Chapter 2
Billiard Systems as the Models for the Rigid Body Dynamics

Victoria V. Fokicheva and Anatoly T. Fomenko

Abstract Description of the rigid body dynamics is a complex problem, which goes back to Euler and Lagrange. These systems are described in the six-dimensional phase space and have two integrals the energy integral and the momentum integral. Of particular interest are the cases of rigid body dynamics, where there exists the additional integral, and where the Liouville integrability can be established. Because many of such a systems are difficult to describe, the next step in their analysis is the calculation of invariants for integrable systems, namely, the so called Fomenko–Zieschang molecules, which allow us to describe such a systems in the simple terms, and also allow us to set the Liouville equivalence between different integrable systems. Billiard systems describe the motion of the material point on a plane domain, bounded by a closed curve. The phase space is the four-dimensional manifold. Billiard systems can be integrable for a suitable choice of the boundary, for example, when the boundary consists of the arcs of the confocal ellipses, hyperbolas and parabolas. Since such a billiard systems are Liouville integrable, they are classified by the Fomenko–Zieschang invariants. In this article, we simulate many cases of motion of a rigid body in 3-space by more simple billiard systems. Namely, we set the Liouville equivalence between different systems by comparing the Fomenko–Zieschang invariants for the rigid body dynamics and for the billiard systems. For example, the Euler case can be simulated by the billiards for all values of energy integral. For many values of energy, such billard simulation is done for the systems of the Lagrange top and Kovalevskaya top, then for the Zhukovskii gyrostat, for the systems by Goryachev–Chaplygin–Sretenskii, Clebsch, Sokolov, as well as expanding the classical Kovalevskaya top Kovalevskaya–Yahia case.

V.V. Fokicheva (✉) · A.T. Fomenko (✉)
Lomonosov Moscow State University, Leninskie Gory, 1, Moscow, Russia
e-mail: arinir@yandex.ru

A.T. Fomenko
e-mail: atfomenko@mail.ru

V.A. Sadovnichiy and M.Z. Zgurovsky (eds.), *Advances in Dynamical Systems and Control*, Studies in Systems, Decision and Control 69,
DOI 10.1007/978-3-319-40673-2_2

2.1 Introduction

Definition 2.1 A symplectic structure on a smooth manifold M is a differential 2-form ω satisfying the following two properties:

(1) ω is closed, i.e., $d\omega = 0$;
(2) ω is non-degenerate at each point of the manifold, i.e., in local coordinates, $\det\Omega(x) \neq 0$, where $\Omega(x) = (\omega_{ij}(x))$ is the matrix of this form.

The manifold endowed with a symplectic structure is called symplectic.

Let H be a smooth function on a symplectic manifold M. We define the vector of skew-symmetric gradient sgrad H for this function by using the following identity:

$$\omega(v, \operatorname{sgrad} H) = v(H),$$

where v is an arbitrary tangent vector v. In local coordinates $x^1, \ldots, x^n$, we obtain the following expression:

$$(\operatorname{sgrad} H)^i = \sum \omega^{ij} \frac{\partial H}{\partial x^j},$$

where ω^{ij} are components of the inverse matrix to the matrix Ω.

Definition 2.2 The vector field sgrad H is called a Hamiltonian vector field. The function H is called the Hamiltonian of the vector field sgrad H.

One of the main properties of Hamiltonian vector fields is that they preserve the symplectic structure ω.

Definition 2.3 Dynamical system $\dot{x} = v$ on the smooth manifold M is called Hamiltonian if and only if on the manifold M we can find symplectic structure ω and the function H such that system can be wrote as $v = \operatorname{sgrad} H$.

Definition 2.4 Let f and g be two smooth functions on a symplectic manifold M. By definition, we set $\{f, g\} = \omega(\operatorname{sgrad} f, \operatorname{sgrad} g) = (\operatorname{sgrad} f)(g)$ This operation $\{\cdot, \cdot\} : C^\infty \times C^\infty \to C^\infty$ on the space of smooth functions on M is called the *Poisson bracket*.

Let M^{2n} be a smooth symplectic manifold, and let $v = \operatorname{sgrad} H$ be a Hamiltonian dynamical system with a smooth Hamiltonian H.

Definition 2.5 The Hamiltonian system is called *Liouville integrable* if there exists a set of smooth functions $f_1, \ldots, f_n$ such that

(1) $f_1, \ldots, f_n$ are integrals of v,
(2) they are functionally independent on M, i.e., their gradients are linearly independent on M almost everywhere.

(3) $\{f_i, f_j\} = 0$ for any i and j,
(4) the vector fields sgrad f_i are complete, i.e., the natural parameter on their integral trajectories is defined on the whole real axis.

Definition 2.6 The decomposition of the manifold M^{2n} into connected components of common level surfaces of the integrals $f_1, \ldots, f_n$ is called the *Liouville foliation* corresponding to the integrable system $v = \operatorname{sgrad} H$.

Since $f_1, \ldots, f_n$ are preserved by the flow v, every leaf of the Liouville foliation is an invariant surface. The Liouville foliation consists of regular leaves (filling M almost in the whole) and singular ones (filling a set of zero measure). The Liouville theorem formulated below describes the structure of the Liouville foliation near regular leaves.

Consider a common regular level T_ξ for the functions $f_1, \ldots, f_n$, that is $T_\xi = \{x \in M | f_i(x) = \xi_i, i = 1, \ldots, n\}$. The regularity means that all 1-forms df_i are linearly independent on T_ξ.

Theorem 2.1 (J. Liouville) *Let* $v = \operatorname{sgrad} H$ *be a Liouville integrable Hamiltonian system on* M^{2n}, *and let* T_ξ *be a regular level surface of the integrals* $f_1, \ldots, f_n$. *Then*

(1) T_ξ *is a smooth Lagrangian submanifold that is invariant with respect to the flow* $v = \operatorname{sgrad} H$ *and* $\operatorname{sgrad} f_1, \ldots, \operatorname{sgrad} f_n$.
(2) if T_ξ *is connected and compact, then* T_ξ *is diffeomorphic to the n-dimensional torus* T^n *(this torus is called the Liouville torus);*
(3) the Liouville foliation is trivial in some neighborhood of the Liouville torus, that is, a neighborhood U *of the torus* T_ξ *is the direct product of the torus* T_ξ *and the disc* D^n;
(4) in the neighborhood $U = T^n \times D^n$ *there exists a coordinate system* $s_1, \ldots, s_n$, $\varphi_1, \ldots, \varphi_n$, *(which is called the action-angle variables), where* $s_1, \ldots, s_n$ *are coordinates on the disc* D^n *and* $\varphi_1, \ldots, \varphi_n$ *are standard angle coordinates on the torus, such that*

- $\omega = \Sigma d\varphi_i \wedge ds_i$, *are functions of the integrals,*
- *the action variables* s_i *are functions of the integrals* $f_1, \ldots, f_n$,
- *in the action-angle variables* $s_1, \ldots, s_n, \varphi_1, \ldots, \varphi_n$, *the Hamiltonian flow* v *is straightened on each of the Liouville tori in the neighborhood* U, *that is,*

$$\dot{s}_i = 0, \ \dot{\varphi}_i = q_i(s_1, \ldots, s_n), i = 1, 2, \ldots, n.$$

(this means that the flow v *determines the conditionally periodic motion that generates a rational or irrational rectilinear winding on each of the tori).*

The problems of the rigid body dynamics can be described on the six-dimensional phase manifold, which in some cases is the Poisson manifold. In integrable case we can restrict the system to a submanifold M^4, where it is possible to introduce a symplectic structure. As a result we assume the existence of such four-dimensional symplectic manifold. Thus, the Liouville tori are two-dimensional tori.

