Chapter 2
Why Paraconsistent Logics?

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Dedicated to Jair Minoro Abe for his 60th birthday.

Abstract In this chapter, we briefly review paraconsistent logics which are closely related to the topics in this book. We give an exposition of their history and formal aspects. We also address the importance of applications of paraconsistent logics to engineering.

Keywords Paraconsistent logics · Contradiction inconsistency · Paraconsistency

2.1 Introduction

Paraconsistent logic is a logical system for inconsistent but non-trivial formal theories. It is classified as *non-classical logic* in the sense that it can be employed as a rival to classical logic. Paraconsistent logic has many applications and it can serve as a foundation for engineering because some engineering problems must solve inconsistent information. However standard classical logic cannot tolerate it. In this regard, paraconsistent is promising.

Here, we give a quick review of paraconsistent logic that is helpful to the reader. Let T be a theory whose underlying logic is L. T is called *inconsistent* when it contains theorems of the form A and $\neg A$ (the negation of A), i.e.,

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S. Akama (ed.), *Towards Paraconsistent Engineering*, Intelligent Systems
Reference Library 110, DOI 10.1007/978-3-319-40418-9_2

$$T \vdash_L A \text{ and } T \vdash_L \neg A$$

where $\vdash_L$ denotes the provability relation in L. If T is not inconsistent, it is called *consistent*.

T is said to be *trivial*, if all formulas of the language are also theorems of T. Otherwise, T is called *non-trivial*. Then, for trivial theory T, $T \vdash_L B$ for any formula B. Note that trivial theory is not interesting since every formula is provable.

If L is classical logic (or one of several others, such as intuitionistic logic), the notions of inconsistency and triviality agree in the sense that T is inconsistent iff T is trivial. So, in trivial theories the extensions of the concepts of formula and theorem coincide.

A *paraconsistent logic* is a logic that can be used as the basis for inconsistent but non-trivial theories. In this regard, sentences of paraconsistent theories do not satisfy, in general, the *principle of non-contradiction*, i.e., $\neg(A \wedge \neg A)$.[1]

Similarly, we can define the notions of paracomplete logic and theory. A *paracomplete logic* is a logic, in which the *principle of excluded middle*, i.e., $A \vee \neg A$ is not a theorem of that logic. In this sense, intuitionistic logic is one of the paracomplete logics. A *paracomplete theory* is a theory based on paracomplete logic.

Finally, a logic which is simultaneously paraconsistent and paracomplete is called *non-alethic logic*.

The structure of this paper is as follows. In Sect. 2.2, we describe the history of paraconsistent logic. In Sect. 2.3, major approaches to paraconsistent logic are given with formal descriptions. In Sect. 2.4, other paraconsistent logics are briefly reviewed.

2.2 History

This section surveys the history of paraconsistent logic. Paraconsistent logics have recently proved attracted to many people, but they have a longer history than classical logic. For example, Aristotle developed a logical theory that can be interpreted to be paraconsistent. But, paraconsistent logics in the modern sense were formally devised in the 1950s.

In 1910, the Russian logician Nikolaj A. Vasil'ev (1880–1940) and the Polish logician Jan Łukasiewicz (1878–1956) independently glimpsed the possibility of developing paraconsistent logics. Vasil'ev's *imaginary logic* can be seen as a paraconsistent reformulation of Aristotle's *syllogistic*; see Vasil'ev [54].

It was here pointed out that Łukasiewicz's *three-valued logic* is a forerunner of the many-valued approach to paraconsistency, although he did not explicitly discuss paraconsistency; see Łukasiewicz [43].

[1] In fact, in some systems of paraconsistent logic, like da Costa's systems C_n, the "good" propositions do satisfy this principle.

However, we believe that the history of paraconsistent logic started in 1948. Stanislaw Jaśkowski (1896–1965) proposed a paraconsistent propositional logic, now called *discursive logic* (or discussive logic) in 1948; see Jaśkowski [37, 38]. Discursive logic is based on modal logic, and it is classified as the modal approach to paraconsistency.

Independently, some years later, the Brazilian logician Newton C.A. da Costa (1929-) constructed for the first time hierarchies of paraconsistent propositional calculi $C_i (1 \leq i \leq \omega)$ and its first-order and higher-order extensions; see da Costa [28]. da Costa's logics are called the *C-system*, which is based on the non-standard interpretation of negation which is dual to intuitionistic negation.

A different route to paraconsistent logic may be found in the so-called *relevance logic* (or relevant logic), which was originally developed by Anderson and Belnap in the 1960s; see Anderson and Belnap [11] and Anderson, Belnap and Dunn [12]. Anderson and Belnap's approach addresses a correct interpretation of implication $A \to B$, in which A and B should have some connection. Its semantic interpretation raises the issues of paraconsistency, and some (not all) relevance logics are in fact paraconsistent.

The above three approaches are considered the major approaches to paraconsistent logics, many paraconsistent logics have been proposed in the literature. They have been developed from some motivation.

2.3 Approaches to Paraconsistent Logic

The section formally reviews several paraconsistent logics, restricting to the principal paraconsistent logics. But, it is far from complete, and the reader should consult in-depth exposition in the relevant reference.

