

Chapter 2

Relativistic Kinetic Theory of Matter and Radiation

The relativistic kinetic theory of irreversible processes in a system of matter presented in Chap. 1 of this volume is generalized to include radiation in this chapter.¹ The generalization enables us to remove the awkwardness inherent to the nonrelativistic theory of radiation and matter reported in the paper [1] by Eu and Mao and provides kinetic theory foundations for the relativistic irreversible thermodynamics and radiation hydrodynamics for a system of radiation and matter. In this chapter, the relativistic Boltzmann equations—more precisely, a relativistic theory (covariant) form of the Boltzmann–Nordholm–Uehling–Uhlenbeck equations [2–4]—is employed to formulate a theory of transport processes in a system of radiation interacting with matter in a manner consistent with the laws of thermodynamics.

The motivations to have a covariant kinetic theory are following: Since photons are inherently relativistic, the relativity principle requires that governing equations of the system must be Lorentz covariant. In Ref. [1] the nonrelativistic kinetic theory was used to study irreversible processes in a system of radiation interacting with matter and, in particular, to study Doppler shift corrections for photon frequencies, but the nonrelativistic kinetic theory treatment of the subject matter was found often cumbersome and awkward. It was also difficult to make sure that the definitions of statistical mechanical formulas for macroscopic variables and the evolution equations indeed had correct nonrelativistic limits. These weaknesses can be assuredly removed if a covariant theory is formulated. Secondly, there are some problems, especially, in the study of the early epoch of the universe and also in the nuclear physics of high energy heavy ion collisions, for which a relativistic formalism is required since post-collision particles move at high speed. It will be shown that the present covariant kinetic theory recovers all the nonrelativistic evolution equations

¹This chapter is a substantially revised version of the unpublished paper by the author with K. Mao, which was also part of the PhD thesis of K. Mao, McGill University, Montreal, 1993, under supervision of B.C. Eu. The revision is concerned with the theory of irreversible thermodynamics making use of the calortropy and the evolution equations for nonconserved variables. There are other aspects which are significantly revised.

in correct forms with proper relativistic connections and thus verifies the previous nonrelativistic formulation [1]. In any case, in the present chapter we formulate a relativistic kinetic theory for a monatomic gas mixture interacting with photons treated relativistically and quantum mechanically. In the kinetic equations for molecular particles and photons, the collision terms are so modified as to make them more suitable for quantum particles. Since the collision terms in the kinetic equations used in the present chapter are quantum mechanical, they naturally affect the dissipation terms of nonconserved variables in the evolution equations, but we are, nevertheless, able to make use of the approach taken in Chap. 1 of this Volume except for some details related to the quantum collision terms. As a matter of fact, the affected parts of the theory turn out to be all related to transport coefficients originating from the collision terms in the kinetic equations. Therefore, most of evolution equations involved in the theory remain the same as those in Chap. 1 of this Volume except for the dissipation terms and quantities related to them. For this reason we will dispense with the details of derivations of evolution equations, wherever possible without sacrificing the comprehension, but will present the results only. The distribution functions used in the present chapter should be regarded as the Wigner representations of the density matrices. For this part of discussion on the kinetic equation and distribution functions employed for quantum particles, see Chap. 9, Sects. 9.1–9.3, of Ref. [5], where a nonrelativistic theory is used to discuss about Wigner distribution functions.

First, covariant Boltzmann equations are briefly presented for a system consisting of photons and material particles with internal degrees of freedom. The material gas molecules can make transitions between various internal states in interaction with photons by absorbing, emitting, or scattering the latter. These dynamical processes at the particulate level are described by the Boltzmann collision terms in the covariant Boltzmann equations. Consequently, the generalized hydrodynamic equations are derived from the covariant kinetic equations for radiation and matter and made fully consistent with the thermodynamic laws at any degree of removal from equilibrium.

The consistency between the relativistic Boltzmann distribution function and the Planck distribution function has been a point of controversy in the past [6–8], and the present covariant kinetic theory makes it possible to examine the question. As a matter of fact, we show that both distribution functions are consistent with each other within the framework of relativity and that their mutual consistency is intimately related to the H theorem and therefore deeply rooted in the second law of thermodynamics framed in terms of calortropy. In this connection, it is interesting to recall that the original derivation by Planck of the radiation distribution function itself was based on thermodynamics [9].

2.1 Covariant Boltzmann Equations for Matter and Radiation

We assume that the system consists of a gas with internal degrees of freedom (e.g., atoms or diatomic molecules, etc.) which interact with a radiation field (photons) not necessarily in equilibrium. The system is assumed to be free from external fields.

The internal quantum states of material particles will be denoted by i and the particles in different internal quantum states a will be regarded as different species of particles. Therefore, the material gas is considered a mixture of particles with different “colors” distinguished by the value of i . For example, if the particle species is the hydrogen atom, then the index i represents the $1s, 2p, \dots$, states of hydrogen suitably arranged. The mass of the particle species a will be denoted by m_a and its momentum by $\mathbf{p}_a$, the kinetic energy by cp_a^0 with c denoting the speed of light.

It then is convenient, as in Chap. 1 of the present Volume, to enumerate the four-momentum of species a by the formula

$$p_a^\mu = (p_a^0, \mathbf{p}_a), \quad (2.1)$$

where the time component p_a^0 is given by the expression

$$p_a^0 = \sqrt{\mathbf{p}_a \cdot \mathbf{p}_a + m_a^2 c^2}. \quad (2.2)$$

This formula looks like the one holding for structureless particles considered in Chap. 1 of the present Volume, but different masses m_a are assigned to the particles in different internal states. This point can be illustrated in the following way: According to relativistic quantum mechanics, the Dirac equation for the hydrogen atom, for example, gives the energy eigenvalues [10] depending on the two quantum numbers, the principal quantum number n and the angular momentum quantum number J ,

$$E_{nJ} = m_e c^2 \left[1 + Z_e^2 \alpha^2 / (n - \varepsilon_J)^2 \right]^{-1/2}, \quad (2.3)$$

where ε_J is defined by the formula

$$\varepsilon_J = J + \frac{1}{2} - \left[\left(J + \frac{1}{2} \right)^2 - Z_e^2 \alpha^2 \right]^{1/2}; \quad (2.4)$$

α is the fine structure constant

$$\alpha = \frac{e^2}{\hbar} \approx \frac{1}{137};$$

Z_e is the electron number in atom; and m_e is the rest mass of the electron. If we set $E_a^{nJ} = m_a c^2$ where a stands for the set (n, J) , (2.3) suggests that the mass of

the excited particles is not the same as the rest mass of the ground-state particles. Therefore, m_a contains the information on the internal state of the particle. It is easy to show by expansion that approximately

$$m_a = m_e + c^{-2}E_a, \quad (2.5)$$

where E_a is the energy eigenvalue:

$$E_a = -\frac{Z_e^2 \alpha^2}{2a^2} \quad (a = n - \varepsilon_J). \quad (2.6)$$

The time component p_a^0 in the four-momentum in (2.1) is understood in the sense of (2.5) or its precursor (2.3) and will be used accordingly in the calculations made in connection with the two-body collision dynamics in this chapter. The singlet distribution function of species a will be denoted by $f_a(x, p_a)$, the italic subscript r being reserved for radiation.

We consider the following microscopic collision processes:

$$a + b \rightleftharpoons a^* + b^*, \quad (M1)$$

$$a + b \rightleftharpoons k + l, \quad (M2)$$

$$a + \hbar\omega \rightleftharpoons a^* + \hbar\omega^*, \quad (M3)$$

$$a + \hbar\omega \rightleftharpoons b + \hbar\omega^*, \quad (M4)$$

where a, b, k and l stand for the matter species and the asterisk denotes the post-collision quantity. Therefore, for example, $\hbar\omega^*$ means a photon of radiation frequency ω^* after collision. Let subscript a stand for the species of particles including photons, which will be designated the r th species.

In contrast to the four-momentum of matter species a given in (2.1), the photon four-momentum is defined as follows: Since photon momentum $\mathbf{p}_r$ is related to the wave vector $\mathbf{k}_r$ of photon propagation, it can be written

$$\mathbf{p}_r = \hbar\mathbf{k}_r := \hbar k_r \widehat{\mathbf{k}}_r, \quad (2.7)$$

where $\widehat{\mathbf{k}}_r$ is the dimensionless unit vector of the photon wave vector $\mathbf{k}_r$:

$$\widehat{\mathbf{k}}_r = \mathbf{k}_r / |\mathbf{k}_r| = \mathbf{k}_r / k_r. \quad (2.8)$$

If we set

$$k_r = \frac{\omega}{c}, \quad (2.9)$$

where ω is the photon frequency, then the photon momentum $\mathbf{p}_r$ may be written

$$\mathbf{p}_r = \frac{\hbar\omega}{c} \hat{\mathbf{k}}_r. \quad (2.10)$$

We now observe that the photon is massless.² Consistent with this fact, the photon four-momentum p_r^μ is enumerated by

$$p_r^\mu = \hbar (k_r, \mathbf{k}_r)^\mu = \hbar \left(\frac{\omega}{c}, \mathbf{k}_r \right)^\mu = \frac{\hbar\omega}{c} (1, \hat{\mathbf{k}}_r)^\mu. \quad (2.11)$$

Therefore we find its scalar product is equal to zero:

$$p_r^\mu p_{r\mu} = \left(\frac{\hbar\omega}{c} \right)^2 (1 - \hat{\mathbf{k}}_r \cdot \hat{\mathbf{k}}_r) = 0. \quad (2.12)$$

This is in contrast to $p_a^\mu p_{a\mu} = m_a^2 c^2$ for the case of matter species a .

It is postulated that the singlet distribution function $f_a(x, p_a)$ for species a obeys the covariant Boltzmann equation³

$$p_a^\nu \partial_\nu f_a(x, p_a) = \mathfrak{R}_a[f_a], \quad (2.13)$$

where collision term $\mathfrak{R}_a[f_a]$, which is decomposable into component collision terms

$$\mathfrak{R}_a[f_a] = \sum_{b=1}^r \mathfrak{R}_{ab}[f_a f_b], \quad (2.14)$$

²Later, this statement is shown to be true from the thermodynamic consideration.

³This is a covariant kinetic equation for particles free from an external force field F^μ . This restriction, however, can be relaxed for the case of the Lorentz force, provided that the external force leaves the rest mass of the particle unaltered at the end of the particle collision

$$p^\mu F_\mu = 0.$$

Furthermore, it is assumed that the particles having momenta in the range $\Delta^3 p$ at the beginning of the proper time interval $\Delta\tau$ occupy an equal momentum range at the end of the interval. This would true if the force has the property

$$\frac{\partial}{\partial p^\mu} F^\mu(x, p) = 0.$$

These properties are met by the Lorentz force. For such an external force the covariant kinetic equation may be modified to the form

$$p_a^\nu \partial_\nu f_a(x, p_a) + m F_a^\mu f_a(x, p_a) = \mathfrak{R}_a[f_a],$$

where F_a^μ is external force on particle a .

is given by the Wang Chang–Uhlenbeck collision term [2] with the quantum effects put in as in the Boltzmann–Nordholm–Uehling–Uhlenbeck (BNUU) equation [3, 4]

$$\mathfrak{R}_{ab} [f_a f_b] = \left[C(f_a f_b) + \sum_{k=i}^r \sum_{l=i}^r C_r (f_a f_b | f_k^* f_l^*) \right]. \quad (2.15)$$

Here the subscript b, k, l runs over all species including matter particles and photons; that is, $b = 1, 2, \dots, m, r$, where m is the number of material species. The index r is reserved for photons. We refer the reader to Ref. [5] where a nonrelativistic theory of irreversible processes is discussed, using the BNUU equation for a dilute quantum gas mixture, since the present covariant kinetic equation is a relativistic generalization of the BNUU equation. The Boltzmann collision integrals—more precisely, relativistic BNUU collision integrals— $C(f_a f_b)$ and $C_r (f_a f_b | f_k^* f_l^*)$ are given by the expressions for the matter part

$$\begin{aligned} C(f_a f_b) = & G_a \int d^3 \bar{p}_a^* \int d^3 \bar{p}_b^* \int d^3 \bar{p}_b W^{(e)} (p_a p_b | p_a^* p_b^*) \\ & \times [f_a^* f_b^* (1 + \epsilon_a f_a) (1 + \epsilon_b f_b) - f_a f_b (1 + \epsilon_a f_a^*) (1 + \epsilon_b f_b^*)] \\ & + \sum_{k=a}^r \sum_{l=a}^r G_a \int d^3 \bar{p}_k^* \int d^3 \bar{p}_l^* \int d^3 \bar{p}_b W^{(in)} (p_a p_b | p_k^* p_l^*) \\ & \times [f_k^* f_l^* (1 + \epsilon_a f_a) (1 + \epsilon_b f_b) - f_a f_b (1 + \epsilon_k f_k^*) (1 + \epsilon_l f_l^*)] \end{aligned} \quad (2.16)$$

and for the radiation part

$$\begin{aligned} C_r (f_a f_b | f_k^* f_l^*) = & G_r \int d^3 \bar{p}_k^* \int d^3 \bar{p}_l^* \int d^3 \bar{p}_b W^{(r)} (p_a p_b | p_k^* p_l^*) \times \\ & \times [f_k^* f_l^* (1 + \epsilon_a f_a) (1 + \epsilon_b f_b) - f_a f_b (1 + \epsilon_k f_k^*) (1 + \epsilon_l f_l^*)] \\ & + \sum_{k=i}^r \sum_{l=i}^r G_a \int d^3 \bar{p}_k^* \int d^3 \bar{p}_l^* \int d^3 \bar{p}_b W^{(in)} (p_a p_b | p_k^* p_l^*) \\ & \times [f_k^* f_l^* (1 + \epsilon_a f_a) (1 + \epsilon_b f_b) - f_a f_b (1 + \epsilon_k f_k^*) (1 + \epsilon_l f_l^*)]. \end{aligned} \quad (2.17)$$

Here $d^3 \bar{p}_a^* = d^3 \mathbf{p}_a / p_a^0$, etc. and

$$\epsilon_a, \epsilon_k = \begin{cases} +1 & \text{for fermions} \\ -1 & \text{for bosons} \\ 0 & \text{Boltzmann particles} \end{cases}.$$

The symbols G_a and G_r stand for the statistical weights for the material particles and photons. The $W^{(e)}$, $W^{(in)}$, and $W^{(r)}$ denote the transition probabilities for elastic,

inelastic and “reactive” collisions, respectively. Since the photons do not interact with each other the transition probabilities for photon-photon interactions are equal to zero: $W^{(e)}(p_r p_r | p_r^* p_r^*) = 0$, $W^{(in)}(p_r p_r | p_r^* p_r^*) = 0$, and $W^{(r)}(p_r p_r | p_r^* p_r^*) = 0$. These transition probabilities can be calculated from quantum mechanics for a given dynamic mechanism. We assume that they are known through few-body mechanics. In kinetic theory, such few-body dynamical information is assumed known. Writing the kinetic equations for material particles and photons as in (2.13), we have assumed both the material gas and photon gas act as a dilute gas mixture.

We close this section by adding that the transition probabilities have the following symmetry properties

$$W^{(s)}(p_a p_b | p_k^* p_l^*) = W^{(s)}(p_k^* p_l^* | p_a p_b), \text{ etc. } (s = e, in, r). \quad (2.18)$$

These relations originate from the microscopic reversibility of microscopic dynamic events and are crucially important for proving the H theorem.

2.2 Boltzmann Entropy and the H Theorem

The covariant kinetic equation postulated here satisfies the H theorem, which enables us to determine uniquely the equilibrium solution of the kinetic equation as in the case of the covariant kinetic equation discussed in Chap. 1 of this volume. The H theorem also guides us to construct a thermodynamic theory of irreversible processes occurring in the system as will be shown. To state the H theorem explicitly we first must introduce the Boltzmann entropy four-flow $S^\mu(x)$ for the system of radiation and matter by the formula

$$S^\mu(x) = -k_B \sum_{a=1}^r \langle p_a^\mu [f_a \ln f_a - \epsilon_a (1 + \epsilon_a f_a) \ln (1 + \epsilon_a f_a)] \rangle, \quad (2.19)$$

where $f_a = f_a(x, p_a)$. The angular brackets $\langle \dots \rangle$ abbreviate the integration over momentum space:

$$\langle A_a(x, p_a) \rangle = G_{ac} \int d^3 \bar{p}_a A_a(x, p_a) \quad (a = 1, \dots, r) \quad (2.20)$$

with the abbreviation

$$d^3 \bar{p}_a = d^3 p_a / p_a^0.$$

Recall that the subscript a is for either matter or photons, but the subscript r stands for radiation (photon). By differentiating $S^\mu(x)$ and using the covariant kinetic equation, we obtain the balance equation for the Boltzmann entropy four-flow

$$\partial_\mu S^\mu = \sigma_{\text{ent}}(x), \quad (2.21)$$

where the Boltzmann entropy production $\sigma_{\text{ent}}(x)$ is given by the statistical mechanical formula

$$\sigma_{\text{ent}}(x) = -k_B \sum_{a=1}^r \langle \{\ln f_a(x, p_a) - \ln [1 + \epsilon_a f_a(x, p_a)]\} \Re_a[f_a] \rangle. \quad (2.22)$$

It is easy to show that $\sigma_{\text{ent}}(x)$ is positive semidefinite and equal to zero, only if the system is at equilibrium:

$$\begin{aligned} \sigma_{\text{ent}}(x) &= \frac{1}{4} k_B \sum_{a,b=1}^r \langle \ln \{ f_a^* f_b^* (1 + \epsilon_a f_a) (1 + \epsilon_b f_b) \\ &\quad / f_a f_b (1 + \epsilon_a f_a^*) (1 + \epsilon_b f_b^*) \} \Re_{ab}[f_a f_b] \rangle \\ &\geq 0. \end{aligned} \quad (2.23)$$

This is the local form of the H theorem for the covariant kinetic equation postulated, (2.13). The equality holds at equilibrium reached in long time. As in the case of a nonrelativistic gas and a relativistic single-component gas discussed in the previous chapter of this volume, the H theorem may be regarded as a stability theorem of Lyapounov [11], which concerns the stability of the equilibrium solution and the state of equilibrium.

The scalar density $\widehat{S}$ of Boltzmann entropy four-flow S^μ is defined by contracting S^μ with the hydrodynamic velocity U^μ :

$$\mathcal{S}(x) := \rho \widehat{S} = c^{-2} U_\mu S^\mu(x). \quad (2.24)$$

Here again we take Eckart's definition of hydrodynamic velocity; see (1.27), Chap. 1 of this Volume. The balance equation of $\widehat{S}$ is then obtained by operating the convective time derivative operator D on this expression and using the same procedure as used for the balance equation for $\widehat{S}$ in (1.149)–(1.153), Chap. 1, Vol. 2:

$$\rho D \widehat{S} = -\partial_\mu J_s^\mu + \sigma_{\text{ent}}(x), \quad (2.25)$$

where Boltzmann entropy flux J_s^μ is given by

$$J_s^\mu = \Delta_\nu^\mu S^\nu \quad (2.26)$$

with Δ_ν^μ denoting a projector.⁴

⁴Here we would like to emphasize that the hydrodynamic velocity is defined according to Eckart, but for the mixture of matter and photons:

Owing to the H theorem, the equilibrium distribution functions satisfying the covariant kinetic equation can be uniquely determined with the help of the Clausius entropy (i.e., the entropy of reversible processes) and the second law of thermodynamics. To implement this step we need to develop the theory a little further. We will presently return to this task to construct the equilibrium solutions more explicitly later.

2.2.1 Equilibrium Solution of the Covariant Kinetic Equation

One of the most important results provided by the H theorem is that the equilibrium solution of the covariant kinetic equation is uniquely determined by the vanishing Boltzmann entropy production:

$$\begin{aligned} \sigma_{\text{ent}}(x)|_{\text{eqil}} &= \frac{k_B}{4} \sum_{a,b=1}^r \left\langle \ln \left[\frac{(\epsilon_a + f_a^{e-1})(\epsilon_b + f_b^{e-1})}{(\epsilon_a + f_a^{e*-1})(\epsilon_b + f_b^{e*-1})} \right] \mathfrak{R}_{ab}[f_a^e, f_b^e] \right\rangle \\ &= 0, \end{aligned} \quad (2.27)$$

which implies the logarithmic factor in it is a collision invariant. The superscript e denotes equilibrium. That is, $\ln(\epsilon_a + f_a^{e-1})$ must be a summation invariant:

$$\ln(\epsilon_a + f_a^{e-1}) + \ln(\epsilon_b + f_b^{e-1}) = \ln(\epsilon_a + f_a^{e*-1}) + \ln(\epsilon_b + f_b^{e*-1}). \quad (2.28)$$

This in turn means that the logarithmic functions $\ln(\epsilon_a + f_a^{e-1})$ must be also a linear combination of basic collision invariants. Therefore, the equilibrium distribution function may be written in the form⁵

$$\ln(\epsilon_a + f_a^{e-1}) = \beta_e (p_a^\nu U_\nu - \mu_a^e). \quad (2.29)$$

Rearranging this expression we find the equilibrium distribution function in the form

$$f_a^e = \frac{1}{e^{\beta_e (p_a^\nu U_\nu - \mu_a^e)} - \epsilon_a} \quad (a = 1, 2, \dots, m, r). \quad (2.30)$$

(Footnote 4 continued)

$$U^\mu = \frac{cN^\mu}{\sqrt{N^\nu N_\nu}},$$

where N^μ is the sum of N_i^μ with the index i running for entire species including photons according to our model.