Liouville foliation provides a lot of information about the solutions of the system. In fact, according to the Liouville theorem, the solutions on each torus, are its rectilinear windings. The manifold of the parameters of the integrals, where the rectilinear winding is rational (the case of the so-called resonant torus) has measure zero. Thus, for almost all values of the additional integral the closure of the solution forms the Liouville torus. If you change the initial data the change entails the change the Liouville torus, which makes it possible to describe the behavior of the solutions of the system. This weakening of the orbital equivalence is called the Liouville equivalence, see below.

Definition 2.7 Let $(M_1^4, \omega_1, f_1, g_1)$ and $(M_2^4, \omega_2, f_2, g_2)$ be two Liouville integrable systems on symplectic manifolds M_1^4 and M_2^4. Consider the isoenergy surfaces $Q_1^3 = \{x \in M_1^4 : f_1(x) = c_1\}$ $Q_2^3 = \{x \in M_2^4 : f_2(x) = c_2\}$, endowed with the Liouville foliations. Integrable systems on these 3-manifolds are said to be Liouville equivalent if there exists a leafwise diffeomorphism $Q_1^3 \to Q_2^3$, preserving the orientation of the 3-manifolds Q_1^3 and Q_2^3 and of all critical circles.

Let (M^4, ω, f_1, f_2) be Liouville integrable system on symplectic manifolds M^4. The manifold $Q^3 = \{x \in M^4 : f_1(x) = c_1\}$ is foliated into tori and singular leaves. Consider the base of the Liouville foliation on Q^3. This is a one-dimensional graph W called the Kronecker–Reeb graph of the function $f_2|_{Q^3}$. The structure of a foliation in a small neighborhood of the singular leaf corresponding to a vertex of the graph is described by a combinatorial object, called atom. A graph each of whose vertices is assigned an atom is called a Fomenko invariant (rough molecule). At the vertices of "atoms" are placed; they describe the corresponding bifurcations of the Liouville tori.

We now describe the atoms we need.

The minimax 3-atom A. Topologically, this 3-atom is presented as a solid torus foliated into concentric tori, shrinking into the axis of the solid torus. In other words, the 3-atom A is the direct product of a circle and a disc foliated into concentric circles (see Fig. 2.1). From the viewpoint of the corresponding dynamical system, A is a neighborhood of a stable periodic orbit.

The saddle 3-atoms without stars. Consider an arbitrary 2-atom without stars, i.e., a two-dimensional oriented compact surface P with a Morse function $f : P \to \mathbb{R}$ having just one critical value. The corresponding 3-atom is the direct product $U = P \times S^1$. An example is shown in Fig. 2.1: this is the simple 3-atom B.

The simple 3-atom A^* with star is presented in Fig. 2.1.

The molecule W contains a lot of essential information on the structure of the Liouville foliation on Q^3. However, this information is not quite complete. We have to add some additional information to the molecule W, namely, the rules that clarify how to glue the isoenergy surface Q^3 from individual 3-atoms.

To this end, cut every edge of the molecule in the middle. The molecule will be divided into individual atoms. From the point of view of the manifold Q^3 this operation means that we cut it along some Liouville tori into 3-atoms. Imagine that we want to make the backward gluing. The molecule W tells us which pairs of boundary atoms we have to glue together. To realize how exactly they should be glued, for every edge of W, we have to define the gluing matrix C, which determines the isomorphism between the fundamental groups of the two glued tori. To write down this matrix, we have to fix some coordinate systems on the tori. As usual, by a coordinate system on the torus, we mean a pair of independent oriented cycles (λ, μ) that are generators of the fundamental group $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ (or, what is the same in this case, of the one-dimensional homology group). Geometrically, this simply means that the cycles λ and μ are both nontrivial and are intersected transversely at a single point. According to the fixed rules for each type of 3-atom we must choose a special coordinate system on the boundary tori of the atom (see [1]) which will be called admissible.

To the gluing matrix $C_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ on the edge e_i we assign two following numerical marks.

Definition 2.8 The mark r_i on the edge e_i of the molecule W is:

$$r_i = \begin{cases} \frac{\alpha_i}{\beta_i} \mod 1 \in \mathbb{Q}/\mathbb{Z}, & \text{if } \beta_i \neq 0, \\ \text{symbol } \infty, & \text{if } \beta_i = 0. \end{cases}$$

Definition 2.9 The mark ε_i on the edge e_i of the molecule W is:

$$\varepsilon_i = \begin{cases} \operatorname{sign} \beta_i, & \text{if } \beta_i \neq 0, \\ \operatorname{sign} \alpha_i, & \text{if } \beta_i = 0. \end{cases}$$

First, we need some preliminary construction. An edge of the molecule with mark r_i equal to ∞ is said to be it an infinite edge. The other edges are called *finite*. Let

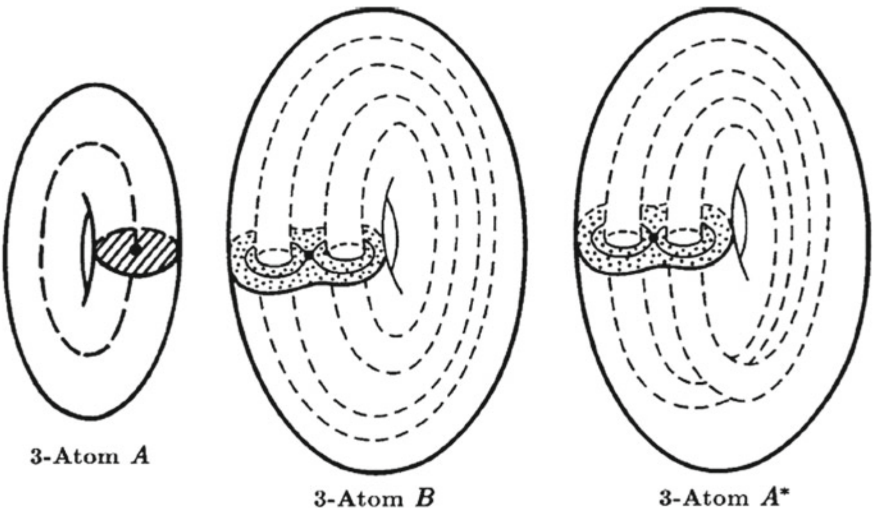


Fig. 2.1 The simple 3-atoms A, B and A^*.

us cut the molecule along all the finite edges. As a result, the molecule splits into several connected pieces.

Definition 2.10 Those pieces which do nor contain atoms of type A are said to be families. For example, if all the edges of a molecule are finite, then each of its saddle atoms is a family by definition.

Consider a single family $U = U_k$. All its edges can be divided into three classes: incoming, outgoing, and interior.