We can list the three logics as the major approaches:

- Discursive logic
- C-systems
- Relevant (relevance) logic

Discursive logic, also known as discussive logic, was proposed by Jaśkowski [37, 38], which is regarded as a non-adjunctive approach. *Adjunction* is a rule of inference of the form: from $\vdash A$ and $\vdash B$ to $\vdash A \wedge B$. Discursive logic can avoid explosion by prohibiting adjunction.

It was a formal system J satisfying the conditions: (a) from two contradictory propositions, it should not be possible to deduce any proposition; (b) most of the classical theses compatible with (a) should be valid; (c) J should have an intuitive interpretation.

Such a calculus has, among others, the following intuitive properties remarked by Jaśkowski himself: suppose that one desires to systematize in only one deductive system all theses defended in a discussion. In general, the participants do not confer the same meaning to some of the symbols.

One would have then as theses of a deductive system that formalize such a discussion, an assertion and its negation, so both are "true" since it has a variation in the sense given to the symbols. It is thus possible to regard discursive logic as one of the so-called *paraconsistent logics*.

Jaśkowski's D_2 contains propositional formulas built from logical symbols of classical logic. In addition, the possibility operator $\Diamond$ in S5 is added. Based on the possibility operator, three discursive logical symbols can be defined as follows:

discursive implication: $A \rightarrow_d B =_{def} \Diamond A \rightarrow B$
discursive conjunction: $A \wedge_d B =_{def} \Diamond A \wedge B$
discursive equivalence: $A \leftrightarrow_d B =_{def} (A \rightarrow_d B) \wedge_d (B \rightarrow_d A)$

Additionally, we can define discursive negation $\neg_d A$ as $A \rightarrow_d$ *false*. Jaśkowski's original formulation of D_2 in [38] used the logical symbols: $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, $\wedge$, $\neg$, and he later defined $\wedge_d$ in [38].

The following axiomatization due to Kotas [42] has the following axioms and the rules of inference.

Axioms
(A1) $\Box(A \rightarrow (\neg A \rightarrow B))$
(A2) $\Box((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
(A3) $\Box((\neg A \rightarrow A) \rightarrow A)$
(A4) $\Box(\Box A \rightarrow A)$
(A5) $\Box(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))$
(A6) $\Box(\neg\Box A \rightarrow \Box\neg\Box A)$

Rules of Inference
(R1) substitution rule
(R2) $\Box A, \Box(A \rightarrow B)/\Box B$
(R3) $\Box A/\Box\Box A$
(R4) $\Box A/A$
(R5) $\neg\Box\neg\Box A/A$

There are other axiomatizations of D_2, but we omit the details here. Discursive logics are considered weak as a paraconsistent logic, but they have some applications, e.g. logics for vagueness.

C-systems are paraconsistent logics due to da Costa which can be a basis for inconsistent but non-trivial theories; see da Costa [28]. The important feature of da Costa systems is to use novel interpretation, which is non-truth-functional, of negation avoiding triviality.

Here, we review C-system C_1 due to da Costa [28]. The language of C_1 is based on the logical symbols: $\wedge$, $\vee$, $\rightarrow$, and $\neg$. $\leftrightarrow$ is defined as usual. In addition, a formula A°, which is read "A is well-behaved", is shorthand for $\neg(A \wedge \neg A)$. The basic ideas of C_1 contain the following: (1) most valid formulas in the classical logic hold, (2) the law of non-contradiction $\neg(A \wedge \neg A)$ should not be valid, (3) from two contradictory formulas it should not be possible to deduce any formula.

The Hilbert system of C_1 extends the positive intuitionistic logic with the axioms for negation.

da Costa's C_1

Axioms

(DC1) $A \to (B \to A)$
(DC2) $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
(DC3) $(A \wedge B) \to A$
(DC4) $(A \wedge B) \to B$
(DC5) $A \to (B \to (A \wedge B))$
(DC6) $A \to (A \vee B)$
(DC7) $B \to (A \vee B)$
(DC8) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
(DC9) $B^\circ \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
(DC10) $(A^\circ \wedge B^\circ) \to (A \wedge B)^\circ \wedge (A \vee B)^\circ \wedge (A \to B)^\circ$
(DC11) $A \vee \neg A$
(DC12) $\neg\neg A \to A$

Rules of Inference

(MP) $\vdash A,\ \vdash A \to B\ \Rightarrow \vdash B$

Here, (DC1)–(DC8) are axioms of the positive intuitionistic logic. (DC9) and (DC10) play a role for the formalization of paraconsistency.