⁵This form is without a constant factor that has to do with the normalization of f_a^e . See Chap. 1 where the case for classical particles is described in connection with equilibrium distribution functions.

The equilibrium distribution function f_a^e contains as-yet undetermined parameters β_e and μ_a^e which must be determined upon corresponding the statistical mechanical formula for the equilibrium Boltzmann entropy to their phenomenological counterpart, namely, the Clausius entropy. This can be achieved if the equilibrium Boltzmann entropy density $\widehat{S}^e$, which is identical with the Clausius entropy for reversible processes in the equilibrium system, is evaluated with the equilibrium solution f_a^e . Since $\mathcal{S}^e = c^{-2} U_\mu \mathcal{S}^{e\mu}(x)$ is given by the statistical mechanical formula

$$\mathcal{S}^e = k_B \sum_{a=1}^r c^{-2} U_\mu \langle p_a^\mu [f_a^e \ln(\epsilon_a + f_a^{e-1}) + \epsilon_a \ln(1 + \epsilon_a f_a^e)] \rangle, \quad (2.31)$$

inserting f_a^e we obtain

$$\mathcal{S}^e = k_B \sum_{a=1}^r c^{-2} U_\mu \langle p_a^\mu [f_a^e \beta_e (p_a^\nu U_\nu - \mu_a^e) + \epsilon_a \ln(1 + \epsilon_a f_a^e)] \rangle. \quad (2.32)$$

This expression upon use of the statistical mechanical formulas for the internal energy—energy-momentum tensor—and density yields

$$\mathcal{S}^e = \beta_e k_B \rho \mathcal{E} - \beta_e k_B \rho \sum_{a=1}^r \mu_a^e \epsilon_a + k_B \sum_{a=1}^r \epsilon_a \langle c^{-2} U_\mu p_a^\mu \ln(1 + \epsilon_a f_a^e) \rangle. \quad (2.33)$$

Here the internal energy density $\rho \mathcal{E}$ is calculated from the projection of the energy-momentum tensor

$$\rho \mathcal{E} = \sum_{a=1}^r c^{-2} U_\mu T^{e\mu\nu} U_\nu = \sum_{a=1}^r c^{-2} U_\mu \langle f_a^e p_a^\mu p_a^\nu \rangle U_\nu \quad (2.34)$$

and the density ρ_a of species a with the statistical mechanical formula

$$\rho = \sum_{a=1}^r \rho_a = c^{-2} \sum_{a=1}^r U_\mu N_a^\mu = c^{-2} \sum_{a=1}^r U_\mu \langle f_a^e p_a^\mu \rangle. \quad (2.35)$$

The ratio of densities ρ_a and ρ provides the statistical mechanical formula for the density fraction $\epsilon_a = \rho_a / \rho$ —concentration of a . The last term on the right of (2.33) is identified⁶ with the statistical mechanical expression for relativistic hydrostatic pressure

$$p^e v_a = \beta_e^{-1} k_B \epsilon_a \langle c^{-2} U_\mu p_a^\mu \ln(1 + \epsilon_a f_a^e) \rangle, \quad (2.36)$$

⁶It should be noted that $p^e v_a$ on the left of (2.36) defines macroscopic parameter p^e by the statistical mechanical formula on the right, which will turn out to be hydrostatic (i.e., equilibrium) pressure, when $\mathcal{S}^e$ calculated therewith is identified with the equilibrium entropy on correspondence with the phenomenological Clausius entropy deduced from the second law of thermodynamics.

where $v_a = 1/\rho_a$ is the specific volume. This formula is the equivalent of the non-relativistic formula for hydrostatic pressure in the grand canonical theory to which it reduces since $c^{-2}U_\mu p_a^\mu \rightarrow m_a$ in the limit of $u/c \rightarrow 0$. Formula (2.36) is the necessary and sufficient condition for $\mathcal{S}^e$ to become a bilinear form in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. In this connection it should be noted that if equilibrium grand partition function Ξ^e were constructed for the mixture of matter and photons such that the equation of state is given by the formula

$$p^e V = \beta_e^{-1} \ln \Xi^e, \quad (2.37)$$

then there would be the correspondence

$$\Xi^e \Leftrightarrow k_B \epsilon_a \int_V d\mathbf{r} \langle c^{-2} U_\mu p_a^\mu \ln(1 + \epsilon_a f_a^e) \rangle. \quad (2.38)$$

In summary, the equilibrium Boltzmann entropy density is given by the formula

$$\mathcal{S}^e = \beta_e k_B \rho \mathcal{E}^e - \beta_e k_B \rho \sum_{a=1}^r \mu_a^e c_a + \beta_e k_B \rho p^e v. \quad (2.39)$$

On the other hand, if we calculate the equilibrium calortropy density differential from the balance equation (2.25), the following equation is obtained at equilibrium

$$\begin{aligned} \rho D\widehat{\mathcal{S}}^e &= k_B \sum_{a=1}^r \partial_\mu \beta_e \Delta_\nu^\mu \langle f_a^e p_a^\mu (p_a^\nu U_\nu - \mu_a^e) \rangle \\ &\quad - k_B \sum_{a=1}^r \left\langle (1 + \epsilon_a f_a^e)^{-1} \Delta_\nu^\mu p_a^\mu \partial_\mu f_a^e \right\rangle \\ &= \beta_e k_B \sum_{a=1}^r (-\partial_\mu Q_e^\mu - \rho \mu_a^e Dc_a) \end{aligned} \quad (2.40)$$

for the following reasons: Firstly, at equilibrium

$$p_a^\mu \partial_\mu f_a^e = 0; \quad (2.41)$$

secondly, Q_e^μ is the heat four-flow defined by projection of the energy-momentum tensor $T_e^{\mu\nu}$

$$Q_e^\mu = -\Delta_\nu^\mu T_e^{\mu\nu} U_\nu = -\Delta_\nu^\mu \langle f_a^e p_a^\mu p_a^\nu \rangle U_\nu; \quad (2.42)$$

and thirdly, the following identification is possible:

$$\Delta_\nu^\mu N_a^\nu = \Delta_\nu^\mu \langle f_a^e p_a^\mu \rangle,$$

which may be rearranged to the form

$$\begin{aligned}\Delta_\nu^\mu N_a^\nu &= (g_\nu^\mu - c^{-2} U^\mu U_\nu) N_a^\nu = N_a^\mu - c^{-2} U^\mu U_\nu N_a^\nu \\ &= N_a^\mu - \rho_a U^\mu.\end{aligned}\quad (2.43)$$

The symbol $\mathfrak{c}_a$ is the number fraction defined earlier: $\mathfrak{c}_a = \rho_a/\rho$. Therefore, $\Delta_\nu^\mu N_a^\nu$ may be regarded as the flux J_a^μ of N_a^μ relative to $\rho_a U^\mu$ in the number fraction balance equation

$$\rho D\mathfrak{c}_a = -\partial_\mu J_a^\mu = -\partial_\mu \Delta_\nu^\mu N_a^\nu. \quad (2.44)$$

Using the energy balance equation for an equilibrium process, we then obtain

$$\begin{aligned}\rho D\mathcal{E}^e &= -\partial_\mu Q_e^\mu - p^e \nabla_\mu U^\mu = -\partial_\mu Q_e^\mu + p^e \rho^{-1} D\rho \\ &= -\partial_\mu Q_e^\mu - \rho p^e Dv,\end{aligned}\quad (2.45)$$

where $v = 1/\rho$. Therefore, we finally obtain from (2.40) the equilibrium Boltzmann entropy differential

$$D\widehat{\mathcal{S}}^e = \beta_e k_B \left(D\mathcal{E} + p^e Dv - \sum_{a \neq r} \mu_a^e D\mathfrak{c}_a \right). \quad (2.46)$$

Here the photon is excluded from the sum over species because for photons $\mu_r^e = 0$ as will be shown shortly when we discuss the Boltzmann entropy of photons. This differential form in comparison with (2.39) implies that the integrability condition of (2.46) is

$$\mathcal{E} D(\beta_e k_B) - \sum_{a=1}^r \mathfrak{c}_a D(\beta_e k_B \mu_a^e) + v D(\beta_e k_B p^e) = 0, \quad (2.47)$$

which is the equilibrium Gibbs–Duhem equation, provided

$$\beta_e = \frac{1}{k_B T^e}. \quad (2.48)$$

That is, making correspondence between the statistical mechanical expression (2.39) with the corresponding thermodynamic (Clausius) entropy for a reversible process within a constant, what is the same thing, comparing (2.47) with the thermodynamic (phenomenological) equilibrium Gibbs–Duhem equation

$$\mathcal{E} D T^{e-1} - \sum_{a=1}^r \mathfrak{c}_a D \left(\frac{\mu_a^e}{T^e} \right) + v D \left(\frac{p^e}{T^e} \right) = 0, \quad (2.49)$$

we find the parameter β_e as in (2.48) in terms of the absolute temperature T^e of the equilibrium system and the parameter μ_a^e as the equilibrium chemical potential of

species a . This way, the parameters β_e (or T^e), p^e , and μ_a^e are fully identified with their thermodynamic counterparts in (statistical mechanical) Boltzmann entropy $\widehat{S}^e$ and the equilibrium distribution function f_a^e . This of course means that the Boltzmann entropy at equilibrium is identical with the Clausius entropy of reversible processes.

Now we would like to show the equilibrium distribution function of radiation is given by the expression

$$f_r^e = \left(e^{\beta_e p_r^\nu U_\nu} - 1 \right)^{-1}, \quad (2.50)$$

if the photon number is variable [1]. Since f_r^e is a Lorentz scalar, which can be demonstrated by the same method as shown in Chap. 1 of volume 2, it is permissible to use the local rest frame formula for U_ν :

$$U_\nu = (c, 0, 0, 0). \quad (2.51)$$

Then, if the wave vector of radiation is denoted $\mathbf{k}_r = k_r \widehat{\mathbf{k}}_r$ where $k_r = |\mathbf{k}_r| = \omega/c$ (ω = frequency) and $\widehat{\mathbf{k}}_r$ is the unit vector in the direction of propagation, the covariant form of the equilibrium photon distribution function is given by a familiar looking expression

$$f_r^e = \left(e^{\beta_e \hbar \omega - \mu_r^e} - 1 \right)^{-1}. \quad (2.52)$$

We now show that $\mu_r^e = 0$ if the photon number is variable. The equilibrium number density is given by the formula

$$\begin{aligned} \rho_r &= N_r / V = \langle f_r^e(\omega) \rangle \\ &= \pi^{-2} c^{-3} \int_0^\infty d\omega \omega^2 \left\{ \exp \left[\beta_e (\hbar \omega - \mu_r^e) \right] - 1 \right\}^{-1}. \end{aligned} \quad (2.53)$$

Since the local grand canonical partition function for photons Γ_r^e is given by

$$\Gamma_r^e = - \left(V / \pi^2 c^3 \beta_e \right) \int_0^\infty d\omega \omega^2 \ln \left\{ 1 - \exp \left[\beta_e (\mu_r^e - \hbar \omega) \right] \right\}, \quad (2.54)$$

we find

$$\rho_r = - \left(\frac{\partial \ln \Gamma_r^e}{\partial \mu} \right)_{T^e, V}, \quad (2.55)$$

where μ is the chemical potential. If we set $\mu = \mu_r^e$ and differentiate (2.54) with μ_r^e , there follows (2.53) exactly. Thus we have identified μ_r^e with the photon chemical potential. Now since

$$\mu_r^e = -T^e \left(\frac{\partial \mathcal{S}_r^e}{\partial \rho_r} \right)_{T^e, V} = \left(\frac{\partial \mathcal{A}_r^e}{\partial \rho_r} \right)_{T^e, V}, \quad (2.56)$$

where $\mathcal{S}_r^e$ and $\mathcal{A}_r^e$ are, respectively, the equilibrium Boltzmann entropy density and work function for photon. If $\mathcal{A}_r^e$ is required to be at a minimum at equilibrium as it is varied with respect to ρ_r , it follows that

$$\left(\frac{\partial \mathcal{A}_r^e}{\partial \rho_r} \right)_{T^e, V} = 0$$

and hence the equilibrium radiation has a vanishing chemical potential:

$$\mu_r^e = 0. \quad (2.57)$$

Therefore the equilibrium radiation distribution function is given by

$$f_r^e = (e^{\beta_e \hbar \omega} - 1)^{-1}. \quad (2.58)$$

The meaning of β_e is the same as (2.48) because according to the Stefan–Boltzmann law [12] the radiation energy density is given by

$$\mathcal{E}_r^e = 4 \frac{a_{\text{SB}} T^e{}^4}{c}, \quad (2.59)$$

where a_{SB} is the Stefan–Boltzmann constant. If $\mathcal{E}_r$ is calculated by using f_r^e in (2.58) and equated with the phenomenological $\mathcal{E}_r$ in (2.59) we conclude

$$(\beta_e T^e)^{-4} = \frac{60 \hbar c^3}{\pi^2} a_{\text{SB}}.$$

The left hand side, $(\beta_e T^e)^{-4}$, must be a universal constant since the Stefan–Boltzmann constant is universal as verified by experiments, i.e., the Kirchhoff law [9, 13]. The universal constant turns out to be the Boltzmann constant raised to the fourth power. Hence we find that the parameter β_e in f_r^e is given by the same formula as (2.48). We therefore conclude that both radiation and matter distribution functions share a single temperature parameter characterized by the black body radiation in thermal equilibrium with matter. Furthermore, they are mutually consistent, and their mutual consistency is demanded by the H theorem.

It is interesting to note that Einstein derived (2.52) from the equilibrium condition between radiation and material particles in his famous paper of 1917 [14]. The present approach elucidates the kinetic theory and dynamic bases which are absent in his theory. In the recent past, Boyer [6] claimed that the Boltzmann distribution functions for relativistic material particles and the Planck distribution function are not consistent within the framework of quantum theory, and this has been a point of controversy [7, 8]. The derivation of (2.30) and (2.52) presented earlier shows that they are completely consistent with each other, provided that the irreversible covariant kinetic equation (2.13) satisfying the H theorem is postulated. Moreover, it shows that the Planck distribution law is deeply rooted in the H theorem and thus

the second law of thermodynamics for radiation in equilibrium with matter. It also points out that it is not possible to think of a radiation distribution function without taking into account the corresponding material particle distribution function and the interaction between radiation and matter. It therefore may be said that this is the first practical and useful result of the present covariant kinetic theory formulation of the theory of radiation and matter.

2.2.2 *Equilibrium Energy-Momentum Tensors of Radiation and Matter*

We define the energy-momentum tensor of matter or radiation in equilibrium by the statistical mechanical formula

$$T_{ei}^{\mu\nu} = \langle p_i^\mu p_i^\nu f_i^e(p_i) \rangle, \quad (i = a, r). \quad (2.60)$$

With the help of the projector defined by

$$\Delta^{\mu\nu} = g^{\mu\nu} - c^{-2} U^\mu U^\nu \quad (2.61)$$

the equilibrium energy-momentum tensor can be decomposed into component as follows:

$$T_{ei}^{\mu\nu} = p_i^e \Delta^{\mu\nu} + c^{-2} \rho_i \mathcal{E}_i^e U^\mu U^\nu, \quad (2.62)$$

where the energy density $\mathcal{E}_i^e$ is defined by

$$\rho_i \mathcal{E}_i^e = c^{-2} U_\mu T_{ei}^{\mu\nu} U_\nu = c^{-2} \langle (p_i^\mu U_\mu) (p_i^\nu U_\nu) f_i^e(p_i) \rangle \quad (2.63)$$

and the hydrostatic pressure p_i^e is given by

$$p_i^e = \frac{1}{3} \Delta_{\mu\nu} T_{ei}^{\mu\nu} = \frac{1}{3} \Delta_{\mu\nu} \langle p_i^\mu p_i^\nu f_i^e(p_i) \rangle. \quad (2.64)$$

Therefore, the total material internal energy and total material pressure are given, respectively, by the formulas

$$\rho_m \mathcal{E}_m^e = \sum_{a \neq r} \rho_a \mathcal{E}_a^e, \quad p_m^e = \sum_{a \neq r} p_a^e = \rho_e k_B T^e. \quad (2.65)$$

Note that the summation sign here excludes photons. The last equality is easy to show in the local rest frame [15]. The second equation of (2.65) is a well known result for gases consisting of structureless particles. It also holds for dilute gases consisting of

particles with an internal structure. By using the definition of the projector $\Delta_{\mu\nu}$ and the identity

$$p_r^\mu p_{r\mu} = 0$$

for photons, we can show from (2.63) and (2.64)

$$\mathfrak{p}_r^e = \frac{1}{3} \rho_r \mathcal{E}_r^e. \quad (2.66)$$

In fact, this relation is indeed verifiable by using the local rest frame. By using the energy-momentum tensor for radiation, we can also derive the Stefan–Boltzmann law [12],

$$\rho_r \mathcal{E}_r^e = a_{SB} T^{e4}, \quad (2.67)$$

where a_{SB} is the Stefan–Boltzmann constant:

$$a_{SB} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3}. \quad (2.68)$$

In connection with this result, we recall that the parameter β_e in the radiation distribution function is determined such that the equilibrium radiation energy is given by (2.67). The β_e so determined exactly coincides with the one given in (2.48). This coincidence is a result of the demand made by the H theorem that the material gas and radiation be in equilibrium. Other thermodynamic quantities for radiation and matter can be calculated with the equilibrium distribution functions determined here.

2.3 Number and Energy-Momentum Tensor Balance Equations

The covariant kinetic equation postulated for matter and radiation gives rise to the particle number and energy-momentum tensor balance equations as does the relativistic kinetic equation considered in the previous chapter of this volume. They will be derived for matter and radiation in the following, but if derived separately for matter and radiation treating them as different species, they give rise to additional source terms because of interactions between molecules and photons, which may be regarded as “chemical reactions” in the present model.

Let us define the number four-flow of species i by the statistical representation

$$N_i^\mu = \langle p_i^\mu f_i(x, p_i) \rangle \quad (2.69)$$

and the energy-momentum tensor of species i by

$$T_i^{\mu\nu} = \langle p_i^\mu p_i^\nu f_i(x, p_i) \rangle, \quad (2.70)$$

where index i runs over matter species $(1, \dots, m)$ and similar expressions for radiation specifically designated by index r . Therefore the total energy-momentum tensor of matter as a whole is given by

$$T_m^{\mu\nu} = \sum_{i=1}^m T_i^{\mu\nu} = \sum_{i=1}^m \langle p_i^\mu p_i^\nu f_i(x, p_i) \rangle, \quad (2.71)$$

and the energy-momentum tensor for radiation by

$$T_r^{\mu\nu} = \langle p_r^\mu p_r^\nu f_r(x, p_r) \rangle. \quad (2.72)$$

The number four-flow for the entire matter species is then given by

$$N_m^\mu = \sum_{i=1}^m N_i^\mu = \sum_{i=1}^m \langle p_i^\mu f_i(x, p_i) \rangle \quad (2.73)$$

and for radiation by the statistical mechanical formula

$$N_r^\mu = \langle p_r^\mu f_r(x, p_r) \rangle. \quad (2.74)$$

By using the kinetic equation, the balance equation for N_a^μ for matter species a can be shown given by the equation

$$\partial_\mu N_a^\mu = \Lambda_a^{(n)}, \quad (2.75)$$

where

$$\Lambda_a^{(n)} = \langle \mathfrak{R}_a[f_a] \rangle. \quad (2.76)$$

This source term $\Lambda_a^{(n)}$ is not equal to zero because of the interactions of matter particles with photons and making transition to other matter species as assumed in the present model. However, when summed over all matter species, that is, for N_m^μ

$$\partial_\mu N_m^\mu = 0, \quad (2.77)$$

because the overall matter is conserved and hence

$$\sum_{i=1}^m \Lambda_i^{(n)} = 0. \quad (2.78)$$

Similarly to the balance equation for N_a^μ . Hence the energy-momentum balance equation acquires a source term:

$$\partial_\nu T_m^{\mu\nu} = \Lambda_m^\mu, \quad (2.79)$$

where the source term is given by

$$\Lambda_m^\mu = \sum_{i=1}^m \langle p_i^\mu \mathfrak{R}_i [f_i] \rangle. \quad (2.80)$$

Therefore, the energy-momentum tensor for matter alone is not a conserved quantity, because it is not a collision invariant. For radiation we obtain the number balance equation

$$\partial_\mu N_r^\mu = \Lambda_r^{(n)}, \quad (2.81)$$

and the energy momentum balance equation

$$\partial_\nu T_r^{\mu\nu} = \Lambda_r^\mu \quad (2.82)$$

with the source terms

$$\Lambda_r^{(n)} = \langle \mathfrak{R}_r [f_r] \rangle \quad (2.83)$$

and

$$\Lambda_r^\mu = \langle p_r^\mu \mathfrak{R}_r [f_r] \rangle, \quad (2.84)$$

respectively. For the system of matter and radiation as a whole the number four-flow is then given by the sum

$$N^\mu = N_m^\mu + N_r^\mu. \quad (2.85)$$

With the total number four-flow N^μ we define hydrodynamic four-velocity U^μ by the Eckart convention

$$U^\mu = \frac{c N^\mu}{\sqrt{N^\nu N_\nu}}. \quad (2.86)$$

Hydrodynamic number density ρ_i of species i is then given by

$$\rho_i = c^{-2} U_\mu N_i^\mu \quad (i = 1, 2, \dots, m, r). \quad (2.87)$$

The total number density therefore is given by the sum

$$\rho = \rho_m + \rho_r = \sum_{i=1}^{\{m,r\}} \rho_i = c^{-2} U_\mu N^\mu, \quad (2.88)$$

where the upper limit $\{m, r\}$ of the sum means that the sum is over all matter particle species $(1, \dots, m)$ and radiation r .

