential ψ is given by

$$\psi = \psi(\theta),$$

and the rate of dissipation ξ is given by

$$\xi = \xi(\theta, \boldsymbol{A}_1) = \frac{\mu(\theta)}{2}|\boldsymbol{A}_1|^2. \tag{2.4.23}$$

Rajagopal and Srinivasa [234] have shown that rate type viscoelastic fluid models such as the Maxwell fluid, the Oldroyd-B fluid, and a variety of other popular models can be derived within such a framework (see Murali Krishnan and Rajagopal [154] for a derivation of Burgers' model). The model takes into account how energy is stored by the body, how this energy that is stored is released (whether it can be recovered in a purely mechanical process or whether one needs to use a thermodynamic process to recover the energy), how the body produces entropy, conducts heat, etc. Rao and Rajagopal [241] have shown that the integral K-BKZ model and generalizations of it can also be obtained within that framework. However, fluids of the differential type such as grade-two fluids do not seem to fall within the class of models that can be obtained directly using the above thermodynamic framework. We need to make certain modifications to the theory in order to obtain the grade-two fluid (see Rajagopal and Srinivasa [234]). These modifications lead to results that are completely in keeping with the results that arise from enforcing the Clausius–Duhem inequality.

If one makes the choice that the Helmholtz potential ψ is given by

$$\psi = \psi(\theta, \boldsymbol{A}_1) = \hat{\psi}(\theta) + \frac{\alpha_1}{4\varrho}|\boldsymbol{A}_1|^2 \text{ with } \alpha_1 > 0,$$

and the rate of dissipation ξ is given by (2.4.23) then one obtains (see [234]) that the stress $\bar{\boldsymbol{T}}$ is

$$\bar{\boldsymbol{T}} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\dot{\boldsymbol{A}}_1. \tag{2.4.24}$$

We note that the model (2.4.24) for $\bar{\boldsymbol{T}}$ is not frame-indifferent as $\dot{\boldsymbol{A}}_1$ is not frame-indifferent. However, it is possible to obtain a properly frame-indifferent model, with the same associated rate of dissipation as $\bar{\boldsymbol{T}}$, that leads to the model for a grade-two fluid. Indeed, on adding the term

$$\hat{\boldsymbol{T}} = \alpha_1(\boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1) - \alpha_1\boldsymbol{A}_1^2,$$

to (2.4.24), we immediately obtain a thermodynamically compatible model for a grade-two fluid wherein the stress is given by (2.4.19) with $\alpha_1 = -\alpha_2$. It is important to note that

$$\hat{\boldsymbol{T}} \cdot \boldsymbol{L} = \boldsymbol{0},$$

and thus there is no rate of dissipation associated with $\hat{\boldsymbol{T}}$. Hence the stress $\boldsymbol{T} = \bar{\boldsymbol{T}} + \hat{\boldsymbol{T}}$ has the same associated rate of dissipation as the stress $\boldsymbol{T}$ which is the model for a grade-two fluid.

When substituting (2.4.20) into the well-known balance of linear momentum:

$$\varrho \frac{dv}{dt} = \operatorname{div} \boldsymbol{T} + \varrho\, \boldsymbol{b}, \tag{2.4.25}$$

after dividing by the density, making obvious changes in notation, and denoting the gradient by ∇, we obtain the equation of motion of a grade-three fluid

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{u} - \alpha_1 \varDelta\, \boldsymbol{u}) - \mu\, \varDelta\, \boldsymbol{u} + \mathbf{curl}(\boldsymbol{u} - (2\alpha_1 + \alpha_2)\varDelta\, \boldsymbol{u}) \times \boldsymbol{u} \\ - (\alpha_1 + \alpha_2)\big(\varDelta([\nabla\, \boldsymbol{u}]\boldsymbol{u}) - 2[\nabla(\varDelta\, \boldsymbol{u})]\boldsymbol{u}\big) - \beta_3 \operatorname{div}(|\boldsymbol{A}_1|^2 \boldsymbol{A}_1) \\ + \nabla\big(p - (2\alpha_1 + \alpha_2)(\boldsymbol{u} \cdot \varDelta\, \boldsymbol{u} + \frac{1}{4}|\boldsymbol{A}_1|^2) + \frac{1}{2}|\boldsymbol{u}|^2\big) = \boldsymbol{f}. \end{aligned} \tag{2.4.26}$$

If we set $\beta_3 = 0$, then (2.4.26) reduces to

$$\begin{aligned} \frac{\partial}{\partial t}(\boldsymbol{u} - \alpha_1 \varDelta\, \boldsymbol{u}) - \mu\, \varDelta\, \boldsymbol{u} + \mathbf{curl}(\boldsymbol{u} - (2\alpha_1 + \alpha_2)\varDelta\, \boldsymbol{u}) \times \boldsymbol{u} \\ - (\alpha_1 + \alpha_2)\big(\varDelta([\nabla\, \boldsymbol{u}]\boldsymbol{u}) - 2[\nabla(\varDelta\, \boldsymbol{u})]\boldsymbol{u}\big) \\ + \nabla\big(p - (2\alpha_1 + \alpha_2)(\boldsymbol{u} \cdot \varDelta\, \boldsymbol{u} + \frac{1}{4}|\boldsymbol{A}_1|^2) + \frac{1}{2}|\boldsymbol{u}|^2\big) = \boldsymbol{f}, \end{aligned} \tag{2.4.27}$$

which is the equation of motion of a grade-two fluid and of course can also be obtained by substituting (2.4.19) into (2.4.25).

2.4.1 Special Motions

There are special flows in which it is impossible to distinguish a general simple fluid from a fluid of complexity three, i.e., in such flows any simple fluid can be described by an appropriate fluid of complexity three. One such class is called monotonous flows (cf. Noll [198]) and many flows such as shear flows fall into this category. A motion is said to be monotonous if and only if there is an orthogonal tensor $\boldsymbol{Q}(t)$, a scalar κ, and a constant tensor $\boldsymbol{N}_0$ such that (see (2.4.1) with $\hat{\tau} = t - s$)

$$\boldsymbol{F}_0(\hat{\tau}) = \boldsymbol{Q}(\hat{\tau})\mathrm{e}^{\hat{\tau}\kappa \boldsymbol{N}_0}, \;\; \boldsymbol{Q}(0) = \boldsymbol{I}, \;\; |\boldsymbol{N}_0| = 1. \tag{2.4.28}$$

In such flows, a simple fluid (2.4.1) has a much simpler representation. Its stress is completely determined by the first three Rivlin–Ericksen tensors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$; other information concerning the history of the relative deformation gradient is unnecessary.

A special subclass of monotonous flows are viscometric flows. These are monotonous flows wherein $\boldsymbol{N}_0^2 = \boldsymbol{0}$. In such flows, a simple fluid cannot be distinguished from a fluid of complexity two, i.e., the first two Rivlin–Ericksen tensors are sufficient to determine the stress. In fact, in viscometric flows, the stress is completely determined by three functions referred to as viscometric functions (cf. Coleman et al. [67]). To illustrate this fact, let us consider the steady linear flow (cf. (2.2.1))

$$\boldsymbol{v} = u(y)\boldsymbol{i},$$

and recall that

$$\kappa := u'(y).$$

Further, define the three viscometric functions by

$$\tau(\kappa) := T_{xy},\ \sigma_1(\kappa) := T_{xx} - T_{zz},\ \sigma_2(\kappa) := T_{yy} - T_{zz}.$$

Then, it follows from (2.4.28) that the extra stress $\boldsymbol{S}$ is given by (see Noll [198] for details)

$$\boldsymbol{S} = \tau(\kappa)(\boldsymbol{N} + \boldsymbol{N}^T) + \sigma_1(\kappa)\boldsymbol{N}^T\boldsymbol{N} + \sigma_2(\kappa)\boldsymbol{N}\boldsymbol{N}^T, \tag{2.4.29}$$

where

$$\boldsymbol{N} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that (2.4.29) is equivalent to (cf. Criminale et al. [71])

$$\boldsymbol{S} = \frac{\tau(\kappa)}{\kappa}\boldsymbol{A}_1 + \frac{1}{2}\frac{[\sigma_1(\kappa) - \sigma_2(\kappa)]}{\kappa^2}\boldsymbol{A}_2 + \frac{\sigma_2(\kappa)}{\kappa^2}\boldsymbol{A}_1^2.$$

It is obvious from the above expression for the extra stress that in viscometric flows, the stress for a simple fluid takes the same form as that of a fluid of complexity two.

2.4.2 Secondary Flows in Pipes

Most non-Newtonian fluids cannot flow unidirectionally along an axis of a pipe of noncircular cross-section, in marked contrast to the Navier–Stokes fluid. In fact, the similarity in the secondary flow structure between the laminar flow of viscoelastic fluids and the turbulent flow of a Navier–Stokes fluid have led to the development of models for turbulent flows of the Navier–Stokes fluid (see Rivlin [245] and Huang and Rajagopal [136]). The problem of the flows of fluids of grade n in noncircular pipes has been studied at great length ever since the study of Ericksen [94] that first investigated this issue.

Fosdick and Serrin [102] have obtained necessary and sufficient conditions for the development of secondary flows in simple fluids. It immediately follows from these conditions that a fluid of grade three cannot flow in a straight path along the axis of a noncircular pipe while a fluid of grade two can. Fosdick and Serrin study the class of simple fluids whose viscometric functions satisfy

1. $\tilde{\mu}(\kappa)$ is of class $\mathcal{C}^2$,
2. $\tilde{\mu}(\kappa)$ and $\sigma_1(\kappa)/\kappa^2$ are of class $\mathcal{C}^3$ near $\kappa = 0$ and their first derivatives and third derivatives vanish at $\kappa = 0$,
3. $\kappa\tilde{\mu}(\kappa)$ is an increasing function of κ.

They show that if

$$\sigma_1(\kappa) \neq C\kappa^2\tilde{\mu}(\kappa),$$

where C is a constant, then rectilinear flow along the axis of a cylindrical tube is possible only in circular tubes. In addition, such a flow is possible in the annular region between two circular concentric tubes. Ericksen [94] had originally conjectured that rectilinear flows would be possible in a tube, the boundary of its cross-section being comprised of parts of circles and straight lines. The analysis of Fosdick and Serrin [102] proved the conjecture false.

In the case of a fluid of grade two, we have in simple shear flows

$$\tilde{\mu}(\kappa) = \mu,\ \sigma_1(\kappa) = (2\alpha_1 + \alpha_2)\kappa^2,\ \sigma_2(\kappa) = \alpha_2\kappa^2.$$

Thus

$$\sigma_1(\kappa) = \frac{2\alpha_1 + \alpha_2}{\mu}\kappa^2\mu.$$

However, in the case of simple shearing flow of a fluid of grade three

$$\tilde{\mu}(\kappa) = \mu + 2(\beta_2 + \beta_3)\kappa^2,\ \sigma_1(\kappa) = (2\alpha_1 + \alpha_2)\kappa^2,\ \sigma_2(\kappa) = \alpha_2\kappa^2,$$

thus

$$\sigma_1(\kappa) \neq C\kappa^2\tilde{\mu}(\kappa),$$

and hence we will have secondary flow.

Assuming that the driving force, i.e., the pressure gradient along the axis of the pipe, is small, Truesdell and Noll [276] study the structure of the secondary flow using a perturbation approach, the perturbation parameter being the driving force. Thus, the basic solution that is being perturbed is the null solution. It turns out that under such a perturbation scheme, the secondary flow manifests itself only at fourth order. Normal tractions, not required in a Navier–Stokes fluid are required in this case (see Pipkin and Rivlin [212]).

We could ask a different question concerning secondary flows in pipes: what is the nature of the secondary flow when the bounding surface of the pipe has a small departure from circularity, the driving force being large? In this case, we cannot study the problem via a perturbation of the driving force; we need to use a domain perturbation technique (see Hadamard [129]). It then transpires that in this case, the secondary flow manifests itself at first order for a fluid of grade three. The zeroth order solution in this case is the Poiseuille flow in a pipe of circular cross-section (see Huang and Rajagopal [136] and Mollica and Rajagopal [192]).

2.4.3 Stability to Finite Disturbances

As explained in the introduction, we are not overly concerned about thermodynamic considerations. However, we find that the consequences of such considerations lead to the fluid exhibiting reasonable response characteristics, while violation of such considerations lead to unacceptable behavior. An important response characteristic is the flow's stability. In this section, we establish results that are similar in nature to the seminal contributions of Orr, Hopf, Kampe de Feriet and Thomas concerning the stability of flows of the Navier–Stokes fluid. The results we establish will leave no doubt as to the proper signs of the material coefficients that appear in the model for fluids of grade two and three. Before we can establish the stability results, we will have to put into place certain identities.

The starting point for the study of stability to finite disturbances is the Power Theorem. Recall the balance of linear momentum (2.4.25)

$$\varrho \frac{dv}{dt} = \operatorname{div} \boldsymbol{T} + \varrho\, \boldsymbol{b}.$$

On taking the scalar product of both the left-hand and right-hand sides of the above equation with the velocity $\boldsymbol{v}$ and integrating over the flow domain, integrating by parts, using the divergence theorem, and the conservation of mass, we obtain

$$\int_{\partial \Omega_t} \boldsymbol{t} \cdot \boldsymbol{v}\, da + \int_{\Omega_t} \varrho\, \boldsymbol{b} \cdot \boldsymbol{v}\, dv = \int_{\Omega_t} \frac{\varrho}{2} \frac{d}{dt}(|\boldsymbol{v}|^2) dv + \int_{\Omega_t} \boldsymbol{T} \cdot \boldsymbol{L}\, dv,$$

where $\boldsymbol{t} = \boldsymbol{T}^T \boldsymbol{n}$ is the surface traction vector, $\boldsymbol{n}$ is the unit outward normal to the boundary, and recall that $|\cdot|$ denotes the Euclidean norm. The first term in the right-hand side denotes the time rate of change of the kinetic energy, while the second term denotes the stress power (the rate at which work is done by the stress on the part Ω_t). This equality is usually referred to as the power theorem and is the starting point of all nonlinear stability analysis.

Owing to the conservation of mass, we have for any function φ

$$\int_{\Omega_t} \varrho\, \varphi\, dv = \int_{\Omega_0} \varrho_0 \varphi\, |\det \boldsymbol{F}|\, dv,$$

where ϱ_0 is independent of t.[12] Therefore,

$$\frac{d}{dt}\int_{\Omega_t} \varrho\,\varphi\,dv = \int_{\Omega_0} \varrho_0 \frac{d}{dt}\varphi\,dv = \int_{\Omega_t} \varrho \frac{d}{dt}\varphi\,dv.$$

Applying this equality with $\varphi = |\boldsymbol{v}|^2$, we derive

$$\int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{v}\,da - \int_{\Omega_t} \boldsymbol{T}\cdot\boldsymbol{L}\,dv + \int_{\Omega_t} \varrho\,\boldsymbol{b}\cdot\boldsymbol{v}\,dv = \frac{d}{dt}\int_{\Omega_t} \frac{\varrho}{2}|\boldsymbol{v}|^2 dv. \tag{2.4.30}$$

If the body is mechanically isolated, then

$$\int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{v}\,da + \int_{\Omega_t} \varrho\,\boldsymbol{b}\cdot\boldsymbol{v}\,dv = 0,$$

and thus we find that (2.4.30) reduces to

$$-\int_{\Omega_t} \boldsymbol{T}\cdot\boldsymbol{L}\,dv = \frac{d}{dt}\int_{\Omega_t} \frac{\varrho}{2}|\boldsymbol{v}|^2 dv. \tag{2.4.31}$$

In the case of the Navier–Stokes fluid, where $\boldsymbol{T}$ is given by (2.4.17), we immediately obtain the rather weak result

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho\,|\boldsymbol{v}|^2 dv = -\frac{1}{2}\int_{\Omega_t} \mu|\boldsymbol{A}_1|^2 dv, \tag{2.4.32}$$

(for a tensor $|\cdot|$ denotes the Frobenius norm), and we conclude that the kinetic energy is nonincreasing provided $\mu \geq 0$. Of course, if $\mu < 0$, we would conclude that the kinetic energy is increasing, an absurd situation that leads us to conclude that the assumption $\mu \geq 0$ is sensible. Our conclusions concerning the material modulus α_1 can be based on similar grounds, without any appeal to thermodynamic arguments.

We shall derive the energy identity (2.4.31) for the model (2.4.20) and by setting $\beta_3 = 0$, we shall recover the energy identity for a fluid of grade two. On substituting (2.4.20) into the energy identity (2.4.31) and using the fact that the density is constant, we obtain the energy identity (that can also be obtained by taking the scalar product of both sides of (2.4.26) by $\boldsymbol{v}$)

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} |\boldsymbol{v}|^2 dv + \frac{\alpha_1}{2\varrho}\frac{d}{dt}\int_{\Omega_t} |\boldsymbol{A}_1|^2 dv + \frac{\mu}{\varrho}\int_{\Omega_t} |\boldsymbol{A}_1|^2 dv&\\ + \frac{\alpha_1+\alpha_2}{\varrho}\int_{\Omega_t} (\mathrm{tr}\,\boldsymbol{A}_1^3) dv + \frac{\beta_3}{\varrho}\int_{\Omega_t} |\boldsymbol{A}_1|^4 dv = 0.&\end{aligned} \tag{2.4.33}$$

On setting $\beta_3 = 0$ in the above equation, and using $\alpha_1 + \alpha_2 = 0$, we find that

[12] More precisely, in the second integral φ stands for $\varphi(\boldsymbol{x}(\boldsymbol{X}, 0), 0)$.

$$\frac{d}{dt}\left(\int_{\Omega_t}|\boldsymbol{v}|^2 dv + \frac{\alpha_1}{2\varrho}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv\right) = -\frac{\mu}{\varrho}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv. \tag{2.4.34}$$

We hasten to add that even if $\alpha_1 + \alpha_2 \neq 0$, we can show that the rest state of the fluid is unstable if $\alpha_1 < 0$. For the moment, let us suppose that $\alpha_1 > 0$. We now define a functional $E(t)$ by

$$E(t) := \int_{\Omega_t}|\boldsymbol{v}|^2 dv + \frac{\alpha_1}{2\varrho}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv. \tag{2.4.35}$$

We note that since $\alpha_1 > 0$ and $\varrho > 0$, then $E(t)$ is zero if and only if $\boldsymbol{v} = \boldsymbol{0}$. Thus, (2.4.34) reads

$$\frac{d}{dt}E(t) = -\frac{\mu}{\varrho}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv, \tag{2.4.36}$$

and since $\mu > 0$ and $\varrho > 0$, we have that the energy is nonincreasing, a result similar to that for the Navier–Stokes fluid (2.4.32). This result is not sufficiently strong. We next show that the energy decays to zero with time, bounded by an exponential, a result due to Dunn and Fosdick [87]. It is stated in a compact domain in which the flow satisfies

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{0} \text{ on } \partial\Omega_t.$$

Within a purely mechanical context, this implies that there is no velocity on the boundary, i.e., the body is not supplied energy from the environment.

Theorem 2.4.2 *Let a homogeneous incompressible fluid of grade two for which* (2.4.22) *holds,*

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \alpha_1 + \alpha_2 = 0,$$

undergo a flow in a compact domain and meet the "no-slip" boundary condition. Then

$$0 \leq E(t) \leq E(0)\mathrm{e}^{-\frac{t}{\tau}} \quad \text{with } \tau = \frac{\alpha_1 + \varrho\, S_{0,2}}{2\mu},$$

where $S_{0,2}$ is the Poincaré constant for the domain Ω (see (3.1.5) in Chapter 3 below).