A semantics for C_1 can be given by a two-valued valuation; see da Costa and Alves [29]. We denote by $\mathcal{F}$ the set of formulas of C_1. A valuation is a mapping v from $\mathcal{F}$ to $\{0, 1\}$ satisfying the following:

$$v(A) = 0 \ \Rightarrow\ v(\neg A) = 1$$
$$v(\neg\neg A) = 1 \ \Rightarrow\ v(A) = 1$$
$$v(B^\circ) = v(A \to B) = v(A \to \neg B) = 1 \ \Rightarrow\ v(A) = 0$$
$$v(A \to B) = 1 \ \Leftrightarrow\ v(A) = 0 \text{ or } v(B) = 1$$
$$v(A \wedge B) = 1 \ \Leftrightarrow\ v(A) = v(B) = 1$$
$$v(A \vee B) = 1 \ \Leftrightarrow\ v(A) = 1 \text{ or } v(B) = 1$$
$$v(A^\circ) = v(B^\circ) = 1 \Rightarrow v((A \wedge B)^\circ) = v((A \vee B)^\circ) = v((A \to B)^\circ) = 1$$

Note here that the interpretations of negation and double negation are not given by biconditional. A formula A is *valid*, written $\models A$, if $v(A) = 1$ for every valuation v. Completeness holds for C_1. It can be shown that C_1 is complete for the above semantics.

Da Costa system C_1 can be extended to C_n $(1 \le n \le \omega)$. Now, A^1 stands for A° and A^n stands for $A^{n-1} \wedge (A^{(n-1)})^\circ$, $1 \le n \le \omega$.

Then, da Costa system C_n $(1 \le n \le \omega)$ can be obtained by (DC1)–(DC8), (DC12), (DC13) and the following:

(DC9n) $B^{(n)} \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
(DC10n) $(A^{(n)} \wedge B^{(n)}) \to (A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \to B)^{(n)}$

Note that the da Costa system C_ω has the axioms (DC1)–(DC8), (DC12) and (DC13). Later, da Costa investigated first-order and higher-order extensions of C-systems.

Relevance logic, also called *relevant logic* is a family of logics based on the notion of relevance in conditionals. Historically, relevance logic was developed to avoid the *paradox of implications*; see Anderson and Belnap [11, 12].

Anderson and Belnap formalized a relevant logic R to realize a major motivation, in which they do not admit $A \rightarrow (B \rightarrow A)$. Later, various relevance logics have been proposed. Note that not all relevance logics are paraconsistent but some are considered important as paraconsistent logics.

Routley and Meyer proposed a basic relevant logic B, which is a minimal system having the so-called *Routley-Meyer semantics*. Thus, B is an important system and we review it below; see Routley et al. [51].

The language of B contains logical symbols: $\sim$, &, $\vee$ and $\rightarrow$ (relevant implication). A Hilbert system for B is as follows:

Relevant Logic B
Axioms
(BA1) $A \rightarrow A$
(BA2) $(A\&B) \rightarrow A$
(BA3) $(A\&B) \rightarrow B$
(BA4) $((A \rightarrow B)\&(A \rightarrow C)) \rightarrow (A \rightarrow (B\&C))$
(BA5) $A \rightarrow (A \vee B)$
(BA6) $B \rightarrow (A \vee B)$
(BA7) $(A \rightarrow C)\&(B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
(BA8) $(A\&(B \vee C)) \rightarrow (A\&B) \vee C)$
(BA9) $\sim\sim A \rightarrow A$
Rules of Inference
(BR1) $\vdash A, \vdash A \rightarrow B \Rightarrow \vdash B$
(BR2) $\vdash A, \vdash B \Rightarrow \vdash A\&B$
(BR3) $\vdash A \rightarrow B, \vdash C \rightarrow D \Rightarrow \vdash (B \rightarrow C) \rightarrow (A \rightarrow D)$
(BR4) $\vdash A \rightarrow \sim B \Rightarrow \vdash B \rightarrow \sim A$

A Hilbert system for Anderson and Belnap's R is as follows:

Relevance Logic R
Axioms
(RA1) $A \rightarrow A$
(RA2) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow C \rightarrow B))$
(RA3) $(A \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow B)$
(RA4) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)$
(RA5) $(A\&B) \rightarrow A$
(RA6) $(A\&B) \rightarrow B$
(RA7) $((A \rightarrow B)\&(A \rightarrow C)) \rightarrow (A \rightarrow (B\&C))$
(RA8) $A \rightarrow (A \vee B)$
(RA9) $B \rightarrow (A \vee B)$
(RA10) $((A \rightarrow C)\&(B \vee C)) \rightarrow ((A \vee B) \rightarrow C))$
(RA11) $(A\&(B \vee C)) \rightarrow ((A\&B) \vee C)$
(RA12) $(A \rightarrow \sim A) \rightarrow \sim A$

(RA13) $(A \to \sim B)) \to (B \to \sim A)$
(RA14) $\sim\sim A \to A$
Rules of Inference
(RR1) $\vdash A, \vdash A \to B \;\Rightarrow \vdash B$
(RR2) $\vdash A, \vdash B \;\Rightarrow \vdash A \& B$

Routley et al. considered some axioms of R are too strong and formalized rules instead of axioms. Notice that B is a paraconsistent but R is not.

Next, we give a Routley-Meyer semantics for B. A model structure is a tuple $\mathcal{M} = \langle K, N, R, *, v \rangle$, where K is a non-empty set of worlds, $N \subseteq K$, $R \subseteq K^3$ is a ternary relation on K, $*$ is a unary operation on K, and v is a valuation function from a set of worlds and a set of propositional variables $\mathcal{P}$ to $\{0, 1\}$.

