The Degree of Irreversibility in Deterministic Finite Automata

Holger Bock Axelsen[1], Markus Holzer[2(✉)], and Martin Kutrib[2]

[1] Department of Computer Science, University of Copenhagen, Universitetsparken 5, Copenhagen, Denmark
funkstar@di.ku.dk

[2] Institut für Informatik, Universität Giessen, Arndtstr. 2, 35392 Giessen, Germany
{holzer,kutrib}@informatik.uni-giessen.de

Abstract. Recently, Holzer *et al.* gave a method to decide whether the language accepted by a given deterministic finite automaton (DFA) can also be accepted by some *reversible* deterministic finite automaton (REV-DFA), and eventually proved NL-completeness. Here, we show that the corresponding problem for *nondeterministic* finite state automata (NFA) is PSPACE-complete. The recent DFA method essentially works by minimizing the DFA and inspecting it for a *forbidden pattern*. We here study the *degree of irreversibility* for a regular language, the minimal number of such forbidden patterns necessary in *any* DFA accepting the language, and show that the degree induces a strict infinite hierarchy of languages. We examine how the degree of irreversibility behaves under the usual language operations union, intersection, complement, concatenation, and Kleene star, showing tight bounds (some asymptotically) on the degree.

1 Introduction

In computation theory, reversibility is the property that computations are both forward and backward deterministic. In the context of finite state models, reversibility can usually be verified by simple inspection of the transition function, ensuring that the induced computation step relation is an injective function on configurations. Despite the apparent simplicity of reversible computations, reversibility is an interesting property that has been studied in a wide array of contexts, including the thermodynamics of computation [7], across a wide array of automata models [9], and even in robotics [10].

It is well-known that the reversibly regular languages, i.e., the languages accepted by reversible deterministic finite automata (REV-DFA), form a strict subclass of the regular languages, see, e.g., [6]. However, the exact cost of reversibility is still not well-understood: for example, changing from one-way

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to two-way tapes is sufficient to collapse the classes [5]. Likewise, adding a reversible transducer in front of the REV-DFA also collapses to the regular languages [1]. This motivates further study into the relationship between the regular and reversibly regular languages, and in particular towards developing methods to understand and bridge the gap in terms of the internal structure of deterministic finite automata (DFA). In this paper, we take steps in this direction.

Recently, Holzer *et al.* showed a method for deciding if the language accepted by a given DFA can also be recognized by some REV-DFA [4]. It was also shown that this is an NL-complete problem, and a decision method was given, which essentially works by minimizing the DFA and inspecting it for the presence of a *forbidden pattern*. If this pattern is present in the minimal DFA, then there is *no* REV-DFA that can accept the same language, and if not, then there *is*. What makes this particularly interesting is that the pattern is structurally more complex than the simplest violation of reversibility (see Sect. 2 for details). This suggests that the forbidden pattern captures an essential aspect of irreversibility, and offers an approach to studying the gap between the reversibly regular and regular languages based on the absence, presence, and count of occurrences, of this pattern.

Our contributions are as follows. We show that the generalization of the problem studied in [4] to *nondeterministic* finite automata (NFA), i.e., the *regular reversibility problem* of whether the language accepted by a given NFA is reversibly regular, is PSPACE-complete. Turning to DFAs, we introduce the notion of *degree of irreversibility* for DFAs, essentially the number of occurrences of the forbidden pattern in a given DFA, and extend this to (regular) languages by minimizing over all DFAs accepting the language. Finally, we show that the degree of irreversibility induces a strict, infinite hierarchy of languages. We then proceed to show exact bounds on the degree of irreversibility under the common language operations union, intersection, and complement, and asymptotically tight bounds for concatenation and Kleene star.

The paper is organized as follows. Section 2 covers the necessary preliminaries. In Sect. 3 we show that the regular reversibility problem is PSPACE-complete. Section 4 defines the degree of irreversibility, and shows the related hierarchy. We present the degree complexity results for common language operations in Sect. 5. Most proofs are omitted due to space constraints, and will be given in the full version of the paper.

2 Preliminaries

An *alphabet* Σ is a non-empty finite set, its elements are called *letters* or *symbols*. We write Σ^* for the *set of all words* over the finite alphabet Σ.