The scalar energy density $\mathcal{E}_i$ of species i (either matter or radiation) is defined by a projection of energy-momentum tensor of the species:

$$E_i = \rho_i \mathcal{E}_i = c^{-2} U_\mu T_i^{\mu\nu} U_\nu. \quad (2.89)$$

Other projections of the energy-momentum tensor are the heat flux (four-flow) Q_i^μ and the stress tensor $P_i^{\mu\nu}$, respectively, defined by

$$Q_i^\mu = U_\nu T_i^{\nu\sigma} \Delta_\sigma^\mu, \quad (2.90)$$

$$P_i^{\mu\nu} = \Delta_\sigma^\mu T_i^{\sigma\tau} \Delta_\tau^\nu, \quad (2.91)$$

where Δ_σ^μ is a projection operator as defined in the previous chapter. Therefore the energy-momentum tensor of a species is decomposed into four components according to the projections given above:

$$\begin{aligned} T_i^{\mu\nu} = & c^{-2} \rho_i \mathcal{E}_i U^\mu U^\nu + c^{-2} (Q_i'^\mu U^\nu + U^\mu Q_i'^\nu) \\ & + c^{-2} (\mathfrak{J}_i^\mu U^\nu + U^\mu \mathfrak{J}_i^\nu) + P_i^{\mu\nu}, \end{aligned} \quad (2.92)$$

where $Q_i'^\mu$ is the net heat flux above and beyond energy carried by matter

$$Q_i'^\mu = Q_i^\mu - \hat{h} J_i^\mu \quad (2.93)$$

and $\mathfrak{J}_i^\mu$ is the diffusion of energy attributable to particle diffusion:

$$\mathfrak{J}_i^\mu = \hat{h} J_i^\mu \quad (2.94)$$

with $\hat{h}$ denoting the enthalpy density and J_i^μ the number diffusion four-flow defined by

$$J_i^\mu = N_i^\mu - c_a N^\mu. \quad (2.95)$$

The index i runs for all matter species and radiation since the decomposition holds both for matter and radiation. From the number and energy-momentum balance equations derived earlier follow various balance equations for conserved variables—number, momentum, energy—when various projections of the energy-momentum tensor balance equations and the number balance equations are taken. The procedure is described in the previous chapter of this volume. Since it basically remains the same, we will simply present the results only, but separately for matter and radiation because we are interested in hydrodynamics and transport processes of radiation and matter separately.

For the purpose in mind we recall that because $N^\mu \nabla_\mu = 0$ in Eckart's definition of hydrodynamic velocity the substantial time derivative $\mathfrak{D} = \rho D + N^\mu \nabla_\mu$ is simply given by

$$\mathfrak{D} = \rho D. \quad (2.96)$$

Therefore the substantial time derivative of macroscopic quantity (density) ψ may be written as

$$\rho D\psi = \partial_\mu (\psi N^\mu). \quad (2.97)$$

Using this identity various evolution equations for macroscopic properties follow straightforwardly with the help of their statistical mechanical definitions.

2.3.1 Equation of Continuity

From the total number density balance equation follows the equation of continuity for the system

$$D\rho = -\rho\nabla_\mu U^\mu. \quad (2.98)$$

It should be noted that this is for the totality of material particles and photons in the system. With the definition of specific volume $v = 1/\rho$, (2.98) is equivalently written

$$\rho Dv = \nabla_\mu U^\mu. \quad (2.99)$$

2.3.2 Balance Equations for Conserved Variables of Matter

Various balance equations for conserved variables follow for matter species similarly to (2.98).

Density Fraction Balance Equation

The density fraction (concentration) balance equation follows from (2.75)

$$\rho Dc_a = -\partial_\mu J_a^\mu + c^{-2} J_a^\mu D U_\mu + \Lambda_a^{(n)}, \quad (2.100)$$

where diffusion four-flow J_a^μ is defined by (2.95). The term $\Lambda_a^{(n)}$ appears because the number of particle species a is not conserved owing to its transition to another material species upon interaction with photons; see the collision processes (M1)–(M4) assumed for the model for collisions.

Momentum Balance Equation

Operating the projector Δ_ν^μ on the energy-momentum balance equation for matter we obtain the momentum balance equation

$$\begin{aligned} c^{-2} \mathcal{E}_m \rho D U^\mu &= -\nabla^\mu p_m - \Delta_\omega^\mu \nabla_\nu \bar{P}_m^{\omega\nu} \\ &\quad + c^{-2} \left(P_m^{\mu\nu} D U_\nu - \Delta_\omega^\mu D Q_m^\omega \right. \\ &\quad \left. - Q_m^\mu \nabla_\nu U^\nu - Q_m^\nu \nabla_\nu U^\mu \right) + \Delta_\nu^\mu \Lambda_m^{(n)\nu}, \end{aligned} \quad (2.101)$$

for which we have made use of the decomposition of energy-momentum tensor $T_m^{\mu\nu}$:

$$T_m^{\mu\nu} = c^{-2} \rho_m \mathcal{E}_m U^\mu U^\nu + c^{-2} \left(Q_m^\mu U^\nu + U^\mu Q_m^\nu \right) + P_m^{\mu\nu}. \quad (2.102)$$

This decomposition of $T_i^{\mu\nu}$ follows from summing components (2.92) over matter species. Consequently, the total energy of matter (the total internal energy of material gas) is given by

$$\rho_m \mathcal{E}_m = \sum_{a \neq r}^m \rho_a \mathcal{E}_a \quad (2.103)$$

with

$$\rho_m = \sum_{a=1}^m \rho_a; \quad (2.104)$$

the heat flux four-flow Q_m^μ by

$$Q_m^\mu = U_\mu T_m^{\sigma\tau} \Delta_\tau^\mu; \quad (2.105)$$

and the pressure tensor $P_m^{\mu\nu}$ by

$$P_m^{\mu\nu} = \sum_{a \neq r} P_a^{\mu\nu} = \Delta_\sigma^\mu T_m^{\sigma\tau} \Delta_\tau^\nu. \quad (2.106)$$

In (2.101)

$$p_m := p_m^e = \frac{1}{3} \Delta_{\mu\nu} T_{em}^{\nu\mu}. \quad (2.107)$$

The pressure tensor $\bar{P}_m^{\sigma\nu}$ is further decomposable into excess trace ($\tilde{\Delta}_m$) and traceless symmetric ($\Pi_m^{\sigma\nu}$) parts:

$$\bar{P}_m^{\sigma\nu} = \left(\frac{1}{3} T_{m\mu}^\mu - p_m \right) \Delta^{\sigma\nu} + \Pi_m^{\sigma\nu} := p_m \Delta^{\sigma\nu} + \tilde{\Delta}_m \Delta^{\sigma\nu} + \Pi_m^{\sigma\nu}, \quad (2.108)$$

where

$$\tilde{\Delta}_m = \frac{1}{3} \Delta_{\mu\nu} P_m^{\mu\nu} - p_m^e := p - p_m^e, \quad (2.109)$$

$$\Pi_m^{\mu\nu} = P_m^{\mu\nu} - \frac{1}{3} \Delta_{\sigma\tau} T_m^{\sigma\tau} \Delta_{\mu\nu}. \quad (2.110)$$

Therefore, $\tilde{\Delta}_m$ is the excess normal stress above and beyond the hydrostatic pressure p_m . It consequently vanishes at equilibrium. It should be noted that we have used the identity $\Delta_\nu^\mu \nabla_\sigma U^\nu = \nabla_\sigma U^\mu$ for the derivation of (2.101).

Energy Balance Equation

The energy balance equation also follows upon contracting the energy-momentum balance equation (2.79) with $c^{-2} U_\mu$ and using some of the identities listed in (1.78). We thereby obtain the energy balance equation

$$\rho D\mathcal{E}_m = -\partial_\mu Q_m^\mu + P_m^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_m^\mu D U_\mu + c^{-2} U_\nu \Lambda_m^\nu. \quad (2.111)$$

The last term in this equation appears because of the interaction of particle species with photons making transition to different particle species—or acquiring a spectrum of “colors”, figuratively speaking. The term $2c^{-2} Q_m^\mu D U_\mu$ vanishes as $u/c \rightarrow 0$ in the nonrelativistic limit.

2.3.3 Balance Equations for Conserved Variables of Radiation

The balance equations for conserved variables of radiation can be derived from (2.81) and (2.82) in the same manner as for material balance equations.

Radiation Density Fraction Balance Equation

With the definition of radiation (photon number) fraction as ratio of ρ_r to ρ

$$c_r = \frac{\rho_r}{\rho}, \quad (2.112)$$

we obtain, by making use of identities (1.78), the radiation fraction balance equation

$$\rho Dc_r = -\partial_\nu J_r^\nu + c^{-2} J_r^\nu D U_\nu + \Lambda_r^{(n)\mu}. \quad (2.113)$$

The term $c^{-2} J_r^\nu D U_\nu$ vanishes in the nonrelativistic limit as $u/c \rightarrow 0$.

Radiation Energy Balance Equation

Contracting the energy-momentum balance equation for radiation (2.82) with $c^{-2} U_\mu$ the radiation energy balance equation is obtained:

$$\rho D\mathcal{E}_r = -\partial_\nu Q_r^\nu + P_r^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_r^\nu D U_\nu + c^{-2} U_\mu \Lambda_r^\mu. \quad (2.114)$$

Here $\mathcal{E}_r$ is the radiation energy density and the equation obtained above describes the evolution of the radiation energy density. We have not listed the radiation momentum balance equation here, since it is related to the radiation heat flux evolution equation, which will be included in the flux evolution equations for nonconserved variables given below.

2.4 Evolution Equations for Nonconserved Variables

We have already seen that there appear macroscopic variables such as J_i^μ , Q_i^μ , and $P_i^{\mu\nu}$ in the balance equations presented for conserved variables, which do not as yet have their own evolution equations. They are, as a matter of fact, examples for what we

call nonconserved variables. The variables J_i^μ , Q_i^μ , and $P_i^{\mu\nu}$ are the leading elements of a hierarchy of macroscopic nonconserved moments, which can be generated from the kinetic equation. We have seen such examples in nonrelativistic kinetic theories and also in the relativistic kinetic theory of pure gases discussed in the previous chapter. We derive them from the covariant kinetic equation by using the moments denoted by $h_i^{(q)\sigma\cdots\alpha}$ for the generic molecular symbol for nonconserved macroscopic variables. The subscript i of moment $h_i^{(q)\sigma\cdots\alpha}$ denotes the species $i \in (1, \dots, m, r)$ and the superscript q denotes the q th element of the moment set: $q = 1, 2, 3, 4, \dots$. The set of moments is constructed by using the conserved moments as seed moments, which have appeared in the balance equations of conserved variables. The higher-order moments following the seed moments are hierarchically constructed by using the Schmidt orthogonalization method employed for constructing an orthogonal set of moments. This method was used in the nonrelativistic kinetic theory of the present work and also in the previous chapter on the relativistic kinetic theory of matter of this volume. The same procedure is employed for the present covariant kinetic equation. The reader is referred to the sections for the Schmidt orthogonalization method of constructing an orthogonal moment set in the previous chapters of this work. We present the matter and radiation parts separately.

2.4.1 Moment Set for Matter

The leading elements of the moment set for matter species are represented by the tensor polynomials $h_a^{(q)\alpha\cdots\omega}$, which may be chosen to be orthogonal tensor polynomials of momentum p_a^σ and ordered as follows:

$$h_a^{(1)\mu\nu} = c^2 (U_\lambda p_a^\lambda)^{-1} \left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3} \Delta_{\sigma\tau} \Delta^{\mu\nu} \right) p_a^\sigma p_a^\tau, \quad (2.115)$$

$$h_a^{(2)} = \frac{1}{3} c^2 (U_\lambda p_a^\lambda)^{-1} \Delta_{\mu\nu} p_a^\mu p_a^\nu - p_a^e / \rho_a + (\vartheta^{-1} - 1) (\bar{E} - E_a), \quad (2.116)$$

$$h_a^{(3)\mu} = c^2 (U_\lambda p_a^\lambda)^{-1} \left\{ \Delta_\sigma^\mu p_a^\sigma p_a^\sigma U_\nu - (\hat{h}_a + m_a c^2) [p_a^\mu - c^{-2} (U_\lambda p_a^\lambda) U^\mu] \right\}, \quad (2.117)$$

$$h_a^{(4)\mu} = c^2 (U_\lambda p_a^\lambda)^{-1} [p_a^\mu - c^{-2} (p_a^\lambda U_\lambda) U^\mu], \quad \text{etc.}, \quad (2.118)$$

where ρ_a and $\hat{h}_a$ are the density and the enthalpy density of “color” species a , respectively; E_a is the energy of the internal degrees of freedom of particle a ;

$$\bar{E} = \sum_{a \neq r} E_a, \quad (2.119)$$

$$\vartheta^{-1} = C_{\text{int}} / C_v \quad (C_v = C_{\text{tr}} + C_{\text{int}}) \quad (2.120)$$

with C_{tr} and C_{int} denoting the translational and the internal specific heat per molecule at constant volume, respectively; and $p_a := p_a^e$ for notational brevity. The set of moments $\{h_a^{(q)\alpha\cdots\omega}\}$ is constructed in the same spirit as taken for it in Chap. 1 such that its elements reduce to the same nonrelativistic limits as for the nonrelativistic theory counterparts in the limit of $u/c \rightarrow 0$. In the case of gases without an internal structure the last term in (2.116) for $h_a^{(2)}$ vanishes and the expression for $h_a^{(2)}$ of a monatomic gas used in the previous chapter is recovered. Note that the quantum aspect does not appear explicitly in the kinematic terms even though particles behave quantum mechanically, at least in their collision processes, in the present theory. The set is ordered as presented, because the present ordering is most suitable for the theory presented below in this work. We also note that the moment set also can be expressible in terms of relativistic peculiar velocity $\mathfrak{C}_a^\sigma$ as was done in Chap. 1 of this Volume. Such a representation would be most convenient when their nonrelativistic limits are discussed. The physical meanings of the tensor moments $h_a^{(q)\alpha\cdots\omega}$ will become evident as the theory is formulated in the following.

To begin the derivation of evolution equations, the macroscopic four-flow tensor (i.e., supermoment) $\psi_a^{(q)\sigma\mu\cdots\nu}$ is defined by the statistical mechanical average of flow $p_a^\sigma h_a^{(q)\mu\cdots\nu}$ of molecular tensor moment $h_a^{(q)\mu\cdots\nu}$:

$$\psi_a^{(q)\sigma\mu\cdots\nu} = \langle p_a^\sigma h_a^{(q)\mu\cdots\nu} f_a(x, p_a) \rangle. \quad (2.121)$$

We then define macroscopic flux tensor $\Phi_a^{(q)\mu\cdots\nu}$ for nonconserved fluxes by taking contraction of supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$ with hydrodynamic velocity U_σ

$$\Phi_a^{(q)\mu\cdots\nu} = c^{-2} U_\sigma \psi_a^{(q)\sigma\mu\cdots\nu} = c^{-2} U_\sigma \langle p_a^\sigma h_a^{(q)\mu\cdots\nu} f_a(x, t) \rangle. \quad (2.122)$$

Therefore summing it over all matter species we obtain the macroscopic moment for matter as a whole

$$\Phi_m^{(q)\mu\cdots\nu} := \rho \hat{\Phi}_m^{(q)\mu\cdots\nu} = \sum_{a=1(a \neq r)}^m \Phi_a^{(q)\mu\cdots\nu} \quad (q = 1, 2, \dots, 4). \quad (2.123)$$

The macroscopic moment tensor $\Phi_a^{(q)\mu\cdots\nu}$ will be sometimes referred to as the macroscopic flux of order q . Here the subscript m stands for matter in order to distinguish the material part from the radiation part of the flux. The leading examples for macroscopic tensor moments $\Phi_a^{(q)\mu\cdots\nu}$ are stress tensors, heat fluxes, diffusion fluxes.

2.4.2 Evolution Equations for Nonconserved Moments for Matter

By using (2.121), (2.123), and the substantial time derivative⁷ $\mathfrak{D} = \rho D$, where $D := U^\nu \partial_\nu$, we easily obtain the evolution equation for $\widehat{\Phi}_a^{(q)\mu\cdots\nu}$ of matter. The procedure is illustrated below as a generic example.

Operating ρD on $\widehat{\Phi}_a^{(q)\mu\cdots\nu} := \Phi_a^{(q)\mu\cdots\nu}/\rho$, we obtain

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \partial_\sigma (\widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma), \quad (2.124)$$

which can be rearranged to the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma (\psi_a^{(q)\sigma\mu\cdots\nu} - \widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma) + \partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu}. \quad (2.125)$$

The flux of $\widehat{\Phi}_a^{(q)\mu\cdots\nu}$ may be defined by the formula

$$\Delta_\gamma^\mu \psi_a^{(q)\sigma\gamma\cdots\nu} = (\psi_a^{(q)\sigma\mu\cdots\nu} - \widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma). \quad (2.126)$$

Upon use of the covariant kinetic equation, we obtain the evolution equation for supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$

$$\partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu} = \langle f_a(x, p_a) p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} \rangle + \Lambda_a^{(q)\mu\cdots\nu}(x), \quad (2.127)$$

where the dissipation term $\Lambda_a^{(q)\mu\cdots\nu}(x)$ is defined by the kinetic theory formula

$$\Lambda_a^{(q)\mu\cdots\nu}(x) = \langle h_a^{(q)\mu\cdots\nu} \mathfrak{R}_a[f_a(x, p_a)] \rangle. \quad (2.128)$$

Therefore, we are finally able to write the generic evolution equation for $\Phi_a^{(q)\mu\cdots\nu}$ in the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma (\Delta_\gamma^\mu \psi_a^{(q)\sigma\gamma\cdots\nu}) + Z_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x). \quad (2.129)$$

In this equation the term $Z_a^{(q)\mu\cdots\nu}$ is defined by the formula

$$Z_a^{(q)\mu\cdots\nu} = \langle p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} f_a(x, p_a) \rangle, \quad (2.130)$$

which is called the kinematic term of the evolution equation for $\Phi_a^{(q)\mu\cdots\nu}$. Equation (2.129) is the generic form of evolution equation for all species and orders ($q = 1, 2, \dots$). As in the case of generic evolution equation for matter alone, the kinematic term is modified by adding to it the contribution from the divergence term, and the

⁷If we choose the Eckart convention [16] for hydrodynamic velocity, the convective time derivative D is identical with the substantial time derivative. For this, see Chap. 1 of this volume.

generic evolution equation reads

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x), \quad (2.131)$$

where $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ and $\Theta_a^{(q)\sigma\mu\cdots\nu}$, similarly to (1.123) and (1.126), are given respectively by the formulas

$$\mathfrak{Z}_a^{(q)\mu\cdots\nu} = Z_a^{(q)\mu\cdots\nu} - c^{-2} U_\sigma D \left(\Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu} \right), \quad (2.132)$$

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}. \quad (2.133)$$

This equation has formally a similar mathematical structure to the evolution equation for nonrelativistic nonconserved flux $\Phi_a^{(q)}$ appearing in nonrelativistic kinetic theories and also to those in the relativistic theory of matter alone. We have repeated about it to set it apart from the relativistic evolution equations for radiation, which are found different from those for matter.

Upon summing (2.129) over all matter species we obtain the generic evolution equation for $\Phi_m^{(q)\mu\cdots\nu}$:

$$\rho D \widehat{\Phi}_m^{(q)\mu\cdots\nu} = -\partial_\sigma \left(\Delta_\gamma^\mu \psi_m^{(q)\sigma\gamma\cdots\nu} \right) + Z_m^{(q)\mu\cdots\nu} + \Lambda_m^{(q)\mu\cdots\nu}(x), \quad (2.134)$$

where

$$\Lambda_m^{(q)\mu\cdots\nu}(x) = \sum_{a \neq r} \langle h_a^{(q)\mu\cdots\nu} \mathfrak{R}_a[f_a(x, p_a)] \rangle, \quad (2.135)$$

$$Z_m^{(q)\mu\cdots\nu} = \sum_{a \neq r} \langle p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} f_a(x, p_a) \rangle. \quad (2.136)$$

Let us recall the decomposition of $T_m^{\mu\nu}$ into the traceless symmetric part $\Pi_m^{\mu\nu}$ of the stress tensor, the excess trace part $\widetilde{\Delta}_m$ of the stress tensor, and the heat flux Q_m^μ as well as diffusion flux $\mathfrak{J}_m^\mu$ and also the definition of diffusion four-flow

$$J_a^\mu = N_a^\mu - c_a N_m^\mu, \quad (2.137)$$

where the statistical mechanical formula for number four-flow is defined by

$$N_m^\mu = \sum_{a \neq r} N_a^\mu = \sum_{a \neq r} \langle p_a^\mu f_a(x, p_a) \rangle. \quad (2.138)$$

Then, the leading members of the macroscopic moment set just defined can be identified with shear stress tensor, excess normal stress, heat flux, and diffusion flux as follows:

$$\begin{aligned}\Phi_m^{(1)\mu\nu} &= \Pi_m^{\mu\nu}; \quad \Phi_m^{(2)} = \tilde{\Delta}_m + \left[\frac{2}{3} (\vartheta^{-1} - 1) \right] \Delta E; \\ \Phi_m^{(3)\mu} &= Q_m^\mu - \mathfrak{a}_m h_m J_m^\mu := Q_m'^\mu; \quad \Phi_m^{(4)\mu} = J_m^\mu, \quad \text{etc.}\end{aligned}\quad (2.139)$$

Here

$$\Delta E = \sum_{a \neq r}^m (\bar{E} - E_a). \quad (2.140)$$

It is important to note that the second member $\Phi_m^{(2)}$ of the macroscopic moment set in (2.139) has an additional term, which is absent in the moment $h_a^{(2)}$ for monatomic gases [1] since the latter does not have the term arising from the internal energy fluctuation, namely, $\left[\frac{2}{3} (\vartheta^{-1} - 1) \right] \Delta E$.