There are some restrictions on. v satisfies the condition that $a \leq b$ and $v(a, p)$ imply $v(b, p) = 1$ for any $a, b \in K$ and any $p \in \mathcal{P}$. $a \leq b$ is a pre-order relation defined by $\exists x(x \in N$ and $Rxab)$. The operation $*$ satisfies the condition $a^{**} = a$.

For any propositional variable p, the truth condition $\models$ is defined: $a \models p$ iff $v(a, p) = 1$. Here, $a \models p$ reads "p is true at a". $\models$ can be extended for any formulas in the following way:

$a \models \sim A \;\Leftrightarrow\; a^* \not\models A$
$a \models A \& B \;\Leftrightarrow\; a \models A$ and $a \models B$
$a \models A \vee B \;\Leftrightarrow\; a \models A$ or $a \models B$
$a \models A \to B \;\Leftrightarrow\; \forall bc \in K(Rabc$ and $b \models A \;\Rightarrow\; c \models B)$

A formula A is *true* at a in $\mathcal{M}$ iff $a \models A$. A is *valid*, written $\models A$, iff A is true on all members of N in all model structures.

Routley et al. provides the completeness theorem for B with respect to the above semantics using canonical models; see [51].

A model structure for R needs the following conditions.

$R0aa$
$Rabc \;\Rightarrow\; Rbac$
$R^2(ab)cd \;\Rightarrow\; R^2a(bc)d \; Raaa$
$a^{**} = a$
$Rabc \;\Rightarrow\; Rac^*b^*$
$Rabc \;\Rightarrow\; a' \leq a \;\Rightarrow\; Ra'bc$

where R^2abcd is shorthand for $\exists x(Raxd$ and $Rxcd)$. The completeness theorem for the Routley-Meyer semantics can be proved for R; see [11, 12].

The reader is advised to consult Anderson and Belnap [11], Anderson et al. [12], and Routley et al. [51] for details. A more concise survey on the subject may be found in Dunn [32].

Belnap proposed a famous *four-valued logic* in Belnap [21, 22], which is closely related to relevant logic and paraconsistent logic. Belnap's four-valued logic aims to formalize the internal states of a computer.

There are four states, i.e. (T), (F), $(None)$ and $(Both)$, to recognize an input in a computer. Based on these states, a computer can compute the following suitable outputs.

(*T*) a proposition is true.
(*F*) a proposition is false.
(*N*) a proposition is neither true nor false.
(*B*) a proposition is both true and false.

Here, (*N*) and (*B*) abbreviate (*None*) and (*Both*), respectively. From the above, (*N*) corresponds to incompleteness and (*B*) inconsistency. Four-valued logic can be thus seen as a natural extension of three-valued logic. In fact, Belnap's four-valued logic can model both incomplete information (*N*) and inconsistent information (*B*).

Belnap proposed two four-valued logics **A4** and **L4**. The former can cope only with atomic formulas, whereas the latter can handle compound formulas. **A4** is based on the *approximation lattice*, which is shown in Fig. 2.1.

Here, B is the least upper bound and N is the greatest lower bound with respect to the ordering $\leq$. Observe that in the lattice *FOUR* in Fig. 2.1, we used $t, f, \perp, \top$ instead of T, F, N, B, respectively.

The logic **L4** has logical symbols; $\sim, \wedge, \vee$. Its truth-values is $\mathbf{4} = \{T, F, N, B\}$ with a different ordering. The lattice **L4** is shown in Fig. 2.2.

One of the features of **L4** is the monotonicity of logical symbols. Let f be a logical operation. It is said that f is monotonic iff $a \subseteq b \Rightarrow f(a) \subseteq f(b)$. To guarantee the monotonicity of conjunction and disjunction, they must satisfy the following:

$a \wedge b = a \Leftrightarrow a \vee b = b$
$a \wedge b = b \Leftrightarrow a \vee b = a$

Logical symbols in **L4** obey th truth-value tables in Table 2.1.

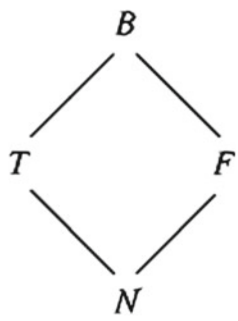


Fig. 2.1 Approximation lattice **L4**

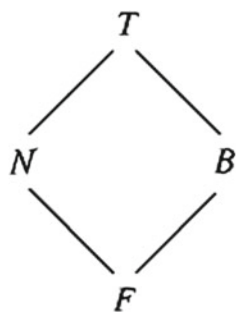


Fig. 2.2 Logical lattice **L4**

Table 2.1 Truth-value tables of **L4**

	N	F	T	B
$\sim$	B	T	F	N

$\wedge$	N	F	T	B
N	N	F	N	F
F	F	F	F	F
T	N	F	T	B
B	F	F	B	B

$\vee$	N	F	T	B
N	N	N	T	T
F	N	F	T	B
T	T	T	T	T
B	T	B	T	B

Belnap gave a semantics for the language with the above logical symbols. A *setup* is a mapping a set of atomic formulas *Atom* to the set **4**. Then, formulas of **L4** are defined as follows:

$s(A \wedge B) = s(A) \wedge s(B)$
$s(A \vee B) = s(A) \vee s(B)$
$s(\sim A) = \sim s(A)$

Further, Belnap defined an entailment relation $\rightarrow$ as follows:

$$A \rightarrow B \Leftrightarrow s(A) \leq s(B)$$

for all setups s. Note that $\rightarrow$ is not a logical connective for implication but an entailment relation. The entailment relation $\rightarrow$ can be axiomatized as follows:

$(A_1 \wedge \ldots \wedge A_m) \rightarrow (B_1 \vee \ldots \vee B_n)$ (A_i shares some B_j)
$(A \vee B) \rightarrow C \leftrightarrow (A \rightarrow C)$ and $(B \rightarrow C)$
$A \rightarrow B \ \Leftrightarrow \sim B \rightarrow \sim A$
$A \vee B \leftrightarrow B \vee A, \ A \wedge B \leftrightarrow B \wedge A$
$A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$
$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$
$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$
$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$
$(B \vee C) \wedge A \leftrightarrow (B \wedge A) \vee (C \wedge A)$
$(B \wedge C) \vee A \leftrightarrow (B \vee A) \wedge (C \vee A)$
$\sim\sim A \leftrightarrow A$
$\sim (A \wedge B) \leftrightarrow \sim A \vee \sim B, \ \sim (A \vee B) \leftrightarrow \sim A \wedge \sim B$
$A \rightarrow B, B \rightarrow C \ \Leftrightarrow \ A \rightarrow C$
$A \leftrightarrow B, B \leftrightarrow C \ \Leftrightarrow A \leftrightarrow C$
$A \rightarrow B \ \Leftrightarrow \ A \leftrightarrow (A \wedge B) \ \Leftrightarrow \ (A \vee B) \leftrightarrow B$

Note here that $(A \wedge \sim A) \rightarrow B$ and $A \rightarrow (B \vee \sim B)$ cannot be derived in this axiomatization. It can be shown that the logic given above is shown to be equivalent to the system of *tautological entailment*; see [11, 12].

An alternative semantics for tautological entailment based on the notion of fact was worked out by van Fraassen [53]. Belnap's **A4** is used as one of the lattice of truth-values as *FOUR*. In this regard, Belnap's four-valued logic is considered as the important background on annotated logics.

2.4 Other Paraconsistent Logics

Although the above three logics are famous approaches to paraconsistent logics, there is a rich literature on paraconsistent logics. Arruda [15] reviewed a survey on paraconsistent logics, and Priest et al. [49] contains interesting papers on paraconsistent logics in the 1980s. For a recent survey, we refer Priest [47]. We can also find a Handbook surveying various subjects related to paraconsistency by Beziau et al. [23].

In 1997, The First World Congress on Paraconsistency (WCP'1997) was held at the University of Ghent, Belgium; see Batens et al. [20]. The Second World Congress on Paraconsistency (WCP'200) was held at Juquehy-Sao Paulo, Brazil; see Carnielli et al. [26].

In the 1990s paraconsistent logics became one of the major topics in logic in connection with other areas, in particular, computer science. Below we review some of those systems of paraconsistent logics.

The modern history of paraconsistent logic started with Vasil'ev's *imaginary logic*. In 1910, Vasil'ev proposed an extension of Aristotle's syllogistic allowing the statement of the form S is both P and not-P; see Vasil'ev [54].

Thus, imaginary logic can be viewed as a paraconsistent logic. Unfortunately, little work has been done on focusing on its formalization from the viewpoint of modern logic. A survey of imaginary logic can be found in Arruda [15].

In 1954, Asenjo developed a calculus of antinomies in his dissertation; see Asenjo [16]. Asenjo's work was published before da Costa's work, but it seems that Asenjo's approach has been neglected. Asenjo's idea is to interpret the truth-value of *antinomy* as both true and false using Kleene's strong three-valued logic.

His proposed calculus is non-trivially inconsistent propositional logic, whose axiomatization can be obtained from Kleene's [39] axiomatization of classical propositional logic by deleting the axiom $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.

In constructivism, an idea of constructing paraconsistent logics may be found. In 1949, Nelson [44] proposed a *constructive logic with strong negation* as an alternative to intuitionistic logic, in which *strong negation* (or constructible negation) is introduced to improve some weaknesses of intuitionistic negation.

Constructive logic N extends positive intuitionistic logic Int^+ with the following axioms for *strong negation* $\sim$:

(N1) $(A \wedge \sim A) \rightarrow B$
(N2) $\sim\sim A \leftrightarrow A$
(N3) $\sim (A \rightarrow B) \leftrightarrow (A \wedge \sim B)$
(N4) $\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(N5) $\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$

In N, intuitionistic negation $\neg$ can be defined as $\neg A \leftrightarrow A \rightarrow (B \wedge \sim B)$. If we delete (N1) from N, we can obtain a paraconsistent constructive logic N^- of Almukdad and Nelson [10]. Akama [3–8] extensively studied Nelson's constructive logics with strong negation; also see Wansing [55].