We recall some definitions on finite automata as contained, for example, in [3]. A *deterministic finite automaton* (DFA) is a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$, where Q is the finite set of *internal states*, Σ is the alphabet of *input symbols*, $q_0 \in S$ is the *initial state*, $F \subseteq Q$ is the set of *accepting states*, and $\delta \colon Q \times \Sigma \to Q$ is the partial *transition function*. Note that here the transition function is not required to be *total*. The *language accepted* by A is $L(A) = \{ w \in \Sigma^* \mid \delta(q_0, w) \in F \}$, where the transition function is recursively extended to $\delta \colon Q \times \Sigma^* \to Q$. By $\delta^R \colon Q \times \Sigma \to 2^Q$, with $\delta^R(q, a) = \{ p \in S \mid \delta(p, a) = q \}$, we denote the *reverse* transition function of δ. Similarly, also δ^R can be extended to words instead of symbols. Two devices A and A' are said to be *equivalent* if they accept the same language, that is, $L(A) = L(A')$.

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting the language L. The set of words $R_{A,q} = \{ w \in \Sigma^* \mid \delta(q, w) \in F \}$ refers to the *right language* of the state q in A. In case $R_{A,p} = R_{A,q}$, for some states $p, q \in Q$, we say that p and q are *equivalent* and write $p \equiv_A q$. The equivalence relation $\equiv_A$ partitions the state set Q of A into equivalence classes, and we denote the equivalence class of $q \in S$ by $[q] = \{ p \in S \mid p \equiv_A q \}$. Equivalence can also be defined between states of different automata: a state p of DFA A and a state q of DFA A' are *equivalent*, denoted by $p \equiv q$, if $R_{A,p} = R_{A',q}$.

A state $p \in Q$ is *accessible* in A if there is a word $w \in \Sigma^*$ such that $\delta(q_0, w) = p$, and it is *productive* if there is a word $w \in \Sigma^*$ such that $\delta(p, w) \in F$. If p is both accessible and productive then we say that p is *useful*. In this paper we only consider automata with all states useful. Let A and A' be two equivalent DFAs. Observe that if p is a useful state in A, then there exists a useful state p' in A', with $p \equiv p'$. A DFA is *minimal* (among all DFAs) if there does not exist an equivalent DFA with fewer states. It is well known that a DFA is minimal if and only if all its states are useful and inequivalent.

Next we define reversible DFAs. Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA. A state $r \in Q$ is said to be *irreversible* if there are two distinct states p and q in Q and a letter $a \in \Sigma$ such that $\delta(p, a) = r = \delta(q, a)$. Then a DFA is *reversible* if it does not contain any irreversible state. In this case the automaton is said to be a *reversible* DFA (REV-DFA). Equivalently the DFA A is reversible, if every letter $a \in \Sigma$ induces an *injective partial mapping* from Q to itself *via* the mapping $\delta_a \colon Q \to Q$ with $p \mapsto \delta(p, a)$. In this case, the reverse transition function δ^R can be seen as a (partial) injective function $\delta^R \colon Q \times \Sigma \to Q$. Notice that if p and q are two distinct states in a REV-DFA, then $\delta(p, w) \neq \delta(q, w)$, for all words $w \in \Sigma^*$. Finally, a REV-DFA is *minimal* (among all REV-DFAs) if there is no equivalent REV-DFA with a smaller number of states.

In [4] the following structural characterization of regular languages that can be accepted by REV-DFAs in terms of their minimal DFAs is given. The conditions of the characterization are illustrated in Fig. 1.

Theorem 1. *Let $A = (Q, \Sigma, \delta, q_0, F)$ be a* minimal *deterministic finite automaton. The language $L(A)$ can be accepted by a reversible deterministic finite automaton if and only if there do not exist useful states $p, q \in Q$, a letter $a \in \Sigma$, and a word $w \in \Sigma^*$ such that $p \neq q$, $\delta(p, a) = \delta(q, a)$, and $\delta(q, aw) = q$.*

Finally we need some notations on computational complexity theory. We classify problems on DFAs with respect to their computational complexity. Consider the complexity class NL (PSPACE, respectively) which refers to the set