Since the generalized hydrodynamic theory arising from the covariant kinetic equation for the matter part is already presented in Chap. 1 of Volume 2, here attention will be more closely paid to the radiation part.

2.4.3 Evolution Equations for Nonconserved Moments for Radiation

We have formulated the radiation part of the kinetic equation with the radiation momenta on equal footing with the particle momenta. However, whereas the material particles obey the relativistic equations of motion, the radiation (photon) obeys a wave equation. According to (2.10) the photon momentum $\mathbf{p}_r$ is related to the wave vector $\mathbf{k}_r$ of radiation by the formula

$$\mathbf{p}_r = \hbar \mathbf{k}_r = \frac{\hbar \omega}{c} \hat{\mathbf{k}}_r. \quad (2.141)$$

The photon momentum is rendered dimensionless if it is multiplied by $c\beta$:

$$\mathbf{q}_r = \beta \hbar \omega \hat{\mathbf{k}}_r. \quad (2.142)$$

With this dimensionless unit vector for the space component, we form the dimensionless four-vector q_r^ν for photon

$$q_r^\mu = \beta \hbar \omega (1, \hat{\mathbf{k}}_r) = c\beta p_r^\mu. \quad (2.143)$$

The trace of this four-vector is then seen to be null, reflecting the fact that the photon has a zero rest mass:

$$(q_r^\mu q_{r\mu}) = 0. \quad (2.144)$$

In addition to this, we find

$$p_r^\mu U_\mu = \hbar\omega\gamma \left(1 - \frac{\mathbf{n} \cdot \mathbf{u}}{c}\right), \quad (2.145)$$

where $\mathbf{n}$ is the unit vector in the direction of photon propagation. With this consideration indicating the distinctive nature of photons and photon momentum $p_r^\mu = q_r^\mu/c\beta$, which we have already alluded to in the early part of this chapter, we construct a set of moments for radiation, which is comparable to the moment set of matter (2.115)–(2.118). The following is the leading elements of the set for radiation:

$$h_r^{(1)\mu\nu} = c^2 (U_\lambda p_r^\lambda)^{-1} \left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3} \Delta_{\sigma\tau} \Delta^{\mu\nu} \right) p_r^\sigma p_r^\tau, \quad (2.146)$$

$$h_r^{(2)} = \frac{1}{3} c^2 (U_\lambda p_r^\lambda)^{-1} \Delta_{\mu\nu} p_r^\mu p_r^\nu - \mathfrak{p}_r / \rho_r, \quad (2.147)$$

$$h_r^{(3)\mu} = c^2 (U_\lambda p_r^\lambda)^{-1} \left\{ \Delta_\sigma^\mu p_r^\nu p_r^\sigma U_\nu - h_r [p_r^\mu - c^{-2} (U_\lambda p_r^\lambda) U^\mu] \right\}, \quad (2.148)$$

$$h_r^{(4)\mu} = c^2 (U_\lambda p_r^\lambda)^{-1} [p_r^\mu - c^{-2} (p_r^\lambda U_\lambda) U^\mu]. \quad (2.149)$$

Here it must be noted that $(p_r^\nu p_{r\nu}) = 0$ which makes it unnecessary to use h_r unmodified in contrast to the case of matter; (2.117). The moment set presented here consists of orthogonal tensor polynomials $h_r^{(\delta)}(q_r)$, which are constructed by means of the Schmidt orthogonalization method similarly to $h_m^{(\delta)}$ presented in (2.115)–(2.118). They can be expressible in terms of isotropic tensors of unit dimensionless four-vectors q_r^ν .

With the hydrodynamic four-velocity defined for matter and radiation together, we define the relativistic peculiar velocity $\mathfrak{C}_a^\sigma$ in the same manner as for case of matter alone. Therefore the index now covers both matter and radiation. With so defined $\mathfrak{C}_a^\sigma$, the moment set for both matter and radiation can be expressed in terms of relativistic peculiar velocities. Such moment sets can be used to examine the nonrelativistic limits of the relativistic generic evolution equations for both matter and radiation in the same line of approach as taken for the case of matter alone in the previous chapter.

As is the case for the matter parts of the macroscopic fluxes, if the radiation part of the macroscopic supermoment $\psi_r^{(q)\sigma\mu\cdots\nu}$ is defined by the statistical mechanical expression

$$\psi_r^{(q)\sigma\mu\cdots\nu} = \langle f_r(x, p_r) p_r^\sigma h_r^{(q)\mu\cdots\nu} \rangle. \quad (2.150)$$

Then the macroscopic fluxes $\Phi_r^{(q)\mu\cdots\nu}$ are obtained by contracting $\psi_r^{(q)\sigma\mu\cdots\nu}$ with U^σ —projecting $\psi_r^{(q)\sigma\mu\cdots\nu}$ onto hydrodynamic velocity U^σ :

$$\Phi_r^{(q)\mu\cdots\nu} = c^{-2} U_\sigma \psi_r^{(q)\sigma\mu\cdots\nu}. \quad (2.151)$$

As is for material particles, the leading macroscopic moments (i.e. fluxes) $\Phi_r^{(q)\mu\cdots\nu}$ for radiation (photon) are identified with the radiation shear stress tensor $\Pi_r^{\mu\nu}$; the excess trace part of the radiation stress tensor $\tilde{\Delta}_r$; the heat flux Q_r^μ ; the number flux J_r^μ . Then their statistical definitions can be obtained if the energy-momentum tensor of radiation is made use of

$$T_r^{\mu\nu} = \langle p_r^\mu p_r^\nu f_r(x, p_r) \rangle. \quad (2.152)$$

Recall that since $T_r^{\mu\nu}$ is decomposable into $\mathcal{E}_r$, $Q_r'^\mu$, $\mathfrak{J}_r^\mu := hJ_r^\mu$, and $P_r^{\mu\nu}$ with the help of the projection operators. It is then straightforward to find the desired statistical mechanical formulas for the decomposed components. Note also that as is for matter the radiation stress tensor is decomposable as

$$P_r^{\mu\nu} = p_r^e \Delta^{\mu\nu} + \tilde{\Delta}_r \Delta^{\mu\nu} + \Pi_r^{\mu\nu}, \quad (2.153)$$

where p_r^e is the equilibrium radiation pressure and

$$\tilde{\Delta}_r = \frac{1}{3} \Delta_{\mu\nu} P_r^{\mu\nu} - p_r^e = \mathfrak{p}_r - p_r^e, \quad (2.154)$$

$$\Pi_r^{\mu\nu} = P_r^{\mu\nu} - \frac{1}{3} \Delta_{\sigma\tau} T_r^{\sigma\tau} \Delta^{\mu\nu} \quad (2.155)$$

with $\mathfrak{p}_r$ denoting the nonequilibrium radiation pressure—the trace part of the radiation pressure tensor $P_r^{\mu\nu}$. The excess normal stress of radiation $\tilde{\Delta}_r$ defined by (2.154) presumes the possibility of radiation pressure $\mathfrak{p}_r$ fluctuating from p_r^e if the radiation is in nonequilibrium.

Therefore, as for the case of matter, we can easily find the physical meanings of $\Phi_r^{(q)\mu\cdots\nu}$ ($q = 1, 2, \dots, 4$) as follows:

$$\begin{aligned} \Phi_r^{(1)\mu\nu} &= \Pi_r^{\mu\nu}; \quad \Phi_r^{(2)} = \tilde{\Delta}_r; \\ \Phi_r^{(3)\mu} &= Q_r^\mu - hJ_r^\mu := Q_r'^\mu; \quad \Phi_r^{(4)\mu} = J_r^\mu; \text{ etc.} \end{aligned} \quad (2.156)$$

The evolution equations for $\Phi_r^{(q)}$ can also be derived from the covariant Boltzmann equation (2.13) by using the same method as for the matter parts. They may be written in the form

$$\rho D \hat{\Phi}_r^{(q)\mu\cdots\nu} = -\partial_\sigma (\Delta_\gamma^\mu \psi_r^{(q)\sigma\gamma\cdots\nu}) + Z_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu}, \quad (2.157)$$

where

$$Z_r^{(q)\mu\cdots\nu} = \langle p_r^\sigma \partial_\sigma h_r^{(q)\mu\cdots\nu} f_r(x, p_r) \rangle, \quad (2.158)$$

$$\Lambda_r^{(q)\mu\cdots\nu} = \langle h_r^{(q)\mu\cdots\nu} \mathfrak{R}_r[f_r(x, p_r)] \rangle \quad (q = 1, 2, \dots). \quad (2.159)$$

Equation (2.157) can be also put into a modified form

$$\rho D \widehat{\Phi}_r^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_r^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu} \quad (2.160)$$

together with the modified kinematic term $\mathfrak{Z}_r^{(q)\mu\cdots\nu}$ and the higher order term $\Theta_r^{(q)\sigma\mu\cdots\nu}$ in the divergence term similar to (2.132) and (2.133):

$$\mathfrak{Z}_r^{(q)\mu\cdots\nu} = Z_r^{(q)\mu\cdots\nu} - c^{-2} U_\sigma D \left(\Delta_\omega^\sigma \psi_r^{(q)\omega\mu\cdots\nu} \right), \quad (2.161)$$

$$\Theta_r^{(q)\sigma\mu\cdots\nu} = \Delta_\omega^\sigma \psi_r^{(q)\omega\mu\cdots\nu}. \quad (2.162)$$

The evolution equations for conserved variables and nonconserved variables for both matter and radiation are what we immediately need to formulate a theory of irreversible macroscopic processes in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ for the system of matter and radiation under consideration. The procedure used for evaluation of modified kinematic terms $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ ($a = 1, 2, \dots, m, r; q = 1, 2, \dots$) is the same as for the generic evolution equation studied in Chap. 1 of this volume. We collect and summarize them in the following before proceeding to investigate them with regard to irreversible thermodynamics undergoing in the system.

2.5 Summary of Macroscopic Evolution Equations

Summarized in subsections below are the balance equation for conserved and non-conserved variables derived from the covariant kinetic equations for both matter and radiation.

2.5.1 Equation of Continuity

The equation of continuity holds for the combined system of matter and radiation

$$D\rho = -\rho \nabla_\mu U^\mu. \quad (2.163)$$

This equation in essence is the evolution equation for density.

2.5.2 Conservation Laws for Matter

The evolution equations for the concentration fraction of matter, the hydrodynamic velocity, and the energy of matter are listed in the order as below:

$$\rho D\mathfrak{c}_a = -\partial_\mu J_a^\mu + c^{-2} J_a^\mu D U_\mu + \Lambda_a^{(n)}, \quad (2.164)$$

$$\begin{aligned} c^{-2} \mathcal{E}_m \rho D U^\mu &= -\nabla^\mu p_m - \Delta_\sigma^\mu \partial_\nu (\Pi_m^{\sigma\nu} + \tilde{\Delta}_m \Delta^{\sigma\nu}) \\ &\quad + c^{-2} (P_m^{\mu\nu} D U_\nu - \Delta_\omega^\mu D Q_m^\omega - Q_m^\mu \nabla_\nu U^\nu - Q_m^\nu \nabla_\nu U^\mu) \\ &\quad + \Delta_\nu^\mu \Lambda_m^{(n)\nu}, \end{aligned} \quad (2.165)$$

$$\rho D \mathcal{E}_m = -\partial_\mu Q_m^\mu + P_m^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_m^\mu D U_\mu + c^{-2} U_\nu \Lambda_m^\nu. \quad (2.166)$$

2.5.3 Conservation Laws for Radiation

The evolution equations are listed for the concentration fraction for radiation and the radiation energy in the following order:

$$\rho D\mathfrak{c}_r = -\partial_\nu J_r^\nu + c^{-2} J_r^\nu D U_\nu + \Lambda_r^{(n)\mu}, \quad (2.167)$$

$$\rho D \mathcal{E}_r = -\partial_\nu Q_r^\nu + P_r^{\mu\nu} \nabla_{\nu\mu} + c^{-2} Q_r^\nu D U_\nu + c^{-2} U_\mu \Lambda_r^\mu. \quad (2.168)$$

2.5.4 Evolution Equations for Nonconserved Variables for Matter

The generic evolution equations for nonconserved variables for matter species a can be given in generic form

$$\rho D \widehat{\Phi}_a^{(q)\sigma\mu\cdots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x), \quad (2.169)$$

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \Delta_\gamma^\mu \psi_a^{(q)\gamma\sigma\cdots\nu}, \quad (2.170)$$

where $\Theta_a^{(q)\sigma\mu\cdots\nu}$ is the flux of supermoment defined earlier, which may be regarded as the flux of macroscopic moment $\Phi_a^{(q)\mu\cdots\nu}$. It is a moment one-order higher than $\Phi_a^{(q)\mu\cdots\nu}$ in the language of moment methods. Because of $\Theta_a^{(q)\sigma\mu\cdots\nu}$ the evolution equation (2.169) represents an open hierarchy of evolution equations for macroscopic moments. The index a runs for all matter species in the system, and $q = 1, 2, 3, 4$, etc. For the entire matter as a whole the generic evolution equation reads

$$\rho D \widehat{\Phi}_m^{(q)\sigma\mu\cdots\nu} = -\nabla_\sigma \Theta_m^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_m^{(q)\mu\cdots\nu} + \Lambda_m^{(q)\mu\cdots\nu}(x), \quad (2.171)$$

$$\Theta_m^{(q)\sigma\mu\cdots\nu} = \Delta_\gamma^\mu \psi_m^{(q)\gamma\sigma\cdots\nu}. \quad (2.172)$$

The kinematic terms and $\Theta_a^{(q)\sigma\mu\cdots\nu}$ can be evaluated in terms of variables belonging to the manifold $\mathfrak{P}$ by using exactly the same procedure and method and the same expressions as described in the previous chapter. The dissipation terms are calculated with the collision integral $\mathfrak{R}[f_a]$ of the covariant kinetic equation (2.13). Although

the details of $\mathfrak{R}[f_a]$ is different from the kinetic equation used in the previous chapter, the formal structure of the dissipation terms remains the same as those in the previous chapter with regard to the dependence of generalized potentials and ultimately macroscopic moments $\Phi_a^{(q)\mu\cdots\nu}$. Therefore, the results of evaluation of the kinematic terms of the material part are not presented here to avoid repetition.

2.5.5 Evolution Equations for Nonconserved Variables for Radiation

The generic evolution equation for nonconserved variables for radiation is given by formally the same equation as the material part (2.171):

$$\rho D \widehat{\Phi}_r^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_r^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu}(x), \quad (2.173)$$

$$\Theta_r^{(q)\sigma\mu\cdots\nu} = \Delta_\gamma^\mu \psi_r^{(q)\gamma\sigma\cdots\nu}, \quad (2.174)$$

but the kinematic terms $\mathfrak{Z}_r^{(q)\mu\cdots\nu}$ will be worked out since the details of the moments $h_r^{(q)\mu\nu}$ in (2.146)–(2.149) are slightly different from their matter counter parts $h_a^{(q)\mu\nu}$ given in (2.115)–(2.118). They constitute counterparts to the kinematic terms for relativistic evolution equations of matter given in Chap. 1 of this volume. They will be explicitly presented in Sect. 2.12.2 of this chapter. For more explicit formulas for dissipation terms $\Lambda_a^{(q)\mu\cdots\nu}$ and $\Lambda_r^{(q)\mu\cdots\nu}$ some suitable approximations will be required, so that they remain consistent with the laws of thermodynamics. Since a thermodynamic theory of irreversible processes is required to discuss the thermodynamic consistency of the evolution equations formulated here we would like turn our attention to the formulation of theory of irreversible thermodynamics.

2.6 Calortropy and Its Balance Equation

To make use of the laws of thermodynamics in developing a kinetic theory of irreversible transport processes it is necessary to link up the kinetic theory with the laws of thermodynamics by some means. For this purpose the conventional trail of thought followed in kinetic theory is to regard the Boltzmann entropy and, particularly, the H theorem as the statistical mechanical representations for the Clausius entropy and the second law of thermodynamics, respectively. We have avoided this line of thinking for the reason clarified in the previous chapter. We will follow the same line of approach as for the previous chapters of this work.

Since the second law of thermodynamics is entirely phrased in terms of macroscopic observables for the system of interest, which are described by a system of Pfaffian differential forms, the Boltzmann entropy must seamlessly fit in such mathematical structures in the appropriate macroscopic variable manifold, if we are to

reach the desired goal. To be more specific, the differential of Boltzmann entropy, in particular, must be an exact Pfaffian differential in the manifold of macroscopic variables appropriate for thermodynamic description of irreversible processes in the system. We have seen in the previous chapters of this work that this, unfortunately, is not the case for the Boltzmann entropy and the H function. This particular feature would not change for the system of matter and radiation, if the system is away from equilibrium. We, therefore, must look for an alternative to the Boltzmann entropy that will provide us a desired theory of irreversible thermodynamics also for matter and radiation. The mathematical framework for the desired theory is again provided by a quantity called *calortropy*, if the system is away from equilibrium, as we have seen in the previous chapters on the nonrelativistic and relativistic kinetic equations for dilute gases.

We introduce the calortropy four-flow $\Psi^\mu(x)$ by the statistical mechanical formula

$$\Psi^\mu(x) = -k_B \sum_{a=1}^r \langle p_a^\mu [f_a \ln f_a^c - \epsilon_a (1 + \epsilon_a f_a) \ln (1 + \epsilon_a f_a^c)] \rangle, \quad (2.175)$$

where $f_a^c(x, p_a)$ is called the nonequilibrium canonical form, which is not necessarily equal to $f_a(x, p_a)$ at all states of the system except at equilibrium. A more concrete form of nonequilibrium canonical form is given under the functional hypothesis for the distribution function as will be discussed presently. For the moment it will be sufficient to know there exists a nonequilibrium canonical form $f_a^c(x, p_a)$, which is generally not equal to $f_a(x, p_a)$ except at equilibrium. The scalar calortropy density is then obtained if the calortropy four-flow $\Psi^\mu(x)$ is contracted with U^μ :

$$\Psi(x) := \rho \hat{\Psi}(x) = c^{-2} U_\mu \Psi^\mu(x). \quad (2.176)$$

Geometrically, this is a projection of Ψ^μ onto the direction of U^μ . The balance equation for the calortropy density is obtained from the formal expression (2.175) by following the procedure used to obtain the balance equation (2.25) for $\hat{S}$.

According to the aforementioned procedure, we begin with the substantial derivative of $\hat{\Psi}$ expressed in the form

$$\rho D \hat{\Psi} = -\partial_\mu (\Psi^\mu - \hat{\Psi} N^\mu) + \partial_\mu \Psi^\mu, \quad (2.177)$$

which follows upon operating on $\hat{\Psi}$ the substantial time derivative operator $\mathfrak{D} = N^\mu \partial_\mu = \rho D + N^\mu \nabla_\mu = \rho D$. We then define the calortropy flux four-flow J_c^μ by the formula

$$J_c^\mu = \Psi^\mu - \hat{\Psi} N^\mu. \quad (2.178)$$

If the definitions of number four-flow N^μ and the projector $\Delta^{\mu\nu}$ are made use of, the calortropy flux four-flow J_c^μ can be written as a projection of Ψ^ν in the direction perpendicular to U^ν :

$$J_c^\mu = \Delta^\mu_\nu \Psi^\nu. \quad (2.179)$$

Substituting (2.175) into this expression, we obtain the statistical mechanical formula for the calortropy flux four-flow

$$J_c^\mu = -k_B \sum_{a=1}^r \Delta_\nu^\mu \langle p_a^\nu [f_a \ln f_a^c - \epsilon_a (1 + \epsilon_a f_a) \ln (1 + \epsilon_a f_a^c)] \rangle. \quad (2.180)$$

For the balance equation for Ψ^μ , upon operating ∂_μ on (2.175) we obtain the calortropy balance equation for calortropy four-flow Ψ^μ

$$\partial_\mu \Psi^\mu(x) = \Sigma_c + \sigma_c, \quad (2.181)$$

where σ_c is the calortropy production defined by the statistical mechanical formula

$$\sigma_c(x) = k_B \sum_{a=1}^r \langle \ln(\epsilon_a + f_a^{c-1}) \Re_a[f_a] \rangle, \quad (2.182)$$

and Σ_c a kinematic term defined by the formula

$$\Sigma_c = k_B \sum_{a=1}^r \langle p_a^\mu [f_a \partial_\mu \ln(1 + \epsilon_a f_a^{c-1}) + \epsilon_a \partial_\mu \ln(f_a^c + \epsilon_a)] \rangle. \quad (2.183)$$

Substituting (2.179), (2.182), and (2.183) into (2.177) we obtain the calortropy density balance equation

$$\rho D \widehat{\Psi} = -\partial_\mu J_c^\mu + \Sigma_c + \sigma_c(x). \quad (2.184)$$

Although reminiscent of the Boltzmann entropy balance equation (2.25), this balance equation is clearly different from it in its mathematical structure, as we have seen in the previous chapter for the case of matter alone. We, in fact, notice that the calortropy balance equation is in the ranks of nonconserved variables such as $\Phi_a^{(q)}$ or $\Phi_r^{(q)}$ —see (2.129) or (2.157) for the evolution equation for macroscopic moment $\Phi_a^{(q)}$ —because the term Σ_c may be regarded as the kinematic term in the evolution equation for $\widehat{\Psi}$, and $\sigma_c(x)$ its dissipation term. Furthermore, $f_a^c(x, p_a)$ is, evidently, not an exact solution of the covariant kinetic equation in the sense that it will be found to be a projection of $f_a(x, p_a)$ onto the thermodynamic manifold characterizing the system at the level of thermodynamic description of the irreversible processes. In fact, f_a^c may be qualified for an approximate solution of the kinetic equation, which may be sought as closely to the exact solution f_a as possible. Equation (2.184), together with (2.182) and (2.183), forms the basis of formulating a thermodynamic theory of irreversible processes as will be shown presently.