Definition 2.11 To each of these edges e_i, we assign an integer number Θ_i by the following rule:

$$\Theta_i = \begin{cases} [\frac{\alpha_i}{\beta_i}], & \text{if } e_i - \text{ outgoing edge,} \\ [-\frac{\delta_i}{\beta_i}], & \text{if } e_i - \text{ incoming edge,} \\ [-\frac{\gamma_i}{\alpha_i}], & \text{if } e_i - \text{ interior edge.} \end{cases}$$

For every family U_k, we define an integer number n_k by setting

$$n_k = \sum \Theta_i,$$

where the sum is taken over all edges of the given family, and k is the number of the family.

Definition 2.12 The molecule W endowed with the marks r_i, ε_i and n_k is called a *marked molecule*. We denote it by

$$W^* = (W, r_i, \varepsilon_i, n_k).$$

Theorem 2.2 (A.T. Fomenko, X. Zieschang) *Two integrable Hamiltonian systems on the isoenergy surfaces* $Q_1^3 = \{x \in M_1^4 : f_1(x) = c_1\}$ *and* $Q_2^3 = \{x \in M_2^4 : f_2(x) = c_2\}$ *are Liouville equivalent if and only if their marked molecules coincide.*

2.2 The Rigid Body Dynamics

The classical Euler–Poisson equations [10, 11], that describe the motion of a rigid body with a fixed point in the gravity field, have the following form (in the coordinate system whose axes are directed along the principal moments of inertia of the body):

$$A\dot{\omega} = A\omega \times \omega - Pr \times \nu,$$

$$\dot{\nu} = \nu \times \omega. \tag{2.1}$$

Here ω and ν are phase variables of the system, where ω is the angular velocity vector, ν is the unit vector for the vertical line. The parameters of (2.1) are the diagonal matrix

$A = diag(A_1, A_2, A_3)$ that determines the tensor of inertia of the body, the vector r joining the fixed point with the center of mass, and the weight P of the body. Notation $a \times b$ is used for the vector product in $\mathbb{R}^3$. The vector $A\omega$ has the meaning of the angular momentum of the rigid body with respect to the fixed point.

N.E. Zhukovskii studied the problem on the motion of a rigid body having cavities entirely filled by an ideal incompressible fluid performing irrotational motion [12]. In this case, the angular momentum is equal to $A\omega + \lambda$, where λ is a constant vector characterizing the cyclic motion of the fluid in cavities. The angular momentum has a similar form in the case when a flywheel is fixed in the body such that its axis is directed along the vector λ. Such a mechanical system is called a gyrostat. The motion of a gyrostat in the gravity field, as well as some other problems in mechanics (see, for instance, [13]), are described by the system of equations

$$A\dot{\omega} = (A\omega + \lambda) \times \omega - Pr \times \nu,$$

$$\dot{\nu} = \nu \times \omega, \tag{2.2}$$

whose particular case for $\lambda = 0$ is system (2.1).

Another generalization of Eq. (2.1) can be obtained by replacing the homogeneous gravity field with a more complicated one. The equations of motion of a rigid body with a fixed point in an arbitrary potential force field were obtained by Lagrange. If this field has an axis of symmetry, then this axis can be assumed to be vertical, and the equations become

$$A\dot{\omega} = A\omega \times \omega + \nu \times \frac{\partial U}{\partial \nu},$$

$$\dot{\nu} = \nu \times \omega, \tag{2.3}$$

where $U(\nu)$ is the potential function, and $\frac{\partial U}{\partial \nu}$ denotes the vector with coordinates $(\frac{\partial U}{\partial \nu_1}, \frac{\partial U}{\partial \nu_2}, \frac{\partial U}{\partial \nu_3})$. For $U = P\langle r, \nu\rangle$ we obtain system (2.1). By $\langle a, b\rangle$ we denote the standard Euclidean inner product in $\mathbb{R}^3$.

The generalized Eqs. (2.2) and (2.3) can be combined by considering, the motion of a gyrostat in an axially symmetric force field. The most general equations that describe various problems in rigid body dynamics have the following form (see, for example, Kharlamov's book [14]):

$$A\dot{\omega} = (A\omega + \kappa) \times \omega + \nu \times \frac{\partial U}{\partial \nu},$$

$$\dot{\nu} = \nu \times \omega, \tag{2.4}$$

where $\kappa(\nu)$—is the vector function whose components are the coefficients of a certain closed 2-form on the rotation group $SO(3)$, the so-called form of gyroscopic forces. Moreover, $\kappa(\nu)$ is not arbitrary, but has the form

$$\kappa = \lambda + (\Lambda - \mathrm{div}\lambda \cdot E)\nu, \tag{2.5}$$

where $\lambda(\nu)$—is an arbitrary vector function, $\mathrm{div}\lambda = \frac{\partial\lambda_1}{\partial\nu_1} + \frac{\partial\lambda_2}{\partial\nu_2} + \frac{\partial\lambda_3}{\partial\nu_3}$, and $\Lambda = \left(\frac{\partial\lambda_i}{\partial\nu_j}\right)^T$ is the transposed Jacobi matrix. Obviously, systems (2.1)–(2.3) are particular cases of (2.4).

System (2.4) always possesses the geometrical integral

$$F = \langle\nu, \nu\rangle = 1$$

and the energy integral

$$E = \frac{1}{2}\langle A\omega, \omega\rangle + U(\nu).$$

If the vector function $\kappa(\nu)$ has the form (2.5) then there exists another integral the so-called area integral

$$G = \langle A\omega + \lambda, \nu\rangle.$$

It can be shown that Eqs. (2.4), (2.5) are Hamiltonian on common four-dimensional levels of the geometrical and area integrals. Moreover, (2.4) and (2.5) can be represented as the Euler equations for the six-dimensional Lie algebra $e(3)$ of the group of isometrical transformations (motions) of three-dimensional Euclidean space.

On the dual space $e(3)^*$, there is the standard Lie-Poisson bracket defined for arbitrary smooth functions f and g:

$$\{f, g\}(x) = x([d_x f, d_x g]),$$

where $x \in e(3)^*$, [,] denotes the commutator in the Lie algebra $e(3)$, and $d_x f$ and $d_x g$—are the differentials of f and g at the point x. These differentials in fact belong to the Lie algebra $e(3)$ after standard identification of $e(3)^{**}$ with $e(3)$. In terms of the natural coordinates

$$S_1, S_2, S_3, R_1, R_2, R_3$$

on the space $e(3)^*$ this bracket takes the form:

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k,\ \{R_i, S_j\} = \varepsilon_{ijk} R_k,\ \{R_i, R_j\} = 0, \tag{2.6}$$

where $\{i, j, k\} = \{1, 2, 3\}$, and $\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$.

A Hamiltonian system on $e(3)^*$ relative to the bracket (2.6), i.e. the so-called Euler equations, by definition has the form:

$$\dot{S}_i = \{S_i, H\},\ \dot{R}_i = \{R_i, H\},$$

where H is a function on $e(3)^*$ called the Hamiltonian of the system. By introducing the vectors

$$S = (S_1, S_2, S_3) \text{ and } R = (R_1, R_2, R_3),$$

these equations can be rewritten in the form of the generalized Kirchhoff equations:

$$\dot{S} = \left(\frac{\partial H}{\partial S}\right) \times S + \left(\frac{\partial H}{\partial R}\right) \times R,\ \dot{R} = \left(\frac{\partial H}{\partial S}\right) \times R. \tag{2.7}$$

Proposition 2.1 *The mapping $\varphi : \mathbb{R}^6(\omega, \nu) \to \mathbb{R}^6(S, R)$, given by the formulas*

$$S = -(A\omega + \lambda), R = \nu, \tag{2.8}$$

establishes an isomorphism between system (2.4), (2.5) and system (2.7) with the Hamiltonian

$$H = \frac{(S_1 + \lambda_1)^2}{2A_1} + \frac{(S_2 + \lambda_2)^2}{2A_2} + \frac{(S_3 + \lambda_3)^2}{2A_3} + U, \tag{2.9}$$

where the parameters A_1, A_2, A_3 and the function $\lambda_1, \lambda_2, \lambda_3, U$ are taken from (2.4), (2.5), but the functions are defined not on the space $\mathbb{R}^3(\nu)$, but on $\mathbb{R}^3(R)$.