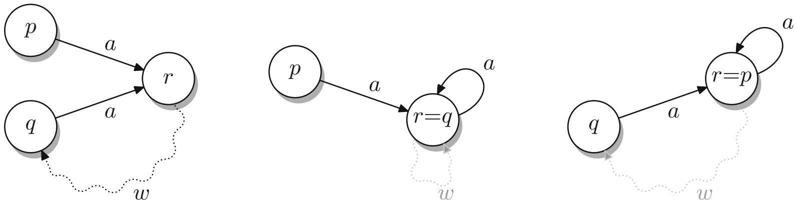

Fig. 1. The "forbidden pattern" of Theorem 1: the language accepted by a minimal DFA A can be accepted by a REV-DFA if and only if A does not contain the structure depicted on the left. Here the states p and q must be distinct, but state r could be equal to state p or state q. The situations where $r = q$ or $r = p$ are shown in the middle and on the right, respectively—here the word w and its corresponding path are grayed out because they are not relevant: in the middle, the word w that leads from r to q is not relevant since it can be identified with the a-loop on state $r = q$. Also on the right hand side, word w is not important because we can simply interchange the roles of the states q and $r = p$.

of problems accepted by nondeterministic logspace bounded (polynomial space, respectively) Turing machines. Further, hardness and completeness is always meant with respect to deterministic logspace bounded reducibility, unless otherwise stated.

3 Complexity of the Regular Reversibility Problem

In [4] it was shown that the regular language reversibility problem—given a DFA A, decide whether $L(A)$ is accepted by any REV-DFA—is NL-complete. If the regular language is given by an NFA or a regular expression, the problem becomes intractable.

Theorem 2. *The regular language reversibility problem is* PSPACE*-complete, if the language is given as a nondeterministic finite automaton or a regular expression.*

Before we can prove this result we need a technical lemma, which will be used in the PSPACE-hardness argument later.

Lemma 3. *Let $A = (Q, \Sigma, \delta, q_0, F)$ be a minimal* DFA*. If there is a state $q \in Q$, other than the initial state, such that $R_{A,q} = \Sigma^*$, then $L(A)$ is irreversible.* □

Let $L \subseteq \Sigma^*$. Then the *left derivative* of L with respect to the letter a in Σ is the set $a^{-1} \cdot L = \{ w \mid aw \in L \}$. This notation generalizes to words. By this definition, there is an obvious relation between these left derivative set and the states of the minimal finite automaton A accepting L. To be more precise, the set $u^{-1} \cdot L$, for $u \in \Sigma^*$, is a description of the state $q_u = \delta(q_0, u)$, where $A = (Q, \Sigma, \delta, q_0, F)$, and *vice versa*. Now we are ready to proof Theorem 2 in a convenient way.

Proof (of Theorem 2). The containment within PSPACE is easily seen. For the hardness we reduce the PSPACE-complete universality problem for regular expressions [8] to the reversibility problem for NFAs or regular expressions. Let the regular expression r be an instance of the universality problem. We may assume that r is an expression over the alphabet $\Sigma = \{a, b\}$. Then we construct the expression

$$s = a \cdot r + b \cdot \Sigma^* + \lambda$$

or equivalently the NFA depicted in Fig. 2 in deterministic logspace. Now assume that $L(r) = \Sigma^*$. Then it is easy to see that $L(s) = \Sigma^*$, too, and therefore a reversible language. On the other hand, if $L(r) \neq \Sigma^*$, then there is a word $u \notin L(r)$. From this it follows that $au \notin L(s)$ but $bu \in L(s)$. Thus we conclude that the states $a^{-1} \cdot L(s)$ and $b^{-1} \cdot L(s)$ are not equivalent in the DFA accepting $L(s)$. Moreover, in that DFA states $L(s)$ and $b^{-1} \cdot L(s)$ are not equivalent, too. Note that the former state is the initial state of the DFA that accepts $L(s)$. Since the right language of the state $b^{-1} \cdot L(s)$ is equal to Σ^* and it is not equal to the initial state, Lemma 3 applies, and the language $L(s)$ is *not* reversible. This proves PSPACE-hardness. $\square$

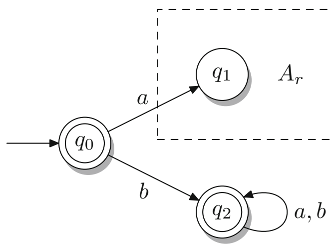

Fig. 2. Finite automaton that accepts the language $L(s)$. It is built from the regular expression r, where A_r is an NFA with initial state q_1 that accepts the language $L(r)$.