2.7 Relative Boltzmann Entropy and Its Balance Equation

Since the calortropy four-flow is obviously not the same as the Boltzmann entropy four-flow we may examine their difference

$$S_r^\mu [f|f^c](x) = \Psi^\mu(x) - S^\mu(x), \quad (2.185)$$

which we call the relative Boltzmann entropy. It contains the portion of molecular theoretic information lost by the nonequilibrium canonical form f_a^c and will tell us of the extent of information shedding incurred by replacing f_a with f_a^c in $\ln f_a$ and $\ln(1 + \epsilon_a f_a)$ in the expression for $S^\mu(x)$.

On inserting the statistical mechanical formulas for S^μ and Ψ^μ into (2.185) the statistical mechanical formula for the relative Boltzmann entropy is obtained:

$$S_r^\mu [f|f^c](x) = k_B \sum_{a=1}^r \left\langle p_a^\mu \left[f_a \ln \left(\frac{f_a}{f_a^c} \right) - \epsilon_a (1 + \epsilon_a f_a) \ln \left(\frac{1 + \epsilon_a f_a}{1 + \epsilon_a f_a^c} \right) \right] \right\rangle. \quad (2.186)$$

The scalar relative Boltzmann entropy density is obtained by contracting four-flow $S_r^\mu [f|f^c]$ with U_μ :

$$\mathcal{S}_r[f|f^c] = \rho \widehat{\mathcal{S}}_r[f|f^c] = c^{-2} U_\mu S_r^\mu [f|f^c]. \quad (2.187)$$

Its balance equation can be easily found to have the form

$$\rho D \widehat{\mathcal{S}}_r[f|f^c] = -\partial_\mu J_r^\mu [f|f^c] + \Sigma_c(x) + [\sigma_c(x) - \sigma_{\text{ent}}(x)]. \quad (2.188)$$

with the relative Boltzmann entropy flux $J_r^\mu [f|f^c]$ given by the expression

$$\begin{aligned} J_r^\mu [f|f^c] &= \Delta_\nu^\mu S_r^\nu [f|f^c] = \Delta_\nu^\mu (\Psi^\nu - S^\nu) \\ &= k_B \sum_{a=1}^r \Delta_\nu^\mu \left\langle p_a^\nu \left[f_a \ln \left(\frac{f_a}{f_a^c} \right) - \epsilon_a (1 + \epsilon_a f_a) \ln \left(\frac{1 + \epsilon_a f_a}{1 + \epsilon_a f_a^c} \right) \right] \right\rangle. \end{aligned} \quad (2.189)$$

We will return to examine and further elucidate the formal balance equations for $\widehat{\Psi}$ and $\widehat{\mathcal{S}}_r[f|f^c]$ after the nonequilibrium canonical form f_a^c and its attendant theory of macroscopic transport processes are more explicitly developed.

2.8 Functional Hypothesis and Nonequilibrium Canonical Form

We have formally derived a complete set of macroscopic evolution equations from the covariant kinetic equation. The formal evolution equations include the conservation laws as well as the evolution equations for nonconserved variables. By making the macroscopic evolution equations to conform to the principles of thermodynamics, it would be possible to formulate a thermodynamically consistent theory of transport processes and hydrodynamics out of the aforementioned evolution equations, if we adopt the procedure taken in the cases of kinetic equations considered in the previous chapters in this work. Thus, we now would like accordingly to formulate the thermodynamics of irreversible processes and generalized hydrodynamics for a system of radiation and matter. This section is devoted to the preparation to achieve the stated aim.

Since we are interested in thermodynamic description of macroscopic transport processes in the system, any solution of the covariant kinetic equation (2.13), regardless of whether exact or approximate, should be formulated to yield a thermodynamic theory of irreversible processes that is consistent with the laws of thermodynamics. In the absence of, and the unlikelihood of acquiring, an exact solution known to produce such a theory, we must be content to explore an approximate solution.⁸ We would like to look for such an approximate solution under the functional hypothesis as in the case of nonrelativistic kinetic equations and the covariant kinetic equation for a single-component gas discussed in the previous chapters. At the risk of being repetitive, we state the functional hypothesis for the distribution function below:

Functional Hypothesis: *The distribution function obeying the covariant kinetic equation (2.13) evolves as a functional of macroscopic observables spanning the thermodynamic manifold for the system, which obey their own evolution equations descending from the kinetic equation.*

In the functional hypothesis, the aforementioned evolution equations of macroscopic variables serve as field equations through which the evolution of the distribution function is described, since they provide the spacetime dependence of the distribution function. This distribution function denoted by $f_a^c(x, p_a)$, which has already symbolically appeared in calortropy and relative Boltzmann entropy, is called nonequilibrium canonical form. It may be, in fact, regarded as a thermodynamic branch of the distribution $f_a(x, p_a)$ obeying the covariant kinetic equation, since it affords us with thermodynamics of irreversible processes. It should be emphasized that it is not the same as $f_a(x, p_a)$ in general: $f_a^c(x, p_a) \neq f_a(x, p_a)$, except at equilibrium to which the system approaches over a sufficiently long period of time. However, we will require both $f_a(x, p_a)$ and $f_a^c(x, p_a)$ to have the same normalization to the number density. The consequence of this requirement is that

⁸The solution is approximate in the sense that it is given by a projection of the phase space distribution function onto the thermodynamic manifold of macroscopic variables whose dimension is much smaller than the full phase space of the system.

$$\rho_i = c^{-2} U_\mu \langle p_i^\mu f_i \rangle = c^{-2} U_\mu \langle p_i^\mu f_i^c \rangle$$

for both matter species and radiation as was done in the case of matter alone in the previous chapter. This requirement is physically sensible and reasonable.

In the modified moment method [5, 17, 18] used in the present work, $f_a^c(x, p_a)$ is constructed with the nonconserved moments, denoted by tensors $h_a^{(q)\mu\cdots\nu}$, such that the evolution equations of the moments are consistent with the second law of thermodynamics. The relativistic extension of this method for a material particle mixture is presented in Chap. 1 of this Volume. In the present case of radiation and matter the nonequilibrium canonical form assumes distinctive forms for radiation (r) and matter ($a = 1, \dots, m$): $f_i^c(x, p_i)$ ($i = 1, \dots, m, r$).

2.8.1 Nonequilibrium Canonical Form for Material Particles

The nonequilibrium canonical form of distribution may be taken for the material particles in the form

$$f_a^c(x, p_a) = (e^{\beta \mathcal{H}_a(p_a)} - \epsilon_a)^{-1} \quad (a = 1, 2, \dots, m), \quad (2.190)$$

where $\mathcal{H}_a(p_a)$ is given by the formula

$$\mathcal{H}_a(p_a) = p_a^\mu U_\mu + \sum_{q \geq 1} X_{a\alpha\cdots\omega}^{(q)} h_a^{(q)\sigma\cdots\omega} - \mu_a. \quad (2.191)$$

In this expression $\beta := 1/k_B T$, μ_a , $X_{a\alpha\cdots\omega}^{(q)}$, are as-yet-undetermined parameters depending on macroscopic variables and spacetime position x . Their precise meanings and x dependence will be determined as the nonequilibrium (irreversible thermodynamic) theory of macroscopic variables is developed under the functional hypothesis. The parameters β , μ_a , $X_{a\alpha\cdots\omega}^{(q)}$ must be determined such that the second law of thermodynamics is satisfied, and $h_a^{(q)\sigma\cdots\omega}$ are the molecular moments already introduced, which yield macroscopic moments (fluxes), when averaged over the ensemble with the statistical weight $f_a(x, p_a)$. The set of the macroscopic moments so obtained spans the manifold of macroscopic variables necessary to describe macroscopic transport processes in the system. Such macroscopic variables obey the evolution equations of nonconserved variables presented earlier. Henceforth, whenever convenient, the symbol $\odot$ will be used to abbreviate the scalar product of tensors for the sake of notational brevity. Thus with this symbol we write, for example,

$$X_{a\alpha\cdots\omega}^{(q)} h_a^{(q)\sigma\cdots\omega} = X_a^{(q)} \odot h_a^{(q)}$$

for the contraction of tensors.

2.8.2 Nonequilibrium Canonical Form for Radiation

The nonequilibrium canonical form of radiation distribution function is sought after in a form similar to the Planck distribution function—an equilibrium distribution function of radiation. Thus, we take it in the form

$$f_r^c(x, p_r) = \left(e^{\beta \overline{W}_r(p_r)} - 1 \right)^{-1}. \quad (2.192)$$

The function $\overline{W}_r(p_r)$ in the exponent, as yet unspecified, will be also sought after in such a way that it is thermodynamically consistent as is the distribution function for matter $f_a^c(x, p_a)$. This function is sought in a dimensionless form.

We now look for $\overline{W}_r$ in terms of q_r^ν , the dimensionless momentum four-vector defined earlier; see (2.143). It is convenient to define

$$W_r = \beta \overline{W}_r(q_r). \quad (2.193)$$

In view of the well known equilibrium radiation (Planck) distribution function, which has been obtained as an equilibrium solution of the covariant kinetic equation, and the exponential form for $f_a(x, p_a)$ in (2.190), it is sensible to take $\overline{W}_r(q_r)$ in the form

$$\overline{W}_r(q_r) = p_r^\mu U_\mu + H_r^{(1)}(q_r) - \mu_r, \quad (2.194)$$

where μ_r is found to be the chemical potential and $H_r^{(1)}(q_r)$ is the nonequilibrium part that can be determined in a way analogous to the one taken for the matter distribution function f_a^c in (2.190). That is, it is made up of nonequilibrium contributions in the form

$$H_r^{(1)}(q_r) = \sum_{q \geq 1} X_r^{(q)} \odot h_r^{(q)}(q_r), \quad (2.195)$$

where $h_r^{(q)}$ are orthogonal tensor polynomials of $p_r^\nu = q_r^\nu / (c\beta)$, which are also orthogonal to each other and to the conserved moments for radiation. The generalized potentials $X_r^{(q)}$ are the functions of macroscopic variables such as the radiation shear stress, the excess trace part of the radiation stress tensor, radiation heat flux, etc. as well as β and μ_r . The nonequilibrium contribution $\beta H_r^{(1)}$ thus can be expressed as a dimensionless function of the reduced four-vector q_r^ν and dimensionless variables.

In the case of radiation and matter, since the photons are put on equal footing with matter as far as the kinetic processes are concerned and the whole system is considered a mixture of photons and material particles, it is reasonable to define a single temperature for the mixture as has been found to be the case for the equilibrium system considered earlier. The fundamental reason for this point of view is that the photons do not interact with each other and hence cannot come to equilibrium on their own without a help from matter—e.g., material gas. This is the basic reason why the same parameter β , which will turn out to be related to the temperature, is used for both material gases and radiation in the nonequilibrium canonical forms. Consistent

with this viewpoint, we may define the temperature by the statistical mechanical formula for the system⁹

$$\frac{3}{2}\rho_t k_B T = \sum_{a=1}^r \frac{1}{2} \left\langle \left[(p_a^\nu U_\nu)^2 - m_a^2 c^4 \right] f_a(x, p_a) \right\rangle, \quad (2.196)$$

where $m_r = 0$ in the case of photon species and

$$\rho_t = \rho + \frac{2a_{\text{SB}} c^2 T^3}{k_B} \quad (a_{\text{SB}} = \text{Stefan-Boltzmann constant}). \quad (2.197)$$

This definition of temperature is reasonable, especially, for the present dilute gaseous system. It is basically rooted in equipartition law of energy by Tolman [19], which was later discussed by Landsberg [20] and by ter Haar and Wergeland [21]. The second term on the right-hand side (2.196) arises from the recognition that the photon number depends on temperature, but also is not conserved. Writing the nonequilibrium photon number density in such a form means that the temperature is determined such that (2.197) holds for nonequilibrium. Equation (2.196) can be recast into the form

$$\frac{3}{2}\rho_t k_B T = \sum_{a=1}^r \Delta_{\mu\nu} \langle p_a^\nu p_a^\mu f_a(x, p_a) \rangle = \sum_{a=1}^r \Delta_{\mu\nu} T^{\nu\mu}. \quad (2.198)$$

The nonequilibrium canonical forms (2.190) and (2.192) involve parameters β , ρ , ρ_r , and $\mathfrak{p}$, which have been defined statistically, but their operational meanings as thermodynamic variables are not as yet fixed. Their phenomenological meanings can be gained only through corresponding the statistically derived macroscopic variables and their relations to the phenomenological thermodynamic counterparts. This procedure is facilitated by deriving the statistical mechanical relations of macroscopic variables, which are consistent with the laws of thermodynamics. Therefore it is now necessary to formulate a theory of thermodynamics of irreversible processes in the system of radiation and matter. To this end we now closely examine the mathematical aspects of calortropy and the related in the light of the laws of thermodynamics for irreversible processes in the system. The approach to this goal is along the same line as for Chap. 1 of Volume 2 and also for the nonrelativistic kinetic theory discussed in Volume 1 of this work.

2.8.3 Calortropy of Matter and Radiation

The calortropy density $\Psi(x)$ of radiation and matter defined by (2.176) can be rearranged to the formula

⁹This is also for the case of matter consisting of a gas.

$$\begin{aligned}\Psi(x) &:= \rho \widehat{\Psi}(x) \\ &= k_B \sum_{a=1}^r c^{-2} U_\mu \langle p_a^\mu [f_a \ln(\epsilon_a + f_a^{c-1}) + \epsilon_a \ln(1 + \epsilon_a f_a^c)] \rangle.\end{aligned}\quad (2.199)$$

From this formula we obtain the statistical mechanical expressions for calortropy density of matter and radiation separately

$$\Psi_m(x) = k_B \sum_{a=1}^m c^{-2} U_\mu \langle p_a^\mu [f_a \ln(\epsilon_a + f_a^{c-1}) + \epsilon_a \ln(1 + \epsilon_a f_a^c)] \rangle, \quad (2.200)$$

$$\Psi_r(x) = k_B c^{-2} U_\mu \langle p_r^\mu [f_r \ln(\epsilon_r + f_r^{c-1}) + \epsilon_r \ln(1 + \epsilon_r f_r^c)] \rangle. \quad (2.201)$$

Upon inserting f_a^c and f_r^c in (2.190) and (2.192), respectively, and making use of definitions of energy densities of matter and radiation, we obtain Ψ_m and Ψ_r :

$$T \widehat{\Psi}_m(x) = \mathcal{E}_m + \mathfrak{p}_m v_m - \sum_{a=1}^m \mu_a \mathfrak{c}_a + \sum_{a=1}^m \sum_{q \geq 1} X_{a\mu\nu\cdots\sigma}^{(q)} \widehat{\Phi}_a^{(q)\sigma\cdots\mu\nu}, \quad (2.202)$$

$$T \widehat{\Psi}_r(x) = \mathcal{E}_r + \mathfrak{p}_r v_r - \mu_r + \sum_{q \geq 1} X_{r\mu\nu\cdots\sigma}^{(q)} \widehat{\Phi}_r^{(q)\sigma\cdots\mu\nu}, \quad (2.203)$$

where pressures are defined for matter and radiation, respectively, by the statistical mechanical formulas

$$\beta \mathfrak{p}_m = \sum_{a=1}^m \epsilon_a \langle c^{-2} U_\mu p_a^\mu \ln(1 + \epsilon_a f_a^c) \rangle, \quad (2.204)$$

$$\beta \mathfrak{p}_r = \epsilon_r \langle c^{-2} U_\mu p_r^\mu \ln(1 + \epsilon_r f_r^c) \rangle, \quad (2.205)$$

and $v_m = 1/\rho_m$, $v_r = 1/\rho_r$, and we have set

$$\beta = 1/k_B T. \quad (2.206)$$

It should be noted that (2.204) and (2.205) are necessary and sufficient conditions for $\widehat{\Psi}_m$ and $\widehat{\Psi}_r$ to be bilinear forms of macroscopic variables in manifold $\mathfrak{P} \cup \mathfrak{T}$. Thus combining (2.202) and (2.203) we obtain the bilinear form for calortropy density of the system:

$$T \widehat{\Psi}(x) = \mathcal{E} + \mathfrak{p} v - \sum_{a=1}^r \mu_a \mathfrak{c}_a + \sum_{a=1}^r \sum_{q \geq 1} X_{a\nu\cdots\mu\sigma}^{(q)} \widehat{\Phi}_a^{(q)\sigma\mu\cdots\nu} \quad (2.207)$$

with

$$\rho \mathcal{E} = \rho_m \mathcal{E}_m + \rho_r \mathcal{E}_r = \sum_{a=1}^r \rho_a \mathcal{E}_a \quad \text{or} \quad \mathcal{E} = \sum_{a=1}^r \epsilon_a \mathcal{E}_a. \quad (2.208)$$

The physical meanings of parameters T , $\mathfrak{p}$, μ_a , and $X_a^{(q)}$ will be presently elucidated by making thermodynamic correspondence of statistical mechanical result for $\widehat{\Psi}$ with the phenomenological (i.e., thermodynamic) calortropy density.

2.9 Calortropy Differential

The conserved variable balance equations and the flux (moment) evolution equations for radiation and matter derived in the previous sections must be subjected to the demands of the second law of thermodynamics, so that the theory of irreversible processes based them becomes consistent with the second law of thermodynamics. To achieve this aim the calortropy balance equation must be shown integrable in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$.

2.9.1 Pfaffian Form for Calortropy

To prove the integrability of the calortropy density balance equation it is necessary to show the right hand side of (2.184) must be expressible in an integrable differential form in the thermodynamic manifold. If f_a^c is substituted into the $\ln f_a^c$ term in (2.180), the divergence term in question is given by the expression

$$\begin{aligned} \partial_\mu (\Psi^\nu \Delta_\nu^\mu) = & -k_B \sum_{a=1}^r \beta \partial_\mu Q_a^\mu - k_B \sum_{a=1}^r Q_a^\mu \partial_\mu \beta \\ & + k_B \sum_{a=1}^r \beta X_{a\alpha\cdots\sigma}^{(q)} \partial_\mu (\Delta_\nu^\mu \psi_a^{(q)\sigma\cdots\alpha\nu}) \\ & + k_B \sum_{a=1}^r \Delta_\nu^\mu \psi_a^{(q)\sigma\cdots\alpha\nu} \partial_\mu (\beta X_{a\alpha\cdots\sigma}^{(q)}) \\ & + k_B \sum_{a=1}^r \partial_\mu \Delta_\nu^\mu (\epsilon_a p_a^\nu \ln (1 + \epsilon_a f_a^c)). \end{aligned} \quad (2.209)$$

On the other hand, the kinematic term Σ_c of the calortropy balance equation is expressible by the form

$$\begin{aligned}
\Sigma_c = & T^{\mu\nu} U_\nu \partial_\mu T^{-1} + \frac{1}{T} T^{\mu\nu} \partial_\mu U_\nu + \sum_{a=1}^r \sum_{q \geq 1} \psi_a^{(q)\mu} \odot \partial_\mu \left(\frac{X_a^{(q)}}{T} \right) \\
& + \sum_{a=1}^r \sum_{q \geq 1} \frac{X_a^{(q)}}{T} \odot \langle f_a \partial_\mu p_a^\mu h_a^{(q)} \rangle - \sum_{a=1}^r \partial_\mu \left(\frac{\mu_a}{T} N_a^\mu \right) \\
& + k_B \sum_{a=1}^r \epsilon_a \langle p_a^\mu \partial_\mu \ln(1 + \epsilon_a f_a^c) \rangle
\end{aligned} \tag{2.210}$$

and the calortropy production σ_c by a linear combination of dissipation terms $\Lambda_a^{(q)}$ of the nonconserved variable evolution equations:

$$\sigma_c = T^{-1} \sum_{a=1}^r \sum_{q \geq 1} X_a^{(q)} \odot \Lambda_a^{(q)}. \tag{2.211}$$