Table 2.2 Truth-value tables of Kleene's strong three-valued logic

A	$\neg A$
T	F
I	I
F	T

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow_K B$
T	T	T	T	T
T	F	F	T	F
T	I	I	T	I
F	T	F	F	T
F	F	F	F	T
F	I	F	I	T
I	T	I	T	T
I	F	F	I	I
I	I	I	I	I

In 1959, Nelson [45] developed a constructive logic S which lacks contraction $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ and discussed its aspects as a paraconsistent logic. Akama [7] gave a detailed presentation of Nelson's paraconsistent constructive logics. Akama et al. [9] proposed a constructive discursive logic based on Nelson's constructive logic.

In 1979, Priest [46] proposed a *logic of paradox*, denoted *LP*, to deal with the semantic paradox. The logic is of special importance to the area of paraconsistent logics. *LP* can be semantically defined by Kleene's strong three-valued logic whose truth-value tables are as Table 2.2.

Here, T and F denote truth and falsity, and the third truth-value I reads "undefined"; see Kleene [39].

Łukasiewicz's three-valued logic is interpreted by the above truth-value tables of Kleene's three-valued logic except for implication. Let $\rightarrow_L$ be the implication in Łukasiewicz's three-valued logic. Then, the truth-value tables are described as Table 2.3.

Here, the third truth-value reads "possible"; see Łukasiewicz [43]. Kleene's three-valued logic was used as a basis for reasoning about incomplete information in computer science.

Priest re-interpreted the truth-value tables of Kleene's strong three-valued logic, namely read the third-truth value as both true and false (B) rather than neither true nor false (I), and assumed that (T) and (B) are designated values. The idea has already been considered in Asenjo [16] and Belnap [21, 22].

Consequently, ECQ: $A, \sim A \models B$ is invalid. Thus, *LP* can be seen as a paraconsistent logic. Unfortunately, (material) implication in *LP* does not satisfy *modus ponens*. It is, however, possible to introduce relevant implications as real implication into *LP*.

Table 2.3 Truth-value tables of Łukasiewicz's three-valued logic

A	$\sim A$
T	F
I	I
F	T

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow_L B$
T	T	T	T	T
T	F	F	T	F
T	I	I	T	I
F	T	F	F	T
F	F	F	F	T
F	I	F	I	T
I	T	I	T	T
I	F	F	I	I
I	I	I	I	T

Priest developed a semantics for *LP* by means of a truth-value assignment relation rather than a truth-value assignment function. Let $\mathcal{P}$ be the set of propositional variables. Then, an evaluation η is a subset of $\mathcal{P} \times \{0, 1\}$. A proposition may only relate to 1 (true), it may only relate to 0 (false), it may relate to both 1 and 0 or it may relate to neither 1 nor 0. The evaluation is extended to a relation for all formulas as follows:

$\neg A\eta 1$ iff $A\eta 0$
$\neg A\eta 0$ iff $A\eta 1$
$A \wedge B\eta 1$ iff $A\eta 1$ and $B\eta 1$
$A \wedge B\eta 0$ iff $A\eta 0$ or $B\eta 0$
$A \vee B\eta 1$ iff $A\eta 1$ or $B\eta 1$
$A \vee B\eta 0$ iff $A\eta 0$ and $B\eta 0$

If we define validity in terms of truth preservation under all relational evaluations, then we obtain *first-degree entailment* which is a fragment of relevance logics.

Using *LP*, Priest advanced his research program to tackle various philosophical and logical issues; see Priest [47, 48] for details. For instance, in *LP*, the liar sentence can be interpreted as both true and false. It is also observed that Priest promoted the philosophical view called *dialetheism* which claims that there are true contradictions. In fact, dialetheism has been extensively discussed by many people.

Since the beginning of the 1990s, Batens developed the so-called *adaptative logics* in Batens [18, 19]. These logics are considered as improvements of *dynamic dialectical logics* investigated in Batens [17]. *Inconsistency-adaptive logics* as developed by Batens [18] can serve as foundations for paraconsistent and non-monotonic logics.

Adaptive logics formalized classical logic as "dynamic logic". Here, "dynamic logic" is not the family of logics with the same name studied in computer science. A logic is *adaptive* iff it adapts itself to the specific premises to which it is applied. In this sense, adaptive logics can model the dynamics of human reasoning. There are two sorts of dynamics, i.e., *external dynamics* and *internal dynamics*.

The external dynamics is stated as follows. If new premises become available, then consequences derived from the earlier premise set may be withdrawn. In other words, the external dynamics results from the *non-monotonic* character of the consequence relations.

Let $\vdash$ be a consequence relation, Γ, Δ be sets of formulas, and A be a formula. Then, the external dynamics is formally presented as: $\Gamma \vdash A$ but $\Gamma \cup \Delta \nvdash A$ for some Γ, Δ and A. In fact, the external dynamics is closely related to the notion of *non-monotonic reasoning* in AI.

The internal dynamics is very different from the external one. Even if the premise set is constant, certain formulas are considered as derived at some stage of the reasoning process, but are considered as not derived at a later stage. For any consequence relation, insight in the premises is gained by deriving consequences from them.

