Chapter 2
Conductors in Equilibrium

Abstract Conductors are characterized by their containment of electric charges, electrons or ions, which are free to move about inside their body. These are the metals, the electrolytes and the ionized gases. In this chapter, we study the behavior of the conductors under electrostatic conditions, namely constant in time. We shall study electrostatic induction between two and more conductors, the capacitors and the electrostatic shield.

In the first chapter, we studied the electrostatic field in a vacuum generated by charges in fixed positions. In real life, material bodies are always present, containing an enormous number of electric charges. Consequently, under practical circumstances, the electric field is due both to the charges under our control and to the charges of the surrounding bodies. The building blocks of matter, molecules and atoms, contain positive and negative charges in equal quantities and are consequently globally neutral. The electric field generated by these nanoscopic structures is zero at distances that are large compared to their size under unperturbed conditions. Contrarily, an externally applied field may deform and move the molecules, as well as the free charges existing in the conductors, changing the internal charge distribution and contributing the macroscopic field.

The electric field we shall consider is "macroscopic", because our description shall always deal with macroscopic phenomena, which happen on distance scales that are enormous compared to the sub-nanometer scale of the molecules and on time scales quite significantly longer than the characteristic times of atomic phenomena. Consider, for example, a piece of metal between the plates of a capacitor. The field generated by the capacitor is a macroscopic field. If we look into the metal at the atomic scale, we shall see a field that varies very rapidly from point to point; it is very strong between a nucleus and the electrons, but it is almost zero outside the atom. In addition, the field is far from being constant, but varies over very short times depending on the motions of the electrons. At the atomic level, there is never

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an electrostatic field, but rather an electric field that rapidly varies in time and with position. This is an extremely complicated situation, but it is not what we are interested in. We are interested in the field we can measure with our instruments. These instruments are macroscopic bodies with geometrical dimensions much larger than the molecular ones that integrate on time scales much longer than those of the microscopic variations. This is the macroscopic field.

The material bodies can be schematically divided into two classes; the conductors, studied in this chapter, and the dielectrics or insulators, studied in Chap. 4. In this chapter, we shall always deal with static, namely time independent, conditions.

In Sect. 2.1, we shall see how the conductors containing elementary charges can move freely. These charge carriers, as they are called, are electrons in metals, ions in electrolytes and in gases, "holes" and electrons in semiconductors, etc. Under static conditions, the charge carriers randomly move, similarly to the molecules of a gas. There is no ordered motion under these conditions.

In Sect. 2.2, we shall see that when a conductor is brought into an external electric field, some of its free charges move to the surface and arrange themselves in such a way as to cancel, with the field they produce, the applied field in the entire volume of the conductor. Under static conditions, the electric filed is zero in a conductor, which is consequently an equipotential volume. The charge density inside the conductor is also zero, while a surface charge density can be present on the surface.

Suppose we want to have an electric field of certain intensity and certain shape in a region of space. The way to produce it is to build a number of conductors having surfaces of the right shape and to give them the right potentials. In doing that, we control the potentials, not the charges. The underlying mathematical problem is finding the potential in a region of space once the values on the surfaces surrounding the region are fixed. We shall see in Sect. 2.4 that the solution of the problem is unique and we shall study some of its properties. We shall also see that the only available general means to find the solution are not analytical, but numerical.

In Sect. 2.6, we shall consider a system of two conductors and learn the conditions that must be satisfied for it to be a capacitor. Capacitors are important elements of any electronic and electric circuit. We shall study their properties in Sects. 2.7 and 2.8. In Sect. 2.9, we shall extend the study to systems of more than two conductors.

In Sect. 2.10, we shall study the electric field of a system of conductors completely surrounded by a hollow conductor. We shall see how the latter divides the space into two independent regions, as far as electrostatic phenomena are concerned. This is the electrostatic shielding action. This action is a consequence of the inverse square law of the electrostatic force and its study allows for verifying this dependence with extreme precision. We shall give examples of the practical importance of electrostatic shielding.

2.1 Conductors

A body is defined as an electric conductor if it contains, in its bulk or on its surface, electric charge carriers that are free to move. As we know, all bodies contain, or, to be more precise, are made of, charged particles, such as electrons and nuclei. These charged particles are inside atoms or molecules, which are globally neutral and are generally not free to move over macroscopic distances. In the conductors, a small fraction of these charges can move over macroscopic distances. The principal types of conductors are as follows.

Metals.
Metals in their solid phase are made of microcrystals. A small fraction of the electrons, typically one or two per atom, are free to move about inside the metal. These are called free electrons. To fix the orders of magnitude, typical free electron densities in metals are $n_p = 10^{29}$–10^{30}/m^3. With a good approximation, at the usual temperatures, we can think of the free electrons as behaving like a gas, with a velocity distribution similar to that of the molecules of a common gas. Their mean kinetic energy at room temperature is like that of a gas, in a round figure, 1/40 eV. The electrons are free inside the metal, but cannot leave it because they are attracted by the array of positive ions that they have abandoned. The energy needed to be given to an electron to extract it from the metal depends on the metal, but is generally of a few electornvolt, much larger than their mean kinetic energy at room temperature.

Semiconductors.
Different types of semiconductor exist, both natural and artificially produced. We shall only mention the simple example of a pure element, the Si. The Si atom has four valence electrons, namely four electrons taking part in its chemical bonds. In a Si crystal, every atom is linked to four other atoms through covalent bonds. A covalent bond is made of two electrons, one for each partner atom. The bonds are very stable and very few of them break, freeing electrons. Technologies are available to dope the crystal, namely to grow it in the presence of a few impurities.

Doping can be done with a pentavalent element, the atoms of which are fit to substitute for Si atoms in the crystal lattice. These extraneous atoms do form four links with four Si atoms, but are left with an extra valence electron. This electron is only weakly bound and detaches due to the thermic motion already at low temperatures. This is the case of the n-type semiconductors, where “n” stands for negative, because the charge carriers are the electrons. The carrier density depends on the applied doping level, but is always much smaller than in metals, because only a small number of Si atoms is substituted by the dopant. A typical order of magnitude is $n_p = 10^{20}$/m^3.

If Si is doped with a trivalent element, we obtain a p-type semiconductor, where “p” stands for positive, because such is the charge of the carriers. When a trivalent atom substitutes for a Si in the ladder, it can only establish three bonds. The fourth is made by one electron alone. There is a "hole", namely a point where an electron is missing. This situation is not stable, and it soon happens that an electron of a nearby bond jumps into the hole. But, in so doing, another hole is produced and the process repeats and propagates. We can also say that the hole moves in the opposite direction from one position to the other and that a positive charge moves with it. Indeed, holes behave like positively charged particles, with a chaotic motion similar to that of electron gas.

Electrolytes.
Ionic molecules, such as the common salt NaCl, are made by two oppositely charged ions, Na^+ and Cl^- in the example. As we shall see in Chap. 4, the electrostatic force is much weaker in water than in a vacuum. Consequently, the ion bond breaks in water and the two ions become free. A small quantity of ions (of both signs) is always present in water, which is consequently a conductor. The ion carrier density can be reduced by distillation or increased by adding salts.

Gases.
A small fraction of ionized molecules is also always present in gases. Ionization is mainly caused by natural radioactivity from the rocks and from cosmic rays. The charge carriers are ions of both signs.

Other types of conductors are plasmas, some non-metallic substances like graphite, some organic substances, etc.

2.2 Conductors in Equilibrium

In this chapter, we study the properties of the conductors in equilibrium, namely under static conditions. We have already encountered the concepts of charge density and electric field in a vacuum. We now need these concepts in a conductor, which is a material medium. Matter is made of molecules, and inside molecules, there are electrons separated from the nuclei of their atoms by empty spaces much larger than the nuclear diameters. These particles are not at all at rest, but move continuously at high speeds. The charge density that we shall consider is an average taken on "physically infinitesimal" volumes, namely very small relative to the macroscopic dimensions but containing a large number of molecules. This is possible because the atomic radiuses are on the order of a tenth of a nanometer.

Inside the material, the electric field changes by large factors over distances on the order of atomic diameters. Fields are very intense inside an atom, becoming almost zero immediately outside of it. In addition, at the atomic scale, the field varies very rapidly, namely over times on the order of the femtosecond, due to the fast motion of the electrons. This electric field at the nanometer and femtosecond scales is not the field we measure with our macroscopic instruments, as we already stated in Sect. 1.3 when we gave the operational definition of the field. The macroscopic field we shall deal with at point **r** at the instant t is an average on a volume around **r**, very small in macroscopic dimensions but very large on the nanometric scale, and on a time interval around t, small on the macroscopic scale but very long on the femtosecond scale.

The conductors' properties we shall now discuss only hold for homogeneous conductors. This means that temperature, chemical composition, aggregation phase (in short, all their chemical and physical properties) are independent of position.

We shall now discuss four properties. They are very simple, but there is a logical order (that should be remembered) in which they have to be considered.

1. *Within a conductor in equilibrium, the charge density is zero.*

Indeed, if the field is not zero, there are forces acting on the free charges. These would accelerate and the condition would not be static. Let us look more closely at what happens.

Consider, for example, a metal, and let us bring it into an electric field. Initially, the field penetrates inside the metal and exerts forces, in particular, on the free electrons. Being unbound, they start moving in the direction opposite to the field. In this way, two charge accumulations develop, a negative one on the side to which electrons move, and a positive one on the opposite side. To be clear, we anticipate that these accumulations are on the surface of the body, as we shall soon see. These charge densities produce an electric field having a direction opposite to that of the external field and tending to cancel it out. As a matter of fact, the free carriers *must* adjust their position to cancel the external field completely. Otherwise, they have not yet reached the equilibrium state. Note that not all the free charges need to move, a very small fraction being sufficient. Note also that the time needed to reach the equilibrium in a metal is extremely short.

We can easily see that this property does not hold for an inhomogeneous conductor. Consider, for example, a metal bar whose extremes are at different temperatures. The conduction electron gas has a higher temperature, hence a higher mean kinetic energy at one extreme than at the other. The carriers will then also move from the hotter to the colder extreme when no external field is present. Under these conditions, the equilibrium is reached when the field in the conductor has a certain, non-zero, value due to the distribution of its carriers.