Upon substituting the three components (2.209)–(2.211) into the calortropy density balance equation, we obtain the calortropy density differential

$$\begin{aligned}
D\widehat{\Psi} = & T^{-1} \left(D\mathcal{E} + \mathbf{p} Dv - \sum_{a=1}^r \mu_a D\mathbf{c}_a + \sum_{a=1}^r \sum_{q \geq 1} X_a^{(q)} \odot D\widehat{\Phi}_a^{(q)} \right) \\
& + \mathcal{E} D T^{-1} + v D(\mathbf{p} T^{-1}) - \sum_{a=1}^r \mathbf{c}_a D \left(\frac{\mu_a}{T} \right) + \sum_{a=1}^r \sum_{q \geq 1} \widehat{\Phi}_a^{(q)} \odot D \left(\frac{X_a^{(q)}}{T} \right),
\end{aligned} \tag{2.212}$$

if

$$\mathbf{p} = \mathbf{p}_m + \mathbf{p}_r \tag{2.213}$$

and $\mathbf{p}_m$ and $\mathbf{p}_r$ are defined by (2.204) and (2.205), respectively. Therefore, the definitions of macroscopic parameters $\mathbf{p}_m$ and $\mathbf{p}_r$ by the formulas (2.204) and (2.205) are necessary and sufficient conditions for (2.212) to exist. The physical meanings of parameters $\mathbf{p}_m$ and $\mathbf{p}_r$ remain as yet not elucidated except that they are macroscopic parameters. Similarly, the meanings of parameter T , μ_a , and $X_a^{(q)}$ also remain undetermined as macroscopic variables. In any case, to obtain this Pfaffian differential form [22] (2.212) we have made use of the energy balance equations for matter and radiation, and the identities listed below:

$$\begin{aligned}
\partial_\mu \left(\frac{\mu_a}{T} N_a^\mu \right) &= \partial_\mu \left(\frac{\mu_a}{T} \mathbf{c}_a \rho U^\mu \right) \\
&= \rho U^\mu \partial_\mu \left(\frac{\mu_a}{T} \mathbf{c}_a \right) + \frac{\mu_a}{T} \mathbf{c}_a \partial_\mu (\rho U^\mu) \\
&= \rho D(\beta \mu_a \mathbf{c}_a),
\end{aligned} \tag{2.214}$$

$$\partial_\mu (\rho U^\mu) = D\rho + \rho \nabla_\mu U^\mu = 0, \quad (2.215)$$

and

$$\begin{aligned} \partial_\mu \left(\frac{\mathfrak{p}v}{T} \rho U^\mu \right) &= \frac{\mathfrak{p}v}{T} \partial_\mu (\rho U^\mu) + \rho U^\mu \partial_\mu \left(\frac{\mathfrak{p}v}{T} \right) \\ &= \rho U^\mu \partial_\mu \left(\frac{\mathfrak{p}v}{T} \right) \\ &= \rho D \left(\frac{\mathfrak{p}v}{T} \right) \\ &= \frac{\mathfrak{p}}{T} Dv + \rho v D \left(\frac{\mathfrak{p}}{T} \right). \end{aligned} \quad (2.216)$$

The second line of the Pfaffian differential form (2.212) vanishes because if the normalization conditions for the nonequilibrium canonical forms are varied with respect to the parameters β , μ_a , $X_a^{(q)}$, there follows the vanishing differential form

$$\mathcal{E} D T^{-1} + v D (\mathfrak{p} T^{-1}) - \sum_{a=1}^r \mathfrak{c}_a D \left(\frac{\mu_a}{T} \right) + \sum_{a=1}^r \sum_{q \geq 1} \widehat{\Phi}_a^{(q)} \odot D \left(\frac{X_a^{(q)}}{T} \right) = 0. \quad (2.217)$$

This differential form will turn out to be the nonequilibrium Gibbs–Duhem (NGD) equation, when the parameters T^{-1} , $\mathfrak{p}$, μ_a , and $X_a^{(q)}$ are identified with their thermodynamic correspondents. Thus, by virtue of (2.217) the Pfaffian differential form (2.212) reduces to the differential form for $\widehat{\Psi}$

$$D\widehat{\Psi} = T^{-1} \left(D\mathcal{E} + \mathfrak{p} Dv - \sum_{a=1}^r \mu_a D\mathfrak{c}_a + \sum_{a=1}^r \sum_{q \geq 1} X_a^{(q)} \odot D\widehat{\Phi}_a^{(q)} \right), \quad (2.218)$$

which is known as the extended Gibbs relation for $\widehat{\Psi}$. This differential form is an exact differential in the thermodynamic manifold, because, when the nonequilibrium Gibbs–Duhem equation is added side by side to it, (2.218) integrates exactly to a bilinear form in the nonequilibrium Gibbs manifold $\mathfrak{G} = \mathfrak{P} \cup \mathfrak{T} \cup \Psi$ for the calortropy density given in (2.207). For this reason the NGD equation (2.217) becomes the necessary and sufficient condition for the integrability of the extended Gibbs relation (2.218). The differential form (2.218) for $\widehat{\Psi}$ is in the same form as for relativistic gas mixture in the absence of radiation obtained in Chap. 1 of this Volume. This is in contrast to the Boltzmann entropy differential $D\mathcal{S}$, which, unless the system is in equilibrium, cannot be put into an exact Pfaffian differential in the Gibbs manifold, and it gives rise to the conclusion that the extended Gibbs relation does not hold for $\mathcal{S}$ and neither is there the nonequilibrium Gibbs–Duhem equation for $\mathcal{S}$.

When the parameters T , $\mathfrak{p}$, μ_a , and $X_a^{(q)}$ are elucidated of their physical (i.e., thermodynamic) meanings on correspondence with the phenomenological thermodynamics, the pair of differential forms, namely, the extended Gibbs relation and the

nonequilibrium Gibbs–Duhem equation, provides the foundation of thermodynamics of irreversible processes for the system of matter and radiation. The following pair of vanishing circular integrals reinforces the aforementioned statement.

The Pfaffian differential form (2.218) for $\widehat{\Psi}$ gives rise to a vanishing circular (i.e., cyclic) integral when integrated over a path of an irreversible process in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. In a local rest frame it may be written as [5]

$$\oint d\widehat{\Psi} = 0. \quad (2.219)$$

It is a representation of the second law of thermodynamics in an integral form in manifold $\mathfrak{P} \cup \mathfrak{T}$. And in this sense the thermodynamic theory of irreversible processes described in terms of generalized hydrodynamic equations in the Gibbs manifold $\mathfrak{G}$ is consistent with the laws of thermodynamics. In addition to this vanishing integral for the second law, there also holds a vanishing integral of internal energy differential in manifold $\mathfrak{P} \cup \mathfrak{T}$, representing the first law of thermodynamics

$$\oint d\mathcal{E} = 0. \quad (2.220)$$

We have shown that this is the case in the nonrelativistic formalism in the nonrelativistic theory chapters of Volume 1. The same conclusion robustly holds up even if the system is relativistic and away from equilibrium: *The thermodynamic laws thus may be represented by a pair of vanishing circular integrals over an irreversible path in the thermodynamic manifold.*

Further progress in the theory of transport processes in the system can be made if the unknowns $X_a^{(q)}$ are suitably approximated in a thermodynamically consistent manner, such that $D\widehat{\Psi}$ remains an exact Pfaffian differential in the Gibbs manifold $\mathfrak{G} = \mathfrak{P} \cup \mathfrak{T} \cup \Psi$ and the theory of macroscopic irreversible processes remains thereby thermodynamically consistent.

2.9.2 Thermodynamic Correspondence

The extended Gibbs relation (2.218)—a Pfaffian differential form—contains as-yet-undetermined parameters $T, \mathbf{p}, \mu_a, X_a^{(q)}$, which must be elucidated of their operational (physical) meanings, so that it becomes a physically and practically useful equation. This objective is accomplished if we invoke the phenomenological irreversible thermodynamic extended Gibbs relation derived from the laws of thermodynamics and make correspondence of the statistical mechanical extended Gibbs relation (2.218) with the phenomenological one, so that the undetermined parameters therein are determined in correspondence to those in the phenomenological relations. That is, if we make the following correspondence between theoretical and thermodynamic observables

$$\begin{aligned}
\widehat{\Psi}|_{\text{st}} &\Leftrightarrow \widehat{\Psi}|_{\text{th}}, & \mathcal{E}|_{\text{st}} &\Leftrightarrow \mathcal{E}|_{\text{th}}, & v|_{\text{st}} &\Leftrightarrow v|_{\text{th}}, \\
c_a|_{\text{st}} &\Leftrightarrow c_a|_{\text{th}}, & \widehat{\Phi}_a^{(q)}|_{\text{st}} &\Leftrightarrow \widehat{\Phi}_a^{(q)}|_{\text{th}} \\
&& (a = 1, \dots, m, r), &&
\end{aligned} \tag{2.221}$$

then there would hold the correspondence between the theoretical parameters $(T, \mathfrak{p}, \mu_a, X_a^{(q)})$ and the thermodynamic parameters appearing in the phenomenological extended Gibbs relation:

$$T|_{\text{st}} \Leftrightarrow T|_{\text{th}}, \quad \mathfrak{p}|_{\text{st}} \Leftrightarrow \mathfrak{p}|_{\text{th}}, \quad \mu_a|_{\text{st}} \Leftrightarrow \mu_a|_{\text{th}}, \quad X_a^{(q)}|_{\text{st}} \Leftrightarrow X_a^{(q)}|_{\text{th}}. \tag{2.222}$$

In (2.221) and (2.222) the subscripts st and th mean statistical mechanical and thermodynamic, respectively. These correspondences between theoretical (statistical mechanical) and thermodynamic (phenomenological) parameters fully endow the thermodynamic meanings on the corresponded parameters, and the nonequilibrium distribution functions are grounded on the thermodynamic principles and experiments. In the relativistic theory the correspondence presented relies on a conjecture that the phenomenological extended Gibbs relation is experimentally realizable, because the extended Gibbs relation is not as yet established experimentally. However, we may reason that the relativistic extended Gibbs relation has the nonrelativistic limit as $u/c \rightarrow 0$. And the validity of the nonrelativistic limit is sufficiently well demonstrated by numerous applications in hydrodynamics as discussed in Volume 1 of this work. Therefore the relativistic extension of extension of irreversible thermodynamics is plausible and reasonable.

2.9.3 *Nonequilibrium Statistical Thermodynamics for Relativistic Quantum Gases*

The formal theory based on the covariant kinetic equation and the nonequilibrium canonical form taken for the distribution function under the functional hypothesis have enabled us to formulate a formal theory of hydrodynamics and a formal theory of thermodynamics of irreversible processes in support of hydrodynamics for the system removed from equilibrium at arbitrary degree. These macroscopic theories must be provided with a molecular theory for various macroscopic observables on which the theories are built. We have seen in the case of nonrelativistic theories discussed in Volume 1 of this work that the stated goal is achieved if nonequilibrium statistical thermodynamics is formulated on the basis of nonequilibrium canonical form and the extended Gibbs relation descending therefrom, since the equilibrium theory is then seamlessly recovered from the nonequilibrium theory as the system tends to equilibrium. In this section we show that the stated goal can be achieved for relativistic quantum dilute gases with the help of the extended Gibbs relation (2.218) if nonequilibrium statistical thermodynamics is formulated for the gases. If

we review the theory of a nonrelativistic noninteracting gas mixture presented in kinetic theory chapters of Volume 1 and the relativistic theory of relativistic gases in Chap. 1 of this Volume and compare them, it is clear how we may proceed in formulating the theory in the case of the system of matter and radiation, bearing in mind that we are dealing with quantum gas particles.

First, let us have a quick review of how equilibrium statistical thermodynamics is formulated for a system consisting of a mixture of quantum ideal gases. If the system consists of a mixture of noninteracting (uncorrelated) gases of N particles ($N = N_1 + N_2 + \cdots + N_r$), the total equilibrium density matrix—an equilibrium grand canonical form—is given by the formula

$$F_e^{(N)} = \exp \left(-\beta \mathcal{H}^{(N)} - \ln \Gamma^e \right), \quad (2.223)$$

where Γ^e is the local grand partition function

$$\Gamma^e = \text{Tr} \prod_{a=1}^r \prod_j \exp \left(-\beta \mathcal{H}_j^{(1)} \right) \quad (2.224)$$

with the Trace operation denoting integration over the entire N particle momentum space, and $\mathcal{H}_j^{(1)}$ are the single particle energy operators

$$\mathcal{H}_j^{(1)} = \varepsilon_{aj} - \mu_a \quad (2.225)$$

with ε_{aj} and μ_a denoting the j th energy eigenvalue and chemical potential of species a , respectively. If the tracing operation is performed according to the Pauli exclusion principle with regard to fermions and bosons, Γ^e can be written in the following products of Fermi–Dirac (FD) and Bose–Einstein contributions:

$$\Gamma^e = \text{Tr} \prod_{a \in FD}^r \prod_j \left(1 + e^{-\beta(\varepsilon_{ja} - \mu_a)} \right) \times \prod_{a \in BE}^r \prod_j \left(1 - e^{-\beta(\varepsilon_{ja} - \mu_a)} \right)^{-1}. \quad (2.226)$$

Since the distribution functions normalize to density, in relativistic theory it is necessary to insert the factor $c^{-2} U_\mu p_a^\mu$ and trace with the integrals over momentum spaces of particles as follows:

$$\text{Tr} A \Rightarrow \langle c^{-2} U_\mu p_a^\mu A \rangle,$$

the angular brackets denoting integration over the momentum space of particle species a , weighted with equilibrium distribution function. For ideal (noninteracting) gases, Γ^e can then be written as

$$\begin{aligned} \ln \Gamma^e &= \sum_{a \in FD} \left\langle c^{-2} U_\mu p_a^\mu \ln \left(1 + e^{-\beta \mathcal{H}_a^{(1)}} \right) \right\rangle \\ &+ \sum_{a \in BE} \left\langle c^{-2} U_\mu p_a^\mu \ln \left(1 - e^{-\beta \mathcal{H}_a^{(1)}} \right)^{-1} \right\rangle \end{aligned} \quad (2.227)$$

in the limit of treating the discrete sum as an integral. Here $\mathcal{H}_a^{(1)}$ now stands for $\mathcal{H}_a^{(1)} = p_a^\nu U_\nu - \mu_a$ in the case of relativistic gases.

If the nonequilibrium canonical form f_a^c is used, the exponential factors in (2.227) can be written as

$$1 + e^{-\beta \mathcal{H}_a^{(1)}} = \frac{1}{1 + \epsilon_a f_a^e} \quad (\epsilon_a = -1 \text{ for fermions; } \epsilon_a = +1 \text{ for bosons}).$$

Hence, it is now possible to write $\ln \Gamma^e$ in a simple unified form

$$\ln \Gamma^e = \sum_{a=1}^r \left\langle c^{-2} U_\mu p_a^\mu \epsilon_a \ln (1 + \epsilon_a f_a^e) \right\rangle, \quad (2.228)$$

where the index a stands for fermion or boson species. Therefore if we define the global form of $\ln \Gamma^e$ by the relation

$$\ln \Xi_e = \int_V d\mathbf{r} \rho \sum_{a=1}^r \ln \Gamma_a^e, \quad (2.229)$$

we find

$$\begin{aligned} k_B T \ln \Xi_e &= \mathfrak{p}^e V \\ &= \int_V d\mathbf{r} \sum_{a=1}^r \left\langle c^{-2} U_\mu p_a^\mu \epsilon_a \ln (1 + \epsilon_a f_a^e) \right\rangle, \end{aligned}$$

and hence

$$\mathfrak{p}^e v = k_B T \sum_{a=1}^r \ln \Gamma_{ea}^{(N)} = k_B T \sum_{a=1}^r \left\langle c^{-2} U_\mu p_a^\mu \epsilon_a \ln (1 + \epsilon_a f_a^e) \right\rangle \quad (2.230)$$

for the statistical mechanical expression for relativistic equilibrium pressure. This confirms (2.204) and (2.205), which have turned out to be statistical mechanical formulas for pressures of matter and radiation, respectively, from the standpoint of the ensemble theory of statistical mechanics. The distribution function (2.223) also gives rise to the equilibrium Gibbs relation and the accompanying theory of equilibrium statistical thermodynamics.

In the case of nonequilibrium relativistic ideal gases, in view of the fact that the nonequilibrium canonical form f_a^c taken under the functional hypothesis gives rise

to the nonequilibrium Gibbs relation, which also tends to the equilibrium Gibbs relation, the nonequilibrium grand canonical form for the N noninteracting quantum particles may be sought for in the form

$$F_c^{(N)} = \exp \left(-\beta \mathcal{H}^{(N)} - \ln \Gamma \right), \quad (2.231)$$

where $\mathcal{H}^{(N)}$ now contains nonequilibrium contributions

$$\begin{aligned} \mathcal{H}^{(N)} &= \sum_{a=FD \text{ or } BE} \sum_j \mathcal{H}_{aj}^{(1)} \\ &= \sum_{a=FD \text{ or } BE} \sum_j \left(p_{aj}^\nu U_\nu + \sum_{q \geq 1} X_{a\alpha \dots \omega}^{(q)} h_{aj}^{(q)} - \mu_a \right) \end{aligned} \quad (2.232)$$

in the notation already defined earlier. Here FD and BE stand for Fermi–Dirac and Bose–Einstein particles. In (2.231) Γ is the local nonequilibrium grand partition function

$$\Gamma = \text{Tr} \prod_{a=1}^r \prod_j \exp \left(\beta \mathcal{H}_{aj}^{(1)} \right) \quad (2.233)$$

with the trace operation denoting integration over all N particle momentum space. If tracing operation is performed according to the Pauli exclusion principle applied to Fermi–Dirac (FD) and Bose–Einstein particles, $\Gamma^{(N)}$ can be written in the following products of fermion and boson contributions:

$$\Gamma = \text{Tr} \prod_{a \in FD}^r \prod_j \left(1 + e^{-\beta \mathcal{H}_{aj}^{(1)}} \right) \times \prod_{a \in BE}^r \prod_j \left(1 - e^{-\beta \mathcal{H}_{aj}^{(1)}} \right)^{-1}. \quad (2.234)$$

Since the distribution functions normalize to density, in the relativistic theory formulation it is necessary to insert the factor $c^{-2} U_\mu p_a^\mu$ and to replace trace with the integrals over momentum spaces of particles as follows:

$$\text{Tr} A \Rightarrow \langle c^{-2} U_\mu p_a^\mu A \rangle$$

the angular brackets denoting integration over momentum space of particle species a . Thus we have

$$\begin{aligned} \ln \Gamma &= \sum_{a \in FD} \left\langle c^{-2} p_a^\mu U_\mu \ln \left(1 + e^{-\beta \mathcal{H}_{aj}^{(1)}} \right) \right\rangle \\ &\quad + \sum_{a \in BE} \left\langle c^{-2} p_a^\mu U_\mu \ln \left(1 - e^{-\beta \mathcal{H}_{aj}^{(1)}} \right)^{-1} \right\rangle \end{aligned} \quad (2.235)$$

in the limit of treating the discrete sum as an integral.

If the nonequilibrium canonical form f_a^c is represented by the form

$$f_a^c = \frac{1}{e^{\beta \mathcal{H}_{aj}^{(1)}} - \epsilon_a}, \quad (2.236)$$

where

$$\epsilon_a = \begin{cases} -1 & \text{for fermions} \\ +1 & \text{for bosons} \end{cases},$$

then $\ln \Gamma$ in (2.235) can be written in a simple unified form

$$\ln \Gamma = \sum_{a=1}^r \epsilon_a \langle c^{-2} p_a^\mu U_\mu \ln (1 + \epsilon_a f_a^c) \rangle, \quad (2.237)$$

where the index a stands for both fermion species or boson species. Therefore if the global form of $\ln \Gamma$ defined by relation

$$\ln \Xi = \int_V d\mathbf{r} \rho \sum_{a=1}^r \ln \Gamma_a \quad (2.238)$$

we find the equation of state for nonequilibrium relativistic pressure p is given by the formula

$$\begin{aligned} k_B T \ln \Xi &= pV \\ &= \int_V d\mathbf{r} \sum_{a=1}^r \epsilon_a \langle c^{-2} p_a^\mu U_\mu \ln (1 + \epsilon_a f_a^c) \rangle. \end{aligned} \quad (2.239)$$

It should be noted that the thermodynamic correspondence has been already made between the statistical thermodynamic parameters and thermodynamic parameters and, consequently, the thermodynamic meanings of various parameters have been established. On the basis of the thermodynamic correspondence we are identifying p with the nonequilibrium thermodynamic pressure.

If the Legendre transformation

$$pv = \sum_{a=1}^r \mu_a \varsigma_a + T \widehat{\Psi} - \sum_{a=1}^r \sum_{q \geq 1} X_{a\alpha \dots \sigma}^{(q)} D \widehat{\Phi}_a^{(q)\sigma \dots \alpha} - \mathcal{E} \quad (2.240)$$

is made, then pv may be regarded as a thermodynamic potential, so that the extended Gibbs relation (2.218)—the differential form—for $\widehat{\Psi}$ is transformed to the exact differential form for the thermodynamic potential pv :

$$D(pv) = \widehat{\Psi}DT + pDv + \sum_{a=1}^r c_a D\mu_a - \sum_{a=1}^r \sum_{q \geq 1} \widehat{\Phi}_a^{(q)} \odot DX_a^{(q)}. \quad (2.241)$$

Making use of this differential form for thermodynamic potential pv or its statistical mechanical expression (grand partition function), which is an extended Gibbs relation (2.241), we obtain the thermodynamic variables in the thermodynamic manifold in the local rest frame in terms of derivatives of $\ln \Gamma$ —namely, nonequilibrium partition function—as below:

$$\widehat{\Psi} = \left(\frac{\partial k_B T \ln \Gamma}{\partial T} \right)_{v, \mu, X} = k_B \ln \Gamma + k_B T \left(\frac{\partial \ln \Gamma}{\partial T} \right)_{v, \mu, X}, \quad (2.242)$$

$$p = k_B T \left(\frac{\partial \ln \Gamma}{\partial v} \right)_{T, \mu, X}, \quad (2.243)$$

$$c_a = k_B T \left(\frac{\partial \ln \Gamma}{\partial \mu_a} \right)_{T, v, \mu', X}, \quad (2.244)$$

$$\widehat{\Phi}_a^{(q)} = -k_B T \left(\frac{\partial \ln \Gamma}{\partial X_a^{(q)}} \right)_{T, v, \mu, X'} \quad (a = 1, \dots, m, r). \quad (2.245)$$

Together with the differential form (2.241), the relations (2.242)–(2.245) enable us to compute the variables $\widehat{\Psi}$, p , c_a , $\widehat{\Phi}_a^{(q)}$ and any other macroscopic variables of the Gibbs manifold $\mathfrak{G}$ from the knowledge of the nonequilibrium grand partition function Γ . The aforementioned derivative set, together with the extended Gibbs relation for $D(pv)$, therefore forms the foundation of the nonequilibrium statistical thermodynamics of a relativistic gas mixture of matter and radiation, since by utilizing relations (2.230)–(2.244) and Legendre transformations of variables in the thermodynamic manifold as well as accompanying Maxwell's relations between derivatives [23], it is possible to calculate any variables belonging to the Gibbs manifold in terms of the grand partition function. Thus the irreversible thermodynamic formalism of ideal relativistic gas mixtures of matter and radiation is now in place for nonequilibrium processes. And we can apply it to study irreversible thermodynamic phenomena in the system considered in the regime far removed from equilibrium at arbitrary degree.