In the absence of a positive test, this results in the internal dynamics. Namely, in the internal dynamics, reasoning has to adapt itself by withdrawing an application of the previously used inference rule, if we infer a contradiction at a later stage. Adaptive logics are logics based on the internal dynamics.

An Adaptive Logic *AL* can be characterized as a triple:

(i) A *lower limit logic* (LLL)
(ii) A set of *abnormalities*
(iii) An *adaptive strategy*

The lower limit logic *LLL* is any monotonic logic, e.g., classical logic, which is the stable part of the adaptive logic. Thus, *LLL* is not subject to adaptation. The set of abnormalities Ω comprises the formulas that are presupposed to be false, unless and until proven otherwise.

In many adaptive logics, Ω is the set of formulas of the form $A \wedge \sim A$. An adaptive strategy specifies a strategy of the applications of inference rules based on the set of abnormalities.

If the lower limit logic *LLL* is extended with the requirement that no abnormality is logically possible, one obtains a monotonic logic, which is called the *upper limit logic ULL*. Semantically, an adequate semantics for the upper limit logic can be obtained by selecting that lower limit logic models that verify no abnormality.

The name "abnormality" refers to the upper limit logic. *ULL* requires premise sets to be normal, and 'explodes' abnormal premise sets (assigns them the trivial consequence set).

If the lower limit logic is classical logic *CL* and the set of abnormalities comprises formulas of the form $\exists A \wedge \exists \sim A$, then the upper limit logic obtained by adding to *CL* the axioms $\exists A \rightarrow \forall A$. If, as is the case for many inconsistency-adaptive logics, the lower limit logic is a paraconsistent logic *PL* which contains *CL*, and the set of abnormalities comprises the formulas of the form $\exists(A \wedge \sim A)$, then the upper limit logic is *CL*. The adaptive logics interpret the set of premises 'as much as possible' in agreement with the upper limit logic; it avoids abnormalities 'in as far as the premises permit'.

Adaptive logics provide a new way of thinking of the formalization of paraconsistent logics in view of the dynamics of reasoning. Although inconsistency-adaptive logic is paraconsistent logic, applications of adaptive logics are not limited to paraconsistency. From a formal point of view, we can count adaptive logics as promising paraconsistent logics.

However, for applications, we may face several obstacles in automating reasoning in adaptive logics in that proofs in adaptive logics are dynamic with a certain adaptive strategy. Thus, the implementation is not easy, and we have to choose an appropriate adaptive strategy depending on applications.

Carnelli proposed the *Logics of Formal Inconsistency* (LFI), which are logical systems that treat consistency and inconsistency as mathematical objects; see Carnelli et al. [27]. One of the distinguishing features of these logics is that they can internalize the notions of consistency and inconsistency at the object-level.

And many paraconsistent logics including da Costa's C-systems can be interpreted as the subclass of LFIs. Therefore, we can regard LFIs as a general framework for paraconsistent logics.

A Logic of Formal Inconsistency, which extends classical logic C with the consistency operator $\circ$, is defined as any explosive paraconsistent logic, namely iff the classical consequence relation $\vdash$ satisfies the following two conditions:

(a) $\exists \Gamma \exists A \exists B (\Gamma, A, \neg A \nvdash B)$
(b) $\forall \Gamma \forall A \forall B)(\Gamma, \circ A, A, \neg A \vdash B)$.

Here, Γ denotes a set of formulas and A, B are formulas. With the help of $\circ$, we can express both consistency and inconsistency in the object-language. Therefore, LFIs are general enough to classify many paraconsistent logics.

For example, da Costa's C_1 is shown to be an LFI. For every formula A, let $\circ A$ be an abbreviation of the formula $\neg(A \wedge \neg A)$. Then, the logic C_1 is an LFI such that $\circ(p) = \{\circ p\} = \{\neg\neg(p \wedge \neg p\}$ whose axiomatization as an LFI contains the positive fragment of classical logic with the axiom $\neg\neg A \rightarrow A$, and some axioms for $\circ$.

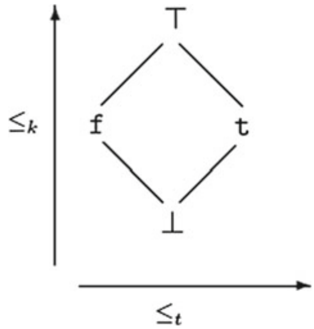
(bc1) $\circ A \rightarrow (A \rightarrow (\neg A \rightarrow B))$
(ca1) $(\circ A \wedge \circ B) \rightarrow \circ(A \wedge B)$
(ca2) $(\circ A \wedge \circ B) \rightarrow \circ(A \vee B)$
(ca3) $(\circ A \wedge \circ B) \rightarrow \circ(A \rightarrow B)$

In addition, we can define classical negation $\sim$ by $\sim A =_{def} \neg A \wedge \circ A$. If needed, the inconsistency operator $\bullet$ is introduced by definition: $\bullet A =_{def} \neg \circ A$.