Proof By Poincaré's inequality, we have

$$\int_{\Omega_t}|\boldsymbol{v}|^2 dv \leq \frac{S_{0,2}}{2}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv.$$

Therefore

$$E(t) \leq \frac{1}{2}(S_{0,2} + \frac{\alpha_1}{\varrho})\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv.$$

On the other hand, (2.4.36) gives

$$\dot{E}(t) + \frac{\mu}{\varrho}\int_{\Omega_t}|\boldsymbol{A}_1|^2 dv = 0,$$

so that,

$$\dot{E}(t) + \frac{2\mu}{\alpha_1 + \varrho\, S_{0,2}} \int_{\Omega_t} E(t) \leq 0,$$

The result follows by integrating in time this last inequality. □

We next ask the question if disturbances to a state of rest of a fluid die down in finite time. It follows from (2.4.36) and (2.4.35) that for any $\lambda > 0$,

$$\dot{E}(t) + \lambda E(t) = \lambda \int_{\Omega_t} |\boldsymbol{v}|^2 dv + (\frac{\lambda\alpha_1}{2\varrho} - \frac{\mu}{\varrho}) \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv. \tag{2.4.37}$$

In particular,

$$\dot{E}(t) + \lambda E(t) \geq \frac{\lambda\alpha_1 - 2\mu}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv,$$

and thus, there exists $\lambda = 2\mu/\alpha_1$ such that

$$\dot{E}(t) + \lambda E(t) \geq 0.$$

Hence the energy cannot decay to zero in finite time! The energy $E(t)$ is bounded both above and below

$$E(t) \geq E(0)\mathrm{e}^{-2\frac{\mu}{\alpha_1}t}.$$

2.4.4 Instability

There are many reasons why α_1 cannot be negative, thermodynamic issues aside. Here, we discuss a compelling reason for the nonnegativity of the modulus α_1. We first establish results for a fluid of grade two with $\mu \geq 0$, $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 = 0$. We then relax the condition $\alpha_1 + \alpha_2 = 0$ and consider fluids wherein $\mu > 0$ and $\alpha_1 < 0$. Once again, our starting point is (2.4.34) which now reads

$$\frac{d}{dt}\left(\int_{\Omega_t} |\boldsymbol{v}|^2 dv - \frac{|\alpha_1|}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv\right) = -\frac{\mu}{\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv.$$

Let us define the quantity $N(t)$ by

$$N(t) := \frac{|\alpha_1|}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv - \int_{\Omega_t} |\boldsymbol{v}|^2 dv. \tag{2.4.38}$$

We recognize that $N(t)$ is not a Lyapunov function; it is not necessarily nonnegative. It follows from (2.4.38) that

$$\dot{N}(t) = \frac{\mu}{\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv, \tag{2.4.39}$$

and thus

$$\dot{N}(t) \geq 0.$$

Of course, we do not know if $N(0)$ is positive for the domain under consideration. Let us investigate this more closely. From the definition of $N(t)$ in (2.4.38), we have

$$N(t) \geq \frac{|\alpha_1| - \varrho\, S_{0,2}(\Omega_t)}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv, \qquad (2.4.40)$$

where $S_{0,2}(\Omega_t)$ is the maximum in time of the Poincaré constant for the domain. We immediately see that if

$$S_{0,2}(\Omega_t) < \frac{|\alpha_1|}{\varrho}, \quad \text{then } N(0) > 0.$$

The inequality (2.4.40) can be interpreted in two ways. Given a fluid with some values for α_1 and ϱ, we can always find a sufficiently small domain Ω_t in which $N(0)$ is positive. Also, given a domain, it is always possible to find a fluid with sufficiently large $|\alpha_1|$ such that $N(0)$ is positive. We shall use the first interpretation, namely given a fluid there are domains in which $N(0)$ is positive. It follows from (2.4.38) and (2.4.39) that

$$\dot{N}(t) - \lambda N(t) = \frac{\mu}{\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv - \frac{\lambda|\alpha_1|}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv + \lambda \int_{\Omega_t} |\boldsymbol{v}|^2 dv. \qquad (2.4.41)$$

In particular, for any $\lambda > 0$,

$$\dot{N}(t) - \lambda N(t) \geq \frac{2\mu - \lambda|\alpha_1|}{2\varrho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv.$$

Thus, if we pick

$$\lambda = \frac{2\mu}{|\alpha_1|},$$

then

$$\dot{N}(t) - \lambda N(t) \geq 0,$$

and in domains wherein $\varrho\, S_{0,2}(\Omega_t) < |\alpha_1|$, then $N(t) \to \infty$ at least exponentially with time. Thus, the difference between the averaged stretching of the fluid (which is an imprecise interpretation of the first integral in (2.4.38)) and the averaged kinetic energy of the fluid becomes unbounded as $t \to \infty$. This essentially implies that disturbances do not die down with time.

A few words about the above instability analysis are in order, as it is not the usual type of analysis that is carried out to show that the rest-state of a fluid is unstable. We are not showing that certain "small disturbances" tend to blow up in time: in fact we show that there exist domains where *all* disturbances blow up in time.

When $\mu > 0$, $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 \neq 0$, we cannot establish as strong a result as the result above. We can show that for any fluid whose material moduli meet the above restrictions, there exist rigid containers (filled with the above fluid) wherein, *any initial disturbance to the rest-state at initial time* is such that

$$\int_{\Omega_t} |\boldsymbol{A}_1|^3 dv$$

must become larger than any pre-assigned positive real number M, provided the fluid is sufficiently viscous. In fact, the larger the viscosity, the more the disturbance grows! (cf. Fosdick and Rajagopal [100] for a detailed discussion of these issues). We can also show that there exist domains in which any sufficiently smooth initial disturbance can never subside in the sense that

$$\lim_{t\to\infty} \{ \sup_{\boldsymbol{x}\in\Omega_t} |\boldsymbol{A}_1(\boldsymbol{x}, t)| \} > 0.$$

Here, we shall rest content proving the first of the above two results. In order to show that $\int_{\Omega_t} |\boldsymbol{A}_1|^3 dv$ is unbounded, we need the following elementary lemma whose proof can be found in [100].

Lemma 2.4.3 *Let* $\boldsymbol{A}$ *be any symmetric traceless tensor and let* α *be any real number. Then,*

$$-\frac{|\alpha|}{\sqrt{6}}|\boldsymbol{A}|^3 \leq \alpha(\mathrm{tr}\,\boldsymbol{A}^3) \leq \frac{|\alpha|}{\sqrt{6}}|\boldsymbol{A}|^3. \tag{2.4.42}$$

Moreover, there exist tensors $\boldsymbol{A}$ *for which* (2.4.42) *reduces to an equality.*

As $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 \neq 0$, it follows from (2.4.33) with $\beta_3 = 0$ that

$$\frac{d}{dt}\left(\int_{\Omega_t} |\boldsymbol{v}|^2 dv - \frac{|\alpha_1|}{2\varrho}\int_{\Omega_t} |\boldsymbol{A}_1|^2 dv\right) = -\frac{\mu}{\varrho}\int_{\Omega_t} |\boldsymbol{A}_1|^2 dv - \frac{\alpha_1+\alpha_2}{\varrho}\int_{\Omega_t} (\mathrm{tr}\,\boldsymbol{A}_1^3) dv.$$

Let $\lambda \in (0, \frac{2\mu}{|\alpha_1|})$. Then, it follows from (2.4.38) and (2.4.42) that

$$\dot{N}(t) - \lambda N(t) \geq -\frac{|\alpha_1+\alpha_2|}{\varrho\sqrt{6}}\int_{\Omega_t} |\boldsymbol{A}_1|^3 dv. \tag{2.4.43}$$

Now, suppose that

$$\forall t \in (0, \infty), \int_{\Omega_t} |\boldsymbol{A}_1|^3 dv \leq M < \infty. \tag{2.4.44}$$

Then (2.4.43) implies that

$$\dot{N}(t) - \lambda N(t) \geq -\frac{|\alpha_1+\alpha_2|}{\varrho\sqrt{6}} M,$$

which in turn implies that,

$$\forall \lambda \in (0, \frac{2\mu}{|\alpha_1|}),\; N(t) \geq \left(N(0) - \frac{|\alpha_1 + \alpha_2| M}{\lambda \varrho \sqrt{6}} \right) e^{\lambda t}.$$

We know that there exist domains for which $N(0) > 0$. Now, if we can pick a λ sufficiently large then the term multiplying the exponential is positive. In fact, the larger the viscosity of the fluid, the larger the λ that we can pick and the faster $N(t)$ will blow up! Thus, (2.4.38) implies that

$$\lim_{t \to \infty} \int_{\Omega_t} |\boldsymbol{A}_1|^2 dv = \infty.$$

An application of Hölder's inequality then implies that

$$\lim_{t \to \infty} \int_{\Omega_t} |\boldsymbol{A}_1|^3 dv = \infty,$$

but this violates our assumption (2.4.44); hence $\int_{\Omega_t} |\boldsymbol{A}_1|^3 dv$ cannot be less than any preassigned number. Thus, even when $\alpha_1 + \alpha_2 \neq 0$, if $\mu > 0$ and $\alpha_1 < 0$, the fluid does not exhibit desirable stability characteristics.

2.5 Boundary Conditions for Fluids of the Differential Type

While there has been intense activity in mechanics in the development of constitutive equations for the bulk material, little attention has been paid to the "determination" of boundary conditions which are also constitutive specifications. In fact, the nature of the boundary, or a narrow region adjacent to the boundary, depends on the material on either side of the boundary. Boundaries are rarely, if ever, sharp and what needs to be prescribed as a boundary condition depends on the structure of the material that is being enveloped by the boundary as well as the structure of the environment. That Stokes [259] was fully cognizant of the need to arrive at boundary conditions based on the nature of the bodies adjacent to the boundaries is made evident by his remark: "Besides the equations which must hold in the interior of the mass, it will be necessary to form the equations which must also be satisfied at its boundaries." Stokes [259] developed a variety of boundary conditions that ought to hold between two fluids, a fluid and a solid, etc.

On the basis of experiments of "slow" flows in channels, Du Buat [85] offered the opinion that the fluid adheres to the solid boundary adjacent to which it flows. However, this suggestion was not immediately accepted and there was considerable debate concerning what one obtains at a boundary. On the basis of arguments, Navier [193] obtained a formula for the "slip" that would take place at the wall, while Poisson [213] suggested that a condition similar to that derived by Navier [193] would hold, not at the wall, but at the edge of a narrow layer adjacent to the wall. The early debate concerning what ought to be enforced at the boundary involved the likes of Navier, Girard, Coulomb, Poisson, Stokes and many others. About the end of the nineteenth century, the adherence ("no-slip") boundary condition was adopted, supposedly on the basis of experimental evidence pertaining to the flow of fluids under moderate pressures and velocities. Stokes is generally credited with the notion that the no-slip boundary condition is appropriate for fluids like water. But his own comments belie this belief when he writes: "Du Buat found by experiment that when the mean velocity of water flowing through a pipe is about less than one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me, and which are included in those given by supposing $\nu = \infty \ldots$." This implies no-slip. He further states that: "I have said that when the velocity is not very small, the tangential force called into action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity ...."

Recent experiments, that are capable of measuring velocities very close to the boundary, question the correctness of this conventional wisdom for at least a class of polymeric fluids. Also, the use of such a boundary condition for the flows of thin films or droplets seems inappropriate. We shall not get into a discussion of the merits or demerits of the adherence condition, or detail the types of slips that can occur. The interested reader can find a clear and succinct account of the early history of the same in Goldstein [123]. More recent developments are discussed at length in Le Roux [157] and in the review article by Le Roux et al. [158].

We shall be concerned with classical boundary conditions. However, we document below some of the nonstandard boundary conditions that might be useful. To our knowledge, while there are few rigorous mathematical results concerning the flows of non-Newtonian fluids under such boundary conditions, they have been considered in some detail, within the context of the Navier–Stokes fluid.

On the basis of molecular considerations, Navier [193] proposed that the slip velocity (i.e., the component of the velocity in the tangential direction at the boundary) is directly proportional to the shear stress in the fluid

$$\boldsymbol{v} \cdot \boldsymbol{\tau} = -k(\boldsymbol{T}^T \boldsymbol{n} \cdot \boldsymbol{\tau}) \text{ on } \partial\Omega, \tag{2.5.1}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{n}$ are the unit tangent and normal vectors to the boundary and $k > 0$ is a constant. The above slip condition is also written in the form

$$(1-\theta)(\boldsymbol{v} \cdot \boldsymbol{\tau}) + \theta(\boldsymbol{T}^T \boldsymbol{n} \cdot \boldsymbol{\tau}) = 0 \text{ on } \partial\Omega,$$

where $\theta = k/(1+k)$. The "no-slip" condition corresponds to $\theta = 0$ (or $k = 0$) and the "free-slip" condition corresponds to $\theta = 1$ (or $k = \infty$).[13] Threshold-type slip conditions seem to be particularly suitable for the flows of polymeric melts. Such fluids

[13] An interesting discussion of the difficulties inherent to the no-slip boundary condition can be found in Frehse and Málek [104].

seem to flow only when a certain threshold for the shear stress is overcome (we have to bear in mind that by definition a fluid cannot resist shear, and thus, it is merely the time scale of the experiment that prevents the observer from noticing the flow that is taking place). The boundary condition that is appropriate for the flow of polymeric melts is

$$|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau}| \le G|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{n}| \Rightarrow \boldsymbol{v}\cdot\boldsymbol{\tau} = 0,$$

$$|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau}| > G|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{n}| \Rightarrow \boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau} = -N(|\boldsymbol{v}\cdot\boldsymbol{\tau}|, |\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{n}|)\frac{\boldsymbol{v}\cdot\boldsymbol{\tau}}{|\boldsymbol{v}\cdot\boldsymbol{\tau}|},$$

where G is a given constant threshold and N is a given nonlinear function. The above boundary condition states that if the shear stress is below a certain value (which depends on the normal stress), then the fluid will not flow in the direction tangent to the boundary, i.e., the "no-slip" condition holds. However, if the shear stress exceeds this value, the fluid will slip past the boundary, the slip velocity depending on the normal stress. We expect that the higher the normal stress, the lower will be the slip velocity. Usually, a simpler form of threshold-type slip is used

$$|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau}| \le g \Rightarrow \boldsymbol{v}\cdot\boldsymbol{\tau} = 0, \tag{2.5.2}$$

$$|\boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau}| > g \Rightarrow \boldsymbol{T}^T\boldsymbol{n}\cdot\boldsymbol{\tau} = -\frac{\boldsymbol{v}\cdot\boldsymbol{\tau}}{|\boldsymbol{v}\cdot\boldsymbol{\tau}|}, \tag{2.5.3}$$

where g is a constant threshold that is independent of the normal stress.

Berker [27] derived an elegant expression for the surface traction $\boldsymbol{t}$ on a solid surface to which the fluid adheres. He showed that

$$\boldsymbol{t} = [-p + (2\alpha_1 + \alpha_2)|\boldsymbol{\omega}|^2]\boldsymbol{n} + \mu\,\boldsymbol{\omega}\times\boldsymbol{n} + \alpha_1\frac{\partial\boldsymbol{\omega}}{\partial t}\times\boldsymbol{n}, \tag{2.5.4}$$

which, by virtue of $\alpha_1 + \alpha_2 = 0$, reduces in the case of a steady flow to

$$\boldsymbol{t} = [-p + \alpha_1|\boldsymbol{\omega}|^2]\boldsymbol{n} + \mu\,\boldsymbol{\omega}\times\boldsymbol{n}. \tag{2.5.5}$$

Later, we shall find this formula particularly useful.

In the case of a thermodynamically compatible fluid of grade three, the traction $\boldsymbol{t}$ on a solid surface to which the fluid adheres is given by (cf. Rajagopal [221])

$$\boldsymbol{t} = [-p + (2\alpha_1 + \alpha_2)|\boldsymbol{\omega}|^2]\boldsymbol{n} + (\mu + \beta_3 \mathrm{tr}\,\boldsymbol{A}_1^2)\boldsymbol{\omega}\times\boldsymbol{n} + \alpha_1\frac{\partial\boldsymbol{\omega}}{\partial t}\times\boldsymbol{n}. \tag{2.5.6}$$

One of the great breakthroughs in fluid mechanics was to determine what would be appropriate as boundary condition at solid boundaries. In the case of the Euler fluid (that is a perfectly elastic fluid), the requirement of impenetrability at an impervious boundary, i.e., $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on the boundary, is sufficient to render the problem well posed. But an Euler fluid slips while it flows past a solid boundary, it was felt, based on many experimental observations, that the requirements that a viscous fluid adheres to the solid boundary surface would be a reasonable assumption; and this is indeed the case for the classical Navier–Stokes fluid. Interestingly, this condition is sufficient to render problems to be well-posed even in the case of a fluid of complexity one (such as power-law fluids, etc.). However, careful experiments suggest that even in the case of fluids that are considered to be of the Navier–Stokes type, the fluid may slip at the boundary or stick-slip at the boundary. Also, if we do not have a solid boundary but a porous boundary or a free surface, modifications have to be made to the adherence boundary condition.

When we leave the domain of the Navier–Stokes theory (or the theory of fluids of complexity one) and enter into the realm of fluids of higher complexity, we immediately get bogged down into a quagmire. For instance, consider the existence of solutions to fluids of grade greater than one, subject to the no-slip boundary condition. It is not surprising that one is able to prove existence of solutions to equations of order greater than two subject to the no-slip boundary condition, a condition which is sufficient for establishing existence of solutions to second-order partial differential equations. To make matters clear, it is best to consider the possibility that in special flows a third-order partial differential equation (namely, the equation governing the flow of a fluid of grade two) could reduce to a third-order ordinary differential equation, and the no-slip boundary condition is not sufficient to fully determine the solution to the problem; in fact an additional condition is necessary. In other words, a one-parameter family of solutions might be possible.[14] What is however remarkable is that for a class of flows, not only can we show existence of solutions, but we can also show that such solutions are unique. These uniqueness results are by and large possible because such results have been established for small data. This is also true for the Navier–Stokes equations, but there is a marked difference between considering small data for the Navier–Stokes equations, versus those for a fluid of grade two. In the case of the Navier–Stokes equations, linearizing the problem does not lead to a reduction in order of the partial differential equation. However, in the case of the equations of motion for a fluid of grade two, linearization leads to a reduction of the order of the equation. In fact, this is the crux of the matter. By considering small data, the higher order nonlinear terms are neglected, thereby allowing the no-slip boundary condition to be sufficient for establishing uniqueness! However, if we are not able to "control" those higher order nonlinearities, we cannot establish such results. We can understand these issues within the context of specific examples wherein these higher order nonlinear terms reduce to higher order linear terms.

Here are a couple of specific problems that illustrate certain subtle issues concerning boundary conditions for fluids of grade two and higher. Let us consider the flow of a grade-two fluid past an infinite porous plate, at which we can either inject or suck in the fluid. Let us seek a special solution of the form

$$\boldsymbol{v}(x, y, z, t) = u(y)\boldsymbol{i} + v(y)\boldsymbol{j}. \tag{2.5.7}$$

[14]More generally, in the flow of a fluid of grade n, the equations of motion are partial differential equations of order $n + 1$. However, in view of thermodynamic restrictions, it is possible that the equations are of lower order as is the case of third-grade fluids.