2. *A conductor is an equipotential volume.*

This is an immediate corollary of the previous property. Indeed, the potential difference between two points of the conductor is the line integral of the field between those points. If the points are inside, we can always find a line completely inside the conductor, on which the field is zero. Hence, all internal points have the same potential. The same is true, by continuity, on the surface. We shall see important applications of this property.

If we want to generate an electric field with a certain shape, we need to fix the potential values in certain space regions. We can do that by building metallic elements of the right shape and giving them the right potentials.

3. *The charge density is zero inside a conductor in equilibrium.*

Let us apply the Gauss law to an arbitrary closed surface Σ completely inside the conductor. The field is zero at all the points of Σ, and consequently the flux is zero. If ΔV is the volume surrounded by Σ, the Gauss law gives us

$$\int_{\Delta V} \rho \, dV = 0.$$

ΔV being arbitrary, ρ must be identically zero in the entire inside volume

$$\rho(x, y, z) = 0 \text{ inside.} \tag{2.1}$$

Clearly, this does not mean that there is no charge in the conductor; rather, it means that the net charge is zero.

Note that the just-stated theorem does not hold on the surface. Indeed, a surface element cannot be enclosed in a Gauss surface completely inside the conductor. Indeed, a net free charge exists on the surface in the presence of an external electric field or when the conductor is charged.

We can say that, in general, a charge of density σ, which is a function of the position, is present on the surfaces of the conductors. The integral of σ over the surface is the net charge of the conductor.

4. *The electric field immediately outside the surface of a conductor in equilibrium with surface density σ is normal to the surface and has a normal component equal to σ/ε_0.*

Indeed, all the field components are zero on the internal face of the surface. Through the surface, the tangent ones are continuous, while the normal one has a discontinuity σ/ε_0. More precisely, if $\mathbf{n}$ is the unit normal outgoing vector, the field on the outside face of the surface is

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\mathbf{n}. \tag{2.2}$$

Note that the field is directed outward if $\sigma > 0$, and inward if $\sigma < 0$.

We now describe electrostatic induction. The phenomenon, not to be confused with electromagnetic induction, which will be discussed in Chap. 7, can be observed with elementary means. Figure 2.1 represents a metallic conductor on an insulating support. Let us bring a charged object, for example, a glass bar charged by friction (the charge is positive in this example), near to an extreme of the conductor. Under these conditions, some negative free charges move to the region of the conductor nearer to the positively charged body. Correspondingly, a net positive charge density develops on the opposite side. This induced charge rearrangement is such as to cancel the electric field in the entire volume of the conductor. However, the net charge of the conductor remains zero, being insulated.

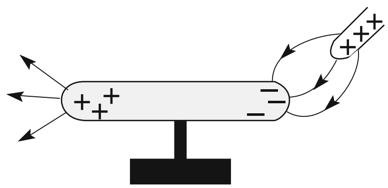


Fig. 2.1 Producing electrostatic induction on an isolated conductor

Consequently, if we now turn away the bar, the positive and negative induced charges in the conductor, so to speak, recombine and the charge density goes back to zero through the entire volume.

A slightly different set-up is shown in Fig. 2.2. Here, the conductor is made of two parts, both on an insulating support. Initially, the two parts are in contact and behave as a single conductor. The effect of the glass charged bar in Fig. 2.2a is equal to that in Fig. 2.1. With the bar still in position, we now separate the two parts, as in Fig. 2.2b, touching the insulating supports (our body is a conductor). When we turn the bar away, the induced negative and positive charges can no longer recombine. In each part, the charge redistributes in order to cancel the field inside the conductor. We end up with two conductors whose surfaces are charged with opposite sign surface charge densities. We say that the bodies have been charged by induction.

The induction phenomenon is easily observed with a gold-leaf electroscope. If we place a body charged by friction, like a glass or plastic bar, near the small sphere of the instrument, without touching it, we observe the leaves opening up, as in Fig. 2.3. If we remove the charged body, the leaves close back up. The phenomenon produced by the charged bar is clearly the electrostatic induction we have just described. If the bar has positive charges, the charges induced on the leaves, which are in the farther side of the electroscope conductor, are positive too, while negative charges are induced on the upper sphere. When we take the charged body away, the induced charges recombine and the leaves discharge.

Figure 2.4 shows how to charge an electroscope permanently by induction. To do that, we just touch the sphere of the electroscope with a finger when the inducing bar is still present. Our body being a conductor, this action electrically connects the sphere with the ground. The charges on the sphere, opposite to those on the inducing body, run away as far as they can, namely to the ground. We now remove

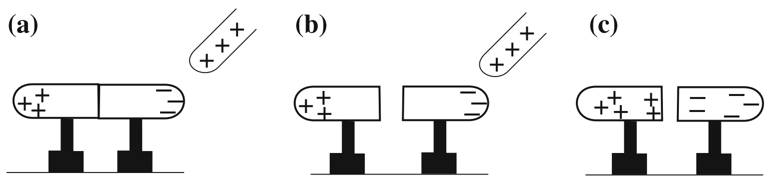


Fig. 2.2 Charging two insulated conductors by induction

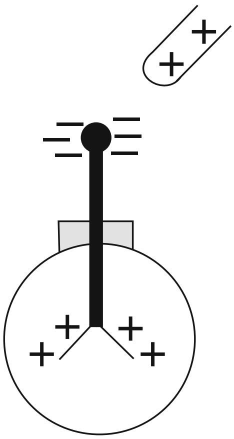


Fig. 2.3 Inducing charges on an electroscope

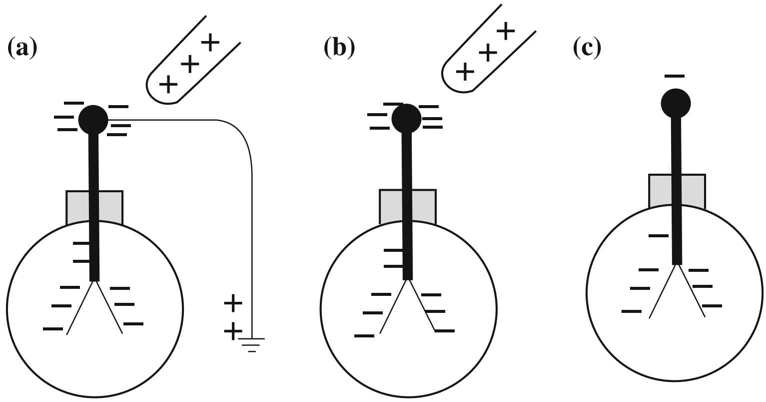


Fig. 2.4 Charging an electroscope by induction

our finger from the sphere and then take the inducing body away. The electroscope is now permanently charged with the sign opposite to that of the inducing body. We can check that by bringing back the charged bar. We see the leaves closing down.

If we observe the leaves of an electroscope we have charged, we see them gradually closing down. The electroscope is discharging because the air in its glass container is not a perfect insulator due to the presence of ions. Ions are continuously formed in air through natural radioactivity. As a matter of fact, we can measure the induced activity rate in air by measuring the discharge time of a properly built electroscope. Here, we open a parenthesis to summarize how cosmic rays were discovered. In 1910, Domenico Pacini (Italy, 1878–1934) developed techniques for measuring the discharge rate of an electroscope underwater. He took measurements three meters underwater in a lake and in the sea, namely under a water thickness sufficient to absorb the largest fraction of the radiation from the ground. The observed discharge rate had decreased, compared to on shore, but it was still relevant. Pacini concluded that an ionization source different from those in the rocks had to exist. The extraterrestrial origin of the source was established by Victor Hess (Austria, 1883–1964), with a series of balloon ascensions between 1911 and 1912. Hess found that the ionization rate in the atmosphere, as measured by the discharge time of his electroscope, was constant or slightly decreasing up to about 2000 m, somewhat equivalent to Pacini's 3 m of water. Above 2000 m, the ionization rate monotonically increased up to the maximum altitude of 5000 m that he was able to reach. Hess concluded that the source of the ionizing radiation found by Pacini had an extraterrestrial origin. These are the cosmic rays, as Robert Millikan (USA, 1868–1953) named them, high energy charged particles, protons, nuclei and electrons, coming from the universe.

2.3 Surface Charges on a Conductor

As we have just seen, the electrostatic field just outside the surface of a conductor at a point at which the surface density is σ is normal to the surface and has magnitude σ/ε_0. Consider now the force exerted by the field on the surface charges. Notice first that the force is, in any case, directed outward, namely tending to rip away the charges. Indeed, where σ is positive, the field is directed outwards and the force has its direction, while where σ is negative, the field is inward but the force is opposite to it. As we are dealing with the charge per unit surface, we should consider the force per unit surface as well. This has the dimensions of a pressure and we shall indicate it with P.

One might think this pressure to be the product of the field σ/ε_0 and the charge on the unit surface σ, namely σ^2/ε_0, but it is not so. As a matter of fact, the "surface" occupied by the charges is not a geometrical one of zero thickness, but has a finite thickness smaller than the atomic diameters. The electric force acting on the most external layer of charges is, indeed, σ^2/ε_0, but it is zero on the innermost layer, with intermediate values in between. Consequently, we take a mean value, namely one half of the external field. The pressure on the surface charges is then

$$P = \frac{\sigma^2}{2\varepsilon_0}. \tag{2.3}$$

Note that the pressure might be large enough to pull the charges out. The pressure is higher where the surface curvature is larger, as we shall now see.

Let us start by considering the simple system in Fig. 2.5, consisting of two metal spheres, one larger (radius R) and one smaller (radius r), joined by a metal wire. The system is a single conductor.