2.10 Cumulant Expansion for the Dissipation Terms

2.10.1 Cumulant Expansion for the Calortropy Production

The description of generalized hydrodynamic equations is not as yet complete since the dissipation terms $\Lambda_a^{(\alpha)}$ of the nonconserved variable evolution equations still remain formally defined, but not explicitly evaluated in terms of variables in $\mathfrak{P} \cup \mathfrak{T}$. Therefore, the formal development in the theory of irreversible thermodynamics

presented up to this point has not acquired desired explicit forms for them in terms of variables of $\mathfrak{P} \cup \mathfrak{T}$. However, if the transport processes are to be studied in the thermodynamic manifold their dependence on the variables in manifold $\mathfrak{P} \cup \mathfrak{T}$ must be explicitly given.

Since the dissipation terms are directly related to the calortropy production in the system, they must be evaluated such that the second law of thermodynamics is rigorously satisfied by suitable forms to be taken for them. Such a procedure is possible to attain if a cumulant expansion method is used [5, 18] for the calortropy production. Since the present covariant kinetic equations of radiation and matter are similar in their structure to the covariant kinetic equations for the case of matter alone discussed in Chap. 1 of this Volume, the cumulant expansion method is expected to be also similar. For this reason we will be brief and present only the final results with necessary definitions in Chap. 1 mentioned.

Let us define the symbol g by the formula

$$g = \frac{(mc^2/k_B T)^4}{c\lambda^2\rho^2}, \quad (2.246)$$

where λ is a parameter of dimension length—it may be taken for the mean free path or the interaction range or the mean size parameter of particles. This parameter g has a dimension of volume $\times$ time. If the calortropy production is scaled by k_B/g it can be rendered dimensionless:

$$\widehat{\sigma}_c = \frac{g}{k_B} \sigma_c \quad (2.247)$$

and similarly for the Boltzmann entropy production σ_{ent} :

$$\widehat{\sigma}_{\text{ent}} = \frac{g}{k_B} \sigma_{\text{ent}}. \quad (2.248)$$

Then analysis can be carried on for $\widehat{\sigma}_c$ and $\widehat{\sigma}_{\text{ent}}$ in the same manner as in the previous chapter. For example, a cumulant expansion method may be applied and a formally identical result is obtained.

To the first-order cumulant approximation, the reduced calortropy production is given by

$$\widehat{\sigma}_c = \kappa \sinh \kappa \geq 0, \quad (2.249)$$

where κ^2 is the Rayleigh dissipation function given by a quadratic form of generalized potentials $X_a^{(q)}$. Since κ is a positive quadratic form of the generalized potentials $X_a^{(\alpha)}$, the first-order cumulant approximation for $\widehat{\sigma}_c$ in (2.249) remains positive semidefinite for all approximations made for the generalized potentials, which may be determined from (2.245), for example. If the right hand side of (2.245) is calculated from the knowledge of Γ and then the relation is inverted, $X_a^{(\alpha)}$ is obtained as a function of $\Phi_a^{(q)}$. In general, their relation may be expressed as

$$X_a^{(\alpha)} = -g_a^{(q)} \Phi_a^{(q)}, \quad (2.250)$$

where $g_a^{(q)}$ is a scalar function of variables spanning the manifold $\mathfrak{P} \cup \mathfrak{T}$. In the lowest order of approximation $g_a^{(q)}$ is a scalar function of the variables of manifold $\mathfrak{T}$ but independent of the variables of manifold $\mathfrak{P}$. See (2.274) below for $g_a^{(q)}$. In the first-order cumulant approximation (2.249) for $\widehat{\sigma}_c$, the cumulant κ is given by the formula

$$\kappa = \left[\sum_{a=1}^r \sum_{b=1}^r \sum_{\alpha \geq 1} \sum_{\gamma \geq 1} X_{a\mu\cdots\sigma}^{(\alpha)} R_{ab}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} X_{b\nu\cdots\omega}^{(\gamma)} \right]^{1/2}. \quad (2.251)$$

Here $R_{ab}^{(\alpha\gamma)}$ are given by the collision bracket integrals of the collision integrals $\mathfrak{R}_a[f_a]$ and $\mathfrak{R}_r[f_r]$ of the covariant kinetic equation for matter and radiation. They can be explicitly determined in terms of particulate dynamical information (e.g., interactions and collision cross sections of particles) if we make use of the linear steady-state evolution equations for nonconserved variables, namely, the linearized steady-state generalized hydrodynamic equations. In fact, so determined $R_{ab}^{(\alpha\gamma)}$ are found to be exactly the same as the collision bracket integrals appearing, for example, in the first-order Chapman–Enskog method [24] for the covariant kinetic equation. Their statistical formulas are as follows. With the notations for reduced momenta

$$\pi_a = c\beta p_a \quad (2.252)$$

and the abbreviation $[A \odot B]_{ab}$ for the integral

$$[A \odot B]_{ab} = G_a G_b \sum_s \int d^3\bar{\pi}_a \int d^3\bar{\pi}_b \int d^3\bar{\pi}_k^* \int d^3\bar{\pi}_l^* f_a^e f_b^e \bar{W}_{ab;kl}^{(s)} \xi_{abkl} A \odot B, \quad (2.253)$$

where $\bar{W}_{ab;kl}^{(s)}$ is a reduced transition rate

$$\bar{W}_{ab;kl}^{(s)} = \left(\frac{mk_B T}{\lambda^2} \right) W_{ab;kl}^{(s)} \quad (2.254)$$

and

$$\xi_{abkl} = \left[(1 + \epsilon_a f_a^e) (1 + \epsilon_b f_b^e) (1 + \epsilon_k f_k^e) (1 + \epsilon_l f_l^e) \right]^{-1}, \quad (2.255)$$

the collision bracket integrals are given by expressions

$$R_{aa}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} = \frac{1}{4} \beta^2 \left[\left(h_a^{(\alpha)} + h_{a'}^{(\alpha)} - h_a^{(\alpha)*} - h_{a'}^{(\alpha)*} \right)^{\sigma\cdots\mu} \right. \\ \left. \times \left(h_a^{(\gamma)} + h_{a'}^{(\gamma)} - h_a^{(\gamma)*} - h_{a'}^{(\gamma)*} \right)^{\omega\cdots\nu} \right]_{aa'}$$

$$+ \frac{1}{2} \beta^2 \sum_{b \neq a} \left[\left(h_a^{(\alpha)} - h_{a'}^{(\alpha)*} \right)^{\sigma \cdots \mu} \left(h_b^{(\gamma)} - h_{b'}^{(\gamma)*} \right)^{\omega \cdots \nu} \right]_{ab}, \quad (2.256)$$

$$R_{ab}^{(\alpha\gamma)\sigma \cdots \mu \omega \cdots \nu} = \beta^2 \sum_{a \neq b} \left[\left(h_a^{(\alpha)} - h_a^{(\alpha)*} \right)^{\sigma \cdots \mu} \left(h_b^{(\gamma)} - h_b^{(\gamma)*} \right)^{\omega \cdots \nu} \right]_{ab} \quad (2.257)$$

$(a \neq b),$

$$R_{rr}^{(\alpha\gamma)\sigma \cdots \mu \omega \cdots \nu} = \frac{1}{2} \beta^2 \sum_{a \neq r} \left[\left(h_r^{(\alpha)} - h_r^{(\alpha)*} \right)^{\sigma \cdots \mu} \left(h_r^{(\gamma)} - h_r^{(\gamma)*} \right)^{\omega \cdots \nu} \right]_{ai}, \quad (2.258)$$

$$R_{ra}^{(\alpha\gamma)\sigma \cdots \mu \omega \cdots \nu} = \beta^2 \sum_{a \neq r} \left[\left(h_r^{(\alpha)} - h_i^{(\alpha)*} \right)^{\sigma \cdots \mu} \left(h_r^{(\gamma)} - h_i^{(\gamma)*} \right)^{\omega \cdots \nu} \right]_{ra}. \quad (2.259)$$

These tensor integrals can be simplified to products of scalar integrals and isotropic tensors as will be shown in the following.

2.10.2 Dissipation Terms in the First-Order Cumulant Approximation

Comparing (2.249) with (2.211) and with the help of the expression for the first-order cumulant κ , we now find approximate dissipation terms consistent with the inequality (2.249) for the calortropy production

$$\Lambda_a^{(q)\sigma \cdots \mu} = (\beta g)^{-1} \sum_{b=1}^r \sum_{\gamma \geq 1} R_{ab}^{(q\gamma)\sigma \cdots \mu \omega \cdots \nu} X_{b\nu \cdots \omega}^{(\gamma)} (\sinh \kappa / \kappa). \quad (2.260)$$

This is the first-cumulant approximation for $\Lambda_a^{(q)}$, which is also consistent with the H theorem in the limit of vanishing fluctuations of distribution function f_a from f_a^c : $\Delta f_a = f_a - f_a^c = 0$, because, in that case, $\sigma_{\text{ent}} = \sigma_c \geq 0$ according to (2.249).

Higher order cumulant approximations for the dissipation terms may be obtained if the corresponding cumulant approximations are used for the reduced entropy production. It is straightforward to obtain them by following the procedure described in the literature [5, 18, 25, 26]. There now remains the task of determining the unknowns $X_a^{(\alpha)}$ in order to complete the formulation of thermodynamically consistent generalized hydrodynamic equations.

To proceed further it is useful to recast the collision bracket integrals $R_{aa}^{(\alpha\gamma)}$, etc., which are given above in unwieldy tensor product forms, to more computationally practical scalar forms. If we exploit the properties of isotropic tensors [18] this aim can be easily achieved for collision bracket integrals considered here, which involve tensors of rank 4 at most. We observe that $R_{ab}^{(q\gamma)}$ are expandable into isotropic tensors with scalar collision bracket integrals appearing as their coefficients; see, for example, Chap. 5, Ref. [18]. Thus we obtain

$$R_{ab}^{(11)\mu\nu\sigma\omega} = \mathcal{R}_{ab}^{(11)} \left[\frac{1}{2} (\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right]. \quad (2.261)$$

The coefficient $\mathcal{R}_{ab}^{(11)}$ is then a scalar obtained by contracting $R_{ab}^{(11)\mu\nu\sigma\omega}$ with the isotropic tensor $\left[\frac{1}{2} (\Delta_{\mu\sigma} \Delta_{\nu\omega} + \Delta_{\mu\omega} \Delta_{\nu\sigma}) - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\sigma\omega} \right]$:

$$\begin{aligned} \mathcal{R}_{ab}^{(11)} &= \frac{1}{5} R_{ab}^{(11)\mu\nu\sigma\omega} \left[\frac{1}{2} (\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right] \\ &= \frac{1}{10} \beta^2 \sum_{a \neq b} \left[(h_a^{(\alpha)} - h_a^{(\alpha)*}) : (h_b^{(\gamma)} - h_b^{(\gamma)*}) \right]_{ab} \quad (a \neq b) \end{aligned} \quad (2.262)$$

and similarly

$$\begin{aligned} \mathcal{R}_{aa}^{(11)} &= \frac{1}{5} R_{aa}^{(11)\mu\nu\sigma\omega} \left[\frac{1}{2} (\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right] \\ &= \frac{1}{10} \beta^2 \left\{ \frac{1}{2} \left[\Delta (h_a^{(1)} + h_{a'}^{(1)}) : \Delta (h_a^{(1)} + h_{a'}^{(1)}) \right]_{aa'} \right. \\ &\quad \left. + \sum_{b \neq a} \left[(h_a^{(1)} - h_{a'}^{(1)*}) : (h_b^{(1)} - h_{b'}^{(1)*}) \right]_{ab} \right\}, \end{aligned} \quad (2.263)$$

where

$$\Delta (h_a^{(1)} + h_{a'}^{(1)}) = h_a^{(1)} + h_{a'}^{(1)} - h_a^{(1)*} - h_{a'}^{(1)*}.$$

Here the prime on the subscript a of $[A]_{aa'}$ means another particle of species a and the colon: between the tensors means contraction of rank 2 tensors. Since the collision bracket integral $R_{ab}^{(22)}$ is scalar for the case of $q = s = 2$, we simply obtain scalar collision bracket integrals:

$$\begin{aligned} \mathcal{R}_{ab}^{(22)} &= R_{ab}^{(22)} \\ &= \beta^2 \sum_{a \neq b} \left[(h_a^{(2)} - h_a^{(2)*}) (h_b^{(2)} - h_b^{(2)*}) \right]_{ab} \quad (a \neq b), \end{aligned} \quad (2.264)$$

$$\begin{aligned} \mathcal{R}_{aa}^{(22)} &= R_{aa}^{(22)} \\ &= \frac{1}{2} \beta^2 \left\{ \frac{1}{2} \left[\Delta (h_a^{(2)} + h_{a'}^{(2)}) \Delta (h_a^{(2)} + h_{a'}^{(2)}) \right]_{aa'} \right. \\ &\quad \left. + \sum_{b \neq a} \left[(h_a^{(\alpha)} - h_{a'}^{(\alpha)*}) (h_b^{(s)} - h_{b'}^{(s)*}) \right]_{ab} \right\}, \end{aligned} \quad (2.265)$$

where

$$\Delta (h_a^{(2)} + h_{a'}^{(2)}) = h_a^{(2)} + h_{a'}^{(2)} - h_a^{(2)*} - h_{a'}^{(2)*}.$$

For $q, s = 3$ or 4 , namely, for vector processes such as heat flow and diffusion the collision bracket integral $R_{ab}^{(q\gamma)\mu\nu}$ is expressible as

$$R_{ab}^{(q\gamma)\mu\nu} = \mathcal{R}_{ab}^{(q\gamma)} \Delta^{\mu\nu} \quad (2.266)$$

and hence upon contracting with $\Delta^{\mu\nu}$ we obtain

$$\begin{aligned} \mathcal{R}_{ab}^{(qs)} &= R_{ab}^{(qs)\mu\nu} \Delta^{\mu\nu} \\ &= \frac{1}{6} \beta^2 \sum_{a \neq b} \left[\left(h_a^{(q)} - h_a^{(q)*} \right) \cdot \left(h_b^{(s)} - h_b^{(s)*} \right) \right]_{ab} \quad (a \neq b), \end{aligned} \quad (2.267)$$

$$\begin{aligned} \mathcal{R}_{aa}^{(qs)} &= \frac{1}{6} \beta^2 \left\{ \frac{1}{2} \left[\Delta \left(h_a^{(q)} + h_{a'}^{(q)} \right) \cdot \Delta \left(h_a^{(s)} + h_{a'}^{(s)} \right) \right]_{aa'} \right. \\ &\quad \left. + \sum_{b \neq a} \left[\left(h_a^{(q)} - h_{a'}^{(q)*} \right) \cdot \left(h_b^{(s)} - h_{b'}^{(s)*} \right) \right]_{ab} \right\} \quad (q, s = 3, 4), \end{aligned} \quad (2.268)$$

where the symbol $\cdot$ means the scalar product of vectors, namely, contraction of rank 1 tensors. These procedures of tensor contractions in the collision bracket integrals can be extended to the cases of higher rank tensors with progressively complicated results.¹⁰ In the present chapter we are limiting the calculations up to rank 4 tensors, in other words, shear stresses $\Phi_a^{(q)\mu\nu}$, since higher rank tensors than 4 are not required in practice. It is, in principle, possible to cast the first-order cumulant κ into the quadratic form of generalized potentials $X_a^{(q)}$:

$$\kappa = \left[\sum_{a=1}^r \sum_{b=1}^r \sum_{q \geq 1} \sum_{s \geq 1} \mathcal{R}_{ab}^{(q\gamma)} \left(X_a^{(q)} \odot X_b^{(s)} \right) \right]^{1/2}, \quad (2.269)$$

where $X_a^{(\alpha)} \odot X_b^{(s)}$ is a scalar:

$$X_a^{(q)} \odot X_b^{(s)} = X_a^{(q)\mu\nu\cdots\sigma\omega} X_{b\omega\sigma\cdots\nu\mu}^{(s)}. \quad (2.270)$$

The generalized potentials $X_a^{(q)}$ in κ and f_a^c now must be determined, so that the whole structure of the theory we are pursuing becomes free from undetermined parameters. It is discussed in the next section.

¹⁰For higher rank tensors isotropic tensors of higher rank, more complicated basis sets would be required. For the basis sets for higher order isotropic Cartesian tensors, see pages 97–98, Chap. 5, Ref. [18].

2.11 Generalized Potentials

We have managed the formulation of the theory, deferring the determination of the generalized potentials $X_a^{(q)}$ making up the nonequilibrium canonical forms (2.190) and (2.192). We have now reached the point to determine $X_a^{(q)}$ in terms of variables of manifold $\mathfrak{P}$.

The formal relations of nonequilibrium quantities are indicated by various relations of the macroscopic irreversible thermodynamics developed with the nonequilibrium canonical forms. One of them can be made use of to obtain the desired relation of $X_a^{(q)}$ to $\Phi_a^{(q)}$; see, especially, (2.245), which may be regarded as one of the most important among them in the present nonequilibrium statistical mechanical theory. Another mathematically satisfactory method of determining $X_a^{(q)}$ was discussed in Chap. 3 of Volume 1 and we may apply it to calculate it in terms of $\widehat{\Phi}_a^{(q)}$. Here we make use of (2.245) for the purpose in mind.

Expanding $\ln \Gamma_a$ in power series in $X_a^{(q)}$, we obtain

$$k_B T \left(\frac{\partial \ln \Gamma_a}{\partial X_a^{(q)}} \right)_{T, v, \mu, X'} = \langle c^{-2} U_\mu p_a^\mu f_a^e (h_a^{(q)} + \beta h_a^{(q)} h_a^{(q)} \odot X_a^{(q)} + \dots) \rangle, \quad (2.271)$$

which, to the lowest nonvanishing order in $X_a^{(q)}$, yields the formula

$$k_B T \left(\frac{\partial \ln \Gamma_a}{\partial X_a^{(q)}} \right)_{T, v, \mu, X'} = \beta \langle c^{-2} U_\mu p_a^\mu f_a^e h_a^{(q)} h_a^{(q)} \odot X_a^{(q)} \rangle + O(X_a^{(q)2}). \quad (2.272)$$

Thus finally, to the lowest nonvanishing order the generalized potentials are given by the linear relation

$$X_a^{(q)} = - (1/g_a^{(q)}) \Phi_a^{(q)}, \quad (2.273)$$

where

$$g_a^{(q)} = \frac{1}{2} I_s \beta \langle f_a^e c^{-2} U_\mu p_a^\mu h_a^{(q)} \odot h_a^{(q)} \rangle \quad (a = 1, \dots, m, r) \quad (2.274)$$

with I_s denoting symmetry-related factor for integrals of tensors

$$I_s = \begin{cases} \frac{1}{5} & \text{for second-rank tensors} \\ \frac{1}{3} & \text{for first-rank tensors} \\ 1 & \text{for scalar} \end{cases}. \quad (2.275)$$

It should be noted that the rank of tensor $X_a^{(q)}$ is the same as that of $\Phi_a^{(q)}$. The approximation for $X_a^{(q)}$ (2.273) leaves the first-order cumulant approximation for the calortropy entropy production positive semidefinite. In this manner, the moment method employed here for relativistic Boltzmann–Nordholm–Uehling–Uhlenbeck

equation is made complete to an approximation consistent with $\sigma_c \geq 0$. When the approximate relation for $X_a^{(q)}$ is used in the dissipation terms, the evolution equations formally obtained earlier are closed with respect to $\{\Phi_a^{(q)}\}$ and ready for solution because all the undetermined parameters are given formal relations to $\{\Phi_a^{(q)}\}$, by means of which we can determine them in terms of variables spanning the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. The transport properties then can be calculated from the solutions of the evolution equations—i.e., the generalized hydrodynamic equations for fluxes and conserved variables. The generalized hydrodynamic equations thus obtained, therefore, provide a mathematical framework—in the form of closed field equations—to describe transport processes attendant on irreversible processes in a system of radiation and matter in a manner consistent with the thermodynamic laws.