We shall show that the problem admits an infinity of solutions of the form (2.5.7) that satisfy the no-slip boundary condition. Of course, it is possible that the equations might admit solutions other than those of the form (2.5.7), but this emphasizes the fact that the problem has nonunique solutions that satisfy the no-slip boundary condition. The incompressibility constraint (2.4.16) gives

$$v(y) = v_0 = \text{ a constant},$$

while (2.4.25) with $\boldsymbol{b} = \boldsymbol{0}$ and (2.4.19) reduce to (cf. Rajagopal and Gupta [230]), where the prime denotes derivation with respect to y

$$\begin{aligned}\mu\, u'' + \alpha_1 v_0 u''' - \varrho\, v_0 u' &= \frac{\partial p}{\partial x},\\ (2\alpha_1 + \alpha_2)[(u')^2]' &= \frac{\partial p}{\partial y},\\ 0 &= \frac{\partial p}{\partial z}.\end{aligned}$$

We recognize that if $v_0 > 0$, we have injection, and if $v_0 < 0$, we have suction at the plate. On defining

$$\hat{p} = p - (2\alpha_1 + \alpha_2)(u')^2,$$

the problem reduces to

$$\mu\, u'' + \alpha_1 v_0 u''' - \varrho\, v_0 u' = \frac{\partial \hat{p}}{\partial x},\ \frac{\partial \hat{p}}{\partial y} = \frac{\partial \hat{p}}{\partial z} = 0. \tag{2.5.8}$$

We observe that (2.5.8) is one order higher than that which corresponds to the Navier–Stokes fluid. As it is linear, it is trivial to establish that its exact solution is

$$u(y) = C_1 \mathrm{e}^{m_1 y} + C_2 \mathrm{e}^{m_2 y} + C_3,$$

where

$$m_{\frac{1}{2}} = \frac{1}{2}\left\{\frac{\mu}{\alpha_1 v_0} \pm \left[\left(\frac{\mu}{\alpha_1 v_0}\right)^2 + \frac{4\varrho}{\alpha_1}\right]^{\frac{1}{2}}\right\}.$$

Since $\alpha_1 > 0$, $\varrho > 0$, $\mu > 0$, and $v_0 > 0$, we note that $m_1 \neq m_2$; without loss of generality, let us assume that $m_1 > m_2$. We use the boundary conditions in order to determine the constants. For the problem under consideration, no-slip at the porous plate implies that

$$u(0) = 0. \tag{2.5.9}$$

Let us suppose that the free stream velocity is U, i.e.,

$$\lim_{y\to\infty} u = U. \tag{2.5.10}$$

In general, these two boundary conditions are not sufficient to completely determine the solution to the problem, as is shown in Rajagopal and Gupta [230].

The above example considers a flow in an unbounded domain and there might be a misapprehension that the difficulty is due to the unboundedness of the domain. In fact, the results are worse in the case of a domain bounded in one dimension. If the flow were to take place between two porous disks, then it transpires that the equation governing the problem are exactly the same as (2.5.8) and once again, we have the boundary condition (2.5.9). However, in contrast to the flow in an unbounded domain, we cannot take recourse to any asymptotic condition such as (2.5.10). We are thus at an impasse, as the boundary condition (2.5.9) is not sufficient to determine the solution to the problem. A similar difficulty presents itself in the case of the radial flow of a fluid of grade two in the annular region between two concentric cylinders (cf. Bernstein and Fosdick [35], Rajagopal and Kaloni [232]).

Next, let us consider the steady radial flow of a fluid of grade two in an annulus bounded by two porous co-axial cylinders. We shall assume a velocity field of the form (in cylindrical polar coordinates)

$$\boldsymbol{v}(r, \theta, z, t) = u(r)\boldsymbol{e}_r + v(r)\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_r$ is the unit vector in the radial direction and $\boldsymbol{e}_\theta$ in the angular direction. The equations of motion in the θ direction reduce to (the prime denotes derivation with respect to r)

$$(v' - \frac{v}{r}) + \frac{T}{r}(v' - \frac{v}{r})' + \mathrm{Re}\frac{v}{r} = \frac{D}{\mu r^2},$$

where we have appropriately nondimensionalized the equations (cf. Rajagopal and Kaloni [232]), and

$$\mathrm{Re} = \frac{\varrho V R_1}{\mu}, \ T = \frac{\alpha_1 V R_1}{\mu},$$

where V is a characteristic velocity, R_1 is the radius of the inner cylinder, Re is the Reynolds number, and D is a constant which appears by virtue of our eliminating the pressure and then integrating the equation. The no-slip boundary condition leads to

$$v(R_1) = v(R_2) = 0,$$

where R_2 is the radius of the outer cylinder. If we could determine $v(r)$, then we could use it to find $u(r)$ from the equations of motion in the radial direction. But the problem is that we cannot determine $v(r)$. Indeed, a straightforward computation leads to, if $\mathrm{Re} \neq 2$:

$$v(r) = C_1 r \mathrm{e}^{-(\frac{r^2}{2T})}\chi_1(\frac{r^2}{2T}) + C_2 r \mathrm{e}^{-(\frac{r^2}{2T})}\chi_2(\frac{r^2}{2T}) + \frac{D}{\mu(\frac{\mathrm{Re}}{2} - 1)r},$$

and if $\mathrm{Re} = 2$,

$$v(r) = C_1 r \mathrm{e}^{-(\frac{r^2}{2T})} \chi_1(\frac{r^2}{2T}) + C_2 r \mathrm{e}^{-(\frac{r^2}{2T})} \chi_2(\frac{r^2}{2T}) + \frac{D}{2\mu r^2}\Big[\frac{2T}{r^2}\ln(\frac{r^2}{2T}) - 2\mathrm{e}^{-(\frac{r^2}{2T})} + \frac{1}{r}\mathrm{e}^{(\frac{r^2}{2T})}\Big],$$

where

$$\chi_1(\xi) = \begin{cases} \dfrac{D\mathrm{e}^{\xi}}{4\mu T(\frac{\mathrm{Re}}{2} - 1)\xi} & \text{if } \mathrm{Re} \neq 2, \\ \dfrac{D}{4\mu T\xi}\displaystyle\int \mathrm{e}^{\xi}\ln(\xi)d\xi & \text{if } Re = 2, \end{cases}$$

$$\chi_2(\xi) = \chi_1(\xi)\ln(\xi) + \chi_3(\xi) + \chi_4(\xi),$$

$$\chi_3(\xi) = \sum \frac{\Gamma(a+k)H_k}{k!(1+k)!\Gamma(a)}, \; a = \frac{\mathrm{Re}}{2} - 2, \; \chi_4(\xi) = \Gamma(a-1)\Gamma(a)\frac{1}{\xi},$$

$$H_k = \sum_{\nu=0}^{k-1}\Big[\frac{-1}{a+r} - \frac{1}{r+\nu} - \frac{1}{r+1}\Big],$$

and Γ denotes the usual Gamma function. We have to determine three constants C_1, C_2 and D, but we have only two equations.

The examples considered thus far correspond to

$$\boldsymbol{v} \cdot \boldsymbol{n} \neq 0 \text{ on } \partial\Omega,$$

and thus one might jump to the erroneous conclusion that the nonuniqueness of solutions stems from the boundary not being impervious. Unfortunately, this is not the case, the difficulties being far deeper. To illustrate the possibility of nonunique solutions even when $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, let us consider the flow engendered above an elastic stretching sheet, occupied by a grade-two fluid, due to the sheet being stretched such that the velocity at a point $(x, 0, z)$ is proportional to the x coordinate. Following Rajagopal et al. [233], we seek solutions of the form

$$\boldsymbol{v}(x, y, z, t) = (xf'(y) + g(y))\boldsymbol{i} - f(y)\boldsymbol{j}. \tag{2.5.11}$$

It follows from (2.5.11), (2.4.19) and (2.4.25) that

$$\begin{aligned} f''' + ff'' - (f')^2 + k\big(2f'''f' - (f'')^2 - ff^{iv}\big) &= 0, \\ g''' + fg'' - f'g' + k\big(g'''f' - (f'''g' - f''g'' - fg^{iv})\big) &= 0, \end{aligned} \tag{2.5.12}$$

where prime denotes differentiation with respect to y and k is an appropriate nondimensional variable

$$k = \frac{\alpha_1}{\varrho} \frac{C^2 \mathrm{Re}}{U_\infty^2}, \ \mathrm{Re} = \frac{U_\infty \sqrt{\varrho}}{\sqrt{C\mu}},$$

C is the proportionality constant for the stretching and U_∞ is the free-stream velocity. Thus, the equations reduce to those for the Navier–Stokes fluid when $k = 0$. Otherwise when $k \neq 0$, the equations are of higher order than the equations for a Navier–Stokes fluid. Let us consider the case $g = 0$; this implies that the sheet is stretched in a special manner. The no-slip boundary condition at the stretching sheet implies

$$f(0) = 0, \ f'(0) = 1, \tag{2.5.13}$$

the asymptotic condition is

$$\lim_{y \to \infty} f' = 0, \tag{2.5.14}$$

and we have a fourth-order equation with these three conditions: this problem admits multiple solutions (cf. Chang et al. [60]). In the case of the Navier–Stokes equations, we have third-order equations with these three conditions, and one can show that the solution to the equations subject to these conditions is unique (cf. McLeod and Rajagopal [189], Overman et al. [206]). Of course, we have used a semi-inverse method and thus the full Navier–Stokes equations might admit other solutions.

2.6 Creeping Flows of Fluids of the Differential Type

By "creeping flows" of fluids of the differential type we mean flows of such fluids in which inertial effects are being neglected. Unfortunately, this choice of terminology leaves much to be desired as one associates "slowness" with the word "creeping." In general, this slowness of the velocity will require a variety of nonlinear terms which depend on the velocity to be neglected, depending on the choice of the specific model of the fluid of differential type. Here, we shall ignore the acceleration terms in the balance of linear momentum while retaining all the nonlinear terms that may stem from the divergence of the stress. Thus, the equations governing the creeping flow of a fluid of the differential type have the form

$$-\nabla p + \mathrm{div}[\boldsymbol{f}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)] = \boldsymbol{0}. \tag{2.6.1}$$

Different choices for $\boldsymbol{f}$ lead to different governing equations.

In the case of a fluid of grade one, i.e., the Navier–Stokes fluid, we obtain the Stokes equation

$$-\nabla p + \mu \Delta \boldsymbol{v} = \boldsymbol{0}. \tag{2.6.2}$$

In 1851, in his influential paper concerning the motion of pendulums in a fluid, Stokes [259] also studied the "slow" motion of a sphere in a fluid. He simplified the Navier–Stokes equations by neglecting the inertial terms and obtained an exact solution for the flow under consideration. Using this exact solution, Stokes derived the famous formula for the drag due to the slow movement of a sphere in an infinite fluid medium. In the case of an object moving in a Navier–Stokes fluid, the Stokes approximation leads to the following expression for the drag:

$$\text{Stokes drag} = \int_{\partial\Omega} (-p\,\boldsymbol{n} + \mu\boldsymbol{\omega}\times\boldsymbol{n})\cdot\boldsymbol{\kappa}\,da, \tag{2.6.3}$$

where the object is moving along the $\boldsymbol{\kappa}$ direction, and p and $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{v}$ correspond to the solution $(p, \boldsymbol{v})$ for (2.6.2).

In the case of general fluids of complexity one, the creeping flow equations reduce to

$$-\nabla p + \operatorname{div}[\boldsymbol{f}(\boldsymbol{A}_1)] = \mathbf{0}. \tag{2.6.4}$$

While solutions to (2.6.2) do not usually exhibit pronounced boundary layers, solutions to (2.6.4) can develop sharp boundary layers depending on the form of $\boldsymbol{f}$. We discuss such layers in Section 2.7.

2.6.1 Creeping Flows of Fluids of Grade Two

Unlike the case for the flows of a Navier–Stokes fluid wherein the equations reduce to the linear Stokes equation in the limit of the Reynolds number tending to zero, in the case of a fluid of grade two, the equations reduce to a nonlinear partial differential equation that can admit possible nonunique solutions. To see this, let us first appropriately nondimensionalize the equations for the steady creeping flow of a fluid of grade two. By taking the curl of (2.4.27), the equation becomes

$$\mu\Delta\,\boldsymbol{\omega} + \alpha_1\mathbf{curl}(\Delta\,\boldsymbol{\omega}\times\boldsymbol{v}) - \varrho\,\mathbf{curl}(\boldsymbol{\omega}\times\boldsymbol{v}) = \mathbf{0}. \tag{2.6.5}$$

Let us introduce nondimensional variables

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L},\ \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V}, \tag{2.6.6}$$

where L and V are some characteristic length scale and velocity, respectively. Then, it immediately follows that

$$\Delta_{\bar{x}}\,\bar{\boldsymbol{\omega}} + \frac{Re}{\Gamma}\mathbf{curl}_{\bar{x}}(\Delta_{\bar{x}}\,\bar{\boldsymbol{\omega}}\times\bar{\boldsymbol{v}}) - \mathrm{Re}\,\mathbf{curl}_{\bar{x}}(\bar{\boldsymbol{\omega}}\times\bar{\boldsymbol{v}}) = \mathbf{0}, \tag{2.6.7}$$

where

$$\mathrm{Re} := \frac{\varrho VL}{\mu},\ \ \Gamma = \frac{\varrho L^2}{\alpha_1}, \tag{2.6.8}$$

and the subscript $\bar{\boldsymbol{x}}$ denotes the non-dimensional variable. In deriving the above, we have supposed that $\mu > 0$ and $\alpha_1 > 0$.

The nondimensional quantity Re is the usual Reynolds number. The nondimensional Γ is a measure of the relative strengths of the inertial and normal stress effects. In 1964, Truesdell [273] realized that the term involving α_1 determines the character of the diffusion of vorticity from the boundary and discussed the change of absorption and phase shift with the driving force for simple shearing flows.

We notice that even in the limit of Re tending to zero, if the ratio Re/Γ is finite and possibly large, then the character of the solution will be determined by the terms which this ratio multiplies, as they are the highest order terms in the equation as well as being nonlinear.

Let us suppose that $\mathrm{Re} \to 0$ and Re/Γ is finite. Then (2.6.7) reduces to

$$\Delta_{\bar{x}}\, \bar{\boldsymbol{\omega}} + \frac{\mathrm{Re}}{\Gamma}\mathbf{curl}_{\bar{x}}\,(\Delta_{\bar{x}}\, \bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{v}}) = \mathbf{0}, \tag{2.6.9}$$

which is referred to as the equation for the creeping flow of a fluid of grade two. As this equation is nonlinear, it is possible that it could have more than one solution. On taking the product of (2.6.9) with $\Delta_{\bar{x}}\, \bar{\boldsymbol{\omega}}$ and dropping the overbars for the sake of notational clarity, we obtain

$$|\Delta\, \boldsymbol{\omega}|^2 + \frac{\mathrm{Re}}{\Gamma}[\frac{1}{2}\mathrm{div}(|\Delta\, \boldsymbol{\omega}|^2\boldsymbol{v}) - \boldsymbol{A}_1\Delta\, \boldsymbol{\omega} \cdot \Delta\, \boldsymbol{\omega}] = 0. \tag{2.6.10}$$

On integrating (2.6.10) over the flow domain and using Green's formula, we obtain

$$\int_{\Omega} |\Delta\, \boldsymbol{\omega}|^2\, dv - \frac{\mathrm{Re}}{\Gamma}\int_{\Omega} \boldsymbol{A}_1\Delta\, \boldsymbol{\omega} \cdot \Delta\, \boldsymbol{\omega}\, dv + \frac{\mathrm{Re}}{2\Gamma}\int_{\partial\Omega} |\Delta\, \boldsymbol{\omega}|^2\boldsymbol{v} \cdot \boldsymbol{n}\, da = 0. \tag{2.6.11}$$

Now, let $M > 0$ denote the maximum of the eigenvalues of $\boldsymbol{A}_1$ over the flow domain.Then

$$\boldsymbol{A}_1\Delta\, \boldsymbol{\omega} \cdot \Delta\, \boldsymbol{\omega} \leq M\, |\Delta\, \boldsymbol{\omega}|^2,$$

and hence since Re and Γ are nonnegative, it follows that

$$(1 - \frac{\mathrm{Re}}{\Gamma}M)\int_{\Omega} |\Delta\, \boldsymbol{\omega}|^2\, dv + \frac{\mathrm{Re}}{2\Gamma}\int_{\partial\Omega} |\Delta\, \boldsymbol{\omega}|^2\boldsymbol{v} \cdot \boldsymbol{n}\, da \leq 0. \tag{2.6.12}$$

Now, if the flow is such that

$$\text{(i)}\quad 1 > \frac{R}{\Gamma}M, \qquad \text{(ii)}\quad \int_{\partial\Omega} |\Delta\, \boldsymbol{\omega}|^2\boldsymbol{v} \cdot \boldsymbol{n}\, da = 0,$$

then, it immediately follows that

$$\int_{\Omega} |\Delta\, \boldsymbol{\omega}|^2\, dv = 0. \tag{2.6.13}$$

Conditions (i) and (ii) are quite reasonable and apply to a sufficiently large class of flows. If the absorption number is sufficiently large, then $\Gamma > \mathrm{Re}\, M$. This condition will also be met if the fluid is sufficiently viscous. The second condition (ii) will be met when $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, in particular in all flows that satisfy $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$.

If $\Delta\,\boldsymbol{\omega}$ is continuous, (2.6.13) implies that

$$\Delta\,\boldsymbol{\omega} = \mathbf{0}, \tag{2.6.14}$$

which one obtains by taking the curl of the equation for Stokes flow. This leads us to a rather interesting history concerning the creeping flows of fluids of grade two. In 1966, Tanner [263] remarked that the classical Stokes solution for plane flows is *a* solution to the plane creeping flow of a fluid of grade two.[15] He did not realize that the equations are of different orders: the Stokes flow being of order two and that for a fluid of grade two of order three. Nor did he realize that the equations for the creeping flow of a fluid of grade two are nonlinear and might admit other solutions in addition to the Stokes solution. Most importantly, he did not realize that it is possible that the equations for the creeping flow of a fluid of grade two could admit a solution that is not a solution of the corresponding Stokes problem (see Rajagopal [224]).

Under conditions (i) and (ii), Fosdick and Rajagopal [100][16] showed that the velocity solution for the Stokes problem is the unique velocity solution for a fluid of grade two, provided that the pressure field is appropriately modified. That is, if $(\boldsymbol{v}, \hat{p})$ solves the Stokes problem, then $(\boldsymbol{v}, p)$ is the unique solution to the creeping flow of a fluid of grade two, where

$$\hat{p} = p + \alpha_1(\boldsymbol{v} \cdot \Delta\,\boldsymbol{v} + \frac{1}{4}|\boldsymbol{A}_1|^2), \tag{2.6.15}$$

provided $\boldsymbol{v}$ is such that the conditions (i) and (ii) are met. For more details, the interested reader can refer to [100].

It is important to recognize that many flows do not meet conditions (i) and (ii). For instance, $\boldsymbol{v} \cdot \boldsymbol{n} \neq 0$ in many flows past a porous boundary, and it is highly unlikely that (ii) is met. Similarly, (i) will not be met if the flow is sufficiently fast (but in this case, the creeping flow equations themselves will not hold).