Corollary 2.1 *Condition (2.5) imposed on the vector function $\kappa(\nu)$ is equivalent to the fact that (2.4) is isomorphic to the Euler equations (2.7) on $e(3)^*$ with the quadratic (in variables S) Hamiltonian of the form*

$$H = \langle CS, S\rangle + \langle W, S\rangle + V, \tag{2.10}$$

where is a constant symmetric 3×3-matrix, $W(R)$ is a vector function, and $V(R)$ is a smooth scalar function.

Under mapping (2.8), the integrals $F = \langle \nu, \nu\rangle$ and $G = \langle A\omega + \lambda, \nu\rangle$ transform into the invariants of the Lie algebra $e(3)$

$$f_1 = R_1^2 + R_2^2 + R_3^2,\ f_2 = S_1R_1 + S_2R_2 + S_3R_3,$$

and the energy integral $E = \frac{1}{2}\langle A\omega, \omega\rangle + U(\nu)$ transforms into Hamiltonian (2.9). System (2.7) is Hamiltonian on common four-dimensional level surfaces of the two invariants f_1 and f_2:

$$M^4_{c,g} = \{f_1 = R_1^2 + R_2^2 + R_3^2 = c,\ f_2 = S_1R_1 + S_2R_2 + S_3R_3 = g\}. \tag{2.11}$$

For almost all values of c and g, these common levels are non-singular smooth submanifolds in $e(3)^*$. In what follows, we shall assume that c and g are such regular values.

It is easily seen that these symplectic 4-manifolds $M^4_{c,g}$ are diffeomorphic (for $c > 0$) to the cotangent bundle TS^2 of the 2-sphere S^2. The symplectic structure on $M^4_{c,g}$ is given by the restriction of the Lie-Poisson bracket onto $TS^2 = M^4_{c,g}$ from the ambient six-dimensional space $e(3)^*$. Since the linear transformation $S' = S, R' = \gamma R$, where $\gamma =$ const, preserves bracket (2.6), we shall assume in what follows that $c = 1$.

Thus, from now on, we shall consider Eq. (2.7) with Hamiltonian (2.9) on symplectic four-dimensional manifolds $M^4_{1,g} = \{f_1 = 1, f_2 = g\}$ in the six- dimensional space $e(3)^*$. In each specific problem, the phase variables and parameters of the system obtain a concrete physical meaning.

Now we give the list of main integrable cases of Eqs. (2.7), (2.9) with necessary comments. For each case we indicate explicitly the Hamiltonian H and the additional integral K independent of H. Note that sometimes the additional integral K may exist only for exceptional values of the area constant g.

The *Euler case* (1750). The motion of a rigid body about a fixed point that coincides with its center of mass.

$$H = \frac{S_1^2}{2A_1} + \frac{S_2^2}{2A_2} + \frac{S_3^2}{2A_3}, \ K = S_1^2 + S_2^2 + S_3^2. \tag{2.12}$$

The *Lagrange case* (1788). The motion of an axially symmetric rigid body about a fixed point located at the symmetry axis.

$$H = \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{S_3^2}{2B} + aR_3, \ K = S_3. \tag{2.13}$$

The *Kovalevskaya case* (1899). The motion of a rigid body about a fixed point with the special symmetry conditions indicated below.

$$\begin{aligned} H &= \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{S_3^2}{A} + a_1R_1 + a_2R^2, \\ K &= \left(\frac{S_1^2 - S_2^2}{2A} + a_2R_2 - a_1R_1\right)^2 + \left(\frac{S_1S_2}{A} - a_1R_2 - a_2R_1\right)^2. \end{aligned} \tag{2.14}$$

The integral has degree 4. In this case, $A_1 = A_2 = 2A_3$ (in particular, the body is axially symmetric), and the center of mass is located in the equatorial plane related to the coinciding axes of the inertia ellipsoid.

The *Goryachev–Chaplygin* case (1899). The motion of a rigid body about a fixed point with the special symmetry conditions indicated below.

$$\begin{aligned} H &= \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{2S_3^2}{A} + a_1R_1 + a_2R^2, \\ K &= S_3(S_1^2 + S_2^2) - AR_3(a_1S_1 + a_2S_2). \end{aligned} \tag{2.15}$$

The integral has degree 3. In this case, $A_1 = A_2 = 4A_3$, and the center of mass is located in the equatorial plane related to the coinciding axes of the inertia ellipsoid.

In this case, the Poisson bracket of H and is

$$\{H, K\} = (S_1R_1 + S_2R_2 + S_3R_3)(a_2S_1 - a_1S_2).$$

Hence the functions H and K do not commute on all the manifolds $M^4_{1,g}$. Therefore, the system is integrable only on the single special manifold $M^4_{1,0} = \{f_1 = 1, f_2 = 0\}$. This is a case of partial integrability corresponding to the zero value of the area constant f_2.

Each of these four cases admits an integrable generalization the the case of gyroscopic forces.

The *Zhukovskii case* (1885). The motion of a gyrostat in the gravity field when the body is fixed at its center of mass.

$$\begin{aligned} H &= \frac{(S_1+\lambda_1)^2}{2A_1} + \frac{(S_2+\lambda_2)^2}{2A_2} + \frac{(S_3+\lambda_3)^2}{2A_3}, \\ K &= S_1^2 + S_2^2 + S_3^2. \end{aligned} \tag{2.16}$$

This case is a generalization of the classical Euler case (the Euler case is obtained for $\lambda_1 = \lambda_2 = \lambda_3 = 0$).

The *Kovalevskaya–Yahia* case (1986). The Kovalevskaya case with gyrostat.

$$\begin{aligned} H =& \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{(S_3+\lambda)^2}{A} + a_1R_1 + a_2R^2, \\ K =& \left(\frac{S_1^2 - S_2^2}{2A} + a_2R_2 - a_1R_1\right)^2 + \left(\frac{S_1S_2}{A} - a_1R_2 - a_2R_1\right)^2 \\ &- \frac{2\lambda}{A^2}(S_3 + 2\lambda)(S_1^2 + S_2^2) + \frac{4\lambda R_3}{A}(a_1S_1 + a_2S_2). \end{aligned} \tag{2.17}$$

The classical Kovalevskaya case is obtained for $\lambda = 0$.

The *Sretenskii case* (1963). The Goryachev–Chaplygin case with gyrostat.

$$\begin{aligned} H &= \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{2(S_3+\lambda)^2}{A} + a_1R_1 + a_2R^2, \\ K &= (S_3 + 2\lambda)(S_1^2 + S_2^2) - AR_3(a_1S_1 + a_2S_2). \end{aligned} \tag{2.18}$$

If $\lambda = 0$, then we obtain the classical Goryachev–Chaplygin case. This system is integrable on the zero level of the area integral.