Let the system be charged and, at equilibrium, let Q and q be the charges on the large and small spheres, respectively. We want to find the relative values of the two

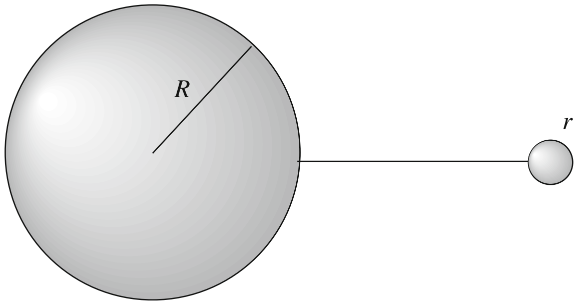


Fig. 2.5 Two spherical conductors joined by a conductive wire

charges. If the spheres are far enough apart from one another, we can think of the field, or the potential, near one of them as not being affected by the field, or the potential, near the other one. The potential on each surface is then the potential of a charged sphere, which is equal to the potential of a point charge in its center. On the other hand, the two spheres are electrically connected and their potentials must be equal. We can the write $Q/(4\pi\varepsilon_0 R) = q/(4\pi\varepsilon_0 r)$, or

$$\frac{Q}{R} = \frac{q}{r}. \tag{2.4}$$

The charge on each sphere is proportional to its radius. The field is proportional to the charge density, which is the charge divided by the surface area, that is proportional to the radius squared. Consequently, the charge densities are inversely proportional to the radiuses, or, we can say, proportional to the curvatures. Calling Σ and σ the charge densities on the large and small spheres, respectively, we have

$$\frac{\Sigma}{\sigma} = \frac{r}{R}. \tag{2.5}$$

Consider now a conductor having regions of different curvature, such as the one shown in Fig. 2.6. In a first approximation, we can consider it similar to the two spheres system and take Eq. (2.5) to be valid near its extremes. The field near the surface, being proportional to the charge density, is proportional to the curvature as well, while the pressure of Eq. (2.3) is proportional to the square of the curvature. Both can be very intense near a tip.

We can verify how the charge is distributed as follows. We use a "spoon" made of an insulating arm finishing in a small conducting sphere. When we touch a point of the conductor with the "spoon", we take out a charge proportional to the density at that point. We bring this charge to an electroscope and see how much the leaves open. We repeat the operation, taking charges from points of different curvature and verifying that they are larger where the curvature is higher.

Notice that surface curvature may be null or even negative. It is negative in any part of a body folded inward. Where the curvature is negative, the charge density is very small. Consider, for example, a metal conductor having the form of a cone, as in Fig. 2.7. With the method we have just described, it is found that almost no

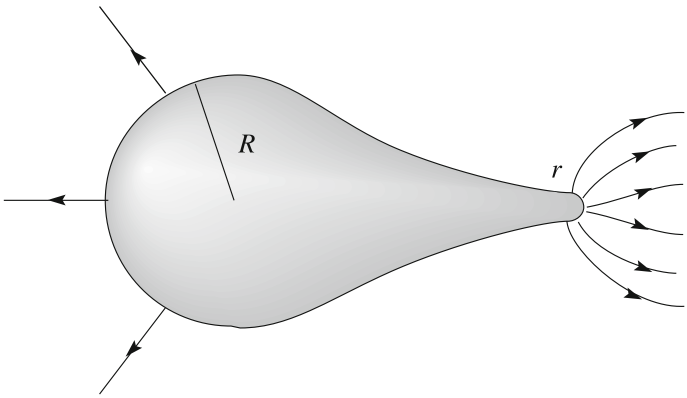


Fig. 2.6 Conductor with different curvatures and lines of field

Fig. 2.7 Demonstration metal cone. The curvature is positive outside, negative inside. Reproduced with permission of the Physics and Astronomy Department of the Padua university

charge is present on the surface inside the cone. We shall see in the next section that the charge density is rigorously zero on the internal surface of a cavity completely enclosed in a conductor.

The above conclusions have relevant practical implications. In several instances, conductors, typically metals, are surrounded by air. If a large potential is given to the conductor, namely if it hosts a large electric charge, and there are points of high curvature (tips) on its surface, the charge density might be extremely high at these points, and so would the electric field just outside the tips. In air, a small number of ions is always present. The ions near the tip are accelerated by the electric field. If the field is high enough, the energy gained by an ion between two collisions with gas molecules may be so large that, in the next collision, the molecule that is hit gets broken. The two parts of the broken molecule have a charge. Each of them will suffer the same fate as the first ion. An avalanche free charge multiplication process is triggered, air becomes a good conductor and an electric spark suddenly develops. The spark can be very dangerous, both for people nearby and for the equipment. When working with high voltages, any tip must be avoided, and all surfaces must have small curvatures and be smoothed.

A lightning strike is triggered by a similar mechanism. An electric field is always present in the atmosphere; its intensity somewhat varies in time at the ground level, being around 100 V/m on a clear day. During a thunderstorm, the field, between the lower parts of the clouds and the ground, grows to 10^4 V/m, two orders of magnitude larger, over flat surfaces. The field is much higher near high curvature points, like bell towers, trees or even the body of a person, when standing on the flat surface of a beach or on a boat. Field intensity may grow enough so as to trigger an avalanche multiplication process of ions present in the atmosphere. Air becomes a good conductor and an extremely intense current develops between the clouds and the ground, for a short duration. This is the lightning strike. During a thunderstorm, it is imprudent to stay near a tree or other pointed objects, to stand on a beach, or, even more so, to hold pointed conductors in one's hand.

The lightning rod, also called a lightning conductor, was invented in 1749 by Benjamin Franklin (USA, 1706–1790), to protect buildings, ships, etc., in the event of a lightning strike. It is a metal rod mounted on top of a structure for protection, connected to the ground (or sea) by thick copper conductors, capable of "attracting" the spark and discharging the high current into the ground, instead of allowing it to pass through the structure.

2.4 Hollow Conductors

Consider a hollow conductor containing an empty closed cavity. We shall now show that the charge surface density in the cavity is zero. The demonstration has two steps, which should not be inverted. In the first step, we use the Gauss law, in the second, the fact that the electrostatic field is conservative.

It is always possible to find a closed surface enclosing the cavity and entirely inside the conductor, as Σ in Fig. 2.8a. If any charge is present inside Σ, it must be on the surface of the cavity. The flux outgoing from Σ is zero because the field is zero at all of its points. Hence, the total charge on the surface of the cavity is zero. This does not mean that there is no charge but that, if there are charges, there should be as many that are positive as are negative. If there were charges on the surface of the cavity, there would be an electric field, like in the example shown in Fig. 2.8b, in which we have drawn a field line. If this were true, we could always find a closed line like Γ in the figure that follows a field line in the cavity and closes back inside the conductor. The line integral of the field around Γ would certainly be positive, because the field has the direction of the line along the entire part inside the cavity

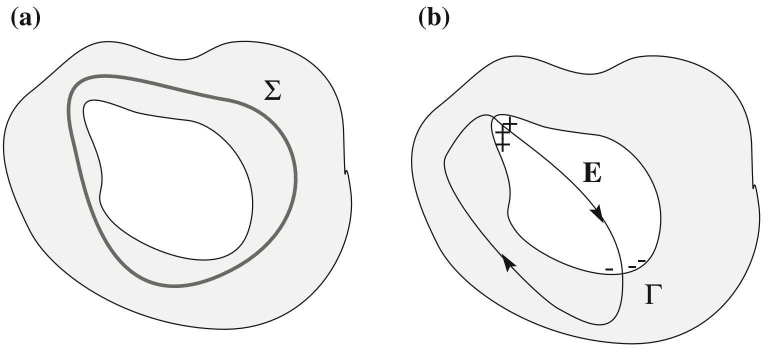


Fig. 2.8 A cavity in a hollow conductor. **a** A surface containing the cavity, **b** a closed path partially in the cavity, partially in the conductor

and is null in the other part. The hypothesis has thus been reduced *ad absurdum* and must be wrong. In conclusion, no charge can be present on the surface of an empty cavity completely enclosed in a conductor in equilibrium. Consequently, the field in the cavity is also zero.

Let us now consider the situation outside the hollow conductor. There, we might have charges of any value in any position outside the conductor and on the conductor itself. The field outside can be anything, while being static. Well, as difficult as it may be to believe, the charges on the external surface of the charged conductor arrange themselves in such a way that no net charge and no field exist, not only in the body of the conductor but even inside the cavity and on its surface. Also, if we put more charge on the conductor of one sign or the other, its potential will rise or fall, but it will never happen, at equilibrium, that any charge or field will appear in the cavity or on its surface. The internal space is completely separated from the external space, for the electrostatic phenomena. This is the electrostatic shielding action. We shall discuss it for cavities that are not empty in Sect. 2.10.

Let us now consider in general terms how one can build an electrostatic field of a desired shape in a given region of space. As we have already mentioned, one can do that by properly shaping a certain number of conductors, metal in general, and giving to each of them a proper potential. Namely, we control the shapes of the equipotential surfaces that are the surfaces of our conductors and their potentials. The charge surface densities, namely how the superficial charges arrange themselves, come about as a consequence. The problem is hence the following: given the surfaces of the conductors and their potentials, find the field and the surface densities. The two quantities are linked. The surface densities are linked to the field intensity on the surface by $\sigma = \varepsilon_0 E$. Notice that the problem is different from the problem we discussed in the first chapter of finding the field of a given charge configuration. Now, the charge distribution is not given, as the charges are free to move outside our control, always arranging in such a way as to have a zero field inside the conductors.

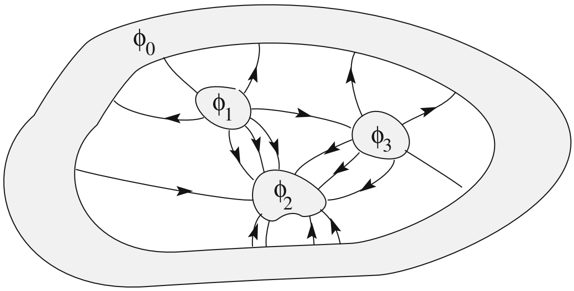


Fig. 2.9 Conductor in a closed cavity of a surrounding conductor

Let us analyze the case of n conductors with charges Q_1, Q_2, … Q_n, respectively, enclosed in the cavity of another conductor, which might be charged too, as shown in Fig. 2.9. Notice that this is the situation usually met in practice, when we conduct experiments in a room in a building. Indeed, walls, ceiling and floor behave, in practice, as conductors for static or electric phenomena not too quickly varying in time. Usually their potential is taken as zero (ground).