The drag on a body moving slowly in a fluid of grade two, the fluid adhering to its boundary, is given by

$$\text{Drag} = \text{Stokes Drag} + \alpha_1 \int_{\partial\Omega} \left(|\boldsymbol{\omega}|^2 - \frac{1}{4}|\boldsymbol{A}_1|^2 \right) \boldsymbol{n} \cdot \boldsymbol{\kappa}\, da, \tag{2.6.16}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{A}_1$ are computed from a velocity field that satisfies the Stokes equations and $\boldsymbol{\kappa}$ is the constant direction along which the body is moving. A straightforward calculation establishes that

[15] In a thermodynamically compatible fluid of grade two, Tanner's comment apply to general three-dimensional flows as $\alpha_1 + \alpha_2 = 0$. If in a plane flow, $\alpha_1 + \alpha_2 \neq 0$, it can be shown that the terms that are multiplied by $\alpha_1 + \alpha_2$ reduce to the gradient of a scalar and hence can be absorbed in the pressure.

[16] Huilgol [137] showed that the Stokes solution is the only solution for the creeping flow equations in the plane flow of a second grade fluid with $\alpha_1 < 0$.

$$|\boldsymbol{\omega}|^2 - \frac{1}{4}|\boldsymbol{A}_1|^2 = \frac{1}{2}|\nabla \boldsymbol{v}|^2 - \frac{3}{2}\mathrm{tr}[(\nabla \boldsymbol{v})^2]. \tag{2.6.17}$$

It can be shown (cf. Fosdick and Rajagopal [99]) that if $\boldsymbol{v}$ is a sufficiently smooth vector field defined on the complement Ω of a smooth open set Ω' in $\boldsymbol{R}^3$, satisfying $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$, then

$$\int_{\partial\Omega} \mathrm{tr}[(\nabla \boldsymbol{v})^2]\boldsymbol{n}\, da = \boldsymbol{0}. \tag{2.6.18}$$

Then (2.6.16)–(2.6.18) immediately imply that

$$\text{Drag} = \text{Stokes Drag} + \alpha_1 \int_{\partial\Omega} |\nabla \boldsymbol{v}|^2 \boldsymbol{n} \cdot \boldsymbol{\kappa}\, da. \tag{2.6.19}$$

Thus, for bodies that possess certain geometric symmetries, say for example

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3),$$

it immediately follows that

$$\int_{\partial\Omega} |\nabla \boldsymbol{v}|^2 \boldsymbol{n} \cdot \boldsymbol{\kappa}\, da = 0,$$

and hence, the drag is the same whether the fluid is a Navier–Stokes fluid or a fluid of grade two! In fact, if the body is star-shaped so that it is symmetric under the transformation $\boldsymbol{x} \mapsto -\boldsymbol{x}$, once again, the drag will be the same in the two fluids.

In the case of a fluid of grade three, the expression for the drag takes the form

$$\text{Drag} = \int_{\partial\Omega} \{-p\boldsymbol{n} + \mu\boldsymbol{\omega} \times \boldsymbol{n} + \alpha_1|\boldsymbol{\omega}|^2\boldsymbol{n} + \beta_3(\mathrm{tr}\, \boldsymbol{A}_1^2)\boldsymbol{\omega} \times \boldsymbol{n}\} \cdot \boldsymbol{\kappa}\, da,$$

where $\boldsymbol{\kappa}$ is the constant direction along which the body is moving and the velocity field $\boldsymbol{v}$ and the pressure p are obtained as solutions to the creeping flow problem for a fluid of grade three, i.e.,

$$-\nabla p + \mu\Delta \boldsymbol{v} + \alpha_1 \Delta \boldsymbol{\omega} \times \boldsymbol{v} + \beta_3(\mathrm{tr}\, \boldsymbol{A}_1^2)\Delta \boldsymbol{v} + \beta_3 \boldsymbol{A}_1 \nabla(\mathrm{tr}\, \boldsymbol{A}_1^2) = \boldsymbol{0}.$$

Early results concerning existence and uniqueness for fluids of grade two (cf. Cioranescu and Ouazar [64, 65], Oskolkov [204], Dunn and Fosdick [87]) were for flows that take place in compact domains. While there has been rigorous studies concerning Stokes flows in exterior domains (cf. [120]), until recently, there have been no studies concerning the creeping flow of a fluid of grade two in an exterior domain. Galdi and Rajagopal [109] show the existence and uniqueness of the solution to the creeping flow of a fluid of grade two in the exterior of a compact body when $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$.

A small perturbation of the "null" solution of the Navier–Stokes equations leads to the Stokes flow equations, and such an approximation is called the Stokes flow approximations. In 1927, Oseen [203] studied a different type of approximation to the

Navier–Stokes equations. He considered perturbations about a uniform solution. Other than a passing reference to such an approximation, nothing has been done concerning the equations within the context of higher grade fluids.

2.7 Boundary Layer Theories for Fluids of the Differential Type

One of the great achievements of Prandtl was the development of the Boundary Layer Theory for Navier–Stokes fluids that made much of the progress in aerodynamics possible. Prandtl recognized that at high Reynolds number, for the flow of a Navier–Stokes fluid past a solid boundary, the vorticity was confined to a narrow region adjacent to the boundary and outside this region the flow was similar to that of an inviscid fluid (i.e., an Euler fluid). Here, we shall be concerned primarily with laminar boundary layers in fluids of the differential type of grade two and three, and also for certain fluids of complexity one.

While in Navier–Stokes fluids, boundary layers develop due to inertial effects at sufficiently large Reynolds numbers, in nonlinear fluids it is possible for vorticity to be confined adjacent to the boundary even in the limit of the Reynolds number tending to zero by virtue of shear-thinning or shear-thickening. Furthermore, in fluids of grade two and three, it is possible that we could have boundary layers with multiple deck structures with different effects confined to different layers. For example in one layer, we could have a concentration of vorticity (due to the effect of viscosity) and in another layer, we could have a concentration of the effects of elasticity (or terms in which α_1 appears).

It is important to discuss another mathematical issue related to boundary layers of the Navier–Stokes fluid that has led to some misunderstanding in applied mathematics. To illustrate the issue, let us consider the steady laminar flow of a Navier–Stokes fluid in dimensionless form (we shall later introduce the nondimensionalization procedure in some detail). It follows that

$$\frac{1}{\mathrm{Re}}\Delta_{\bar{x}}\bar{\boldsymbol{v}} - \frac{P}{\varrho V^2}\mathbf{grad}_{\bar{x}}\,\bar{p} = [\mathbf{grad}_{\bar{x}}\bar{\boldsymbol{v}}]\bar{\boldsymbol{v}},$$

where $\bar{\boldsymbol{v}}$ is the nondimensional velocity, $\bar{p}$ the nondimensional pressure, V is some characteristic velocity for the problem of interest, P is some characteristic pressure and Re stands for the Reynolds number defined by (2.6.8):

$$\mathrm{Re} := \frac{\varrho\, V\, L}{\mu}.$$

We now notice that if $\mathrm{Re} \to \infty$, then the equation loses the highest order terms and we have a singular perturbation. In fact, many of the advances made in singular perturbation theory can be directly traced to important applications in fluid mechanics. Boundary Layer Theory for Navier–Stokes fluids goes far beyond the above observation that as $\mathrm{Re} \to \infty$, the equation essentially reduces to the equation for an Euler fluid. This equation holds outside a narrow region where the effects of viscosity dominate. We see from the definition of the Reynolds number that it is a measure of the ratio of inertial effects to viscous effects, and when viscous effects are dominant, the Reynolds number is small. It is also important to bear in mind that in the limit of the Reynolds number tending to infinity, the usual "no-slip" boundary conditions are more than what is necessary. In Boundary Layer Theory, even in the narrow region adjacent to the boundary layer where the effects of viscosity cannot be neglected, further simplifications are made based on an order of magnitude analysis for the components of the velocity field. In this section, we shall carry out such an analysis for fluids of grades two and three. The important point that we would like to make is that the confinement of vorticity in fluids flowing past solid boundaries can, in the case of nonlinear fluids, occur in the limit of $\mathrm{Re} \to 0$.

2.7.1 Boundary Layers in the Limit of Zero Reynolds Number

To illustrate the confinement of vorticity adjacent to solid boundaries in the limit of zero Reynolds number, let us consider, the flow of a fluid of complexity one of the following type:

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu(\mathrm{tr}\boldsymbol{A}_1^2)^m \boldsymbol{A}_1. \tag{2.7.1}$$

This fluid is a shear-thinning fluid if $m < 0$; we shall see that, due to such shear-thinning, very pronounced boundary layers can develop in the limit of the Reynolds number Re tending to zero.

Let us consider the flow of a fluid whose stress is given by (2.7.1), in the absence of body forces, and in the limit of Re tending to zero. As mentioned in the preceding section, flows in which $\mathrm{Re} \to 0$ are usually referred to as "creeping flows". Substituting (2.7.1) into the balance of linear momentum yields

$$\varrho \frac{dv}{dt} = -\mathbf{grad}\, p + \mu \mathrm{div}\left[(\mathrm{tr}\boldsymbol{A}_1^2)^m \boldsymbol{A}_1\right] + \varrho \boldsymbol{b}.$$

We now have $\boldsymbol{b} = \boldsymbol{0}$, and ignoring the Reynolds number (which is tantamount to neglecting inertial effects) and considering the steady problem, leads to

$$-\,\mathbf{grad}\, p + \mu \mathrm{div}\left[(\mathrm{tr}\boldsymbol{A}_1^2)^m \boldsymbol{A}_1\right] = \boldsymbol{0}. \tag{2.7.2}$$

We shall illustrate our main thesis by means of a special example. Let us consider the Jeffrey–Hamel flow of a fluid modeled by (2.7.2) between two intersecting planes. Following Jeffrey [142] and Hamel [130], we assume a velocity field, in a cylindrical polar coordinate system, of the form

$$\boldsymbol{v} = \frac{F(\theta)}{r}\boldsymbol{e}_r. \tag{2.7.3}$$

We can get rid of the pressure from (2.7.2) by taking the curl of the equation and on substituting (2.7.3) into the resulting equation, we obtain

$$\begin{aligned}(2+4m)\big(2F'[8F^2+2(F')^2]^m\big)+\big(F''[8F^2+2(F')^2]^m\big)'\\ +4m(2+4m)F'[8F^2+2(F')^2]^m-2F\big([8F^2+2(F')^2]^m\big)'\\ +\big(8mF[8F^2+2(F')^2]^m+F'\big([8F^2+2(F')^2]^m\big)'\big)'=0.\end{aligned} \tag{2.7.4}$$

The appropriate boundary conditions for the problem are

$$F(\alpha)=0,\ \ F(-\alpha)=0. \tag{2.7.5}$$

However, these boundary conditions do not suffice to solve (2.7.4), as it is a third-order equation (in eliminating the pressure, we have increased the order of the equation). We fill this lacuna by requiring that

$$\int_0^{\alpha} F(\theta)\,d\theta = Q, \tag{2.7.6}$$

where Q denotes the flow rate. Interestingly, when $m=0$, the above equation reduces to (cf. Birkhoff and Zarantanello [40]):

$$F'''+4F'=0,$$

for which we have an exact solution if we impose the boundary conditions (2.7.5) and the integral condition (2.7.6). We shall not get into a detailed discussion of the nature of this solution. Suffice it is to say that the equations admit very pronounced boundary layers for a certain range of values of m (cf. Mansutti and Rajagopal [176]). Recall that we have neglected inertial effects in deriving (2.7.2), i.e., we have considered the case where the Reynolds number $\mathrm{Re}=0$; yet we have sharp concentration of vorticity adjacent to the boundaries.

2.7.2 Development of Boundary Layers in Flows of Fluids of Grade Two

In a fluid of grade two, it is possible that a boundary layer with a two-deck structure develops, where in one of the layers, the viscous and inertial effects are comparable, while in the other, the effects of the viscous and elastic (to be more precise, the influence of α_1) are comparable; the flow outside this two-deck structure being essentially that of an Euler fluid (cf. Rajagopal et al. [231]). Once again, let us introduce nondimensional quantities through

$$\bar{x}=\frac{x}{L},\ \ \bar{v}=\frac{v}{V},\ \ \bar{p}=\frac{p}{P}.$$

For steady flows, we find

$$\frac{\mu V}{L^2}\Delta_{\bar{x}}\bar{\boldsymbol{v}} + \frac{\alpha_1 V^2}{L^3}(\Delta_{\bar{x}}\bar{\omega}\times\bar{\boldsymbol{v}}) + \mathbf{grad}_{\bar{x}}\left[\frac{V^2\alpha_1}{L^2}\bar{\boldsymbol{v}}\cdot\Delta_{\bar{x}}\bar{\boldsymbol{v}} + \frac{V^2\alpha_1}{4L^2}|\bar{\boldsymbol{A}}_1|^2\right] - \frac{P}{L}\mathbf{grad}_{\bar{x}}\bar{p} = \varrho\frac{V^2}{L}[\mathbf{grad}_{\bar{x}}\bar{\boldsymbol{v}}]\bar{\boldsymbol{v}}.$$

We shall restrict ourselves to two-dimensional flows. On taking the curl of the above equation, we obtain:

$$\Delta_{\bar{x}}\bar{\omega} + \frac{\mathrm{Re}}{\Gamma}\mathrm{curl}_{\bar{x}}(\Delta_{\bar{x}}\bar{\omega}\times\bar{\boldsymbol{v}}) - \mathrm{Re}\,\mathrm{curl}_{\bar{x}}(\bar{\omega}\times\bar{\boldsymbol{v}}) = 0,$$

where Re and Γ are defined by (2.6.8), $\bar{\omega}$ denotes the scalar curl

$$\bar{\omega} = \frac{\partial\bar{v}_2}{\partial\bar{x}_1} - \frac{\partial\bar{v}_1}{\partial\bar{x}_2},$$

and $\bar{\omega}\times\bar{\boldsymbol{v}} = \bar{\omega}(-\bar{v}_2, \bar{v}_1)$. The terms involving Γ can significantly affect the diffusion of the vorticity from the boundary (cf. Truesdell [275]) and can therefore affect the structure of the boundary layer. Dividing both sides by Re, we can rewrite the above equation in the form:

$$\frac{1}{\mathrm{Re}}\Delta_{\bar{x}}\bar{\omega} + \frac{1}{\Gamma}\mathrm{curl}_{\bar{x}}(\Delta_{\bar{x}}\bar{\omega}\times\bar{\boldsymbol{v}}) - \mathrm{curl}_{\bar{x}}(\bar{\omega}\times\bar{\boldsymbol{v}}) = 0. \tag{2.7.7}$$

We now recognize that $\frac{1}{\Gamma}$ is the term that multiplies the highest order term in the partial differential equation. Thus boundary layers due to the dominance of α_1 are possible as $\frac{1}{\Gamma}\to\infty$, and these equations are

$$\mathrm{curl}_{\bar{x}}(\Delta_{\bar{x}}\bar{\omega}\times\bar{\boldsymbol{v}}) = 0. \tag{2.7.8}$$

We also notice that $\bar{\omega} = 0$ automatically satisfies (2.7.8), and thus, we could have a situation such that when $\frac{1}{\Gamma}\to\infty$, we have a narrow region adjacent to the boundary governed by (2.7.8) with the flow being that due to an Euler fluid outside this boundary layer.

We realize issues are rather delicate in the case of equation (2.7.7). For instance, it is possible that $\mathrm{Re}\to 0$, while the ratio $\frac{\mathrm{Re}}{\Gamma}$ is $O(1)$, in which case, (2.7.7) would reduce to

$$\Delta_{\bar{x}}\bar{\omega} + \frac{\mathrm{Re}}{\Gamma}\mathrm{curl}_{\bar{x}}(\Delta_{\bar{x}}\bar{\omega}\times\bar{\boldsymbol{v}}) = 0. \tag{2.7.9}$$

We notice that these different equations are of different orders and great care must be exercised, as the boundary conditions that are necessary to make these equations well posed are quite different. We shall discuss these issues subsequently.

2.7.3 Inertial Boundary Layers

By inertial boundary layers, we mean boundary layers at high Reynolds number. To illustrate the development of inertial boundary layers in fluids of grade two, we shall consider the flow of a grade-two fluid past a wedge. This is the counterpart of the problem studied by Falkner and Skan [96] for the Navier–Stokes fluid. Using the same approximations as that used for the Navier–Stokes fluid, we can reduce the problem to solving (cf. Mansutti et al. [175])

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = V\frac{dV}{dx} + \frac{\partial^2 u}{\partial y^2} + \kappa\Big(\frac{\partial}{\partial x}\big(u\frac{\partial^2 u}{\partial y^2}\big) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + u\frac{\partial^3 u}{\partial y^3}\Big),
\end{aligned} \tag{2.7.10}$$

where V is the free stream velocity, u and v are appropriately nondimensionalized velocities in the x and y directions, respectively, and

$$\kappa := \frac{\alpha_1}{\varrho L^2 \sqrt{\mathrm{Re}}},$$

where L is an appropriate length scale.

The boundary conditions to be enforced are

$$\begin{aligned}
u = 0,\ v = 0 \ \text{ at } y = 0,\\
u \to V \ \text{ as } y \to \infty.
\end{aligned} \tag{2.7.11}$$

Here $y = 0$ is one of the faces of the wedge (see Figure 2.7). But here also these boundary conditions are insufficient, as we have increased the order of the equation by eliminating the pressure. We augment the above boundary conditions with the condition that

$$T_{xy} \to 0 \ \text{ as } y \to \infty, \tag{2.7.12}$$

which reflects the fact that the shear stresses vanish as y tends to infinity. We find that (2.7.10), subject to (2.7.11) and (2.7.12) admits solutions that have very pronounced boundary layers adjacent to the wedge, and the boundary layers become more pronounced (thinner) as Re increases.

In the previous section, we discussed the issue of the lack of boundary conditions for general flows of fluids of grade two. We come across this difficulty here as we notice that (2.7.10) has higher order than the Navier–Stokes equations by virtue of the term $\kappa u \frac{\partial^3 u}{\partial y^3}$. Many studies on such problems use a formal perturbation approach in which the velocity is expressed as a power series in κ that is assumed to be small. The rationale for such a formal approach is that in a "slightly" non-Newtonian fluid, α_1 is small and thus κ is small and the Reynolds number is large. However, it is important to recognize that the small parameter κ multiplies the highest order derivative in the equation, and thus, we have a singular perturbation. It is incorrect to treat it as a regular perturbation

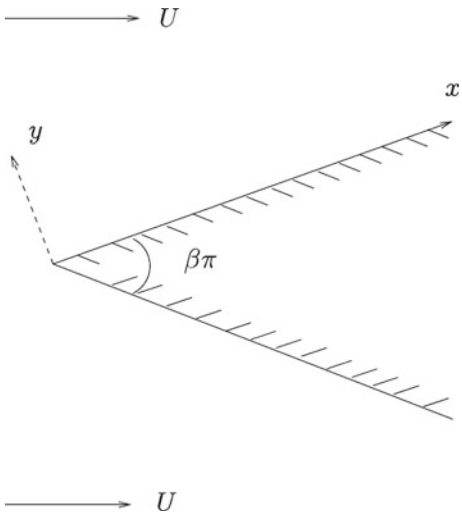


Fig. 2.7 Flow past a wedge

(cf. Bourgin and Tichy [46] for a discussion of the delicate issues concerning the singular nature of the perturbation). The additional boundary condition that we have introduced helps us overcome this difficulty. However, we cannot always resort to such methods to augment boundary conditions. We are able to use such an asymptotic assumption based on "physical" reasoning as we have an unbounded domain. Were the flow taking place past an object inside a pipe, say, we would not be able to augment our boundary condition in such a manner.

2.7.4 Flows of Fluids of Grade Two with a Free Surface

Most of the mathematical results concerning fluids of grade two suppose that the fluid adheres to a solid boundary. We have already discussed the difficulties associated with the flow of fluids of grades two or greater past porous boundaries. We now consider the problem of the flow of a fluid of grade two wherein a part of the boundary is free, i.e., the fluid is free of traction (we are of course neglecting the ambient pressure).