The *Clebsch case* (1871). Motion of a rigid body in a fluid.

$$\begin{aligned} H &= \frac{S_1^2}{2A_1} + \frac{S_2^2}{2A_2} + \frac{S_3^2}{2A_3} + \frac{\varepsilon}{2}(A_1R_1^2 + A_2R_2^2 + A_3R_3^2), \\ K &= \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) - \frac{\varepsilon}{2}(A_2A_3R_1^2 + A_3A_1R_2^2 + A_1A_2R_3^2). \end{aligned} \tag{2.19}$$

The calculation of Fomenko–Zieschang invariants is an effective method for recognizing the Liouville equivalence of the systems. The bifurcations of Liouville tori, bifurcation diagrams, and molecules W for these cases were first calculated by M.P. Kharlamov [14] and A.A. Oshemkov [15–17]. Then the complete invariants of the Liouville foliations (marked molecules W^*) were computed in a series of papers by several authors (A.V. Bolsinov [10], P. Topalov [18], A.V. Bolsinov, A.T. Fomenko [7, 8], O.E. Orel [19], O.E. Orel, S. Takahashi [20]). As a result, a complete classification of the main integrable cases in rigid body dynamics has been obtained up to Liouville equivalence. P. Morozov proved the Liouville equivalence of the Clebsch case [21] and the Sokolov case [22] for certain values of the integrals. In [23], the Liouville equivalence invariants for the Kovalevskaya–Yehia case (this is a generalization of the classical Kovalevskaya top to the case of the problem on the motion of a heavy gyrostat) were calculated.

2.3 Billiard Motion

Let the domain Ω be the domain on the plane $\mathbb{R}^2$ such that its boundary is the piecewise smooth curve and the angle at the corner points equals $\frac{\pi}{2}$. Consider the billiard dynamical system in Ω that describes the motion of a point inside Ω with natural reflection at the boundary $P = \partial\Omega$. At those points where the boundary P is not smooth, the trajectory of the system is extended by continuity: hitting a corner vertex, a material point is reflected back along the same trajectory without losing the rate.

The phase space of the system is the manifold

$$M^4 := \{(x, v)|x \in \Omega, v \in T_x\mathbb{R}^2, |v > 0|\}/\sim$$

where the equivalence relation $\sim$ is defined by

$(x_1, v_1) \sim (x_2, v_2)$ if and only if $x_1 = x_2 \in P$, $|v_1| = |v_2|$ and $v_1 + v_2 \parallel T_xP$. Here, T_xP denotes the tangent to the domain Ω at the point x and $|v|$ is the Euclidean length of the vector v.

Billiard motion has the natural integral—the speed $|v|$ of the material point x. If $|v| > 0$ then we can restrict the system to the isoenergy surface $Q^3 := \{(x, v) \in M^4 : |v| = 1\}$. Such isoenergetic surfaces are homeomorphic to each other and in the subsequent discussion we put $|v| = 1$. Some restriction of the choice of the boundary allows to find the additional integral.

Theorem 2.3 (Jacobi, Chasles [24]) *Given a geodesic curve on a quadric in n-dimensional Euclidean space, tangent lines which are drawn at arbitrary points on the geodesic are tangent both to this quadric and to $n - 2$ confocal quadrics, which are the same for all the points on the geodesic.*

Now fix the family of the confocal quadrics on the plane $\mathbb{R}^2$ and consider the equation

$$(b - \lambda)x^2 + (a - \lambda)y^2 = (a - \lambda)(b - \lambda), \lambda \leq a. \tag{2.20}$$

where $\infty \geq a \geq b > 0$ is the fixed pair of numbers, which describe the family of quadrics, λ is the parameter defining the quadric which belongs to the family.

Suppose that a domain Ω in the plane $\mathbb{R}^2$ is such that its boundary is the union of piecewise smooth curves consisting of arcs of the confocal quadric. This domain will be called *elementary*.

From the Jacobi Chasles theorem it follows that the tangent lines to a billiard trajectory at any point inside a plane two-dimensional domain are tangent to an ellipse or a hyperbola confocal with the family of quadrics forming the boundary of this domain [24].

This implies the integrability of the billiard in a plane domain bounded by arcs of confocal ellipses and hyperbolas. The functions $|v|$—the speed of the material point—and λ—the parameter of the confocal quadric—commutate inside the domain Ω. Thus, they commutate in the boundary P of the domain Ω because they are integrals of the system.

As a result the billiard system which if defined in the plane domain bounded by the arcs of the confocal quadrics has two independent (see [24]) integrals $|v|$ and λ. Function λ sets on the isoenergy surface Q^3 the Liouville foliation which can be described in terms of the Fomenko–Zieshang invariant.

To classify all the domains bounded by ellipses and hyperbolas it is convenient to take the equivalence relation, which would allow, smoothly changing the class of confocal quadrics of the border region, to preserve the Liouville foliation of the billiard motion in it.

Definition 2.13 Elementary domain Ω, bounded by arcs of the confocal family of quadrics (2.20), is called *equivalent* to the other elementary domain Ω', which is bounded by arcs of quadrics from the same family (2.20), if Ω' can be obtained from Ω by the following composition of transformations:

- sequential changing borders by continuous segments deformation in the class of quadrics (2.20), so that the value of the parameter λ of the variable segment of the border did not take the value b;
- symmetry with respect to the axis of the family (2.20).

As a result of such definition of equivalence all elementary domains can be divided into three classes:

- pieces of the ellipse: domain A_2 (bounded by ellipse), A_1 (right part of the A_2), A_0 (rectangle, limited by ellipse and two branches of a hyperbola) and its upper halfs A_2',A_1' and A_0';
- ring-domain C_2, bounded by two ellipses;
- simply connected domain-bands series B, which are parts of the ring-domain C_2.

We can extend the class of elementary domains, adding to them the flat domains that do not have an immersion into the plane. In this case, to have the above-described non-simply connected domains we need to add the domains C_{2n}–n−sheets coverings over the domain C_2 and results of C_n of the quotient by the group $\mathbb{Z}_2$. As for simply connected domain, we must add "prolongations" of the domains B, which are now subsets of relevant domains C_n.

The Fomenko–Zieschang invariants of these systems were calculated by M. Radnovic and V. Dragovic in [25] and V.V. Fokicheva in [26].

The generalized billiard system in a generalized locally flat domain is defined in a similar way as the billiard in domains glued together along common convex segments of their boundaries. In this case, if a point reaches such a segment, its trajectory passes from one elementary domain to another. If a pair of domains is glued together along the common corner (the case of *a conical point*), then, by continuity, the motion must be defined as follows: a point moving on a sheet and hitting the corner is reflected along the same trajectory on the same sheet.

The equivalence relation on the set of generalized domains if taken as a continuation of the equivalence relation on the set of elementary domains. Namely, the domains will be called equivalent if they can be obtained from each other by replacing their constituent elementary domains on their equivalent. All such domains were classified by V. Fokicheva [28].

Obviously, with such a definition phase manifold M^4 preserves integrability of the system, namely, retained additional integral λ—parameter of the confocal quadric, which concerns the billiard trajectory. This is due to the fact that the boundary of any elementary domain Ω_i, which is part of the generalized domain Δ, and in particular, all the gluing edges pass into the arc of the same family of confocal quadrics in the isometric immersion of the field Ω_i or double covering in the plane.