4 On the Degree of Irreversibility

For an automaton A we define its *degree of irreversibility* $d(A)$ as the number of irreversible states that are part of one of the forbidden patterns shown in Fig. 1. Observe, that since our DFAs need not to be complete and only contain useful states, the non-accepting sink state does not count for the degree of irreversibility. This notation is generalized to languages in the usual way. This means, for a regular language $L \subseteq \Sigma^*$ we define its *degree of irreversibility* $d(L)$ as the minimum degree of irreversibility among all equivalent DFAs A, that is,

$$d(L) = \min\{\, d(A) \mid A \text{ is a DFA with } L(A) = L \,\}.$$

The next example explains our notation.

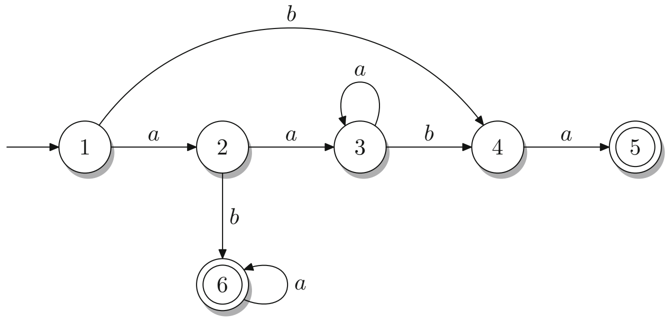

Fig. 3. DFA which accepts $aba^* + a^*ba$ that has irreversibility degree one.

Example 4. Consider the following DFA depicted in Fig. 3, which accepts the union of aba^* and a^*ba. This automaton has irreversibility degree one by state 3. Note that although state 4 has two ingoing b-transitions, this state does not yield a forbidden pattern as shown in Fig. 1. There is no word that leads from state 4 to either state 1 or 3. Moreover, the language accepted by this automaton, which is $aba^* + a^*ba$ is also of irreversibility degree one, since it is *not* reversible by Theorem 1. $\square$

Next we consider the hierarchy on regular languages that is induced by the irreversibility degree. To this end let

$$\text{IREV}_k\text{-DFA} = \{\, A \mid A \text{ is a DFA and } d(A) \leq k \,\}.$$

We have $\text{IREV}_0\text{-DFA} = \{\, A \mid A \text{ is a reversible DFA} \,\}$ and thus the equality $\mathscr{L}(\text{IREV}_0\text{-DFA}) = \mathscr{L}(\text{REV-DFA})$ holds, where the family of all languages accepted by an automaton of some type X is denoted by $\mathscr{L}(X)$. Moreover, by definition the inclusion $\text{IREV}_k\text{-DFA} \subseteq \text{IREV}_{k+1}\text{-DFA}$ follows and therefore the corresponding language classes satisfy $\mathscr{L}(\text{IREV}_k\text{-DFA}) \subseteq \mathscr{L}(\text{IREV}_{k+1}\text{-DFA})$, for $k \geq 0$. By the example above we have

$$\mathscr{L}(\text{REV-DFA}) = \mathscr{L}(\text{IREV}_0\text{-DFA}) \subset \mathscr{L}(\text{IREV}_1\text{-DFA}).$$

Before we show that the degree of irreversibility induces an infinite strict hierarchy we need some tool that allows us to determine the irreversibility degree for an arbitrary regular language. Since for the degree of irreversibility of a language L we quantify over all equivalent DFAs we have to show that we cannot trade more states for less irreversibility. The following example shows that this is in fact not the case in general.

Example 5. Consider the substructure of a DFA as depicted in Fig. 4. It is not hard to see that this pattern may appear in a *minimal* DFA. Both states r_1 and r_2 in the substructure are irreversible. By splitting both of these states, we obtain a connecting structure as shown in Fig. 5. The structure has *one* irreversible state only. Thus, the irreversibility degree of a *minimal* DFA is not necessarily the irreversibility degree of the language under consideration. $\square$

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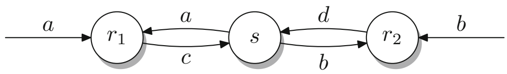

Fig. 4. Substructure of a DFA containing two irreversible states r_1 and r_2.