In the case of a general free surface, we would have to express this traction-free condition in terms of a local coordinate system with one coordinate normal to the surface and the two others lying on the tangent plane. Let $\boldsymbol{n}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ denote the unit normal and any two orthonormal vectors lying on the tangent plane at the point of interest. The fact that the surface is free of traction at that point implies that

$$(\boldsymbol{T}^T\boldsymbol{n})\cdot\boldsymbol{n}=0,\quad (\boldsymbol{T}^T\boldsymbol{n})\cdot\boldsymbol{\tau}_1=0,\quad (\boldsymbol{T}^T\boldsymbol{n})\cdot\boldsymbol{\tau}_2=0. \tag{2.7.13}$$

If the surface were exposed to atmospheric pressure, then the first equation in (2.7.13) would take the form

$$(\boldsymbol{T}^T\boldsymbol{n})\cdot\boldsymbol{n}=p_{\mathrm{atm}},$$

where p_{atm} is the atmospheric pressure. These innocuous looking conditions turn out to be quite complicated expressions, even if the surface is planar, say $x_3 = 0$ in a cartesian system (x_1, x_2, x_3). In this case, the second and third equations in (2.7.13) take the following forms (cf. Galdi and Rajagopal [108]):

$$\begin{aligned}
t_{x_1} &= \mu(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + \alpha_1\Big[\frac{d}{dt}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + 2\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_3}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1}) \\
&+ \frac{\partial u_3}{\partial x_3}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + \frac{\partial u_1}{\partial x_1}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + \frac{\partial u_2}{\partial x_1}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + 2\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3}\Big] \\
&+ \alpha_2\Big[2\frac{\partial u_1}{\partial x_1}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + 2\frac{\partial u_3}{\partial x_3}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1})\Big] \\
&= 0 \text{ on the free surface,}
\end{aligned}$$

and

$$\begin{aligned}
t_{x_2} &= \mu(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + \alpha_1\Big[\frac{d}{dt}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + \frac{\partial u_1}{\partial x_3}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) + 2\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} \\
&+ \frac{\partial u_3}{\partial x_3}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + \frac{\partial u_1}{\partial x_1}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) + \frac{\partial u_2}{\partial x_2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + 2\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\Big] \\
&+ \alpha_2\Big[(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2})(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) + 2\frac{\partial u_2}{\partial x_2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + 2\frac{\partial u_3}{\partial x_3}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2})\Big] \\
&= 0 \text{ on the free surface,}
\end{aligned}$$

where the velocity $\boldsymbol{v} = (u_1, u_2, u_3)$ and $\boldsymbol{n} = \boldsymbol{e}_{x_3}$ is the unit vector in the x_3-direction. On the free surface, we also need to meet the condition

$$u_3 = \boldsymbol{v} \cdot \boldsymbol{n} = 0. \tag{2.7.14}$$

We note that the expressions for t_{x_1} and t_{x_2} are evolution equations, a situation quite unlike the situation one has in the Navier–Stokes theory. We first conclude from (2.7.14) that

$$\frac{\partial u_3}{\partial x_1} = \frac{\partial u_3}{\partial x_2} = 0 \text{ on the free surface.} \tag{2.7.15}$$

Thus, the above equations for t_{x_1} and t_{x_2} simplify to

$$\begin{aligned}
t_{x_1} =&\mu\frac{\partial u_1}{\partial x_3} + \alpha_1\Big[\frac{d}{dt}(\frac{\partial u_1}{\partial x_3}) + 3\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} \\
&+ \frac{\partial u_2}{\partial x_3}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1}) + \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\Big] \\
&+ \alpha_2\Big[2\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_3}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) + 2\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_3}\Big] = 0,
\end{aligned} \tag{2.7.16}$$

and

$$
\begin{aligned}
t_{x_2} =&\mu \frac{\partial u_2}{\partial x_3} + \alpha_1 \left[\frac{d}{dt}(\frac{\partial u_2}{\partial x_3}) + \frac{\partial u_1}{\partial x_3}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) \right. \\
&\left. + 3\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_3} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1} \right] \\
&+ \alpha_2 \left[\frac{\partial u_1}{\partial x_3}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) + 2\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + 2\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_3} \right] = 0.
\end{aligned} \tag{2.7.17}
$$

We notice that in the case of a Navier–Stokes fluid, (2.7.16) and (2.7.17) reduce to

$$
t_{x_1} = \mu \frac{\partial u_1}{\partial x_3} = 0, \; t_{x_2} = \mu \frac{\partial u_2}{\partial x_3} = 0,
$$

and then the free-surface boundary condition can be expressed in terms of the velocity and its gradients, i.e.,

$$
\frac{\partial u_1}{\partial x_3} = 0, \; \frac{\partial u_2}{\partial x_3} = 0 \text{ on the free surface,}
$$

and the first equation in (2.7.13).

It is interesting to note that if the first equation in (2.7.13) holds, then even in the case of a fluid of grade three, the tractions t_{x_1} and t_{x_2} vanish. It can be shown that under certain conditions, even in the case of a fluid of grade two, the first equation in (2.7.13) is equivalent to the tractions t_{x_1} and t_{x_2} vanishing on the free surface (cf. Galdi and Rajagopal [108]).

2.7.5 Universal Flows

A flow is said to be universal for a class of fluids if it is possible in every member of the class, the body forces being held fixed. Thus, a universal flow of a fluid of grade n is a universal flow for a fluid of grade $n-1$. The converse may or may not be true. There has been considerable amount of effort aimed towards determining all the universal solutions of the Navier–Stokes fluid (see the work by Marris and his coworkers [177–181]). The higher grade the fluid is, the fewer the universal flows that are possible in that class of fluids (see Fosdick and Truesdell [103]).

Since Euler fluids are particular examples of fluids of grade n, it is necessary that universal flows of fluids of grade n meet:

$$
\operatorname{skw} \mathbf{grad}\, \ddot{x} = \mathbf{0}, \tag{2.7.18}
$$

(where skw **grad** denotes the skew part of the tensor), i.e., the flows should preserve circulation. For the case of grade-one fluids, i.e., the Navier–Stokes fluid, in addition to (2.7.18), we need to meet

$$\operatorname{skw} \mathbf{grad} \operatorname{div} \boldsymbol{A}_1 = \mathbf{0}. \tag{2.7.19}$$

In the case of a fluid of grade two, it can be shown (cf. Fosdick and Truesdell [103], Truesdell and Rajagopal [277]) that in addition to (2.7.18) and (2.7.19), one needs to satisfy for a potential Ψ

$$\operatorname{div} \boldsymbol{A}_2 = -\mathbf{grad}\, \Psi \tag{2.7.20}$$

for a flow to be universal.

We notice that the velocity field, in order that it be universal, has to meet ever increasing partial differential equations. In the case of a fluid of grade three, in addition to (2.7.18)–(2.7.20), one needs to meet

$$\begin{aligned} \operatorname{skw} \mathbf{grad} \operatorname{div} \boldsymbol{A}_3 &= \mathbf{0}, \\ \operatorname{skw} \mathbf{grad} \operatorname{div}[(\operatorname{tr} \boldsymbol{A}_2)\boldsymbol{A}_1] &= \mathbf{0}, \end{aligned} \tag{2.7.21}$$

in order for the flow to be universal.

The class of universal solutions for grade-three fluids has not been fully delineated, let alone the Navier–Stokes fluid that has a far richer class of universal solutions.

2.8 Rate Type Fluids

The first one-dimensional rate type fluid model was developed in 1866 by Maxwell in his celebrated paper on the dynamical theory of gases [184]. He recognized that there is a concept of the "time of relaxation" that is inherent in every body. His rate type fluid model is based on a superposition of viscous and elastic response, and while it could be thought of in terms of a mechanical analogue consisting of a spring and a dashpot in series, Maxwell himself gave no such analogy. His model allowed one to describe the stress relaxation that is exhibited by many bodies. In 1874, Boltzmann [44] also developed a linear one-dimensional model for describing the viscoelastic response of fluids. Nearly, four decades later, in 1915, Jeffery [142] developed a rate type viscoelastic model that has enjoyed a great deal of popularity. Then in 1939, Burgers [55] introduced the one-dimensional analogue of a model in which the Cauchy stress $\boldsymbol{T}$ has the form:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \tag{2.8.1}$$

$$\boldsymbol{S} + \lambda_1 \overset{\nabla}{\boldsymbol{S}} + \lambda_2 \overset{\nabla\nabla}{\boldsymbol{S}} = 2\eta_1 \boldsymbol{D} + 2\eta_2 \overset{\nabla}{\boldsymbol{D}}, \tag{2.8.2}$$

where $\boldsymbol{S}$ is the constitutively determined part of the stress, $-p\boldsymbol{I}$ is the indeterminate spherical stress due to the constraint of incompressibility, $\lambda_1, \lambda_2, \eta_1, \eta_2$ are constant, and the superscript ∇ is defined by (2.3.27). Several models derived later on are particular cases of (2.8.2) with an appropriate choice of parameters. Indeed, if $\lambda_2 = 0$, then (2.8.1)–(2.8.2) is the Oldroyd-B model (2.8.6), if $\lambda_2 = \eta_2 = 0$, it is the Maxwell model

(2.8.4), and if $\lambda_1 = \lambda_2 = \eta_2 = 0$, then it is the incompressible Navier–Stokes model (2.4.17).

In 1950, Oldroyd [200] was the first to create a systematic framework for rate type models describing the response of viscoelastic fluid within a three-dimensional context. He introduced convective derivatives of appropriate physical quantities that ensured their frame indifference and he allowed for the current stress of the body to depend on the history of its deformation. He also provided explicit formulae for calculating the evolution of material symmetry due to the deformation.[17] This remarkable work of Oldroyd's drew a great deal from the earlier study of Frohlick and Sack [106] who had developed three-dimensional models that were not properly frame-indifferent. Oldroyd generated many models and some of these had more than one characteristic time scale associated with their response. One that subserves several of the popular models is the Oldroyd 8 constant model wherein the Cauchy stress $\boldsymbol{T}$ has the form (2.8.1) with

$$\begin{aligned}\boldsymbol{S} + \lambda_1 \overset{\nabla}{\boldsymbol{S}} + \frac{\mu_0}{2}(\operatorname{tr}\boldsymbol{S})\boldsymbol{A}_1 + \frac{\mu_1}{2}(\boldsymbol{S}\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{S}) + \frac{\nu_1}{2}(\operatorname{tr}(\boldsymbol{S}\boldsymbol{A}_1)\boldsymbol{I}) \\ = \eta_0\left[\boldsymbol{A}_1 + \lambda_2 \overset{\nabla}{\boldsymbol{D}} - \mu_2\boldsymbol{A}_1^2 + \frac{\nu_2}{2}(\operatorname{tr}\boldsymbol{A}_1^2)\boldsymbol{I}\right].\end{aligned} \tag{2.8.3}$$

This model includes the Oldroyd-B, Maxwell and Navier–Stokes fluids as special cases.

While a great variety of rate type models have been used to describe viscoelastic fluids, rigorous mathematical results have been established for only a few of them. As this work is concerned with a rigorous mathematical treatment of non-Newtonian fluids, we shall restrict our discussion to these few models, though we mention in passing closely related models of the rate type.

Few rate type models are such that the differential equation for the extra stress can be integrated to obtain an integral representation for this extra stress. In the case of the Maxwell model which is given by (2.8.1) and

$$\boldsymbol{S} + \lambda_1 \overset{\nabla}{\boldsymbol{S}} = 2\eta_1\boldsymbol{D}, \tag{2.8.4}$$

one can obtain the following equivalent integral representation for the stress:

$$\boldsymbol{S} = 2\frac{\eta_1}{\lambda_1^2}\int_0^\infty e^{-\frac{s}{\lambda_1}}[\boldsymbol{C}_t^{-1}(t-s) - \boldsymbol{I}]\,ds. \tag{2.8.5}$$

The expression for the extra stress in this form leads to an integro-differential equation. An approximating scheme has been developed to express (2.8.3) as an integral model under special flow conditions, but we shall not discuss such integral approximations here (see Huilgol [138] for a detailed discussion).

[17]This evolution of material symmetry was given with regard to physical constants that appear in the constitutive equations and is different from the evolution of material symmetry of materials with multiple natural configurations (cf. Rajagopal and Srinivasa [235] for a discussion of material symmetry in anisotropic liquids).

The most popular model due to Oldroyd is the "upper convected Oldroyd-B model." In this model, the Cauchy stress $\boldsymbol{T}$ is related to the fluid motion through (2.8.1) and

$$\boldsymbol{S} + \lambda_1 \overset{\triangledown}{\boldsymbol{S}} = 2\eta_1(\boldsymbol{D} + \lambda_2 \overset{\triangledown}{\boldsymbol{D}}), \tag{2.8.6}$$

where η_1 is the viscosity and λ_1 and λ_2, satisfying $\lambda_1 > \lambda_2 > 0$, are the *relaxation* and *retardation* time, respectively. As mentioned above, when $\lambda_1 = \lambda_2 = 0$, the above model reduces to the classical incompressible Navier–Stokes fluid model (2.4.17) and when the retardation time λ_2 is set to zero, the model reduces to the Maxwell model (2.8.4).

There is an alternative way for expressing the Cauchy stress in a Maxwell fluid that could prove to have some advantages. This alternate formulation stems from a thermodynamic framework that has been developed for describing the response of bodies such as the Maxwell fluids that are capable of instantaneous elastic response. The stress within such a thermodynamic framework is expressed as

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \boldsymbol{B}_{\kappa_{p(t)}},$$

where $\boldsymbol{B}_{\kappa_{p(t)}}$ is the Cauchy-Green tensor measured from an appropriate stress-free configuration $\kappa_{p(t)}$, that satisfies the evolution equation

$$\overset{\triangledown}{\boldsymbol{B}}_{\kappa_{p(t)}} = \lambda[\boldsymbol{B}_{\kappa_{p(t)}} - \gamma \boldsymbol{I}].$$

We shall not get into a detailed discussion of the thermodynamic framework for materials with multiple natural configurations within which most rate type models that are capable of instantaneous response fit. The interested reader can find the details in Rajagopal and Srinivasa [234]. Here, we provide a redacted version of the same. The models that arise from the use of the framework automatically allow for all the response characteristics of viscoelastic fluids, namely shear-thinning/shear-thickening, creep, normal stress differences and stress relaxation. The methodology adopted in the framework is in keeping with the original ideas of Maxwell in his development of a model for viscoelastic fluids and automatically ensures that the second law is met.[18] One can also make an identification between models developed within this framework and those that use the notion of a conformation tensor. Depending on the choice for the stored energy and the rate of dissipation, the relaxation time can depend on the deformation, a feature that is exhibited by fluids such as blood (see Thurston [267]).

Most bodies have more than one natural (i.e., stress-free) configuration and as the body deforms, the underlying natural configuration changes, as does the response from

[18] It is worth emphasizing that Maxwell [184] developed his original model for a viscoelastic fluid to describe the dynamic response of air, a fluid that can store as well as dissipate energy. The classical Euler fluid is a perfectly elastic fluid incapable of dissipation; it can only store energy, and usually gases are either modeled as ideal fluids or Van der Waal's fluids. Once a choice is made for the rate of dissipation, as it is nonnegative, it serves as a Lyapunov function and decreases with time satisfying the minimum entropy production theorem of Onsager [201] (see also Prigogine [218]) that characterizes the steady states for special choices of the rate of entropy production (see [234]).

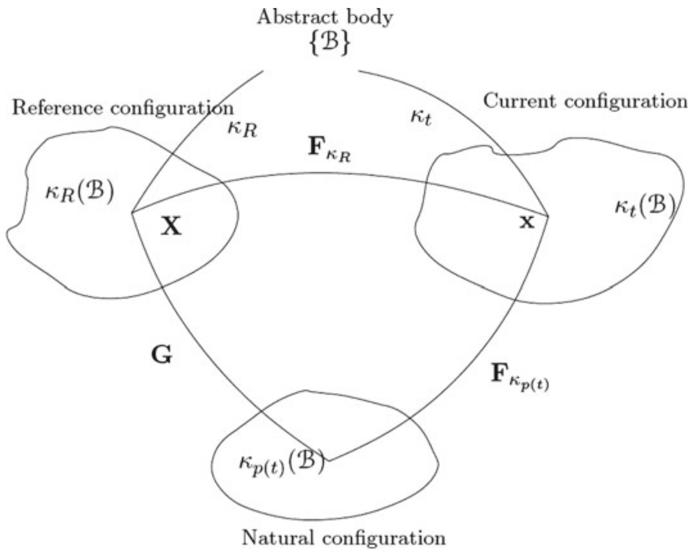


Fig. 2.8 Evolution of stress-free states

these natural configurations. An elastic body has one natural configuration modulo rigid motions while a traditional "elastic-plastic" body has infinity of natural configurations, the elastic response being possibly different from these natural configurations. From now on, when we refer to natural configurations we always refer to it modulo rigid motions.

A solid body that undergoes solid-to-solid phase transitions such as that from Austenite to Martensite has a finite number of natural configurations. Viscoelastic fluids that are capable of instantaneous elastic response can also be thought of as materials with multiple natural configurations, infinity of them. The natural configuration corresponding to a current deformed state is determined by allowing the body to instantaneously stress relax to a stress-free state (see Figure 2.8). The viscoelastic body in question is characterized by a stored energy function ψ which depends on the Cauchy–Green tensor associated with the deformation from the natural configuration (i.e., the stretch associated with the elastic response), and a rate of dissipation function ξ which automatically meets the requirement that it be nonnegative. Thus, the second law is automatically enforced.

The evolution of the natural configuration is accompanied by dissipation and the manner in which the natural configuration evolves is determined by maximizing the rate of dissipation and in a full thermodynamic setting, the rate of entropy production (see [234]). That is, among all possible admissible processes, the one that is chosen is that which maximizes the rate of dissipation (see Rajagopal [225], Rajagopal and Srinivasa [235] for a rationale for such a requirement). For viscoelastic fluids the rate of dissipation depends upon the elastic stretch and the velocity gradient associated with the evolution of the natural configuration. In order to make these ideas clear, we introduce a few kinematical concepts.

As we shall not deal with one reference configuration, but several natural configurations, let us carefully label the various functions based on the reference from which they are measured. Let $\kappa_{p(t)}$ denote the natural configuration corresponding to the current configuration κ_t (see Figure 2.8). If κ_R is the reference configuration and if the body has been subject to an inhomogeneous deformation, then it is not possible to unload to a stress-free configuration that is geometrically compatible (i.e., fits together) in a three-dimensional Euclidean space. However, it is always possible to do so in a non-Euclidean space (see Eckart [92, 235]). But if the deformation is homogeneous, one can unload to a geometrically compatible stress-free state. Moreover, the notion of configuration (see Noll [197]) is local, and hence, we can always locally unload to a stress-free configuration.