The Fomenko–Zieschang invariants of these systems were calculated by V.V. Fokicheva in [28].

For further convenience, we assume the following notation. By Ω will be denoted an elementary domain. Glued from several elementary domains Ω_i the generalized domain is denoted by Δ. For a fixed generalized domain Δ the unification of all the borders of its constituent regions Ω_i, which are not glue arcs will be called a free boundary. By Θ we will denote the domain bounded by arcs of confocal quadrics, without specifying whether it is a elementary (Ω) or generalized (Δ).

The generalized domain without conical points is denoted by Δ_α, with conical points by Δ_β. We distinguish three types of conical points: type x is formed by the intersection of the focal line ($\lambda = b$) and confocal ellipse ($\lambda < b$), type c—at the intersection of the focal line ($\lambda = b$) and confocal hyperbola ($b < \lambda < a$), type y—at the intersection of the confocal ellipse ($\lambda < b$) and confocal hyperbola ($\lambda > b$). In the notation of the generalized domain in parentheses we specify the types of domains that make up this region and generalized types of conical points, if they exist.

We describe several classes of generalized domains and calculate Fomenko–Zieschang invariants that describe the topology of the Liouville foliation of the billiard motion in them. More precisely, we describe the domains of the invariants of the billiard motion that occur in problems of rigid body dynamics.

Proposition 2.2 ([28]) *Let the domain Θ be that, first, the interior of each elementary domain in its composition does not include points of the focal line, and*

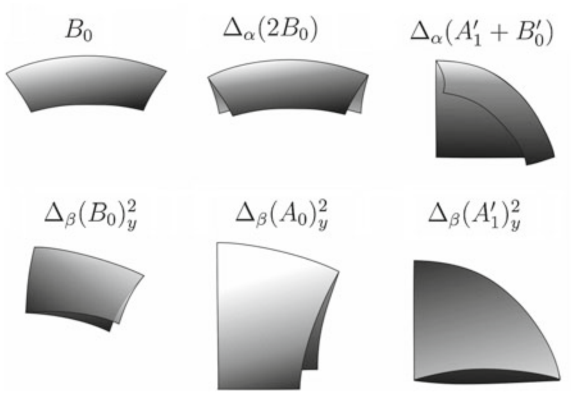


Fig. 2.2 In the *top row* there are domains without conical points at the *bottom*—with one conical point

secondly, any conical point is of the type y (see examples on the Fig. 2.2). Then Fomenko–Zieschang invariant describing the topology of Liouville foliation for the billiard motion in Θ is of the form:

- $A \xrightarrow{r=0,\varepsilon=1} A$*, if the domain Θ does not contain conical points;*
- $A \xrightarrow{r=\frac{1}{2},\varepsilon=1} A$*, if the domain Θ contains conical points.*

Remark 2.1 According to the classification of generalized domains [28] the domain, which has no common points with the focal line, and contains a conical point is arranged as follows: it contains exactly one conical point, with its free boundary homeomorphic to a circle.

Proposition 2.3 ([28]) *Suppose that domain Θ without conical points is such that each elementary domain Ω in its composition does not contain any focuses (see examples on the Fig. 2.3). Then Fomenko–Zieschang invariant describing the topology of Liouville foliation for the billiard motion in Θ is of the form:*

- $A \xrightarrow{r=\infty,\varepsilon=1} B \rightrightarrows {A \atop A}$*, where marks on the right edges are $r=0$, $\varepsilon=1$, if domain Θ is equivalent to B_1, $\Delta_\alpha(2B_1)$, A_0, $\Delta_\alpha(A_0+B_0), \Delta_\alpha(A_0+A'_0), \Delta_\alpha(B_0+A_0+B_0)$, $\Delta_\alpha(A'_0+A_0+B_0)$ or $\Delta_\alpha(A'_0+A_0+A'_0)$, i.e. domain Θ is homeomorphic to a disc and contains only one line segment of the focal line (either only one elementary domain Ω in its composition contains line segment of the focal line or these segments are glued into one along the arcs of the focal line);*
- $A \xrightarrow{r=0,\varepsilon=1} B \rightrightarrows {A \atop A}$*, where marks on the right edges are $r=\infty$, $\varepsilon=1$, if domain Θ is equivalent to $\Delta_\alpha(A_0)^2$ or C_1, i.e. domain Θ is homeomorphic to a cartesian product $S^1\times[0,1]$ and contains only one line segment of the focal line;*

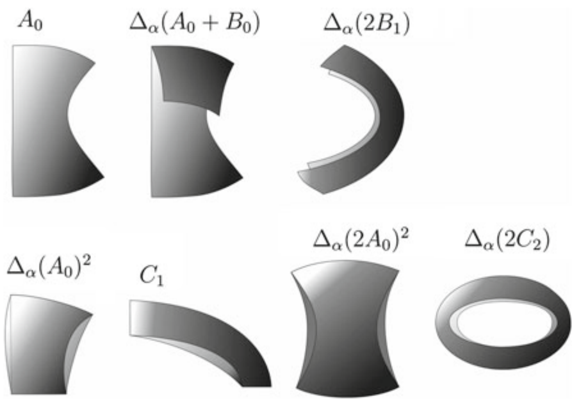


Fig. 2.3 Domains without focuses and conical points

- $\begin{matrix}A\\A\end{matrix} \rightrightarrows C_2 \rightrightarrows \begin{matrix}A\\A\end{matrix}$, *where marks on the left edges are* $r = \infty$, $\varepsilon = 1$, *and on the right edges are* $r = 0$, $\varepsilon = 1$, *if domain* Θ *is equivalent to* $\Delta_\alpha(2A_0)^2$*or* $\Delta_\alpha(2C_2)$, *i.e. domain* Θ *is homeomorphic to a cartesian product* $S^1 \times [0, 1]$ *and contains two line segments of the focal line.*

If the domain contains the focuses, all the edges of the molecule are finite, that makes compute mark n.

Proposition 2.4 ([28]) *Let the domain* Θ *be such that an elementary domain in its composition contains focuses of the confocal family of the domain's border (see examples on the Fig. 2.4). Then Fomenko–Zieschang invariant describing the topology of Liouville foliation for the billiard motion in* Θ *is of the form:*

- $\begin{matrix}A\\A\end{matrix} \rightrightarrows B \to A$, *marks on the all edges are* $r = 0$, $\varepsilon = 1$, *and mark* n *in the family is equal to* 1, *if domain* Θ *is equivalent to* A_2, $\Delta_\alpha(2A_1)$ *or* $\Delta_\alpha(A_2 + C_2)$*;*
- $A \xrightarrow{r=0,\varepsilon=1} A^* \xrightarrow{r=0,\varepsilon=1} A$, *mark* n *in the family is equal to* 0, *if domain* Θ *is equivalent to* A_1 *or* $\Delta_\alpha(A_1 + B_1)$*;*
- $\begin{matrix}A\\A\end{matrix} \rightrightarrows C_2 \rightrightarrows \begin{matrix}A\\A\end{matrix}$, *marks on the all edges are* $r = 0$, $\varepsilon = 1$, *and mark* n *in the family is equal to* 2, *if domain* Θ *is equivalent to* $\Delta_\alpha(2A_2)$*;*
- $\begin{matrix}A\\A\end{matrix} \rightrightarrows B \to A$, *marks on the all edges are* $r = 0$, $\varepsilon = 1$, *and mark* n *in the family is equal to* 2, *if domain* Θ *is equivalent to* $\Delta_\beta(A_1')^2_c, \Delta_\beta((A_1')^2_c + C_1)$ *or* $\Delta_\beta(A_1')^2_x$.