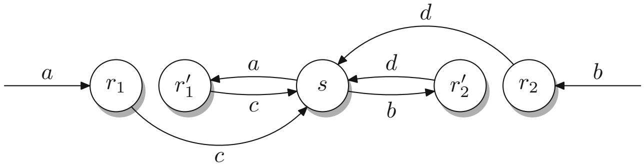

Fig. 5. Substructure of a DFA with just one irreversible state s obtained after splitting both irreversible states.

For a special class of finite automata, we can show that the minimal DFA already gives the degree of irreversibility. A DFA is *simply-irreversible* if all irreversible states are of the form depicted in the middle and right drawing shown in Fig. 1. That is, the irreversibility state is entered by an a-transition and has an a-self-loop, which is the simplest form of irreversibility. For the languages accepted by these automata we can prove the next result.

Theorem 6. *Let L be a regular language and A be its minimal deterministic finite automaton. If A is simply-irreversible, then the degree of irreversibility of A is equal to the irreversibility degree for L. That is $d(L) = d(A)$.* □

Now we are ready to show that the strict hierarchy on regular languages induced by the irreversibility degree is tight and infinite.

Theorem 7. *For all $k \geq 0$, $\mathscr{L}(IREV_k\text{-}DFA) \subset \mathscr{L}(IREV_{k+1}\text{-}DFA)$.*

Proof. Consider the languages L_k over the alphabet $\{a, b\}$ defined as follows: for $k \geq 0$ set

$$L_{2k} = (aa^*bb^*)^k \quad \text{and} \quad L_{2k+1} = (aa^*bb^*)^k aa^*.$$

The language L_k, for $k \geq 0$, is accepted by the DFA $A_k = (Q_k, \{a, b\}, \delta_k, q_0, F_k)$ with $Q_k = \{1, 2, \ldots, k+1\}$, $q_0 = 1$, $F_k = \{k+1\}$, and

$$\delta(i, a) = \begin{cases} i+1 & \text{if } i \text{ is odd and } 1 \leq i < k+1 \\ i & \text{if } i \text{ is even and } 1 < i \leq k+1 \end{cases}$$

and

$$\delta(i, b) = \begin{cases} i+1 & \text{if } i \text{ is even and } 1 \leq i < k+1 \\ i & \text{if } i \text{ is odd and } 1 < i \leq k+1. \end{cases}$$

By construction the DFA A_k is minimal and simply-irreversible. Thus, by the previous theorem the degree of irreversibility of A_k is equal to the irreversibility degree of the language L_k. Since A_k contains exactly k irreversible states, we have $d(L_k) = k$. This shows that $L_k \in \mathscr{L}(\mathrm{IREV}_k\text{-DFA}) \setminus \mathscr{L}(\mathrm{IREV}_{k-1}\text{-DFA})$, for $k \geq 1$. □

Finally, we consider unary regular languages and their irreversibility degree. It is not difficult to see that a unary *complete* DFA consists of a path, which starts from the initial state, followed by a cycle of one or more states. Thus the irreversibility degree of any unary DFA is at most one. Thus, the hierarchy on the irreversibility degree collapses to its second level and $\mathscr{L}(\mathrm{IREV}_1\text{-DFA}) \cap 2^{\{a\}^*}$ is already equal to the class of all unary regular languages. Moreover, we conclude that $\mathscr{L}(\mathrm{IREV}_0\text{-DFA}) \cap 2^{\{a\}^*}$ is the class of languages that contains only finite or cyclic unary regular languages. Here a unary regular language is *cyclic* if it is accepted by a unary DFA which is a cycle of one or more states.

5 Operations on Languages and Degree of Irreversibility

In this section we study the descriptional complexity of the operation problem for reversible languages. We start with the Boolean operations and continue with the concatenation and Kleene star operation.

First we consider the union operation. For the union of two reversible language, the increase of the degree of irreversibility is linear in the sum of the number of states of the involved automata. This can be seen in the next theorem.