Carnielli et al. [27] showed classifications of existing logical systems. For example, classical logic is not an LFI, and Jáskowski's D_2 is an LFI. They also introduced a basic system of LFI, called LFI1 with a semantics and axiomatization.

We can thus see that the Logics of Formal Inconsistency are very interesting from a logical point of view in that they can serve as a theoretical framework for existing paraconsistent logics. In addition, there are tableau systems for LFIs; see

Fig. 2.3 The bilattice *FOUR*



Carnielli and Marcos [25], and they can be properly applied to various areas including computer science and AI.

The above mentioned logics have been worked as a paraconsistent logic. But there are other logics which share the features of paraconsistent logics. The two notable examples are *possibilistic logic* and logics based on *bilattices*. In fact, these logics can properly deal both with incompleteness and inconsistency of information.

A *bilattice* was originally introduced by Ginsberg [35, 36] for the foundations of reasoning in AI, which has two kinds of orderings, i.e., truth ordering and knowledge ordering.

Later, it was extensively studied by Fitting in the context of logic programming in [33] and of theory of truth in [34]. In fact, bilattice-based logics can handle both incomplete and inconsistent information.

A *pre-bilattice* is a structure $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$, where B denotes a non-empty set and $\leq_t$ and $\leq_k$ are partial orderings on B. The ordering $\leq_k$ is thought of as ranking "degree of information (or knowledge)". The bottom in $\leq_k$ is denoted by $\perp$ and the top by $\top$. If $x <_k y$, y gives us at least as much information as x (and possibly more).

The ordering $\leq_t$ is an ordering on the "degree of truth". The bottom in $\leq_t$ is denoted by *false* and the top by *true*. A bilattice can be obtained by adding certain assumptions for connections for two orderings.

One of the most well-known bilattices is the bilattice *FOUR* as depicted as Fig. 2.3. The billatice *FOUR* can be interpreted a combination of Belnap's lattices **A4** and **L4** as is clear from Fig. 2.3.

The bilattice *FOUR* can be seen as Belnap's lattice *FOUR* with two kinds of orderings. Thus, we can think of the left-right direction as characterizing the ordering $\leq_t$: a move to the right is an increase in truth.

The meet operation $\wedge$ for $\leq_t$ is then characterized by: $x \wedge y$ is rightmost thing that is of left both x and y. The join operation $\vee$ is dual to this. In a similar way, the up-down direction characterizes $\leq_k$: a move up is an increase in information. $x \otimes y$ is the uppermost thing below both x and y, and $\oplus$ is its dual.

Fitting [33] gave a semantics for logic programming using bilattices. Kifer and Subrahmanian [41] interpreted Fitting's semantics within generalized annotated logics *GAL*. Fitting [34] tried to generalize Kripke's [40] theory of truth, which is based on Kleene's strong three-valued logic, in a four-valued setting based on the bilattice *FOUR*.

A billatice has a negation operation $\neg$ if there is a mapping $\neg$ that reverse $\leq_t$, leaves unchanged $\leq_k$ and $\neg\neg x = x$. Likewise a bilattice has a *conflation* if there is a mapping—that reverse $\leq_k$, leaves unchanged $\leq_t$. and $- - x = x$. If a bilattice has both operations, they *commute* if $-\neg x = \neg - x$ for all x.

In the billatice *FOUR*, there is a negation operator under which $\neg t = f$, $\neg f = t$, and $\perp$ and $\top$ are left unchanged. There is also a conflation under which $-\perp = \top$, $-\top = \perp$ and t and f are left unchanged. And negation and conflation commute. In any bilattice, if a negation or conflation exists then the extreme elements $\perp$, $\top$, f and t will behave as in *FOUR*.

Bilattice logics are theoretically elegant in that we can obtain several algebraic constructions, and are also suitable for reasoning about incomplete and inconsistent information. Arieli and Avron [13, 14] studied reasoning with bilattices. Thus, bilattice logics have many applications in AI as well as philosophy.

Annotated logic is a logic for paraconsistent logic programming; see Subrahmanian [24, 52]. It is also regarded as one of the attractive paraconsistent logics; see da Costa et al [30, 31]. Note that annotated logic has many applications for several areas including engineering. And Abe studied annotated logic for many years.

Billatice logics described above are seen as a rival to annotated logics. We can also unify annotated logics and billatice logics; see Rico [50]. We will review annotated logic in details in Chap. 5; see Abe, Akama and Nakamatsu [1, 2].

Finally, we make an important remark. The propositional calculus is the basis of the usual classical and non-classical logics; however, a true and strong logical system has to contain quantification and a theory of identity at least, and should in principle incorporate a higher-order logic (a form of higher-order logic, some set theory or some other more or less equivalent logical tool).

The relevance of people like Frege, Russell and Peirce, is that they created quantification theory and other aspects of logic beyond the propositional level. Da Costa was the first logician to present a system of paraconsistent logic in this extended sense.

Acknowledgments The authors would like to thank the referee for constructive remarks.

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Towards Paraconsistent Engineering
Akama, S. (Ed.)
2016, XVI, 234 p. 52 illus., 9 illus. in color., Hardcover
ISBN: 978-3-319-40417-2