Let $\boldsymbol{F}_{\kappa_{p(t)}}$ denote the gradient of the mapping from the configuration $\kappa_{p(t)}$ to κ_t (see Figure 2.8).[19] We define as before the Cauchy–Green tensors $\boldsymbol{B}_{\kappa_{p(t)}}$ and $\boldsymbol{C}_{\kappa_{p(t)}}$ through

$$\boldsymbol{B}_{\kappa_{p(t)}} = \boldsymbol{F}_{\kappa_{p(t)}} \boldsymbol{F}^T_{\kappa_{p(t)}}, \quad \boldsymbol{C}_{\kappa_{p(t)}} = \boldsymbol{F}^T_{\kappa_{p(t)}} \boldsymbol{F}_{\kappa_{p(t)}}. \tag{2.8.7}$$

We denote by $\boldsymbol{G}$ the mapping (see Figure 2.8)

$$\boldsymbol{G} = \boldsymbol{F}_{\kappa_R \mapsto \kappa_{p(t)}} := \boldsymbol{F}^{-1}_{\kappa_{p(t)}} \boldsymbol{F}_{\kappa_R}.$$

Next, we define

$$\boldsymbol{C}_{\kappa_R \mapsto \kappa_{p(t)}} := \boldsymbol{G}^T \boldsymbol{G},$$

and it immediately follows that

$$\boldsymbol{B}_{\kappa_{p(t)}} = \boldsymbol{F}_{\kappa_R} \boldsymbol{C}^{-1}_{\kappa_R \mapsto \kappa_{p(t)}} \boldsymbol{F}^T_{\kappa_R}.$$

The velocity gradient $\boldsymbol{L}_{\kappa_{p(t)}}$ is defined by

$$\boldsymbol{L}_{\kappa_{p(t)}} := \dot{\boldsymbol{G}} \boldsymbol{G}^{-1}, \tag{2.8.8}$$

and the symmetric part of $\boldsymbol{L}_{\kappa_{p(t)}}$ is defined by

$$\boldsymbol{D}_{\kappa_{p(t)}} = \frac{1}{2}[\boldsymbol{L}_{\kappa_{p(t)}} + \boldsymbol{L}^T_{\kappa_{p(t)}}]. \tag{2.8.9}$$

As we shall not provide a detailed derivation but merely indicate the methodology for generating rate type models, also we shall not define a variety of other kinematical quantities and the relationships between them that are necessary for developing the models. For our purposes, the above kinematical definitions suffice.

[19] In general, $\boldsymbol{F}_{\kappa_{p(t)}}$ is a mapping transforming the vectors belonging to the tangent space at a material point of $\kappa_{p(t)}$ into the tangent space at the same material point in the configuration κ_t.

The stored energy ψ is assumed to depend on $\boldsymbol{C}_{\kappa_{p(t)}}$. Since we shall be concerned with isotropic fluids, it follows that ψ can depend on $\boldsymbol{C}_{\kappa_{p(t)}}$ only through its invariants,

$$\begin{aligned} I &:= I_{\boldsymbol{C}_{\kappa_{p(t)}}} = \operatorname{tr} \boldsymbol{C}_{\kappa_{p(t)}} = \operatorname{tr} \boldsymbol{B}_{\kappa_{p(t)}}, \\ II &:= II_{\boldsymbol{C}_{\kappa_{p(t)}}} = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{C}_{\kappa_{p(t)}})^2 - \operatorname{tr} \boldsymbol{C}^2_{\kappa_{p(t)}}\right] = II_{\boldsymbol{B}_{\kappa_{p(t)}}}, \\ III &:= III_{\boldsymbol{C}_{\kappa_{p(t)}}} = \det \boldsymbol{C}_{\kappa_{p(t)}} = III_{\boldsymbol{B}_{\kappa_{p(t)}}}. \end{aligned}$$

Since we are interested in an incompressible fluid, ψ can only depend on I and II, i.e.,

$$\psi = \hat{\psi}(I, II).$$

We shall assume that the rate of dissipation ξ has the following dependence

$$\xi = \hat{\xi}(\boldsymbol{B}_{\kappa_{p(t)}}, \boldsymbol{D}_{\kappa_{p(t)}}).$$

Let us make the special choice

$$\psi = \frac{\mu}{2}(I-3), \;\; \xi = \bar{\eta}\boldsymbol{D}_{\kappa_{p(t)}} \cdot \boldsymbol{B}_{\kappa_{p(t)}}\boldsymbol{D}_{\kappa_{p(t)}},$$

where μ and $\bar{\eta}$ are constants. The above choices mean that we have a mechanical analogue in which the spring stores energy like a neo-Hookean solid and a dashpot which is in series with the spring that dissipates like a Navier–Stokes fluid. The fact that we have a spring and dashpot in series leads to the rate of dissipation depending on $\boldsymbol{D}_{\kappa_{p(t)}}$ and not on $\boldsymbol{D}$. The choice for the stored energy ensures that when there is no elastic deformation, i.e., $\boldsymbol{F}_{\kappa_{p(t)}} = \boldsymbol{I}$, then $I = 3$ thus no energy is stored. Also, when $\boldsymbol{G}$ is a constant, i.e., when the natural configuration does not change, then $\dot{\boldsymbol{G}} = \boldsymbol{0}$ which by virtue of (2.8.8) and (2.8.9) implies that the rate of dissipation $\xi = 0$. It then follows (see [234] for details) that the stress in such a fluid is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{B}_{\kappa_{p(t)}}, \tag{2.8.10}$$

$$-\overset{\nabla}{\boldsymbol{B}}_{\kappa_{p(t)}} = \frac{2\mu}{\bar{\eta}}\left[\boldsymbol{B}_{\kappa_{p(t)}} - \lambda\boldsymbol{I}\right], \tag{2.8.11}$$

where

$$\lambda = \frac{3}{\operatorname{tr} \boldsymbol{B}^{-1}_{\kappa_{p(t)}}}. \tag{2.8.12}$$

Recall that the Cauchy stress in an upper-convected Maxwell fluid is given by (2.8.1) with $\boldsymbol{S}$ defined through (2.8.4)

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \;\; \boldsymbol{S} + \lambda_1 \overset{\nabla}{\boldsymbol{S}} = 2\eta_1\boldsymbol{D}.$$

By virtue of (2.3.28), (2.8.4) can be written as

$$\boldsymbol{S} + \lambda_1 \left[\overset{\nabla}{\boldsymbol{S}} + 2\frac{\eta_1}{\lambda_1} \overset{\nabla}{\boldsymbol{I}} \right] = \boldsymbol{0}.$$

Next, defining $\bar{\boldsymbol{S}}$ through

$$\bar{\boldsymbol{S}} = \boldsymbol{S} + 2\frac{\eta_1}{\lambda_1} \overset{\nabla}{\boldsymbol{I}}, \tag{2.8.13}$$

we can rewrite (2.8.1) and (2.8.4) as

$$\boldsymbol{T} = -\bar{p}\boldsymbol{I} + \bar{\boldsymbol{S}}, \tag{2.8.14}$$

$$\bar{\boldsymbol{S}} + \lambda_1 \overset{\nabla}{\bar{\boldsymbol{S}}} = 2\frac{\eta_1}{\lambda_1}\boldsymbol{I}, \tag{2.8.15}$$

where

$$\bar{p} = p + 2\frac{\eta_1}{\lambda_1}.$$

We notice that (2.8.14), (2.8.15) can be re-written as

$$\boldsymbol{T} = -\bar{p}\boldsymbol{I} + \bar{\bar{\boldsymbol{S}}}, \tag{2.8.16}$$

$$\bar{\bar{\boldsymbol{S}}} + \frac{\eta_1}{2\mu} \overset{\nabla}{\bar{\bar{\boldsymbol{S}}}} = \bar{\lambda}\boldsymbol{I}, \tag{2.8.17}$$

where

$$\bar{\bar{\boldsymbol{S}}} = \mu \boldsymbol{B}_{\kappa_{p(t)}}. \tag{2.8.18}$$

The equations (2.8.1), (2.8.4) and (2.8.16), (2.8.17) have exactly the same form, except that in (2.8.17), $\bar{\lambda}$ is not a constant.

If we make the additional assumption that the elastic strains are small in the sense that

$$\|\boldsymbol{B}_{\kappa_{p(t)}} - \boldsymbol{I}\| = O(\delta), \ \delta \ll 1, \tag{2.8.19}$$

then it immediately follows that:

$$\bar{\lambda} = \mu + O(\delta^2),$$

and for this value of $\bar{\lambda}$, the systems (2.8.1), (2.8.4) and (2.8.16), (2.8.17) are the same. Thus, the rate type model reduces to the Maxwell model in the limit of small elastic strains. That is, the classical Maxwell fluid stores energy like a linearized elastic spring and dissipates energy like a Navier–Stokes fluid.

The above procedure can be used to generate a variety of properly frame-indifferent and thermodynamically compatible models. It is worth noting that the tensor $\boldsymbol{B}_{\kappa_{p(t)}}$ bears a striking similarity in the manner in which it appears in the constitutive relations, to the conformation tensor $\mathcal{C}$ that is associated with the second moment of the end-to-end molecular distance distribution (see Flory [98] and Beris and Edwards [26]). If such an identification is made, then one can immediately see a similarity between the constitutive theories. While $\mathcal{C}$ has some relation to configurations in view of it being a distribution of molecular distance, there is yet no clear identifiable characteristic that $\mathcal{C}$ is a measure of the stretch from a natural configuration, as end-to-end distance can change without the stretch changing and vice-versa. Moreover, there is no clear meaning to the evolution of the conformation tensor $\mathcal{C}$. Also, while the enforcement of the constraint of incompressibility is achieved by requiring that $\det \boldsymbol{B}_{\kappa_{p(t)}} = 1$, requiring the same of the conformation tensor cannot have such a meaning though it is often interpreted in such a manner.

Notice that the model defined through (2.8.10)–(2.8.12) has a relaxation time that depends on the deformation. On the other hand, the models due to Maxwell, Oldroyd, Burgers, and others have a constant relaxation time. In general, all the material moduli such as the viscosity and relaxation time can depend not only on the manner in which the body deforms but also on the pressure. Experiments on blood by Thurston [267] show that the relaxation time of blood depends on the shear rate. Early experiments on asphaltic bitumen by Saal and Koens [250] showed that the viscosity depended on shear stress and the normal stress, while experiments by Bingham and Stephens [38] showed the effect of pressure on the material properties.

Now, we proceed to derive the constitutive representation for the Oldroyd-B fluid within the above thermodynamic framework. Let us keep the same choice of the stored energy ψ:

$$\psi = \frac{\mu}{2}(I - 3),$$

and now choose the rate of dissipation ξ of the form

$$\xi = \bar{\eta} \boldsymbol{D}_{\kappa_{p(t)}} \cdot \boldsymbol{B}_{\kappa_{p(t)}} \boldsymbol{D}_{\kappa_{p(t)}} + \eta_1 \boldsymbol{D} \cdot \boldsymbol{D};$$

this leads to the model (see [234])

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \boldsymbol{B}_{\kappa_{p(t)}} + \eta_1 \boldsymbol{D}, \tag{2.8.20}$$

$$-\overset{\nabla}{\boldsymbol{B}}_{\kappa_{p(t)}} = \frac{2\mu}{\bar{\eta}}[\boldsymbol{B}_{\kappa_{p(t)}} - \lambda \boldsymbol{I}], \tag{2.8.21}$$

where λ is once again given by (2.8.12). When we assume that the elastic deformations are small in the sense of (2.8.19), it can be shown (see [234]) that this model is equivalent to

$$\boldsymbol{T} = -p\boldsymbol{I} + \hat{\boldsymbol{S}}, \tag{2.8.22}$$

$$\hat{\boldsymbol{S}} + \lambda_1 \overset{\triangledown}{\hat{\boldsymbol{S}}} = 2\eta_1(\boldsymbol{D} + \lambda_2 \overset{\triangledown}{\boldsymbol{D}}). \tag{2.8.23}$$

When the elastic deformations are not small, we have in place models (2.8.10)–(2.8.12) and (2.8.20)–(2.8.21) capable of nonlinear elastic response that can shear thin. This is an important advantage over the classical Maxwell and Oldroyd-B models that cannot shear thin or shear thicken. Also, modifications to how the body stores energy and dissipates energy lead to models in which the relaxation and retardation times depend on the deformation.

Despite the equivalence of the models (2.8.10)–(2.8.12) and (2.8.22)–(2.8.23), there is an important philosophical difference that is worth noting. In the model (2.8.22)–(2.8.23), the extra stress $\hat{\boldsymbol{S}}$ is introduced as a quantity that meets (2.8.23). However, no physical meaning whatsoever is given to this quantity other than that it is the constitutively determined part of the stress. It has been shown recently that a splitting of the stress into a constrained part and a constitutively determined part has an inherent nonuniqueness (see [234]). In the thermodynamic approach that leads to (2.8.10)–(2.8.12), a very precise meaning is associated with the extra stress. It is the elastic response of the fluid from the natural configuration $\kappa_{p(t)}$ at time t, to its current configuration κ_t at time t. The configurational tensor approach also assigns a specific meaning to the extra stress and this is to be expected as we have observed earlier that there is a correspondence between the conformation tensor $\mathcal{C}$ and the tensor $\boldsymbol{B}_{\kappa_{p(t)}}$.

We now turn our discussion toward the subtle issue of prescribing boundary conditions for rate type fluids. We first recall that for fluids of grade $n > 1$, the balance of linear momentum is of order $n + 1$. This is a direct consequence of $\boldsymbol{A}_n$ containing a term that has the nth spatial gradient of the velocity $\boldsymbol{v}$. In the case of a fluid of grade two, by virtue of the stress depending on $\boldsymbol{A}_2$, we have a partial differential equation of order three. Now, let us consider the Oldroyd-B fluid defined through (2.8.6). The term $\overset{\triangledown}{\boldsymbol{D}}$ that appears in the equation also involves second spatial derivatives of the velocity, i.e., as far as spatial gradients are concerned, $\boldsymbol{A}_2$ and $\overset{\triangledown}{\boldsymbol{D}}$ are of the same order. In fact, both $\boldsymbol{A}_2$ and $\overset{\triangledown}{\boldsymbol{D}}$ are different but properly invariant temporal derivatives of $\boldsymbol{A}_1$ or $\boldsymbol{D}$. As we have difficulties with prescribing boundary conditions in general for a fluid of grade two, we anticipate similar difficulties for Oldroyd-B fluids. We also note that the definition of a Burgers fluid (2.8.2) also involves the term $\overset{\triangledown}{\boldsymbol{D}}$ which has spatial derivatives of the same order as $\boldsymbol{A}_2$; thus we expect that Burgers' model will require additional boundary conditions than the Navier–Stokes model.

Let us consider a specific flow that of the flow of an Oldroyd-B fluid past a porous plate. We seek a solution in a Cartesian coordinate system for the velocity and extra stress of the form

$$\boldsymbol{v} = u(y)\boldsymbol{i} + v(y)\boldsymbol{j}, \ \ \boldsymbol{S} = \boldsymbol{S}(y). \tag{2.8.24}$$

The constraint of incompressibility implies that

$$v(y) = v_0 \text{ a constant.}$$

Substituting (2.8.24) into the balance of linear momentum (2.4.25), we obtain

$$-\frac{\partial p}{\partial x}+\frac{d\,S_{xy}}{d\,y}=\varrho v_0\frac{d\,u}{d\,y}\quad -\frac{\partial p}{\partial y}+\frac{d\,S_{yy}}{d\,y}=0,\ -\frac{\partial p}{\partial z}=0. \tag{2.8.25}$$

The constitutive relation (2.8.6) implies that

$$\begin{aligned} S_{xy}+\lambda_1 v_0\frac{d\,S_{xy}}{d\,y}&=\lambda_1\frac{d\,u}{d\,y}S_{yy}+\eta_1\frac{d\,u}{d\,y}+\lambda_2\eta_1 v_0\frac{d^2\,u}{d\,y^2},\\ S_{yy}+\lambda_1 v_0\frac{d\,S_{yy}}{d\,y}&=0, \end{aligned} \tag{2.8.26}$$

plus an additional differential equation that determines explicitly S_{xx} once u is known.

Let us consider the case of suction at the plate, i.e., $v_0 < 0$. Since $\lambda_1 > 0$, the second equation in (2.8.26) implies that if we require S_{yy} to be bounded, it has to be zero. It then follows from the first equation in (2.8.26) and (2.8.25) that

$$\lambda_2 v_0\eta_1 u''' + (\varrho\lambda_1 v_0^2-\eta_1)u'' - \varrho\, v_0 u' = 0. \tag{2.8.27}$$

In arriving at (2.8.27), we have used the fact that, since we are looking for bounded solutions, then $S_{yy}=0$ and (2.8.25) leads to $\partial p/\partial x=$ constant. But as the flow is uniform at infinity, i.e., as $u\to U$ as $y\to\infty$, then $\partial p/\partial x$ has to be zero, i.e., the pressure field is a constant. Of course, it is possible that the equations do not admit such a solution, but we shall show below that there do exist solutions with such an asymptotic structure.

We notice that (2.8.27) is a third-order ordinary differential equation. Its solution is

$$u = C_1 + C_2\mathrm{e}^{m_1 y} + C_3\mathrm{e}^{m_2 y}.$$

The adherence boundary condition at the porous plate requires

$$u(0)=0. \tag{2.8.28}$$

Also, as the free stream velocity is U, we have

$$u\to U \text{ as } y\to\infty. \tag{2.8.29}$$

These two conditions are sufficient to find a bounded solution

$$u(y) = U[1-\mathrm{e}^{my}],$$

where

$$m := \frac{1}{2\lambda_2 v_0\eta_1}\left(\eta_1-\lambda_1 v_0^2+\sqrt{(\eta_1-\lambda_1 v_0^2)^2+4\lambda_2 v_0^2\,\eta_1}\right).$$

We are able to obtain this solution with the two conditions above, although this is a third-order differential equation, because we require boundedness of the solution. However, suppose we seek a solution in a bounded domain, which is the case in the flow between two porous parallel plates, one of which is at rest and the other moving with constant speed U, the fluid being either sucked or blown through the porous plates. Then, following the above procedure, we find that the boundary condition (2.8.28) and

$$u(h) = U,$$

are not sufficient to determine the solution. Of course, what we have shown is that we are unable to find solutions of the form (2.8.24) with the boundary conditions (2.8.28) and (2.8.29). It is possible that there could be other solutions to the general equations that satisfy these boundary conditions. However, to our knowledge, we are unaware of any rigorous existence results for the flow of an Oldroyd-B fluid when the normal component of the velocity at the boundary is not zero.

Notice that in the case of a Maxwell model, $\lambda_2 = 0$ and (2.8.27) reduces to

$$(\varrho\lambda_1 v_0^2 - \eta_1)u'' - \varrho\, v_0 u' = 0,$$

which is a second-order equation. Then, the conditions (2.8.28) and (2.8.29) are sufficient to determine the solution u

$$u(y) = U\left[1 - \mathrm{e}^{-\frac{\varrho|v_0|}{(\eta_1 - \varrho\lambda_1 v_0^2)}y}\right],$$

provided $\varrho\lambda_1 v_0^2 < \eta_1$. If $\varrho\lambda_1 v_0^2 \geq \eta_1$, we cannot satisfy the boundary condition and solutions of the form that is sought cannot exist.

Unfortunately, issues concerning boundary conditions for rate type fluids are not as simple as that. This will become clear in what follows. In the case of an Oldroyd-B fluid, let us introduce a new tensor $\hat{\boldsymbol{S}}$ through

$$\boldsymbol{S} = \hat{\boldsymbol{S}} + \gamma \boldsymbol{D}, \;\; \eta > \gamma > 0, \tag{2.8.30}$$

where $\boldsymbol{S}$ satisfies (2.8.6). Then (2.8.1) takes the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \hat{\boldsymbol{S}} + \gamma \boldsymbol{D}, \tag{2.8.31}$$

and, since $\lambda_1 > \lambda_2$, by choosing

$$\gamma = \eta\frac{\lambda_2}{\lambda_1},$$

Equation (2.8.6) can be written as

$$\hat{\boldsymbol{S}} + \lambda_1 \overset{\triangledown}{\hat{\boldsymbol{S}}} = (\eta - \gamma)\boldsymbol{D}. \tag{2.8.32}$$

We notice that the term $\overset{\triangledown}{\boldsymbol{D}}$ does not appear explicitly in the equations, and as far as $\boldsymbol{S}$ or $\hat{\boldsymbol{S}}$ are concerned, the spatial derivative of them are of the same order.