2.4 Main Results

The descriptions of all systems of the rigid body dynamics are fairly complex. It turns out that, in many cases, the Fomenko–Zieschang theorem makes it possible to establish the Liouville equivalence of these systems to certain simpler billiard systems on the four-dimensional phase space M^4.

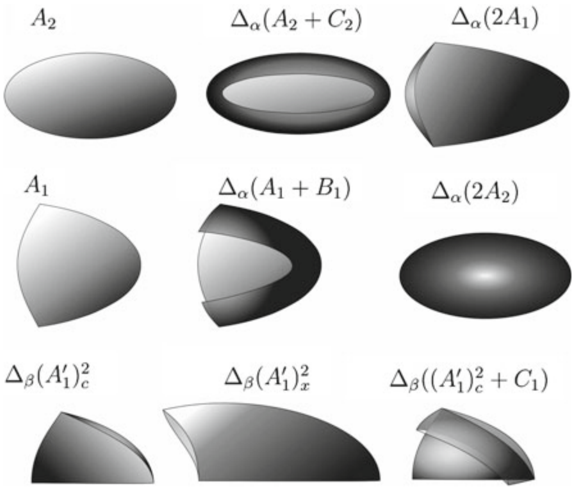


Fig. 2.4 Domains which contains the focuses

Theorem 2.4 ([27]) *The following cases of rigid body dynamics are modeled by (Liouville equivalent to) the following generalized billiards*

- *the Euler case (see [9]) is completly modeled by the billiards in the generalized domains shown in Figs. 2.5 and 2.6;*
- *the Zhukovskii case (see [15]) is modeled by the billiards in the generalized domains shown in Fig. 2.5b (energy zone 11, $Q^3 \simeq RP^3$), Fig. 2.5c (energy zone 2, $Q^3 \simeq S^1 \times S^2$), Fig. 2.5d (energy zone 8, $Q^3 \simeq S^3$), and Fig. 2.5f (energy zone 12, $Q^3 \simeq RP^3$);*
- *the Lagrange case (see [9]) is modeled by the billiards in the generalized domains shown in Fig. 2.5a (energy zone 2, $Q^3 \simeq S^3$) and Fig. 2.5b (energy zone 3, $Q^3 \simeq RP^3$);*
- *the Goryachev–Chaplygin–Sretenskii case (see [19]) is modeled by the billiards in the generalized domains shown in Fig. 2.5c (energy zone 4, $Q^3 \simeq S^1 \times S^2$) and Fig. 2.5g (energy zone 2, $Q^3 \simeq S^3$);*
- *the Kovalevskaya–Yehia case (see [23]) is modeled by the billiards in the generalized domains shown in Fig. 2.5c (energy zone h_{28}, $Q^3 \simeq S^1 \times S^2$) and Fig. 2.5e (energy zone h_{18}, $Q^3 \simeq S^3$);*
- *the Clebsch case (see [21]) is modeled by the billiards in the generalized domains shown in Fig. 2.5e (energy zone 2, $Q^3 \simeq S^3$), Fig. 2.5h (energy zones 10 and 12, $Q^3 \simeq S^1 \times S^2$), and Fig. 2.5i (energy zone 5, $Q^3 \simeq RP^3$);*
- *the Sokolov case (see [22]) is modeled by the billiards in the generalized domains shown in Fig. 2.5e (energy zone B, $Q^3 \simeq S^3$) and Fig. 2.5i (energy zone I, $Q^3 \simeq RP^3$).*

The Liouville equivalence of these billiard systems and integrable systems of the motion of a rigid body follows from the Fomenko–Zieschang theorem and the

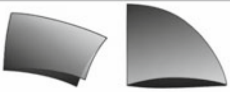
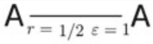
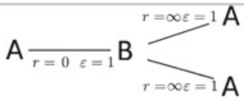
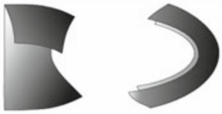
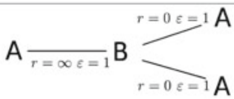
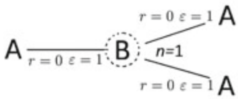
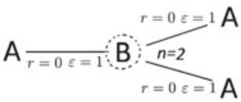
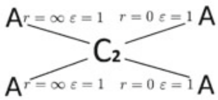
	Generalized billiard domain	The Fomenko-Zieschang invariant describing the billiard in this domain	Equivalent known cases of integrability for a rigid body
a			Lagrange (1), Euler (1)
b			Lagrange (3), Zhukovskii (11)
c			Kovalevskaya (5), Zhukovskii (2), Goryachev—Chaplygin—Sretenskii (4), Kovalevskaya—Yehia (h_{18})
d			Zhukovskii (8)
e			Clebsch (2), Sokolov (B), Kovalevskaya—Yehia (h_{18})
f			Zhukovskii (12)
g			Goryachev—Chaplygin—Sretenskii (2)
h			Euler (2), Clebsch (10, 12)

Fig. 2.5 The left column shows the billiard domains, in the middle column – Fomenko-Zieschang invariants describing the topology of the billiard motion in them. The right column shows the cases of rigid body dynamics Fomenko-Zieschang which also have the form shown in the middle column (in parentheses are the numbers of the isoenergy surfaces in accordance with the numbering of the authors, the data to calculate the invariants)

comparison of the invariants of generalized billiards found by these authors with invariants calculated in the cited works of other authors.

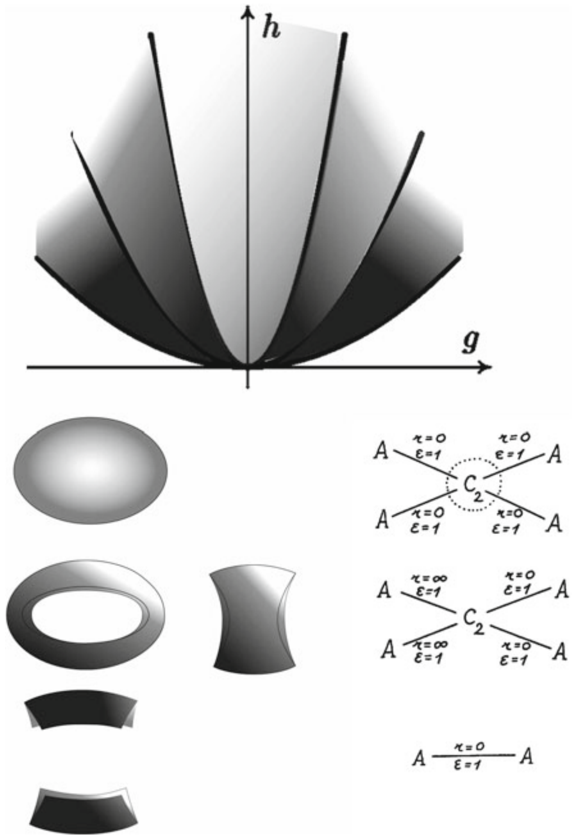


Fig. 2.6 Billiard system and the Euler case of the rigid body dynamics. The motion of a rigid body, the appropriate settings in the shaded gray area on the bifurcation diagram, modeled billiards in the domain shaded in the same shade

The Euler case is Liouville equivalent to the case of the geodesic flow on the ellipsoid [7]. This has been proven by the application of the theory of Fomenko–Zieschang—by calculating and comparing the invariants. On the other hand, the problem of the geodesic flow is closely connected with the integrable billiard problem in the domain bounded by arcs of confocal quadrics—by limiting to zero at the half-axis the ellipsoid becomes the flat ellipse, and geodesic lines on it become straight line segments. However, as can be seen, the billiard in an ellipse will not be Liouville equivalent to the geodesic flow.