We start by observing that a charge must exist on the internal surface of the hollow conductor. This charge, which we call Q_0, is exactly equal and opposite to the sum of the charges of the bodies inside the cavity, namely

$$Q_0 = -Q_1 - Q_2 - \ldots - Q_n.$$

To show that, let us consider a surface Σ enclosing the cavity completely inside the conductor. The field is zero at all its points, and consequently the outgoing flux is zero. For the Gauss law, the net charge inside Σ must be zero.

Note that Q_0 is, as we said, the charge on the internal surface of the hollow conductor. It is not its total charge. More charge might be present on the external surface. However, whatever that charge, the charge on the internal surface is always Q_0.

Question Q 2.1. Two charges Q_1 and Q_2 are at rest at a certain distance. Let F_{12} be the force exerted by Q_1 on Q_2. You now enclose Q_1 in the center of a spherical metal shell, letting Q_2 be outside. How does F_{12} vary? □

Let us now analyze the problem more formally. Under the given conditions, the conductors inside have known potentials ϕ_1, ϕ_2, … ϕ_n, with the hollow one being ϕ_0. We are interested in the internal space, in which the Laplace equation holds

$$\nabla^2\phi = 0 \tag{2.6}$$

with the boundary conditions $\phi\ (x, y, z) = \phi_1$ at the points of the surface Σ_1 of conductor 1, $\phi\ (x, y, z) = \phi_2$ at the points of the surface Σ_2 of conductor 2,… $\phi\ (x, y, z) = \phi_0$ at the points of the surface Σ_0 of the hollow conductor. The problem is now precisely posed and we can ask the questions: does a solution exist? If it exists, is it unique? How can we find it? From a physical point of view, it is obvious that the system will reach a unique equilibrium configuration, in which the potential energy is a minimum. The potentials of the conductors in this configuration are the solution to the problem. This is clearly not a mathematical demonstration. The rigorous demonstration can be found in calculus books. We shall not give it here. Rather, we shall assume that a solution exists and show that it is unique, once the boundary conditions are given.

Let us show a few useful properties of the harmonic functions, as the solution of the Laplace equation are called.

Let V be the space between the conductors and Σ the surface limiting this space, namely the set of the surfaces Σ_0, Σ_1, Σ_n. Let us show that if the function ϕ is harmonic in the volume V and is zero at the points of Σ, then ϕ is identically zero in V. We start from the identity valid for every scalar function ϕ

$$\nabla \cdot (\phi \nabla \phi) = \nabla \phi . \nabla \phi + \phi \nabla^2 \phi, \tag{2.7}$$

which is immediately shown by direct calculation (it is mainly the derivative of a product). If ϕ is now harmonic, the second term on the right-hand side is identically zero and we have

$$\nabla \cdot (\phi \nabla \phi) = |\nabla \phi|^2. \tag{2.8}$$

Let us now integrate $\nabla \cdot (\phi \nabla \phi)$ over the volume V and apply the Gauss divergence theorem, namely

$$\int\limits_V \nabla \cdot (\phi \nabla \phi) dV = \int\limits_\Sigma (\phi \nabla \phi) \cdot \mathbf{n} d\Sigma.$$

The right-hand side of this equation is zero because ϕ is zero at the points of Σ, by assumption. Hence, using Eq. (2.8), we have

$$\int\limits_V |\nabla \phi|^2 dV = 0.$$

The integrand on the left-hand side cannot be negative. Hence, the equation implies that

$$\nabla \phi = 0$$

in the entire V. This means that ϕ is uniform in the entire volume, the surface included. Being zero on the surface, ϕ is zero in the entire volume.

In conclusion, if the electrostatic potentials of the hollow conductor and of all the internal conductors are zero, or equal (considering that the potential is defined modulo an additive constant), then the entire internal region is equipotential and the field is zero. The case of the empty cavity is a particular case. We have retrieved the result discussed at the beginning of the section.

We are now ready to show that the solution of the Laplace equation with given boundary conditions is unique. Let us assume knowing a solution ϕ of the Laplace equation and that another solution, say ψ, exists with the same boundary conditions. Namely, the following conditions are satisfied

$$\begin{aligned} &\phi(x,y,z) = \psi(x,y,z) = \phi_1 \quad \text{on the points of } \Sigma_1 \\ &\phi(x,y,z) = \psi(x,y,z) = \phi_2 \quad \text{on the points of } \Sigma_2 \\ &\ldots\ldots\ldots \\ &\phi(x,y,z) = \psi(x,y,z) = \phi_0 \quad \text{on the points of } \Sigma_0. \end{aligned}$$

Now, the Laplace equation being linear, ϕ-ψ is a solution as well, with boundary conditions

$$\phi(x,y,z) - \psi(x,y,z) = 0 \quad \text{on the points of any } \Sigma_i.$$

Hence, for the just demonstrated theorem, ϕ-ψ is identically zero in the entire V. ϕ and ψ are the same function.

The fact that the solution is uniquely defined by fixing the electrostatic potentials of the conductors has important practical consequences. One of these is the already-mentioned electrostatic shielding. An electrostatic shield, which is a hollow conductor, divides the space, from what concerns the electrostatic phenomena, into two completely separate and independent regions: the internal and the external.

Finally, we notice that the uniqueness theorem also holds when the surface of the enclosing conductor goes to infinity. More precisely, the theorem is also valid when one of the boundary conditions is at infinity.

2.5 Equilibrium in an Electrostatic Field

One might wonder whether is it possible to find any static arrangement of electric charges such as to produce a *stable* equilibrium position. If it was possible, we could put a charge in that position and have it remain there at rest. As we shall now show, the answer is negative. No stable equilibrium position exists in an electrostatic field. Note that this is a consequence of the inverse square dependence of the electrostatic force. As such, the conclusion is also valid for the gravitational force. There is no stable equilibrium position in the gravitational field either. We cannot put a spacecraft at some point and have it standing there in equilibrium.

A stable equilibrium position for a positive or negative charge should be in a minimum or a maximum, respectively, of the potential. We shall now show that such points do not exist for a harmonic function. Saddle points do exist, such as, for example, the point halfway between two equal point charges of the same sign that we noticed with reference to Fig. 1.11b. Indeed, this is an equilibrium position, but the equilibrium is not stable. The potential there has a minimum moving in one direction, and a maximum moving in a direction 90° from it.

Let us assume, in a *reductio ad absurdum* argument, the harmonic function ϕ to have a maximum at the point A. Let $\mathbf{r}$ be the position vector drawn from A. If the assumption is true, we can always find a sphere centered at A such that, at all the points on its surface, which we call S, $\partial\phi/\partial r < 0$. Thus, it will also be

$$\int_S \frac{\partial\phi}{\partial r} dS < 0.$$

Let $\mathbf{n}$ be the unit vector normal to the sphere pointing outside. This is also the direction of $\mathbf{r}$ on the surface, and hence, we have $\partial\phi/\partial r = \nabla\phi\cdot\mathbf{n}$. Applying the divergence theorem, we obtain

$$\int_S \frac{\partial\phi}{\partial r} dS = \int_S \nabla\phi\cdot\mathbf{n}dS = \int_V \nabla\cdot\nabla\phi dV = \int_V \nabla^2\phi dV = 0$$

where, in the last step, we took into account that ϕ is harmonic in V. Hence, the opening statement must be false.

2.6 Electrostatic Capacitance

A conductor, think of a metal to be concrete, is considered isolated if it is far enough from any other conductor and any other charged body (even if it is not a conductor). This condition is very rarely met in practice, but is easy to analyze, and we shall do that as a starting point.

Let Σ be the surface of the conductor and $\mathbf{n}$ the unit vector of the outside normal, as shown in Fig. 2.10. Let us put the charge Q on the conductor and let ϕ_0 be the potential it takes and $\phi(x, y, z)$ the potential in the space outside the conductor. The function $\phi(x, y, z)$ is harmonic with the boundary conditions

$$\phi(x,y,z) = \phi_0 \quad \text{on } \Sigma; \quad \phi(x,y,z) = 0 \quad \text{at infinity.} \tag{2.9}$$

Once $\phi(x, y, z)$ is known, the field on the surface is known too, being given by

$$\mathbf{E} = -\frac{\partial\phi}{\partial n}\mathbf{n}. \tag{2.10}$$

We also know the charge density, namely

$$\sigma = \varepsilon_0 E \tag{2.11}$$

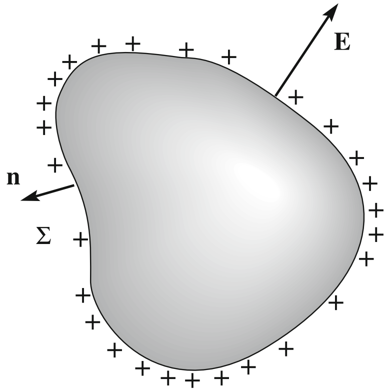


Fig. 2.10 A charged isolated conductor

and the total charge of the conductor

$$Q = \int_{\Sigma} \sigma \, d\Sigma \tag{2.12}$$

The Laplace equation being a linear equation, if ϕ is the solution to a problem with certain boundary conditions, namely certain values of ϕ on the surfaces of the boundary (the surface of our conductor and infinity), a solution is also ϕ multiplied by any constant λ, with boundary values λ times the previous ones. In the new solution, the field will be λ times larger in every point and, and so will the charge density on the surface of the conductor and, finally, so will its charge. Inverting the argument, if we change the charge of the isolated conductor by a factor, its potential will change by the same factor, namely

$$Q = C\phi_0 \tag{2.13}$$

The proportionality constant C is the *electrostatic capacitance* or simply *capacitance* and also *capacity* of the conductor. The name comes from the era in which electric charge was thought to be a sort of fluid and the fact that the higher the capacitance, the higher the "capacity" of the conductor to store charge at a given voltage. To be precise, the higher the capacitance, the lower the potential reached by the conductor for a given charge. The measurement unit for capacitance is the *farad* (F), after Michael Faraday (UK, 1791–1867), who made enormous contributions to all sectors of electromagnetism. The physical dimensions are coulomb per volt, namely C/V. An isolated conductor has a capacitance of one farad if, when charged with one coulomb, it reaches the potential of one volt.