Theorem 8. *Let $m, n \geq 1$ be two integers, A be an m-state and B be an n-state reversible deterministic finite automaton. Then the degree $m+n$ of irreversibility for the language $L(A) \cup L(B)$ is sufficient and necessary in the worst case.*

Proof. Let $A = (Q_A, \Sigma, \delta_A, q_{0,A}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{0,B}, F_B)$. In order to accept the union of $L(A)$ and $L(B)$ we apply the standard cross-product construction. To this end define $C = (Q_C, \Sigma, \delta_C, q_{0,C}, F_C)$, where

$$Q_C = (Q_A \times Q_B) \cup (Q_A \times \{-\}) \cup (\{-\} \times Q_B),$$

$q_{0,C} = (q_{0,A}, Q_{0,B})$, and $F_C = (Q_A \times F_B) \cup (F_A \times Q_B) \cup (F_A \times \{-\}) \cup (\{-\} \times F_B)$. The transition function δ_C is set to

$$\delta_C((p,q),a) = \begin{cases} (\delta_A(p,a), \delta_B(q,a)) & \text{if both } \delta_A(p,a) \text{ and } \delta_B(q,a) \text{ are defined} \\ (\delta_A(p,a), -) & \text{if } \delta_A(p,a) \text{ is defined and } \delta_B(q,a) \text{ is undefined} \\ (-, \delta_B(q,a)) & \text{if } \delta_A(p,a) \text{ is undefined and } \delta_B(q,a) \text{ is defined} \end{cases}$$

and furthermore $\delta_C((p,-),a) = (\delta_A(p,a), -)$, if $\delta_A(p,a)$ is defined, as well as $\delta_C((-,q),a) = (-, \delta_B(q,a))$, if $\delta_B(q,a)$ is defined, for $a \in \Sigma$. So we have

$L(C) = L(A) \cup L(B)$. From the $m \cdot n + m + n$ states of C at most $m + n$ are irreversible. To be more precise, none of the states from $Q_A \times Q_B$ are irreversible. This is seen as follows: consider a state $(r, r') \in Q_A \times Q_B$. Assume to the contrary that (r, r') is irreversible. Then there are different states (p, p') and (q, q') with $\delta_C((p, p'), a) = (r, r') = \delta_C((q, q'), a)$, for some $a \in \Sigma$. Since (p, p') is not equal to (q, q') we have $p \neq q$ or $p' \neq q'$. We only consider the case $p \neq q$ by symmetric reasons. But then we find that r is an irreversible state, because $\delta_A(p, a) = r = \delta_A(q, a)$, for the letter a from above. This is a contradiction, because automaton A is a reversible DFA. It is worth mentioning that a similar argumentation does not apply to states of the form $(r, -)$ or $(-, r)$. This is seen by the counterexample $\delta_C((r, p), a) = (r, -) = \delta_C((r, -), a)$, for some $a \in \Sigma$, which induces only $\delta_A(r, a) = r$ and $\delta_B(p, a)$ is undefined—an analogous example can be given for state of the form $(-, r)$. Hence, this does not contradict the irreversibility of either A or B.

It remains to be shown that the bound $m + n$ is tight. Define the reversible DFA $A = (Q_A, \{a, b\}, \delta_A, q_0, F)$ with $Q_A = \{1, 2, \ldots, m\}$, $q_0 = 1$, $F = \{m\}$, and the transition function is given by $\delta(i, a) = i + 1$, for $1 \leq i < m$, and $\delta(i, b) = i$, for $1 \leq i \leq m$. The automaton B is the same as A, but with n states, and where the letters a and b are interchanged. The automaton C constructed above is easily seen to be minimal.

Finally we show that all states of the form $(i, -)$ and $(-, j)$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, are irreversible and yield a forbidden pattern as shown in Fig. 1. The below given argument shows even more, namely that the automaton C is simply-irreversible. We have already argued that the state (i, n) is accessible. Then it is easy to see that from state (i, n) reading a b the automaton C enters state $(i, -)$, which has a b loop. Therefore state $(i, -)$ is simply-irreversible. A similar argument shows that state $(-, j)$ is simply-irreversible as well. By Theorem 6 the stated claim follows. $\square$

A careful inspection of the previous proof reveals that we can use parts of it for the intersection of two reversible languages. For two reversible automata A and B we construct an automaton C by the cross-product construction described in the proof of Theorem 8 but only using states of the form $Q_A \times Q_B$ and by altering the set of accepting states to be $F = F_A \times F_B$. Then $L(C) = L(A) \cap L(B)$. It was shown that none of the states from $Q_A \times Q_B$ are irreversible. Hence C does not contain any irreversible state. Thus, we have shown the following result.