The introduction of generalized billiards allowed to expand the class of classical billiard systems and successfully simulate not only the case of Euler fixed type isoenergy surfaces, but also to select for each constant-energy surface of a billiard a movement which will simulate the motion of a rigid body fixed at its center of mass.

It turns out that, in a sense, the billiard system is not so simple. However, its complexity lies in the complexity of a generalized billiard table—the more exotic the boundary the more complicated the topology of the Liouville foliation isoenergy surface Q^3.

Thus, as a result of the introduction of generalized billiards we have been able not only fully simulate the Euler case, but also to get a large number of systems, whose Fomenko–Zieschang invariants coincide with those calculated previously for many systems of rigid body dynamics. This has allowed to simulate a wide class of problems of rigid body dynamics, though not completely.

References

1. Fomenko, A.T.: The topology of surfaces of constant energy in integrable Hamiltonian systems, and obstructions to integrability. Math. USSR Izvestija **29**(3), 629–658 (1987)
2. Brailov, A.V., Fomenko, A.T.: The topology of integral submanifolds of completely integrable Hamiltonian systems. Math. USSR-Sb. **62**(2), 373–383 (1989)
3. Matveev, S.V., Fomenko, A.T.: Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds. -. Russ. Math. Surv. **43**(1), 3–24 (1988)
4. Fomenko, A.T., Zieschang H.: On typical topological properties of integrable Hamiltonian systems. USSR-Izv. **32**(2), 385–412 (1989)
5. Fomenko, A.T.: Topological invariants of Hamiltonian systems that are integrable in the sense of Liouville functional. Anal. Appl. **22**(4), 286–296 (1988)
6. Fomenko, A.T.: Symplectic Geometry. – Gordon and Breach. Advanced Studies in Mathematics. vol. 5 (1988)
7. Bolsinov, A.V., Fomenko, A.T.: The geodesic flow of an ellipsoid is orbit ally equivalent to the Euler integrable case in the dynamics of a rigid body. Dokl. Akad. Nauk SSSB. **339**(3), 293–296 (1994)
8. Bolsinov, A.V., Fomenko, A.T.: Orbital classification of the geodesic flows on two- dimensional ellipsoids. The Jacobi problem is orbitally equivalent to the integrable Euler case in rigid body dynamics. Funkts. Analiz i ego Prilozh. **29**(3), 1–15 (1995)
9. Bolsinov, A.V., Fomenko, A.T.: Integrable Hamiltonian Systems: Geometry, Topology, Classification, 1, 2. Regulyarnaya i Khaolichcskaya Dinamika, Izhevsk (1999). [in Russian]
10. Bolsinov, A.V.: Methods of calculation of the Fomenko – Ziesehang invariant. In: Topological Classification of Integrable Systems – Advances in Soviet Mathematics, vol 6, 147–183. AMS, Providence (1991)
11. Arkhangelskii, YuA: Analytical Dynamics of a Rigid Body. Nauka, Moscow (1977)
12. Zhukovskii, N.E.: On the motion of a rigid body having cavities filled with homogeneous liquid. Zh. Russk. Fiz-Khim. Obsch. **17**(6), 81–113; **7**, 145–149; **8**, 231–280 (1885)
13. Kharlamov, P.V.: Lectures on Rigid Body Dynamics. Novosibirsk State University, Novosibirsk (1965)
14. Kharlamov, M.P.: Topological Analysis of Integrable Problems in Rigid Body Dynamics. Leningrad University, Leningrad (1988)
15. Oshemkov, A.A.: Fomenko invariants for the main integrable cases of the rigid body motion equations. In: Topological Classification of Integrable Systems –Advances in Soviet Mathematics, vol. 6, 67–146. AMS, Providence (1991)
16. Oshemkov, A.A.: Topology of isoenergy surfaces and bifurcation diagrams for integrahle cases of rigid body dynamics on $SO(4)$. Uspekhi Mat. Nauk **42**(6), 199–200 (1987)
17. Oshemkov, A.A.: Description of isoenergetic surfaces of some integrable Hamiltonian systems with two degrees of freedom. In: Trudy Semin. po Vektor. Tenzor. Analizu, vol. 23, 122–132. Izdatel'stvo Moskovskogo Universiteta, Moscow (1988)
18. Topalov, P.I.: Calculation of the fine Fomenko-Ziesehang invariant for the main integrable cases in rigid body motion. Matem. Sbornik **187**(3), 143–160 (1996)

19. Orel, O.E.: The rotation function for integrable problems that are reducible to Abel equations. Trajectory classification of Goryachev–Chaplygin systems. Matem. Sbormk **186**(2): 105–128 (1995)
20. Orel, O.E., Takahashi, S.: Trajectory classification of integrable Lagrange and Goryachev Chaplygin problems by computer analysis methods. Mat em. Sbomik **187**(1), 95–112 (1996)
21. Morozov, P.V.: The Liouville classification of integrable systems of the Clebsch case. Sb. Math. **193**(10), 1507–1533 (2002)
22. Morozov, P.V.: Topology of Liouville foliations in the Steklov and the Sokolov integrable cases of Kirchhoff's equations. Sb. Math. **195**(3), 369–412 (2004)
23. Slavina, N.S.: Topological classification of systems of Kovalevskaya-Yehia type. Sb. Math. **205**(1), 101–155 (2014)
24. Kozlov, V.V., Treshchev, D.V.: (Russian) [Billiards] A genetic introduction to the dynamics of systems with impacts. Moskov Gos University, Moscow (1991)
25. Dragovich, V., Radnovich, M.: Integrable Billiards, Quadrics, and Multidymensional Poncelet Porisms. Regulyarnaya i Khaolichcskaya Dinamika, Izhevsk (2010). [in Russian]
26. Fokicheva, V.V.: Description of singularities for billiard system bounded by confocal ellipses and hyperbolas. Moscow Univ. Math. Bull. **69**(4), 148–158 (2014)
27. Fokicheva, V.V., Fomenko, A.T.: Integrable billiards model important integrable cases of rigid body dynamics. Doklady Math. **92**(3), 1–3 (2015). doi:10.7868/S0869565215320055
28. Fokicheva, V.V.: A topological classification of billiards in locally planar domains bounded by arcs of confocal quadrics. Sb. Math. **206**(10), 127–176 (2015). doi:10.1070/SM2015v206nl0ABEH004502

http://www.springer.com/978-3-319-40672-5

Advances in Dynamical Systems and Control
Sadovnichiy, V.A.; Zgurovsky, M.Z. (Eds.)
2016, XXII, 471 p. 71 illus., 39 illus. in color., Hardcover
ISBN: 978-3-319-40672-5