One farad is a very large capacitance. To see that, let us consider a spherical conductor and let us calculate the value of its radius, say R, to have the capacitance of one farad.

Let Q be the charge on the sphere. The potential at a point immediately outside the surface, and, by continuity, on the surface as well, is the potential of a point charge at the center, namely

$$\phi_0(R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}.$$

The capacitance is then

$$C = 4\pi\varepsilon_0 R \tag{2.14}$$

To have C = 1 F, the radius of the sphere must be $R = 9 \times 10^9$ m, namely nine million kilometers. In practice, the submultiples are used (μF, nF, pF, etc.).

QUESTION Q 2.2. Find the capacitance of earth (radius-6400 km). How much does its potential vary if its charge increases by 1 C. □

The concept of capacitance for an isolated conductor has a very limited practical utility, because other conductors are always present, like the walls and the floor of the room and the bodies of the people around. Under these conditions, electrostatic induction takes place. The manner in which charges distribute along the surfaces of the conductors and the potentials the conductors assume influence one another.

The simplest case consists of two conductors near one another. Even in this case, however, the presence of a third conductor, floor and walls, which we collectively call ground, cannot be ignored, as shown in Fig. 2.11. Let us assume the two conductors initially not to have a charge and to charge up them by transferring the charge Q from the conductor we shall call 2 to conductor 1. The charge of 1 and 2 will be Q and $-Q$, respectively. Let ϕ_1 and ϕ_2 be the two potentials relative to ground, whose potential we define as ϕ_0 = 0.

Under these conditions, we have electrostatic induction between the two conductors and between each of them and ground. As opposed to the isolated conductor, there is no proportionality between the potentials of either of the two conductors, or their potential difference, and their charge. Electrostatic induction is

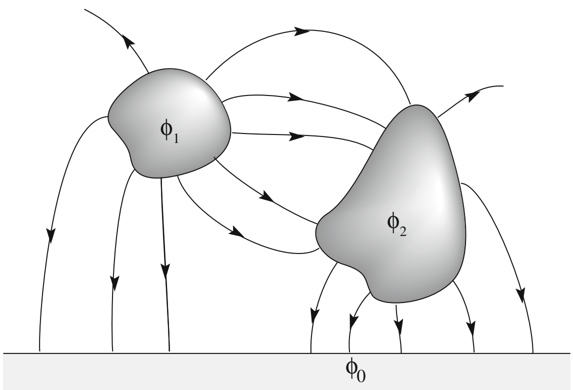


Fig. 2.11 Two conductors near one another

visualized by the behavior of the field lines, as in Fig. 2.11. In particular, not all the field lines leaving conductor 1 reach conductor 2. We say that the induction between them is not complete. The induction between two conductors is complete when all the field lines that exit from one enter the other. The necessary condition for that is that the charges of the two conductors are equal and opposite. The condition is not sufficient, as we have just discussed.

Looking at Fig. 2.11, it is clear that we might have complete induction by moving the pair of conductors very far from the ground. This is, however, impossible in practice, as it was for the isolated conductor. With two conductors, however, we have the possibility, which we did not have with just one. The solution is to have one of the conductors, say 1, be hollow and to lodge the second one in its cavity, as in Fig. 2.12. In this configuration, the Gauss law requires that all the field lines leaving conductor 2 terminate on the internal surface of conductor 2, because the charges on the two surfaces must be equal and opposite. The induction between 1 and 2 is complete. Note that this conclusion is independent of the charge on the external surface of conductor 1 and of the presence of conductors in the external surroundings. All of that is irrelevant for the field and the charges inside the cavity.

A *capacitor* is defined as a system of two conductors between which the electrostatic induction is complete. The two conductors are called the plates of the capacitor. While the word *condenser* is often used as being synonymous with capacitor, we shall use the latter terminology.

Let us now "charge up the capacitor", meaning that we move a certain charge, say Q, from conductor 2 to conductor 1. Their charges will be $Q_1 = Q$ and $Q_2 = -Q$ and their potentials, say, ϕ_1 and ϕ_2. We now show that the potential difference $\phi_2 – \phi_1$ is proportional to Q.

The argument is similar to that which we made for an isolated conductor. Let Σ_1 and Σ_2 be the surfaces of the two conductors. The potential $\phi(x, y, z)$ in the space between them is given by the solution of the Laplace equation with the boundary conditions $\phi(x, y, z) = \phi_1$ on Σ_1 and $\phi(x, y, z) = \phi_2$ on Σ_2. Once $\phi(x, y, z)$ is known, we get the electric field, which is its gradient and the charge densities on the surfaces, say σ_1 and σ_2. The charges on the conductors are then

$$Q_1 = Q = \int_{\Sigma_1} \sigma_1 \, d\Sigma; \quad Q_2 = -Q = \int_{\Sigma_2} \sigma_2 \, d\Sigma \tag{2.15}$$

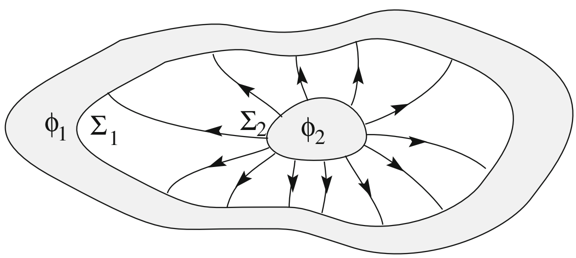


Fig. 2.12 A capacitor, namely two conductors with complete induction

Again, if ϕ is the solution to a problem with the above boundary conditions, $\lambda\phi$ is the solution with boundary conditions $\lambda\phi_1$ on Σ_1 and $\lambda\phi_2$ on Σ_2. The field near the surfaces is λ times larger and so are the surface charge densities, and so, finally, are the charges on the plates, and we can write

$$Q = C(\phi_2 - \phi_1) \tag{2.16}$$

where the constant C is the capacitance of the capacitor (or capacity of the condenser). The measurement unit of the capacitance of a capacitor is obviously the farad.

Capacitors are important elements of electric and electronic circuits. In practice, they are built joining two conductors separated by an insulating sheet (which also has the effect of increasing the capacitance, as we shall see in Chap. 4). The geometric dimensions of the surfaces of the two conductors facing one another are very large compared to their distance, in order to minimize the effects of the lack of complete closure at the borders. Indeed, in these regions, a few field lines might "escape" and end up on another conductor nearby, making the induction incomplete. In practice, however, capacitors can be produced in which these effects are negligible.

Figure 2.13 shows the capacitor of the simplest geometry, namely the parallel-plate capacitor. It is made of two equal metallic plane plates of surface S separated by a small gap of height h, which is much smaller than the diameter of S. We shall make the approximate assumptions that the field is uniform between the plates, with magnitude σ/ε_0, and zero outside, as in Fig. 2.14a.

The potential difference V is the line integral of the field from one plate to the other, which is simply Eh. If Q is the charge (on the positive plate), the charge density is $\sigma = Q/S$. Hence, we have

$$V = \phi_2 - \phi_1 = \frac{\sigma}{\varepsilon_0} h = \frac{Q}{S\varepsilon_0} h. \tag{2.17}$$

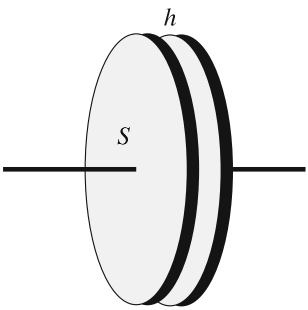


Fig. 2.13 Parallel-plate capacitor

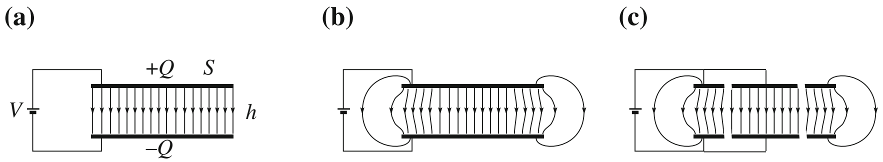


Fig. 2.14 The field of a parallel plate capacitor. **a** Ideal case, **b** as in real life, **c** with the Thomson guard-ring

In conclusion, the capacity of the parallel plate capacitor is

$$C = \frac{S\varepsilon_0}{h}. \tag{2.18}$$

We see that the capacitance is greater the larger the area and the smaller the distance between the plates. To get an idea of the orders of magnitude, let us consider a parallel-plate capacitor of 1 F capacity with the distance between the plates being $h = 0.1$ mm. We immediately see that the surface needed for that is $S = 11 \times 10^6$ m^2, namely a square with more than 3 km sides.

Equation (2.18) tells us that, as we anticipated in Sect. 1.2, we can measure the vacuum permittivity in farads per meter. In round figures, its value is, as in Eq. (1.9),

$$\varepsilon_0 = 8.8 \text{ pF/m}. \tag{2.19}$$

To fix the orders of magnitude, it is good to remember that 1pF is about the capacitance of a parallel-plate capacitor of 1 cm^2 area and 1 mm plate separation.

Let us inquire into the validity of our assumptions. Contrary to them, the actual field does not terminate abruptly at the rim of the plates, but rather extends into the region surrounding the capacitor, as shown in Fig. 2.14b. In addition, the actual field is not uniform between the plates near the rim. Therefore, the solution we have found is not completely correct. We say that there are "fringing effects" at the edge of the capacitor. The smaller the separation h of the plates relative to their area S, the smaller the fringe effects. However, our solution can be approximated very well by the simple modification devised by William Thomson (UK, 1824–1907). We divide both plates into a central part, where the field is uniform, and in an external "guard ring" separated by a very narrow gap. The ring is in the same plane and has the same potential as the nearby plate, as shown in Fig. 2.14c. Our capacitor is now the central part of the system, from which the edge effects are removed. Equation (2.18) holds with a very good approximation.