Theorem 9. *Let $m, n \geq 1$ be two integers, A be an m-state and B be an n-state reversible deterministic finite automaton. Then the language $L(A) \cap L(B)$ is accepted by a reversible deterministic finite automaton.* $\square$

Next we deal with the complementation operation, and show that the degree of irreversibility can be increased by one.

Theorem 10. *Let $n \geq 1$ be an integers and A be an n-state reversible deterministic finite automaton. Then the degree 1 of irreversibility for the complement of $L(A)$ is sufficient and necessary in the worst case.* $\square$

In the remainder of this section we investigate the effect of the concatenation and the Kleene star operation on the degree of irreversibility. First we recall the construction of DFAs for the concatenation [11]. Let $A = (Q_A, \Sigma, \delta_A, q_{0,A}, F_A)$ and $B = (Q_B, \Sigma, \delta_B, q_{0,B}, F_B)$ be two DFAs. As in [11] we construct the DFA $C = (Q_C, \Sigma, \delta_C, q_{0,C}, F_C)$, where

$$Q_C = (Q_A \times 2^{Q_B}) \setminus (F_A \times 2^{Q_B \setminus \{q_{0,B}\}}),$$

the initial state is

$$q_{0,C} = \begin{cases} (q_{0,A}, \emptyset) & \text{if } q_{0,A} \notin F_A \\ (q_{0,A}, \{q_{0,B}\}) & \text{otherwise,} \end{cases}$$

the final states are

$$F_C = \{\, (p, P) \mid (p, P) \in Q_C \text{ and } P \cap F_B \neq \emptyset \,\},$$

and the transition function is defined by $\delta_C((p, P), a) = (q, Q)$, for $a \in \Sigma$, where $q = \delta_A(p, a)$ and

$$Q = \begin{cases} \delta_B(P, a) \cup \{q_{0,B}\} & \text{if } q \in F_A \\ \delta_B(P, a) & \text{otherwise.} \end{cases}$$

Clearly, automaton C accepts $L(A) \cdot L(B)$ and has at most $m \cdot 2^n - 2^{n-1}$ states. Thus, the construction gives rise to an exponential upper bound on the number of irreversible states.

Theorem 11. *Let $m, n \geq 2$ be two integers, A be an m-state and B be an n-state reversible deterministic finite automaton. Then the degree $m \cdot 2^n - 2^{n-1}$ of irreversibility is sufficient for a deterministic finite automaton to accept the language $L(A) \cdot L(B)$.* □

The next theorem gives an exponential lower bound on the degree of irreversibility for the concatenation operation.

Theorem 12. *Let $m, n \geq 2$ be two integers. There are a reversible m-state deterministic finite automaton A and a reversible n-state deterministic finite automaton B such that any deterministic finite automaton accepting $L(A) \cdot L(B)$ has at least the degree $(3m - 2) \cdot 2^{n-2}$ of irreversibility.*

Proof. Define the left automaton to be $A = (Q_A, \{a, b, c, d\}, \delta_A, q_{0,A}, F_A)$ with $Q_A = \{0, 1, \ldots, m-1\}$, initial state $q_{0,A} = 0$, final states $F_A = \{m-1\}$, and the transition function

$$\delta_A(i, a) = \begin{cases} i + 1 & \text{if } 0 \leq i < m - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_A(i, b) = \delta_A(i, c) = \delta_A(i, d) = i$$

for $0 \leq i \leq m - 1$. The right automaton is $B = (Q_B, \{a, b, c, d\}, \delta_B, q_{0,B}, F_B)$ with $Q_B = \{0, 1, \ldots, n-1\}$, initial state $q_{0,B} = 0$, final states $F_A = \{0\}$, and the transition function

$$\delta_B(i,a) = i, \quad \text{for } 0 \le i \le n-1, \text{ and } \quad \delta_B(i,b) = \begin{cases} i+1 & \text{if } 0 \le i < n-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_B(i,c) = i, \quad \text{for } 0 < i \le n-1, \text{ and } \quad \delta_B(i,d) = i, \quad \text{for } i = 0 \text{ or } 2 \le i \le n-1.$$

Both reversible automata are depicted in Fig. 6.