The transformation (2.8.30) that allows us to get rid of the term involving $\overset{\triangledown}{\boldsymbol{D}}$ will not work in general for rate type fluids wherein $\overset{\triangledown}{\boldsymbol{D}}$ appears. For instance, consider the rate type model whose extra stress $\boldsymbol{S}$ satisfies the equation

$$\boldsymbol{S} + \lambda_1 \overset{\triangledown}{\boldsymbol{S}} = \boldsymbol{f}(\boldsymbol{D}, \overset{\triangledown}{\boldsymbol{D}}), \tag{2.8.33}$$

where $\boldsymbol{f}$ is nonlinear in $\overset{\triangledown}{\boldsymbol{D}}$. In this case, (2.8.30) will not allow us to eliminate $\overset{\triangledown}{\boldsymbol{D}}$ from the equation for $\hat{\boldsymbol{S}}$.

The models (2.8.20)–(2.8.21) and (2.8.22)–(2.8.23) are equivalent, as both represent the Oldroyd-B model; but they demand quite different initial and boundary conditions. Indeed, both $\boldsymbol{D}$ and $\overset{\triangledown}{\boldsymbol{D}}$ appear in the representation (2.8.22)–(2.8.23), while only $\boldsymbol{D}$ appears in the representation (2.8.20)–(2.8.21). Thus, the representation (2.8.20)–(2.8.21) for the Oldroyd-B model seems much more suitable for solving problems, as it demands less in terms of boundary conditions. Even in the representation (2.8.31)–(2.8.32), which seems to demand less in terms of boundary conditions for the velocity, we seem to need initial conditions on the extra stress. But a priori, we do not know what this extra stress is other than that it satisfies (2.8.32). However, in the representation (2.8.20)–(2.8.21) that arises from the thermodynamic framework, the extra stress given by (2.8.18) is well-defined, and we have an evolution equation for $\boldsymbol{B}_{\kappa_{p(t)}}$. Now, the initial condition for $\boldsymbol{B}_{\kappa_{p(t)}}$ is known if we know the initial conditions for the displacement field from the appropriate natural configuration. More importantly, we shall see that even for rate type models in which $\overset{\triangledown}{\boldsymbol{D}}$ can appear nonlinearly (see (2.8.33)) and in which we cannot eliminate $\overset{\triangledown}{\boldsymbol{D}}$ in the constitutive expression for the extra stress, the thermodynamic framework will yield equivalent models wherein we will only have an evolution equation for $\boldsymbol{B}_{\kappa_{p(t)}}$. Thus, within the thermodynamic framework we need to consider only the initial condition for the displacement from the appropriate natural configuration.

We end with a discussion of the efficacy of the thermodynamic procedure by considering the status of Burgers' model within the above framework. It will also serve to illustrate the use of more than one natural configuration corresponding to the current configuration of the body. When one considers a body such as asphalt that is essentially composed of a mixture of constituents, then it is necessary to associate natural configurations corresponding to each of the constituents (see Murali Krishnan and Rajagopal [154]). Such models are also necessary to describe viscoelastic fluids whose mechanical analogue involves several springs and dashpots, natural configurations being associated with the undeformed lengths of the various springs.

The mechanical analogue for the Burgers model is made up of two springs and two dashpots; for the linear Burgers model in one dimension, the stress σ and the rate of strain $\dot{\varepsilon}$ follow the relation:

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} = q_1\dot{\varepsilon} + q_2\ddot{\varepsilon},$$

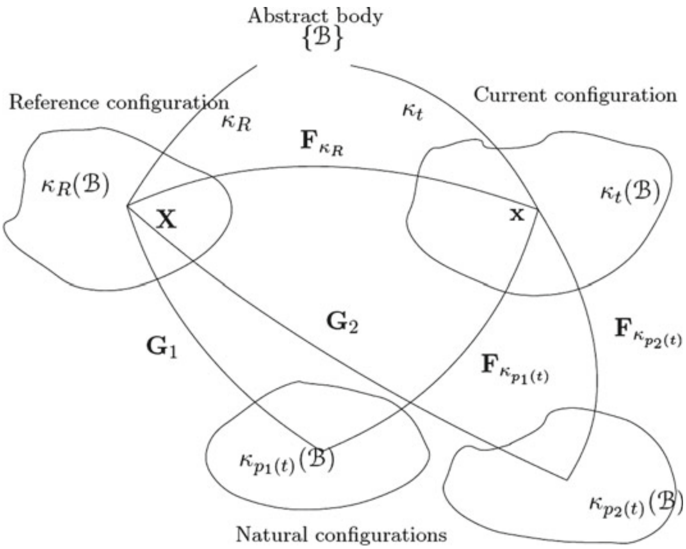


Fig. 2.9 Bodies with multiple relaxation times

where p_1, p_2, q_1, q_2 are material functions that are related to the properties of the two springs and two dashpots.

We now proceed to derive the constitutive relation for Burgers' model within the thermodynamic framework for bodies that have multiple natural configurations. Recall that the Cauchy stress for the Burgers model is given by (2.8.1)–(2.8.2)

$$\boldsymbol{T} = -\boldsymbol{I} + \boldsymbol{S},$$

$$\boldsymbol{S} + \lambda_1 \overset{\nabla}{\boldsymbol{S}} + \lambda_2 \overset{\nabla\nabla}{\boldsymbol{S}} = 2\eta_1 \boldsymbol{D} + 2\eta_2 \overset{\nabla}{\boldsymbol{D}} .$$

Since we have two springs, each of them has an associated natural configuration. Let $\kappa_{p_1(t)}$ and $\kappa_{p_2(t)}$ denote the two natural configurations associated with the configuration κ_t at time t, and let $\boldsymbol{F}_{\kappa_{p_1(t)}}$ and $\boldsymbol{F}_{\kappa_{p_2(t)}}$ denote the deformation gradients from the configurations $\kappa_{p_1(t)}$ and $\kappa_{p_2(t)}$ to the configuration κ_t, respectively (see Figure 2.9). We can define as before the appropriate Cauchy-Green tensor $\boldsymbol{B}_{\kappa_{p_i(t)}}$, $i = 1, 2$ (see [154]) and the tensor

$$\boldsymbol{G}_i := \boldsymbol{F}^{-1}_{\kappa_{p_i(t)}} \boldsymbol{F}_{\kappa_R}, \ i = 1, 2.$$

The velocity gradient $\boldsymbol{L}_{\kappa_{p_i(t)}}$ and its symmetric part $\boldsymbol{D}_{\kappa_{p_i(t)}}$ are defined through

$$\boldsymbol{L}_{\kappa_{p_i(t)}} = \dot{\boldsymbol{G}}_i \boldsymbol{G}_i^{-1}, \ i = 1, 2,$$

$$\boldsymbol{D}_{\kappa_{p_i(t)}} = \frac{1}{2}[\boldsymbol{L}_{\kappa_{p_i(t)}} + \boldsymbol{L}^T_{\kappa_{p_i(t)}}], \ i = 1, 2.$$

Let the stored energy associated with the i-th constituent (spring) depend only on its deformation, i.e., the deformation gradient $\boldsymbol{F}_{\kappa_{p_i(t)}}$. As we are concerned with an incompressible isotropic material, it immediately follows that:

$$\psi_i = \hat{\psi}_i(I_{\boldsymbol{B}_{\kappa_{p_i(t)}}}, II_{\boldsymbol{B}_{\kappa_{p_i(t)}}}), \ i = 1, 2.$$

Let the rate of dissipation associated with the ith constituent (dashpot) ξ_i be given through

$$\xi_i = \hat{\xi}_i(\boldsymbol{B}_{\kappa_{p_i(t)}}, \boldsymbol{D}_{\kappa_{p_i(t)}}).$$

A full thermodynamic treatment that includes the effects of temperature, heat flux, entropy, etc., that arrives at the appropriate form for the stress by requiring that the rate of dissipation be maximized can be found in [154]. For instance, in the case where the stored energy and the rate of dissipation have the special forms

$$\psi_i = \frac{\mu_i}{2}(I_{\boldsymbol{B}_{\kappa_{p_i(t)}}} - 3), \ i = 1, 2,$$

where μ_i, $i = 1, 2$, are constant and

$$\xi_i = \eta_i \boldsymbol{D}_{\kappa_{p_i(t)}} \cdot \boldsymbol{B}_{\kappa_{p_i(t)}} \boldsymbol{D}_{\kappa_{p_i(t)}}, \ i = 1, 2,$$

where η_i, $i = 1, 2$, are constant, a lengthy analysis (see [154]) leads to

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu_1 \boldsymbol{B}_{\kappa_{p_1(t)}} + \mu_2 \boldsymbol{B}_{\kappa_{p_2(t)}}, \tag{2.8.34}$$

with

$$-\frac{1}{2} \overset{\nabla}{\boldsymbol{B}}_{\kappa_{p_i(t)}} = \frac{\mu_i}{\eta_i}\left[\boldsymbol{B}_{\kappa_{p_i(t)}} - \frac{3}{\operatorname{tr} \boldsymbol{B}^{-1}_{\kappa_{p_i(t)}}}\right], \ i = 1, 2. \tag{2.8.35}$$

In general μ_i and η_i are not constants. If the material under consideration strain softens or strain stiffens, then it is necessary to allow the μ_i to be functions of the Cauchy–Green tensors $\boldsymbol{B}_{\kappa_{p_i(t)}}$. Similarly, if the body shear thins or shear thickens, it would be necessary that all viscosities η_i be functions of $\boldsymbol{D}_{\kappa_{p_i(t)}}$. It is also possible that the η_i depend on $\boldsymbol{B}_{\kappa_{p_i(t)}}$ as one cannot always separate the elastic and dissipative responses in the sense that the dissipation in the current configuration could depend on the elastic response from the natural configuration to the current configuration. In fact, materials such as asphalt concrete present very complicated response characteristics. They have completely different responses to compressive and tensile loading. Moreover, their elastic as well as dissipative response changes considerably with the deformation, and they are very sensitive to temperature, an issue that we ignore in this book.

For a viscoelastic material such as asphalt, a simple deformation such as triaxial response cannot be described by a Burgers model in which the material moduli are constant. Murali Krishnan and Rajagopal [153] use shear moduli of the form

$$\mu_i = \bar{\mu}_i \left[1 + \frac{b_i}{n_i} (\text{tr}\, \boldsymbol{B}_{\kappa_{p_i(t)}} - 3) \right]^{n_i - 1}, \tag{2.8.36}$$

and viscosities of the form

$$\eta_i = \bar{\eta}_i \left[1 + N (\text{tr}\, \boldsymbol{B}_{\kappa_{p_i(t)}} - 3)^m) \right]^{n_i - 1}, \tag{2.8.37}$$

where $\bar{\mu}_i$, N and $\bar{\eta}_i$ are constants. When $n_i = 1$ in (2.8.36) and (2.8.37), we recover Burgers' model (2.8.1)–(2.8.2).

It can be shown that (2.8.34) and (2.8.35) are equivalent to

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S},$$

$$\boldsymbol{S} = \sum_{i=1}^{2} \boldsymbol{S}_i, \ \ \boldsymbol{S}_i + \frac{\eta_i}{2\mu_i} \overset{\nabla}{\boldsymbol{S}_i} = \lambda_i \boldsymbol{I}, \ \ i = 1, 2, \tag{2.8.38}$$

where

$$\boldsymbol{S}_i = \mu_i \boldsymbol{B}_{\kappa_{p_1(t)}}, \ \ \lambda_i = \frac{3}{\text{tr}\, \boldsymbol{S}_i^{-1}}, \ \ i = 1, 2. \tag{2.8.39}$$

Since $\boldsymbol{S}_i$ is non-singular, it can be shown that the system (2.8.38)–(2.8.39) is equivalent to

$$\boldsymbol{S} + \alpha \overset{\nabla}{\boldsymbol{S}} + \beta \overset{\nabla\nabla}{\boldsymbol{S}} = \bar{\lambda} \overset{\nabla}{\boldsymbol{I}} + \bar{\lambda} \beta \overset{\nabla\nabla}{\boldsymbol{I}}, \tag{2.8.40}$$

where

$$\alpha = \frac{\eta_1}{2\mu_1} + \frac{\eta_2}{2\mu_2}, \ \ \beta = \frac{\eta_1 \eta_2}{4\mu_1 \mu_2}.$$

By virtue of (2.3.28), (2.8.40) reduces to

$$\boldsymbol{S} + \alpha \overset{\nabla}{\boldsymbol{S}} + \beta \overset{\nabla\nabla}{\boldsymbol{S}} = \gamma_1 \boldsymbol{D} + \gamma_2 \overset{\nabla}{\boldsymbol{D}},$$

which is indeed equivalent to (2.8.2).

The framework within which the rate type models were derived here has very clear thermodynamic underpinnings. Not only does such a framework seem to yield reasonable rate type fluid models, it leads to sensible models in a whole variety of fields including plasticity, twinning, solid-to-solid phase transitions, super plasticity, crystallization of polymer melts, multi-network polymer mechanics, and mechanics of granular materials.

Finally, we describe a rate type model, the Johnson–Segalman fluid, that for a range of values for the material parameters exhibits a marked difference from most other rate type models and the classical Navier–Stokes fluid. It is well-known that for the Navier–Stokes fluid, the shear stress increases monotonically with the shear rate. The Maxwell, Oldroyd-B and Burgers fluids also exhibit such a monotone behavior. Unlike such fluids, the Johnson–Segalman fluid exhibits a non-monotone relationship between the

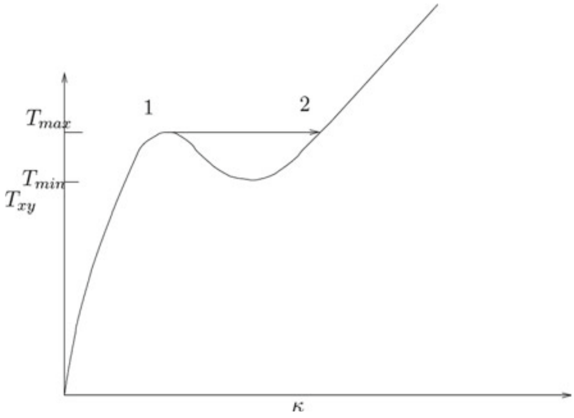


Fig. 2.10 Spurt phenomenon

shear stress and shear rate, for a certain range of parameters. Such a response has been advanced as the cause for many interesting phenomena, such as "spurt," that manifest themselves during the flow of polymeric melts. We shall discuss this phenomenon later.

In a Johnson–Segalman fluid, the Cauchy stress has the form (cf. [146]),

$$\boldsymbol{T} = -p\boldsymbol{I} + \hat{\boldsymbol{S}},$$

$$\hat{\boldsymbol{S}} = \boldsymbol{S} + 2\mu\boldsymbol{D},$$

$$\boldsymbol{S} + \lambda\left[\frac{D\boldsymbol{S}}{Dt} + \boldsymbol{S}(\boldsymbol{W} - a\boldsymbol{D}) + (\boldsymbol{W} - a\boldsymbol{D})^T\boldsymbol{S}\right] = 2\eta_1\boldsymbol{D},$$

where $\boldsymbol{D}$ and $\boldsymbol{W}$ are the symmetric and skew part of the velocity gradient, μ and η are viscosities, λ is the relaxation time and a is called the slip parameter, all the material moduli being constant. When $a = 1$, the above model reduces to an Oldroyd-B model, while if in addition, $\mu = 0$, it reduces to a Maxwell model. When $\mu = 0$ and $\lambda = 0$, the model reduces to the classical Navier–Stokes model. For a certain range of values of the parameter a, the relationship between the shear stress and shear rate in a plane Couette or cylindrical Poiseuille flow, is non-monotonic. This lack of monotonicity is cited as the reason for the phenomenon of "spurt"[20] that has been observed during the flow of some polymeric melts (cf. Kolkka et al. [151], Malkus et al. [172, 173]). As the relationship is non-monotone in a very special manner, as depicted in Figure 2.10, a jump in the shear rate occurs at a constant shear stress at the value when it loses monotonicity. This jump occurs at a critical pressure gradient, and the flow rate versus the pressure gradient curve has a discontinuity in its derivative at the point, the derivative having a pronouncedly larger value as the pressure gradient increases. At those points, the velocity gradient is discontinuous.

[20]The phenomenon of a large increase in the volume flow rate due to a small increase in the driving pressure is referred to as "spurt."

However, it seems more probable that spurt is a consequence of the stick-slip that takes place at the boundary. Indeed, if the cause were the lack of monotonicity, then a variety of other strange phenomena ought to manifest themselves in different flow situations. Nonetheless, the model provides yet another dimension to the description of behavior that does not conform to what is expected in a Navier–Stokes fluid.

Many exact solutions have been established for simple flows of rate type models such as the Maxwell, Oldroyd-B, Burgers, and the Johnson–Segalman fluid undergoing steady and unsteady unidirectional flows or planar flows. Also, some linearized stability analysis of special flows have been carried out. These studies are too numerous to document here, and moreover our interests lie in documenting mathematical results concerning the existence and uniqueness of the flows of such fluids under general conditions.

2.8.1 A Gibbs-Potential-Based Formulation for Obtaining Rate Type Response Functions for Viscoelastic Fluids

Thus far, we have discussed the development of models within the context of a body having a natural configuration that evolves. The procedure rests in choosing appropriate forms for the Helmholtz potential and the rate of dissipation. Recently, Rajagopal and Srinivasa [237] developed a different formulation, namely one based on a choice being made for the Gibbs potential and the rate of dissipation, as one cannot obtain some of the useful rate type fluid models by using the Helmholtz potential formulation. These two ways of generating models go hand in hand and together yield practically all the phenomenological models that are in use, and in addition many new models that can describe experimental data better than those that are currently available. The approach provides a rational means for describing anisotropic fluids without appealing to the use of directors.

We shall not get into a discussion of the thermodynamic procedure other than mentioning that appropriate choices for the Gibbs potential and the rate of dissipation, and the maximization of the rate of dissipation subject to the reduced energy-dissipation equation as a constraint, leads to models proposed by Phan-Thien and Tanner, Metzner, White and Dunn, Giesekus, and several others. We refer to Rajagopal and Srinivasa [237] for details of the same.

2.9 Bingham Fluids

The one-dimensional response of a "perfect" Bingham fluid is portrayed in Figure 2.11 below.