2.7 Calculating Capacitances

In this section, we shall calculate the capacitances of capacitors of two symmetric geometries, spherical and cylindrical.

Spherical capacitor.
Figure 2.15 shows a cross-section through the center of a spherical capacitor, which consists of two concentric spherical conducting shells, one inside the other. A small hole is made in the outer sphere to allow an electric connection with the inner one going through. Let R_1 and R_2 be the radiuses of the inner and outer conductors, respectively, and Q the charge of the capacitor.

The field between the two surfaces is the field of a point charge Q in the center, namely

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\mathbf{u}_r$$

The potential difference is obtained by integrating the field on a line between the plates, which we chose to be along a radius. We then have

$$\phi_2 - \phi_1 = -\int_{R_1}^{R_2} \mathbf{E}\cdot d\mathbf{r} = -\frac{Q}{4\pi\varepsilon_0}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right),$$

which gives the capacitance

$$C = 4\pi\varepsilon_0\frac{R_1R_2}{R_2 - R_1}. \tag{2.20}$$

We observe that if the plates are very near, namely if $R_1 \approx R_2$, and we call $h = R_2 – R_1$ the distance between the conductors, then we have $C = \varepsilon_0 4\pi R^2/h = \varepsilon_0 S/h$, which is the capacitance of the parallel plate capacitor.

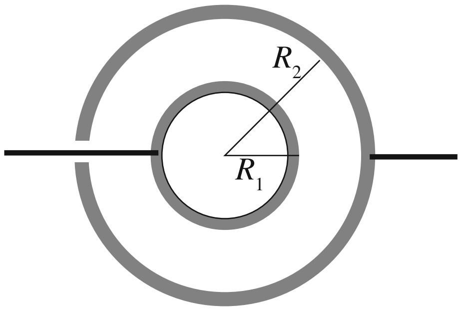


Fig. 2.15 Equatorial cross-section of a spherical capacitor

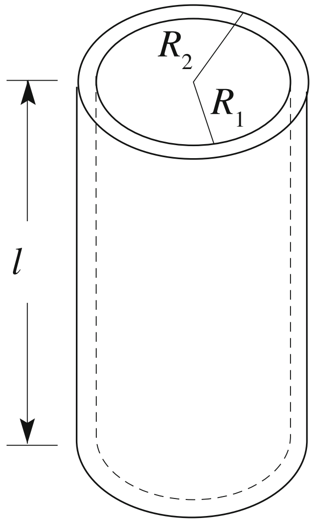


Fig. 2.16 Geometry of a cylindrical capacitor

In order to compare this with the spherical isolated conductor, let us calculate the radius of a spherical capacitor of 1 F with 0.1 mm spacing between the plates. We find $R = 300$ m, which is quite big, but not really enormous.

Cylindrical capacitor.
The cylindrical capacitor is made of two coaxial conducting shells, one inside the other. Let R_1 and R_2 be the radiuses of the inner and outer conductors, respectively, l their height and Q the charge of the capacitor. Figure 2.16 shows the geometry.

The field between the electrodes is the field of a linear charge distribution on the axis with linear density $\lambda = Q/l$. Its magnitude at the distance r' from the axis is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r'}\mathbf{u}_{r'}. \tag{2.21}$$

By integration, we obtain the potential difference

$$\phi_2 - \phi_1 = -\int_{R_1}^{R_2} \mathbf{E}\cdot d\mathbf{r}' = -\frac{\lambda}{2\pi\varepsilon_0} ln\frac{R_2}{R_1} = -\frac{Q}{2\pi\varepsilon_0 l}\ln\frac{R_2}{R_1}.$$

The capacitance is then

$$C = 2\pi\varepsilon_0 l/\ln(R_2/R_1). \tag{2.22}$$

We leave as an exercise to show that if $R_1 \approx R_2$, this expression reduces to the one valid for the parallel plate capacitor with $h = R_2 – R_1$. Use the approximation $\ln(R_2/R_1) = \ln[(R_1 + h)/R_1] = \ln(1 + h/R_1) \approx h/R_1$.

In practice, a cylindrical capacitor is build by overlapping two rectangular flexible metal strips separated by an insulating layer. To have a handy device, one side of the rectangle is short, 1–2 cm, the other very long. The sandwich is then wrapped in a helix to form a cylinder.

In practice, as in the just-considered example, capacitors have an insulator rather than a vacuum between the conductors. Electric insulators are also called dielectrics. We shall study their properties in Chap. 4. We shall see there that the expressions we have found for the capacitances need to be modified, simply changing the vacuum permittivity ε_0 into a constant ε characteristic of the medium, called the permittivity of the material. A connected term is the relative permittivity, also called the dielectric constant that is the ratio $\kappa = \varepsilon/\varepsilon_0$. The dielectric constant is, in any case, larger than one, having values ranging, for different media, from a few units to hundreds of thousands.

Another important feature of the dielectric used to separate and insulate the plates is its dielectric strength. This is the maximum field that the material can withstand without breaking down. Breakdown results in the formation of an electrically conductive path and a discharge through the material. For a solid material, a breakdown generally destroys its insulating capability. Good insulators have dielectric strengths up to tens of MV/m.

2.8 Combining Capacitors

As we already mentioned, capacitors are commonly used in electronic circuits, sometimes in quite complicated combinations. It is thus useful to have a set of rules for finding the *equivalent capacitance* of the different combinations of capacitors. It turns out that we can always find the equivalent capacitance by repeated application of two simple rules. These rules are for the two basic connection types: in series and in parallel.

Figure 2.17 shows n capacitors connected in series. The arrangement forms a line in which the positive plate of a capacitor is connected to the negative plate of the next one. The two plates become a unique conductor. Now, let us charge the line by taking a charge Q from the last plate on the right (which will then have the charge $-Q$) and putting it on the first plate on the left (which will then have the charge $+Q$). The charge on the other plate of the first capacitor is, by induction, $-Q$. No net charge has gone on the conductor made of the right plate of the first capacitor and the left plate of the second. Its net charge is zero. Consequently, the left plate of the second capacitor has the charge $+Q$. The process repeats itself to the end of the line.

The potential difference on the i-th capacitor is $\Delta\phi_i = Q/C_i$. The total potential difference between the extremes of the line is then

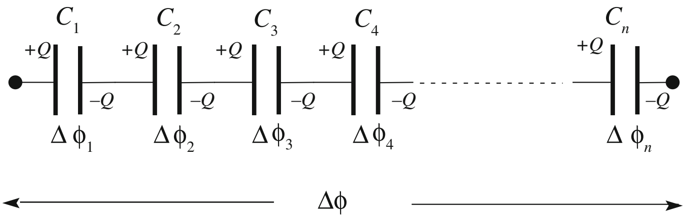


Fig. 2.17 Capacitors connected in series

$$\Delta\phi = \sum_{i=1}^{N} \Delta\phi_i = Q\sum_{i=1}^{N} \frac{1}{C_i},$$

which we see to be proportional to the charge. This means that a series of capacitors behaves like a single capacitor with the equivalence capacitance

$$C = 1/\sum_{i=1}^{N} \frac{1}{C_i}. \tag{2.23}$$

Figure 2.18 shows n capacitors connected in parallel. In this arrangement, all the positive plates are connected together, thus forming a unique conductor, and the negative plates are similarly connected.

Consequently, the potential difference is the same for all capacitors. Let it be $\Delta\phi$ and let Q be the total charge on the connected plates. This charge distributes on the capacitors depending on their capacitance. Indeed, the charge on the ith capacitor is $Q_i = \Delta\phi \times C_i$. Adding them up, we have

$$Q = \Delta\phi \sum_{i=1}^{N} C_i.$$

Even now, the charge is proportional to the potential difference and we can state that a system of capacitors connected in parallel is equivalent to a single capacitor of equivalent capacitance

$$C = \sum_{i=1}^{N} C_i. \tag{2.24}$$

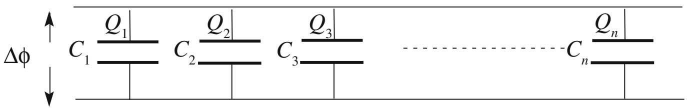


Fig. 2.18 Capacitors connected in parallel

To deal with complicated arrangements of capacitors, one starts by considering subsets in which the capacitors are connected in series or in parallel. To each subset, one substitutes a single equivalent capacitor using the above rules. The arrangement is now simpler and we can apply the same procedure over and over again, since we are left with a single equivalent capacitor.

2.9 Electrostatic Induction Coefficients

In Sect. 2.6, we considered a system consisting of a hollow conductor containing in its cavity a second conductor. We have seen that the electrostatic induction between them is complete and introduced the concept of capacitance. In this section, we shall consider a more general situation in which the hollow conductor contains any number of conductors in its cavity.

To be concrete, we shall consider a system of three conductors in the cavity, without affecting the generality of the argument. Let Q_1, Q_2 and Q_3 be the charges of the three internal conductors and ϕ_1, ϕ_2 and ϕ_3 their potentials. Let Q_0 be the charge on the internal surface of the hollow conductor (the charge on its external surface is irrelevant for the field in the cavity) and ϕ_0 its potential. As we know, $Q_0 = -Q_1 - Q_2 - Q_3$. Let us search for the relation between charges and potentials. We shall use the superposition principle and the uniqueness of the solution of the Laplace equation.