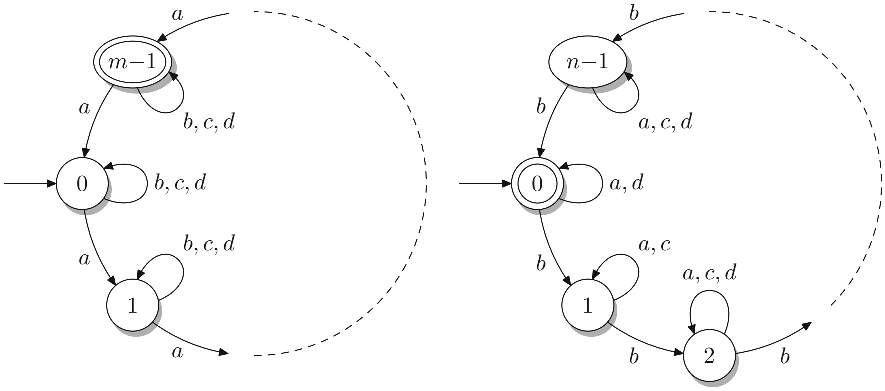

Fig. 6. The reversible automata A (left) and B (right) with m and n states, respectively, that witness the irreversibility degree lower bound for the concatenation operation.

We construct the DFA C for the concatenation $L(A) \cdot L(B)$ as described above. In order to apply Theorem 6 we need to show that C is minimal. Thus, one has to verify that every state in C is useful and defines a distinct equivalence class.

Finally, it remains to determine the lower bound on the irreversibility degree of C. We show that all states of C whose second component does not contain 0 and 1 at the same time are simply-irreversible. We have already seen that all states of the form $(p, P \cup \{0\})$ and $(p, P \cup \{1\})$ are reachable in C. We distinguish two cases:

1. Assume $p = m-1$. Then $0 \in P$, but then by assumption $1 \notin P$. We have $\delta_C((p, P \cup \{1\}), d) = (p, P)$ and $\delta_C((p, P), d) = (p, P)$. Thus (p, P) is simply-irreversible.
2. Let $p = i$ with $0 \le i < m-1$. If $0 \notin P$, then $\delta_C((p, P \cup \{0\}), c) = (p, P)$ and $\delta_C((p, P), c) = (p, P)$. Also in the case $1 \notin P$, the two transitions $\delta_C((p, P \cup \{1\}), d) = (p, P)$ and $\delta_C((p, P), d) = (p, P)$ follow. In both cases the state (p, P) is simply-irreversible.

Next we count the number of simply-irreversible states. The first item above induces 2^{n-2} possibilities, and the second item $3(m-1) \cdot 2^{n-2}$. There are $(m-1)$ choices for p and the number of different sets P that do not contain 0 or 1 is $3 \cdot 2^{n-2}$. For each of the cases (i) both 0 and 1 are not in P, (ii) element 0 is in P but 1 is not, and (iii) element 0 is not in P but 1 is member of P, there are 2^{n-2} possibilities. This results in $3(m-1)\cdot 2^{n-2}$ sets for the second item above. Putting things together results in in at least $(3m-2)\cdot 2^{n-2}$ simply-irreversible states in C. By Theorem 6 the stated claim follows. $\square$

Finally, we consider the Kleene star operation. From [11] the tight worst case bound for a DFA to accept the Kleene closure of an n-state DFA language is $2^{n-1}+2^{n-2}$. Thus, the upper bound for the irreversibility degree for the Kleene closure is exponential.

Theorem 13. *Let $n \geq 2$ be an integers and A be an n-state reversible deterministic finite automaton. Then the degree $2^{n-1}+2^{n-2}$ of irreversibility is sufficient for a deterministic finite automaton to accept the language $L(A)^*$.* $\square$

As in the case of the concatenation operation we can provide an exponential lower bound.

Theorem 14. *Let $n \geq 3$ be an integer. There is a reversible n-state deterministic finite automaton A such that any deterministic finite automaton accepting $L(A)^*$ has at least the degree $3\cdot 2^{n-3}-1$ of irreversibility.* $\square$

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