After a threshold τ_{th} is reached for the shear stress, the body flows at constant shear. Such a response is the counterpart to a rigid-perfectly plastic response of solids. Clearly, with regard to the response depicted in Figure 2.11 below, the stress cannot be expressed

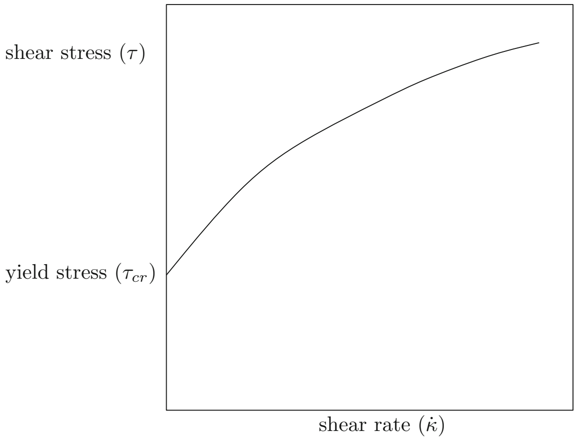


Fig. 2.11 1-D Bingham fluid

as a function of the shear rate. In many instances one needs a truly implicit relationship between the stress and the shear rate. Recently, Rajagopal and Srinivasa [236] have developed a thermodynamic framework within which such response can be described.[21] The thermodynamic approach rests on making an appropriate choice for the rate of entropy production and then carrying out a maximization of the rate of entropy production subject to the reduced energy-dissipation equation holding (see [236] for details). A choice of the rate of dissipation ξ of the form

$$\xi = \frac{\boldsymbol{\tau} \cdot \boldsymbol{\tau}}{2\,\mu},$$

where $\boldsymbol{\tau}$ is the deviatoric part of the stress and $\mu > 0$, leads to the constitutive relation

$$\boldsymbol{D} = \|\boldsymbol{D}\| \frac{\boldsymbol{\tau}}{2\,\mu}. \tag{2.9.1}$$

Thus, if $\boldsymbol{D} \neq \mathbf{0}$, then we can conclude that $\|\boldsymbol{\tau}\| = 2\,\mu$, and if $\|\boldsymbol{\tau}\| < 2\,\mu$, we can conclude that $\boldsymbol{D} = \mathbf{0}$. Hence $\|\boldsymbol{\tau}\| = 2\,\mu$ is the threshold value for the stress, below which there is no flow and when the threshold is reached the body flows. Therefore (2.9.1) describes the response depicted in Figure 2.11. For the one-dimensional response of a linear Bingham fluid see Figure 2.12.

Such a fluid withstands shear stresses up to the threshold level of τ_{th}, and beyond the threshold flows like a linearly viscous fluid. Once again, we notice that $\boldsymbol{\tau}$ is not a function of κ, however, κ is a function of $\boldsymbol{\tau}$ (see Figure 2.13).

[21] Bulíček et al. [53, 54] have provided a detailed mathematical study of the equations governing the flows of such fluids.

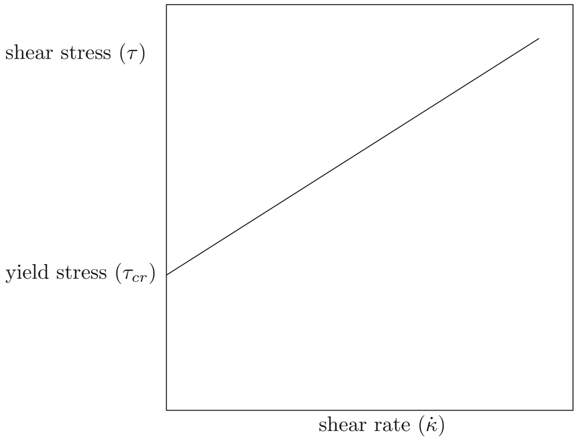


Fig. 2.12 1-D linear Bingham fluid

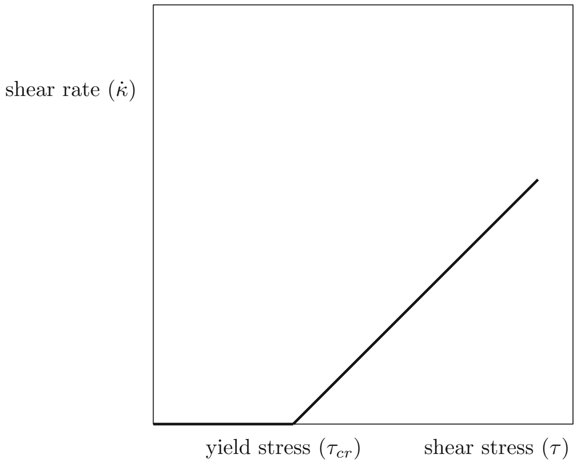


Fig. 2.13 1-D linear Bingham fluid, κ as a function of τ

2.10 Appendix

2.10.1 Integral Type Constitutive Equations

Integral type constitutive equations can be traced back to the work of Boltzman [44]. In fact, though Maxwell [184] developed his model in the rate form, its equivalent integral form can be easily established (see the integral model given by the equation (2.8.5)). We shall not go into a detailed discussion of integral models, as little by way of rigorous mathematical results have been established for properly frame-indifferent nonlinear models.

The most popular integral model is called the general linear integral model, in which the Cauchy stress takes the form:

$$\boldsymbol{T} = -p\boldsymbol{I} + \int_{-\infty}^{t} G(t-\tau)\boldsymbol{D}(\tau)\, d\tau,$$

where $G(t-\tau)$ is called the relaxation modulus. The above model, which was the model of choice several decades ago, has been studied in great detail, and continues to be widely used to describe the response of polymers within their linear range even today. However, it is incapable of describing normal stress differences in simple shear flows. Several mathematical results concerning the above model have been established and we refer the reader to Hrusa et al. [135] for a discussion of, and references to, the same.

Most of the integral models that are currently in vogue are special cases of the K-BKZ model (cf. Kaye [147], Bernstein et al. [36]). The Cauchy stress in a K-BKZ model takes the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \int_{-\infty}^{t} \left[(\frac{\partial U}{\partial I})\boldsymbol{C}_t^{-1}(\tau) - (\frac{\partial U}{\partial II})\boldsymbol{C}_t(\tau) \right] d\tau, \tag{2.10.1}$$

where U is the stored energy that depends on $t-\tau$ and the principal invariants of $\boldsymbol{C}_t(\tau)$, i.e.,

$$\begin{aligned} U &= \hat{U}(t-\tau, I, II), \quad I = \operatorname{tr} \boldsymbol{C}_t(\tau), \\ II &= \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{C}_t(\tau))^2 - \operatorname{tr} \boldsymbol{C}_t^2(\tau) \right] = \operatorname{tr} \boldsymbol{C}_t^{-1}(\tau). \end{aligned} \tag{2.10.2}$$

The last equality in (2.10.2) holds because we are interested in incompressible fluids, and thus, the flows are isochoric. This model is properly frame-indifferent and the statistical mechanical approach to the modeling of polymers leads to models of the form (2.10.2), cf. Doi and Edwards [83]. Different choices for the stored energy U leads to different models. For instance, Wagner [280], based on experimental evidence, made the choice

$$\frac{\partial U}{\partial I} = 0, \ \frac{\partial U}{\partial II} = -\frac{1}{2}\frac{d\,G(s)}{d\,s} \mathrm{e}^{-n\sqrt{II-3}},$$

where $G(s)$ is the shear stress relaxation function and $n = 0.29$, a constant value that he fixed on the basis of experiments on polyethylene melts.

The Lodge model (cf. [167]) corresponds to a choice of U of the form:

$$\frac{\partial U}{\partial I} = \phi_1(t-\tau), \ \frac{\partial U}{\partial II} = \phi_2(t-\tau).$$

In this case, the model reduces to

$$\boldsymbol{T} = -p\boldsymbol{I} + \int_{-\infty}^{t} \left[\phi_1(t-\tau)\boldsymbol{C}_t^{-1}(\tau) - \phi_2(t-\tau)\boldsymbol{C}_t(\tau) \right] d\tau.$$

In addition to models such as (2.10.1) given in terms of a single integral, multiple integral representations for the stress have been proposed (see Green and Rivlin [125]) but little if anything by way of specific initial-boundary value problems or rigorous mathematical results have been established within the context of such models.

2.10.2 Fractional Derivative Models of the Rate and Integral Type

Fractional derivative models have been used to describe the response of viscoelastic materials such as asphalt. They have also been advanced as possible models for describing the response of amorphous polymers near the glassy state. Interestingly, they were introduced to model viscoelastic materials on the basis of models in the field of Psychology! (cf. Grement [111] for a more detailed discussion of fractional derivative models for viscoelastic fluids).

The notions of fractional differentiation and integration were recognized by Leibnitz [159] in 1675 and studied by Euler [95] in 1730. It was given a somewhat rigorous basis by Liouville [165] in 1832 and Riemann [244] in 1876.

Fractional derivatives can be derived in a variety of ways and these definitions are unfortunately not equivalent, see for instance the definition of fractional Sobolev spaces in Subsection 3.1.1. What makes the study of papers using models based on fractional derivatives difficult is that these papers do not use fractional derivatives in the same sense and hence one has to be careful in passing judgements concerning the usefulness, or otherwise, of these models. Fractional derivatives or integrals can be defined without making any explicit use of either derivatives or integrals (see (2.10.3) below). However, it is common to introduce fractional integrals in terms of Liouville or Riemann–Liouville operators by using integrals explicitly. The fractional derivative based on the Riemann–Liouville operator has the shortcoming that the fractional derivative of a constant function is not zero (cf. Oldham and Spanier [199]). By slightly modifying its definition, one can obtain the fractional derivative to be zero. But the price to pay is a lack of consistency: the limit obtained by setting the fractional order of differentiation or integration to zero is not the same. Nevertheless, some authors adopt this definition of fractional derivative to develop rheological models (see Van Arsdale [278]). On the other hand, some rheologists prefer to use a definition for fractional integration that yields a nonzero fractional derivative of a constant (cf. Palade et al. [207]).

There has been no systematic study of any of the fractional derivative or integral models from either a rigorous mathematical or numerical point of view. A few patchy studies that are available do not provide any insight into the usefulness of these models with regard to describing the behavior of viscoelastic fluids. It might be worthwhile to carry out a careful and systematic assessment of the status of such models within the class of models used to describe the response of viscoelastic fluids. There have been several studies concerning fractional derivative models for viscoelastic solids from an engineering standpoint (see Rossikin and Shitikova [249]). However, no rigorous mathematical results have been established concerning these models.

In 1936, Gremant [111], used fractional derivative models to describe experimental results for viscoelastic bodies. In 1947, Blair et al. [41] used fractional derivative models to describe the non-Newtonian response of certain fluids. These early studies by Blair et al. have been followed by numerous attempts at modeling the viscoelastic response of materials to this date (the papers by Bagley [16], Bagley and Torvik [17], Koeller [150], Stastna et al. [257], Palade et al. [207], Palade and Santo [208] and the references cited therein will give the interested reader a reasonable picture of the successes and failures of fractional derivative models).

The derivative or integral of order q of a function is given by (see [199])

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{[\frac{x-a}{N}]^q}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x-j(\frac{x-a}{N}))\right\}, \tag{2.10.3}$$

where a denotes the lower limit of integration, q is arbitrary and Γ denotes the Gamma function defined by

$$\Gamma(x) = \lim_{N\to\infty}\left[\frac{N!N^x}{x(x+1)\ldots(x+N)}\right].$$

In 1985, Van Arsdale [278] introduced a generalization of the Rivlin–Ericksen tensors within the context of fractional derivatives through

$$\boldsymbol{A}_\alpha = \boldsymbol{F}^{-T} D^\alpha(\boldsymbol{F}^T\boldsymbol{F})\boldsymbol{F}^{-1}.$$

This tensor is properly frame-indifferent, i.e., for all $\boldsymbol{Q} \in \mathcal{O}$, $\boldsymbol{A}_\alpha^\star = \boldsymbol{Q}\boldsymbol{A}_\alpha\boldsymbol{Q}^T$. Also, these tensors satisfy the following recursive relation:

$$\boldsymbol{A}_{\alpha+n} = \frac{d}{dt}\boldsymbol{A}_{\alpha+n-1} + \boldsymbol{A}_{\alpha+n-1}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_{\alpha+n-1}.$$

Clearly, one can develop constitutive models based on these Rivlin–Ericksen tensors of fractional order. As an example, [278] discusses the model

$$\boldsymbol{S} = \mu_0\boldsymbol{A}_1 + \mu_1\boldsymbol{A}_\alpha\boldsymbol{A}_1\boldsymbol{A}_\alpha + \mu_2(\boldsymbol{A}_1\boldsymbol{A}_\alpha^2 + \boldsymbol{A}_\alpha^2\boldsymbol{A}_1),$$

that ensures that the stress power is nonnegative provided μ_0 and μ_2 are positive and $|\mu_1| \le \mu_2/2$. A variety of such models can be constructed. We shall not get into a discussion of these fluid models of the differential type.

A general three-dimensional constitutive model based on fractional derivatives that has been used is of the form

$$\left(1+\sum_{k=1}^{K} a_k D^{\beta_k}\right)\left(1+\sum_{p=1}^{P} b_p D^{\gamma_p}\right)\boldsymbol{T} = \left[\left(1+\sum_{p=1}^{P} b_p D^{\gamma_p}\right)\left(\lambda_0+\sum_{j=1}^{J}\lambda_j D^{\alpha_j}\right)(\operatorname{tr}\boldsymbol{\varepsilon})\right]\boldsymbol{I}$$
$$+\,2\left(1+\sum_{k=1}^{K}\alpha_k D^{\beta_k}\right)\left(\mu_0+\sum_{m=1}^{M}\mu_m D^{\delta_m}\right)\boldsymbol{\varepsilon},$$

where $\boldsymbol{\varepsilon}$ is the linearized strain and D denotes the time derivative. Even if one switches D to be a frame-indifferent time derivative such as the upper-convected Oldroyd derivative, since $\boldsymbol{\varepsilon}$ is neither frame-indifferent nor Galilean invariant, the above model is neither frame-indifferent nor Galilean invariant.

A proper frame-indifferent model was introduced in [207] where the extra stress tensor satisfies

$$\begin{aligned}\boldsymbol{S}(t) + \lambda^{\alpha}\boldsymbol{F}(t)\left\{\int_{-\infty}^{t}\mu_1(t-\tau)\boldsymbol{F}^{-1}(\tau)\overset{\nabla}{\boldsymbol{S}}(\tau)\boldsymbol{F}^{-T}(\tau)\,d\tau\right\}\boldsymbol{F}^{T}(t)\\ = G\lambda^{\beta}\boldsymbol{F}(t)\left\{\int_{-\infty}^{t}\mu_2(t-\tau)\boldsymbol{F}^{T}(\tau)\boldsymbol{A}_1(\tau)\boldsymbol{F}(\tau)\,d\tau\right\}\boldsymbol{F}^{T}(t),\end{aligned}\tag{2.10.4}$$

where μ_1 and μ_2 are memory kernels given in terms of fractional derivatives and λ is a relaxation time. It is shown in this paper that if

$$\mu_1(t-\tau) = \frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha},$$

$$\mu_2(t-\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta},$$

then (2.10.4) reduces to a generalization of the Maxwell model in fractional integral form

$$\boldsymbol{S}(t) + \lambda^{\alpha}\int_{-\infty}^{t}\mu_1(t-\tau)\frac{\partial\boldsymbol{S}(\tau)}{\partial\tau}\,d\tau = G\lambda^{\beta}\int_{-\infty}^{t}\mu_2(t-\tau)\boldsymbol{A}_1(\tau)\,d\tau.$$

The constants α and β have physical significance. The constant β represents the slope of the complex modulus G'' in the limit of zero frequency on a log–log plot while $\beta-\alpha$ is related to the slope of the complex modulus G' in the high frequency limit.

Palade et al. [207] have shown that the model (2.10.4), which helped describe the linear viscoelastic behavior of polymers quite well, exhibits peculiar stability characteristics calling into question the specific model that they have used. But this by no means closes the door with regard to the use of models based on fractional derivatives and integrals.

However, before one starts embarking on rigorous mathematical investigations one has to grapple with an important physical issue, that of boundary and initial conditions for fractional derivative models. For instance, when dealing with differential or rate type fractional derivative models, say involving the fractional derivative $\alpha > n$, where

$n = 1$, one has to first determine what boundary conditions need to be prescribed in order to render the problem well posed. We already know for integral values of $n > 1$, we have difficulties concerning fluids of the differential type. Even for the integral type fractional derivative models, if the memory kernel is given in terms of fractional derivatives, it is not clear that the usual initial conditions will suffice.

2.10.3 Free-Surface Flows

A very important class of fluids that is not considered in this book is free-surface flows. Such flows, even within the context of the Navier–Stokes fluid are obviously difficult for a variety of reasons. First, in most such flows the domain in which the flow takes place changes with time. Second, one invariably finds certain incompatibilities with regard to the specification of boundary conditions, for instance, traction may be specified on one part of the boundary and the velocity on another part, leading to an incompatibility where these two boundaries intersect. Third, the boundary of the domain in which the flow takes place is usually not smooth. However, free boundary flows have very important technological significance.

For instance, the problem of "fiber spinning," which is how polyester yarn is produced, or "film blowing," which is how most plastic bags (grocery bags, garbage bags, etc.) are produced lead to free-surface problems. The classical experiments that best elucidate the effect of normal stress differences are, namely "rod climbing" (see Figure 2.5 and "flow down a tilted trough" (see Figure 2.14), which are also used to experimentally determine the material moduli of non-Newtonian fluids.

Another important "free-surface flow" that needs careful study is at the heart of the manufacture of all manners of thin films, tapes, etc. This is the flow that is involved in all manufacturing processes made via extrusion. The flow domain is exceedingly complex, consisting of boundaries that are made up of both free and fixed surfaces (see Figure 2.15). The engineer is in fact faced with the inverse problem of designing the "die" so that the flow that emerges out of the "die" is flat (see Figure 2.16) and without wrinkles (referred to in the engineering literature as the "shark skin effect").

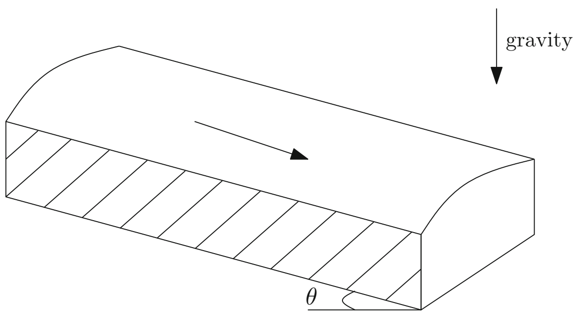


Fig. 2.14 Flow down a tilted trough

Fig. 2.15 Extrusion with die swelling

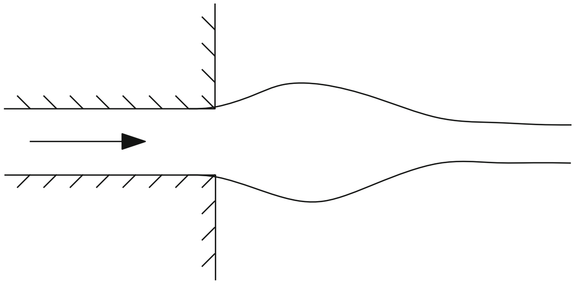
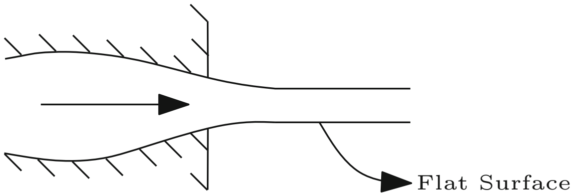


Fig. 2.16 Extrusion without swelling



The "stress-free" ("traction-free") boundary condition that needs to be specified in the case of non-Newtonian fluids are far more complicated than their Navier–Stokes counterpart. Even in the case of a second-grade fluid, it takes a complicated form. The "traction-free" boundary condition for a third-grade fluid would include additional nonlinear terms that would make it most daunting. This notwithstanding, it is necessary to tackle such problems in view of their technological importance; "free-surface" problems for non-Newtonian fluids constitute a very important class of problems that have not received the rigorous mathematical scrutiny that they deserve. Most analysis that has been carried out by the engineering community, in virtue of their technological significance, while physically insightful, lack mathematical rigor and often times are clearly incorrect.

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