We start by considering, one after the other, the three particular arrangements shown in Fig. 2.19. One case is when $\phi_2 = \phi_3 = \phi_0$ and ϕ_1 is arbitrary. We can consider connecting conductors 1 and 2 to the hollow one with two conductive wires. We call this configuration state 1 of the system. Let Q_1^1, Q_2^1 and Q_3^1 be the charges of the three internal conductors (the superscript indicates the state we are considering). One can easily see that the three charges are proportional to $\phi_1 - \phi_0$. The argument is the same one we have already used a few times. If, for example, $\phi_1 - \phi_0$ were to double, the field would double too, and the charges Q_1^1, Q_2^1 and Q_3^1 as well. We can then write

$$Q_1^1 = C_{11}(\phi_1 - \phi_0); \quad Q_2^1 = C_{21}(\phi_1 - \phi_0); \quad Q_3^1 = C_{31}(\phi_1 - \phi_0).$$

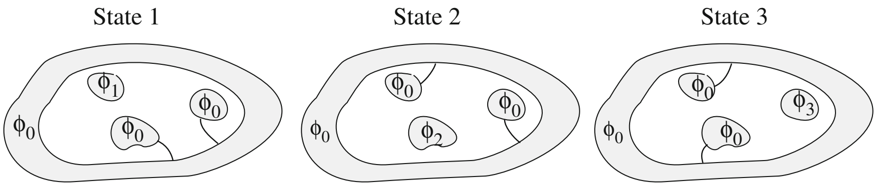


Fig. 2.19 Three states of the system of three conductors in a cavity

Let us now consider State 2, in which $\phi_1 = \phi_3 = \phi_0$ and ϕ_2 is arbitrary. The charges, which we call Q_1^2, Q_2^2 and Q_3^2, are proportional to ϕ_2–ϕ_0, namely

$$Q_1^2 = C_{12}(\phi_2 - \phi_0);\ Q_2^2 = C_{22}(\phi_2 - \phi_0);\ Q_3^2 = C_{32}(\phi_2 - \phi_0).$$

Finally, State 3 is when $\phi_1 = \phi_2 = \phi_0$ and ϕ_3 is arbitrary. The charges, Q_1^3, Q_2^3 and Q_3^3, are proportional to ϕ_3–ϕ_0, namely

$$Q_1^3 = C_{13}(\phi_3 - \phi_0); \quad Q_2^3 = C_{23}(\phi_3 - \phi_0); \quad Q_3^3 = C_{33}(\phi_3 - \phi_0).$$

The proportion coefficients C_{ij} are called electrostatic induction coefficients and have the physical dimensions of a capacitance. If there is only one conductor, say i, we are back to the case in Sect. 2.6. The system is a capacitor and C_{ii} is its capacitance.

The general case in which the potentials are arbitrary is immediately obtained considering the superposition of States 1, 2 and 3. The Laplace equation being linear, the potential difference between any internal point and the hollow conductor is the sum of the corresponding potential differences in the three cases, the field is the sum of the fields and the charge on each surface the sum of the charges. Adding up the three equations and calling $Q_i = Q_i^1 + Q_i^2 + Q_i^3$, we have

$$\begin{aligned} Q_1 &= C_{11}(\phi_1 - \phi_0) + C_{12}(\phi_2 - \phi_0) + C_{13}(\phi_3 - \phi_0) \\ Q_2 &= C_{21}(\phi_1 - \phi_0) + C_{22}(\phi_2 - \phi_0) + C_{23}(\phi_3 - \phi_0)\,. \\ Q_3 &= C_{31}(\phi_1 - \phi_0) + C_{32}(\phi_2 - \phi_0) + C_{33}(\phi_3 - \phi_0) \end{aligned} \tag{2.25}$$

In conclusion, the charges of each conductor depend linearly on the potential differences between them and the external conductor. If we fix the arbitrary constant of the potentials, taking the potential of the hollow conductor as zero, then the charge of each internal conductor is proportional to its potential.

2.10 Electrostatic Shield

Let us again consider a system of n conductors enclosed in the cavity of an external conductor, as shown in Fig. 2.20.

We already know that, once the potentials ϕ_0, ϕ_1, $\phi_2 \ldots \phi_n$ are fixed, the field in the cavity and the charges on the conductors are defined. As we shall now see, this implies that regardless of any change that can be made in the field and in the charges outside the external conductor, no observable change can happen inside the cavity (under static conditions). We might move the charged external bodies, such as the one in the figure, change their charges, or even put the charge from the outside on the external surface of the external conductor. In the latter case, its

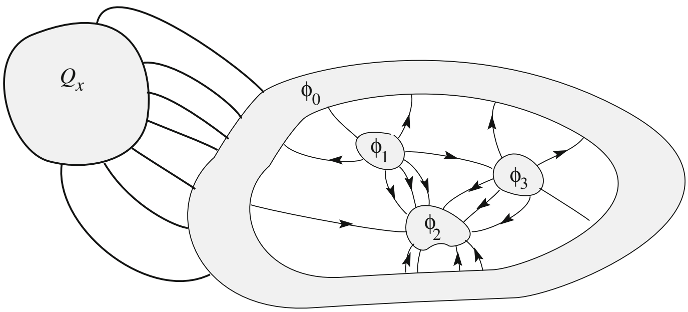


Fig. 2.20 Electrostatic shielding

potential will vary, but the potential differences with the internal conductors will remain exactly the same. We shall now prove the latter sentence.

We start by observing that whatever we can do outside the charge on the surface of the cavity cannot change because it is equal to the opposite of the charges of the internal conductors.

Let us next consider the simplest situation in which there is only one internal conductor. This is the case already studied in Sect. 2.6. The system is a capacitor and the potential difference between internal and external conductors, say $\phi_1 - \phi_0$, is proportional to the charge of the internal one. No external action, in statics, can change $\phi_1 - \phi_0$. We can change from outside the charge of the external conductor and its potential as a consequence, but the potential of the internal conductor will change by the same amount.

We can extend the same argument to the case of a number of conductors in the cavity. Indeed, as we have shown in Sect. 2.9, their charges, which cannot vary for an external action, are linked by linear relations to the potential differences with the external conductor. Hence, any external electrostatic action can only change the potentials of all the conductors, ϕ_0, ϕ_1, $\phi_2 \ldots \phi_n$, by the same quantity.

In the volume of the cavity, the potential $\phi(\mathbf{r})$ is the solution of the Laplace equation with the boundary conditions given by the potentials ϕ_0, ϕ_1, $\phi_2 \ldots \phi_n$ of the conductors, which form the surface of that volume. As we just saw, a change in the external conditions can change the potential, but all by the same quantity, say $\Delta\phi$. On the other hand, $\phi(\mathbf{r}) + \Delta\phi$ is also a solution of the Laplace equation. More precisely, this solution satisfies the new boundary conditions and consequently, the solution being unique, it is *the* solution. In conclusion, a change in the external electrostatic conditions can change the potential at all the points of the cavity by the same additive amount. But any observable effect is due to the field, not to the potential, and the field does not vary if the potential varies at every point by the same additive quantity. This is the action of electrostatic shielding already mentioned.

Notice that the shielding action works in both direction. Namely, whatever we may do inside (electrostatically), like moving charges from one conductor to the other, surface of the cavity included, no effect can be observed outside. In electrostatics, internal and external spaces are completely separate and independent.

We can also express our conclusions by stating that, under static conditions, an observer inside the cavity of a conductor cannot determine with any measurement the potential of the conductor that encloses him/her, which we can call the potential of his/her laboratory. In other words, the laws of the electrostatic phenomena are *invariant* under a change of the potential of the laboratory, namely they have the *same form* for whatever that potential is. This is the particular form met in electrostatics of a general property of electromagnetism called *gauge invariance*.

Electrostatic shielding has several practical applications. Delicate experiments often employ very sensitive equipment that is affected by the presence of conductors in the surroundings, especially when measuring very small charges or potential differences. To protect them, one encloses the instruments in a metallic, grounded structure, often in the shape of a cage, called a Faraday cage, having been invented in 1836 by Michael Faraday (UK, 1791–1867). The walls of the structure can be made of a continuous metal foil but also of a wire mesh. The latter solution works because it can be shown that the influence of the openings between the wires of the mesh only extends over distances comparable with their diameters. Faraday cages are capable of providing a partial shield for fields variable in time as well.

Note that the properties discussed in this chapter ultimately stem from the inverse square dependence of the electrostatic force. If the field of a point charge varied with a power of distance different from −2, even of a very small amount, the Gauss law would not be valid and, in particular, the shielding action would not be complete. This feature provides an extremely sensitive way to measure the difference from −2 of the r exponent.

Suppose that the dependence on the distance of the force is $1/r^{-2+\varepsilon}$. We build a hollow conductor, for example, a spherical shell. We charge it from outside, raising its potential as much as we can, to obtain the maximum effect, if any. We measure the charge on the internal surface with a sensitive electrometer. If we do not find any charge, as is the case, we can say that if any charge is present, it must be smaller than the sensitivity of our instrument (we can never say that it is exactly zero). In parallel, we calculate how much charge we would expect to find as a function of ε. The upper limit found on the charge will then translate into an upper limit of ε. Notice that, while, in principle, any shape of hollow conductor will do, having a symmetric shape, namely a sphere, makes the calculation possible in practice. Notice also that we shall never be able to make a *perfect* sphere, and that, consequently, we need to take that into account in evaluating the experimental uncertainties.

The invention of the method is credited to Joseph Priestley (UK, 1773–1804) in 1767. In that year, which, it should be noted, was 18 years before the Coulomb experiment, the idea already existed that the electrostatic force might have inverse square law dependence in analogy to the gravitational force. Priestley knew the Newton theorem showing that the gravitational field inside a spherical shell is zero (see Vol. 1, Sect. 4.6). He measured the charge inside a spherical shell, finding none, within a limited sensitivity. This was the first historical hint at the inverse square law, which was later established by Coulomb with a direct measurement (Sect. 1.2).

The first precise measurement with Priestley's method was done by Henry Cavendish (UK, 1731–1810), who, in 1773, established the limit $|\varepsilon| \leq 0.02$. One century later, in 1873, James Clerk Maxwell (UK, 1831–1879) improved the limit to $|\varepsilon| \leq 5 \times 10^{-5}$. The limit constantly improved over time until it reached the present value of $|\varepsilon| \leq 6 \times 10^{-17}$. Here, we mention that, in quantum mechanics, the electromagnetic interaction is described as being due to the exchange of photons, which are the quanta of the electromagnetic field. The mass of the photon is rigorously zero if the exponent is exactly -2. Consequently, the upper limit on $|\varepsilon|$ provides an upper limit to the photon mass. Taking the electron mass m_e for comparison, $|\varepsilon| < 6 \times 10^{-17}$ corresponds to the photon mass m_γ, being such that $m_\gamma/m_e < 10^{-21}$.

2.11 The Method of Images

The solution of the Laplace equation is unique not only if the boundary conditions are on surfaces at finite distances, but also if they are at infinity. It can be shown, although we shall not do that, that this is true if both potential and field go to zero fast enough when the distance goes to infinity.

The problem in finding the solution is usually much more difficult, depending on the boundary conditions. As a matter of fact, the only general methods are numerical, using powerful computer codes. In a few particularly simple cases, however, the solution can be found with certain "tricks". One of these is the method of images that we shall now see.

Let us start with an electric field generated by a charge arrangement that we are able to calculate. For example, Fig. 2.21 represents the field of two equal and opposite point charges. Let us now take a metal sheet and give it the form of one of the equipotential surfaces, placing it exactly on that surface and giving it the potential of the surface. The sheet divides the space into two regions, separated from one another by the electrostatic shielding of the sheet. In each of them, the field is exactly the same as before the introduction of the sheet, because the boundary conditions in each of them have not changed. We can now fill one of the two regions with a conducting medium without inducing any change in the other one. We have thus found a solution to the Laplace equation for a system consisting of a point charge near a solid conductor with a certain surface shape (the shape of the equipotential we have chosen) and a certain potential. Considering a number of cases like this, we end up with a collection of solutions to possible problems. If we should encounter one of these problems, we can pick the solution from our collection.

The simplest of such problems is shown in Fig. 2.22. The challenge is to find the field of a point charge q at the distance z_q from a grounded plane conductor. Looking at Fig. 2.21, we see that one of the equipotential surfaces, having null potential, is the middle plane between the two charges. We then place a plane conductor at zero potential (grounded) on that plane. The field in the semispace on

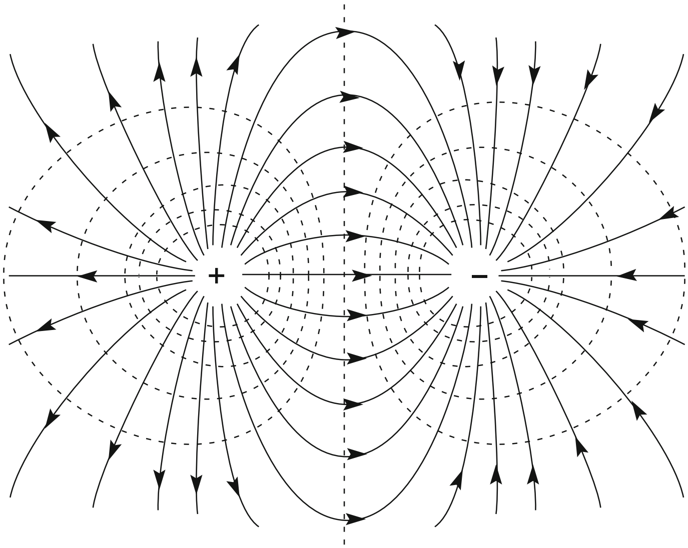
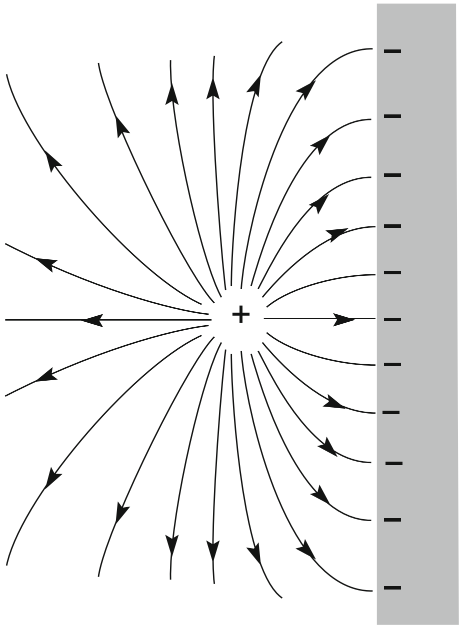


Fig. 2.21 Electric field lines and equipotential surfaces of two equal and opposite charges

Fig. 2.22 Electric field lines for a positive charge and a conductive plane



the left of this conductor is equal to the field that there would be without the plane, but with a charge $-q$ at $-z_q$. This is the specular distance relative to the real charge q and is called an image charge of q. As a matter of fact, the real charge sources of the field are q and the negative charges that move inside the conductor to its surface and arrange themselves to make the field inside null. As we know the field at every point on the surface, we also know the charge density σ, which is the field magnitude divided by ε_0.

Let us take a reference frame with the origin O at the point of the conductor plane below the charge, the z-axis normal to the surface through q, and the x and y axes on the surface. The electric field on the surface has the direction equal and opposite to the z-axis. Let $\rho = \sqrt{x^2 + y^2}$ be the distance from O of a generic point (x, y) of the surface. The field at that point due to q and its image is

$$E_z = -\frac{1}{4\pi\varepsilon_0} \frac{2z_q q}{\left(z_q^2 + \rho^2\right)^{3/2}}.$$

The charge density is then

$$\sigma = -\frac{1}{2\pi} \frac{z_q q}{\left(z_q^2 + \rho^2\right)^{3/2}}. \tag{2.26}$$

We leave as an exercise the verification that the integral of the surface density on the plane is just $-q$.

We finally observe that a charge q facing a conductor is acted upon by an attractive force due to the negative charges it induces on the surface of the conductor. In the case just discussed, the force can be evaluated as the attractive force of the image charge, namely

$$F_z = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{\left(2z_q\right)^2}$$

In similar, more complicated cases, image charges are always present, but in general, their values and their distances from the surface are not equal and opposite to those of the real charge. These quantities must evaluated case by case.

QUESTION Q 2.3. A nucleus of Fe ($Z = 26$) is at rest at 1 μm from a plane grounded conductor. Is there any force on the nucleus? If yes, what is its value? □

Summary

In this chapter, we have learned the following principal concepts:

1. The concept of the electric conductor.
2. The properties of a conductor under static conditions; electric field and charge density inside the conductor and on its surface.

3. The force on the surface of a charged conductor.
4. The properties of hollow conductors; electrostatic shielding.
5. How to calculate the potential, and the field, in a space region between conductors at fixed potentials, when the charges that produce the field are not given.
6. That no stable equilibrium for a charge exists in an electrostatic field.
7. The concept of electrostatic capacitance.
8. Capacitances and their different arrangements in electronic circuits.
9. The method of images.

Problems

2.1. How does the electrostatic field vary doubling the distance from its sources if they are: (a) a point charge, (b) a linear uniform charge distribution, (c) a planar uniform charge distribution?

2.2. Two metal spheres of radiuses R_1 and R_2 at a distance much larger than the radiuses are connected by a conducting wire. The system is charged with the charge Q. Find the charges Q_1 and Q_2 on the two spheres.

2.3. The capacitance of a metallic conductor depends or does not depend on (choose): the metal, the shape, the temperature, the presence and the position of other conductors.

2.4. Does any charge exist inside a conductor? Which is the charge density in a conductor under static conditions?

2.5. You are inside a Faraday cage that is on insulating supports. You know that the cage is connected outside to a constant voltage generator of 100 kV, but you do not know if the generator is on or off. Would you touch the wall of the cage?

2.6. Fig. 2.23 is a cartoon showing hypothetic lines of an electric field in the presence of three conductors, A having a positive charge, B having a negative charge and C having no charge. The figure has 7 different mistakes. Find them and explain.

2.7. The capacitance of a cylindrical capacitance depends or does not depend on (choose): the radiuses of its surfaces, their heights, the metal of which it is composed, its charge, the presence of other conductors.

2.8. Four spherical drops of water are connected to the positive pole of a battery having the potential ϕ and then disconnected. They then merge into a single drop. What is its potential?

2.9. Four plane square conductors form four faces of a cube and have potentials as in Fig. 2.24. Draw the electric field lines.

2.10. Two plane-conducting surfaces are charged with equal surface density. How does the repulsive force between them vary with their distance?

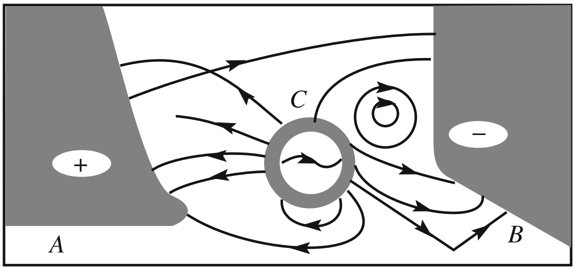


Fig. 2.23 Conductors and field lines

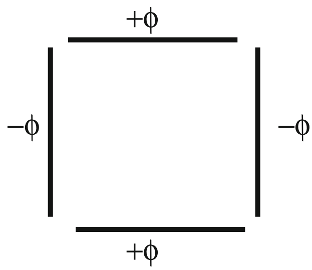


Fig. 2.24 Four plane conductors and their potentials

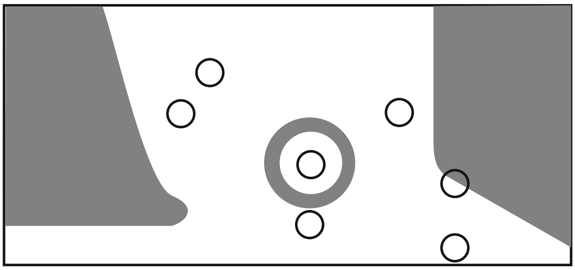


Fig. 2.25 Solution to Problem 2.7

2.11. The point charge Q = 50 nC is at rest at a distance of 1 mm from a grounded plane metal sheet. There are no other materials. What is the force on the charge? What is the charge density on the sheet as a function of the distance of the foot of the normal to the sheet from the charge? What is the induced charge?

2.12. We have two capacitances C_1 = 10 μF and C = 35 μF. We connect them in series and apply to the series the potential difference $\Delta\phi$ = 100 V. What are the potential differences at each capacitor? (Fig. 2.25).

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