2.12 Generalized Hydrodynamics

The combined set of conservation laws, (2.163)–(2.168), and nonconserved variable evolution equations [(2.169) and (2.173)] constitutes the generalized hydrodynamics of radiation and matter. Having been appropriately subjected to the laws of thermodynamics, they are thermodynamically consistent. They are the fruits of labor expended so far to make generalized potentials and dissipation terms explicit with regard to variables spanning manifold $\mathfrak{P} \cup \mathfrak{T}$, so that the generalized hydrodynamic equations properly describe irreversible processes in the system consistently with the thermodynamic laws, given the initial and boundary conditions for variables belonging to $\mathfrak{P} \cup \mathfrak{T}$. They are now ready for hydrodynamic applications. It would be useful to summarize them in the following.

2.12.1 Conservation Laws

The equation of continuity for the total density is

$$D\rho = -\rho\nabla_\mu U^\mu. \quad (2.276)$$

The conserved variables balance equations are:

$$\rho D\mathfrak{c}_m = -\partial_\mu J_m^\mu + c^{-2} J_m^\mu D U_\mu + \Lambda_m^{(n)}, \quad (2.277)$$

$$\rho D\mathcal{E}_m = -\partial_\mu Q_m^\mu + P_m^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_m^\mu D U_\mu + U_\nu \Lambda_m^\nu, \quad (2.278)$$

$$c^{-2} \mathcal{E}_m \rho D U^\mu = \nabla^\mu p_m + \Delta_\sigma^\mu \partial_\nu \Pi_m^{\sigma\nu} + c^{-2} (P_m^{\mu\nu} D U_\nu - \Delta_\nu^\mu D Q_m^\nu - Q_m^\mu \nabla_\nu U^\nu - Q_m^\nu \nabla_\nu U^\mu) \quad (2.279)$$

for matter and

$$\rho D\mathfrak{c}_r = -\partial_\nu J_r^\nu + c^{-2} J_r^\mu D U_\mu + \Lambda_r^{(n)}, \quad (2.280)$$

$$\rho D\mathcal{E}_r = -\partial_\nu Q_r^\nu + P_r^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_r^\nu D U_\nu + U_\mu \Lambda_r^\mu \quad (2.281)$$

for the radiation. Here $\mathcal{E}_r$ is the radiation energy per material particle. We emphasize that the radiation momentum equation is not listed here, since it is related to the radiation heat flux evolution equation which appears as one of the evolution equations for $\widehat{\Phi}_a^{(q)\mu\nu\dots l}$.

2.12.2 Evolution Equations for Nonconserved Variables

Coupled to the balance equations of the conserved variables, there are the evolution equations for nonconserved variables

$$\rho D\widehat{\Phi}_a^{(q)\mu\dots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\dots\nu} + \mathfrak{Z}_a^{(q)\mu\nu\dots l} + \Lambda_a^{(q)\mu\nu\dots l}(x), \quad (2.282)$$

$$\Theta_a^{(q)\sigma\mu\dots\nu} = \Delta_\gamma^\mu \psi_a^{(q)\gamma\sigma\mu\dots\nu} \quad (a = 1, \dots, m, r; q \geq 1) \quad (2.283)$$

which holds for either matter or radiation. This is the generic form of nonconserved variable evolution equations. The kinematic terms and dissipation terms are presented separately for matter and radiation below.

Kinematic Terms for Matter

With the definitions for symbols

$$\begin{aligned} [\nabla U]^{(2)\mu\nu} &= \frac{1}{2} (\nabla^\mu U^\nu + \nabla^\nu U^\mu) - \frac{1}{3} \Delta^{\mu\nu} \nabla^\tau U_\tau, \\ [A \cdot B]^{(2)\mu\nu} &= \frac{1}{2} (A^{\mu\sigma} B_\sigma^\nu + A^{\nu\sigma} B_\sigma^\mu) - \frac{1}{3} \Delta^{\mu\nu} A^{\sigma\tau} B_{\sigma\tau}, \end{aligned} \quad (2.284)$$

the kinematic terms for matter species are collected below:

Modified Kinematic Term for Shear Stress

$$\begin{aligned} \mathfrak{Z}_a^{(1)\mu\nu} &= -2 [P_a \cdot \nabla U]^{(2)\mu\nu} - 2c^{-2} [Q'_r \cdot D U]^{(2)} - 2c^{-2} [\mathfrak{J}_r D U]^{(2)\mu\nu} \\ &\quad - 2 [P_a U \cdot D U]^{(2)} + c^{-2} [U U]^{(2)\mu\nu} P_a^{\tau\varepsilon} \nabla_\varepsilon U_\tau \\ &\quad + c^{-2} (U^\mu P_a^{\nu\omega} D U_\omega + U^\nu P_a^{\mu\omega} D U_\omega) + c^{-2} [U U]^{(2)\mu\nu} Q'_r{}^\omega D U_\omega \\ &\quad - \frac{1}{3} c^{-2} (U^\mu Q'_a{}^\varepsilon \nabla_\varepsilon U^\nu + U^\nu Q'_a{}^\varepsilon \nabla_\varepsilon U^\mu) + c^{-2} [U U]^{(2)\mu\nu} Q'_a{}^\omega D U_\omega \\ &\quad - \frac{1}{3} c^{-2} (U^\mu \mathfrak{J}_a{}^\varepsilon \nabla_\varepsilon U^\nu + U^\nu \mathfrak{J}_a{}^\varepsilon \nabla_\varepsilon U^\mu) + 2c^{-2} [U U]^{(2)\mu\nu} \mathfrak{J}_a{}^\omega D U_\omega \end{aligned}$$

$$\begin{aligned}
& -c^{-4} U^\mu U^\nu Q_a^{\prime\omega} D U_\omega + \frac{1}{3} \left\langle f_a \frac{p_a^\varepsilon (p_a^\sigma p_{a\sigma})}{(U_\lambda p_a^\lambda)^2} \right\rangle (U^\mu \nabla_\varepsilon U^\nu + U^\nu \nabla_\varepsilon U^\mu) \\
& - c^2 \left\langle f_a \frac{p_a^\mu p_a^\nu [p_a p_a]^{(2)\omega\varepsilon}}{(U_\lambda p_a^\lambda)^2} \right\rangle \nabla_\varepsilon U_\omega,
\end{aligned} \tag{2.285}$$

where

$$[p_a p_a]^{(2)\omega\varepsilon} = p_a^\omega p_a^\varepsilon - \frac{1}{3} \Delta^{\mu\nu} (p_a^\sigma p_{a\sigma}) = p_a^\omega p_a^\varepsilon - \frac{1}{3} m_a^2 c^2 \Delta^{\mu\nu}.$$

Modified Kinematic Term for Excess Normal Stress

$$\begin{aligned}
\mathfrak{Z}_a^{(2)} = & -\rho_a D (p_a / \rho_a) - \frac{2}{3} P_a^{\mu\varepsilon} \nabla_\varepsilon U_\mu - \frac{2}{3} c^{-2} Q_a^\omega D U_\omega - J_a^\varepsilon \nabla_\varepsilon (p_a / \rho_a) \\
& - c^{-2} (p_a / \rho_a) J_a^\omega D U_\omega + \frac{1}{3} \left[P_a^{\mu\varepsilon} - c^2 \left\langle f_a \frac{p_a^\omega p_a^\varepsilon (p_a^\mu p_{a\mu})}{(U_\lambda p_a^\lambda)^2} \right\rangle \right] \nabla_\varepsilon U_\omega
\end{aligned} \tag{2.286}$$

where

$$(p_a^\mu p_{a\mu}) = m_a^2 c^2.$$

Modified Kinematic Term for Heat Flux

$$\begin{aligned}
\mathfrak{Z}_a^{(3)\mu} = & P_a^{\mu\omega} D U_\omega - Q_a^{\prime\sigma} \nabla_\sigma U^\mu - P_a^{\mu\sigma} \nabla_\sigma \hat{h}'_a - c^{-2} \mathfrak{Z}_a^\sigma \nabla_\sigma U^\mu \\
& + \hat{h}'_a J_a^\sigma \nabla_\sigma U^\mu + \rho_a (\hat{h}'_a - \mathcal{E}_a) D U^\mu - J_a^\mu D \hat{h}'_a \\
& - c^{-2} U^\mu (Q_a^\sigma D U_\sigma + c^{-2} \mathfrak{Z}_a^\sigma D U_\sigma - \hat{h}'_a J_a^\omega D U_\omega) \\
& - U^\mu (P_a^{\sigma\varepsilon} \nabla_\sigma U_\varepsilon - J_a^\sigma \nabla_\sigma \hat{h}'_a) \\
& + c^2 \hat{h}'_a \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{(U_\lambda p_a^\lambda)^2} \right\rangle \nabla_\sigma U_\omega + \left[P_a^{\mu\sigma} - c^2 \left\langle f_a \frac{p_a^\mu p_a^\sigma}{(U_\lambda p_a^\lambda)^2} \right\rangle \right] \nabla_\sigma \hat{h}'_a.
\end{aligned} \tag{2.287}$$

Modified Kinematic Term for Diffusion Flux

$$\mathfrak{Z}_a^{(4)\mu} = -\rho_a D U^\mu - J_a^\sigma \nabla_\sigma U^\mu - c^{-2} U^\mu J_a^\omega D U_\omega - c^2 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{(U_\lambda p_a^\lambda)^2} \right\rangle \nabla_\sigma U_\omega. \tag{2.288}$$

In this case the divergence term is added to the kinematic term because $\Theta_a^{(4)\mu}$ gives rise to a variable in the manifold $\mathfrak{P} \cup \mathfrak{T}$.

Modified Kinematic Terms for Radiation

The modified kinematic terms for radiation are collected below.

Modified Kinematic Term for Shear Stress

$$\begin{aligned}
\mathfrak{Z}_r^{(1)\mu\nu} = & -2 [P_r \cdot \nabla U]^{(2)\mu\nu} - 2c^{-2} [Q'_r \cdot DU]^{(2)} - 2c^{-2} [\mathfrak{Z}_r DU]^{(2)\mu\nu} \\
& - 2 [P_r U \cdot DU]^{(2)} + c^{-2} [UU]^{(2)\mu\nu} P_r^{\tau\varepsilon} \nabla_\varepsilon U_\tau \\
& + c^{-2} (U^\mu P_r^{\nu\omega} DU_\omega + U^\nu P_r^{\mu\omega} DU_\omega) + c^{-2} [UU]^{(2)\mu\nu} Q_r'^\omega DU_\omega \\
& - c^{-4} U^\mu U^\nu Q_r'^\omega DU_\omega + c^{-2} ([UU]^{(2)\mu\nu} - c^{-2} U^\mu U^\nu) \mathfrak{Z}_r^\omega DU_\omega \\
& - \frac{1}{3} c^{-2} (U^\mu Q_r'^\varepsilon \nabla_\varepsilon U^\nu + U^\nu Q_r'^\varepsilon \nabla_\varepsilon U^\mu) \\
& - \frac{1}{3} c^{-2} (U^\mu J_r^\varepsilon \nabla_\varepsilon U^\nu + U^\nu J_a^\varepsilon \nabla_\varepsilon U^\mu) - c^2 \left\langle f_r \frac{p_r^\mu p_r^\nu p_r^\omega p_r^\varepsilon}{(U_\lambda p_r^\lambda)^2} \right\rangle \nabla_\varepsilon U_\omega.
\end{aligned} \tag{2.289}$$

Modified Kinematic Term for Excess Normal Stress

$$\begin{aligned}
\mathfrak{Z}_r^{(2)} = & -\rho_r D(p_r/\rho_r) - \frac{1}{3} P_r^{\mu\varepsilon} \nabla_\varepsilon U_\mu - J_r^\varepsilon \nabla_\varepsilon (p_r/\rho_r) - c^{-2} (p_r/\rho_r) J_r^\omega DU_\omega \\
& - \frac{2}{3} c^{-2} \mathfrak{Z}_r^\omega DU_\omega - \frac{2}{3} c^{-2} Q_r'^\omega DU_\omega.
\end{aligned} \tag{2.290}$$

Modified Kinematic Term for Heat Flux

$$\begin{aligned}
\mathfrak{Z}_r^{(3)\mu} = & -P_r^{\mu\sigma} \nabla_\sigma h_r - Q_r'^\sigma \nabla_\sigma U^\mu - c^{-2} \mathfrak{Z}_r^\sigma \nabla_\sigma U^\mu \\
& + P_r^{\omega\mu} DU_\omega + \rho_r p_r DU^\mu - J_r^\mu Dh_r + h_r J_r^\sigma \nabla_\sigma U^\mu \\
& - U^\mu P_r^{\sigma\omega} \nabla_\sigma U_\omega - U^\mu Q_r'^\sigma DU_\sigma - c^{-2} U^\mu \mathfrak{Z}_r^\sigma DU_\sigma \\
& + U^\mu J_r^\sigma \nabla_\sigma h_r + c^{-2} U^\mu h_r J_r^\omega DU_\omega + c^2 h_r \left\langle f_r \frac{p_r^\mu p_r^\omega p_r^\sigma}{(U_\lambda p_r^\lambda)^2} \right\rangle \nabla_\sigma U_\omega \\
& + 2h_r \left\langle f_r \frac{p_r^\omega p_r^\sigma}{(U_\lambda p_r^\lambda)} \right\rangle U^\mu \nabla_\sigma U_\omega + \left[P_r^{\mu\sigma} - c^2 \left\langle f_r \frac{p_r^\mu p_r^\sigma}{(U_\lambda p_r^\lambda)} \right\rangle \right] \nabla_\sigma h_r.
\end{aligned} \tag{2.291}$$

The third and fourth lines proportional to U^μ on the right vanish in the local rest frame. They are relativistic effects as are the integrals.

Modified Kinematic Term for Diffusion Flux

$$\mathfrak{Z}_a^{(4)\mu} = -\rho_a DU^\mu - J_a^\sigma \nabla_\sigma U^\mu - c^{-2} U^\mu J_a^\omega DU_\omega - c^2 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{(U_\lambda p_a^\lambda)^2} \right\rangle \nabla_\sigma U_\omega. \tag{2.292}$$

The integrals represent relativistic effects. In this case of $\mathfrak{Z}_r^{(4)\mu}$, the divergence term

$$-\partial_\sigma \Omega_r^{(4)\sigma\mu} = J_r^\mu \nabla_\sigma U^\sigma - c^{-2} U^\mu J_r^\omega D U_\omega + \left\langle f_r \frac{p_r^\omega p_r^\mu}{(p_r^\lambda U_\lambda)} \right\rangle D U_\omega \quad (2.293)$$

has been added to $\mathcal{Z}_r^{(4)\mu}$ to obtain $\mathfrak{Z}_a^{(4)\mu}$ since the term consists of variables in the manifold $\mathfrak{P}$.

The kinematic terms for matter and radiation are different despite the formally rather similar moments $h_i^{(q)\mu\cdots\nu}$ ($a = 1, 2, \dots, m, r$) because $(p_i^\sigma p_{i\sigma}) = m_a^2 c^2$ for matter whereas $(p_i^\sigma p_{i\sigma}) = 0$ for radiation.

Dissipation Terms $\Lambda_a^{(q)\mu\cdots\nu}$

The dissipation terms accompanying the kinematic terms in the generic evolution equation are calculated in the first-order cumulant approximation. The first-order cumulant approximation combined with the first-order approximation for the generalized potentials given in (2.273) is found sufficient for many applications in the case of nonrelativistic generalized hydrodynamic equations applied so far. For this reason it is logical to consider a similar level of approximations in the case of the relativistic theory. With the generalized potentials approximated to the linear order in fluxes as given in (2.273) the dissipation terms are given by the formula

$$\Lambda_a^{(q)\mu\cdots\nu} = - \sum_{b=1}^{\{r,m\}} \sum_{s=1}^4 \left(\beta g g_b^{(s)} \right)^{-1} \mathfrak{R}_{ab}^{(qs)} \Phi_b^{(s)\mu\cdots\nu} (\sinh \kappa / \kappa) \quad (a = 1, \dots, m, r), \quad (2.294)$$

where the coefficients $\mathfrak{R}_{ab}^{(qs)}$ are defined by

$$\mathfrak{R}_{ab}^{(qs)} = g_a^{(q)} \mathcal{R}_{ab}^{(qs)} g_b^{(s)} \quad (2.295)$$

in terms of the collision bracket integrals $\mathcal{R}_{ab}^{(qs)}$ given in (2.262)–(2.265), (2.267), and (2.268). Recalling the definition of the first-order cumulant κ , we now find that κ is given, in the first order approximation (2.273) for generalized potentials, by the quadratic form of $\Phi_b^{(s)}$:

$$\kappa^2 = \sum_{a,b=1}^{\{r,m\}} \sum_{q,s \geq 1} \mathfrak{R}_{ab}^{(qs)} \Phi_a^{(s)\mu\cdots\nu} \Phi_{b\nu\cdots\mu}^{(s)}. \quad (2.296)$$

Note that this is a generalized Rayleigh dissipation function. With the dissipation terms given in (2.294) we are now ready to explore some useful approximate forms for the nonconserved variable evolution equations presented above.

2.12.3 Linear Constitutive Equations and Transport Coefficients

In order to derive the statistical mechanical formulas for transport coefficients of relativistic gases in terms of dynamical quantities such as collision cross sections we must first obtain appropriate constitutive equations for linear transport processes. This aim is achieved if we linearize the nonconserved variable evolution equations presented earlier with respect to the thermodynamic forces and if the open hierarchy of evolution equations is closed by setting $\Theta_i^{(1)\sigma\mu\nu}$, $\Theta_i^{(2)\sigma}$, and $\Theta_i^{(3)\sigma\mu}$ ($i = 1, 2, \dots, m, r$) equal to zero or quantities belonging to the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. By means of such closures, the hierarchy of evolution equations is closed at the tensorial moment of rank 2. The linear transport coefficients can then be identified from the steady-state nonconserved variable evolution equations.

Linear Constitutive Equations

Upon imposing the closure and linearizing the kinematic terms with respect to thermodynamic driving forces—spatial gradients of velocity, temperature, pressure, or concentrations—and fluxes and setting $q_n(\kappa) = 1$ in the dissipation terms of evolution equations, we obtain linearized constitutive equations for fluxes. They take the following forms

$$\rho D \widehat{\Phi}_a^{(1)\mu\nu} = -\chi_a^{(1)\mu\nu} - (\beta g g_a^{(1)})^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab}^{(11)} \Phi_b^{(1)\mu\nu}, \quad (2.297)$$

$$\rho D \widehat{\Phi}_a^{(2)} = -\chi_a^{(2)} - (\beta g g_a^{(2)})^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab} \Phi_b^{(2)}, \quad (2.298)$$

$$\rho D \widehat{\Phi}_a^{(3)\mu} = -\chi_a^{(3)} - (\beta g g_a^{(3)})^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(3s)} \Phi_a^{(s)\mu}, \quad (2.299)$$

$$\rho D \widehat{\Phi}_a^{(4)\mu} = -\chi_a^{(4)} - (\beta g g_a^{(4)})^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(4s)} \Phi_b^{(s)\mu}, \quad (2.300)$$

where the subscript a runs over all species, i.e., $a = 1, \dots, m, r$ and the thermodynamic forces $\chi_a^{(1)\mu\nu}$, etc. are defined by linear thermodynamic gradients

$$\chi_a^{(1)\mu\nu} = 2\mathfrak{p}_a [\nabla U]^{(2)\mu\nu}, \quad (2.301)$$

$$\chi_a^{(2)} = \frac{5}{3} \mathfrak{p}_a \nabla_\mu U^\mu, \quad (2.302)$$

$$\chi_a^{(3)} = \mathfrak{p}_a \widehat{C}_p \nabla^\mu \ln T, \quad (2.303)$$

$$\chi_a^{(4)} = \rho_a \nabla^\mu \ln \mathfrak{c}_a. \quad (2.304)$$

These thermodynamic forces are the linear approximations of the kinematic terms $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ with respect to the thermodynamic gradients. It should be noted that in the case of diffusion flux evolution equations the divergence term $\nabla_\sigma \varphi_a^{(4)\mu\sigma}$ is combined with the kinematic term $\mathfrak{Z}_a^{(4)\mu}$ to obtain the modified kinematic term.

At the steady state of this set of differential equations we obtain the linear steady-state constitutive equations

$$-\chi_a^{(1)\mu\nu} - (\beta g g_a^{(1)})^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab}^{(11)} \Phi_b^{(1)\mu\nu} = 0, \quad (2.305)$$

$$-\chi_a^{(2)} - (\beta g g_a^{(2)})^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab} \Phi_b^{(2)} = 0, \quad (2.306)$$

$$-\chi_a^{(3)} - (\beta g g_a^{(3)})^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(3s)} \Phi_a^{(s)\mu} = 0, \quad (2.307)$$

$$-\chi_a^{(4)} - (\beta g g_a^{(4)})^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(4s)} \Phi_b^{(s)\mu} = 0. \quad (2.308)$$

This set is algebraic with respect to fluxes $\Phi_a^{(q)}$. Since the first two of the set are not coupled to the rest of the equations, they are easily solved for the fluxes. We thus obtain

$$\Phi_a^{(1)\mu\nu} = - \sum_{b=1}^{\{m,r\}} [\mathfrak{B}]_{ab}^{-1} [\nabla U]^{(2)\mu\nu}, \quad (2.309)$$

$$\Phi_a^{(2)} = - \sum_{b=1}^{\{m,r\}} [\mathfrak{V}]_{ab}^{-1} \nabla_\mu U^\mu, \quad (2.310)$$

where $\mathfrak{B}$ and $\mathfrak{V}$ are $r \times r$ square matrices consisting of elements $\mathfrak{R}_{ab}^{(11)}$ and $\mathfrak{R}_{ab}^{(22)}$, respectively:

$$\mathfrak{B}_{ab} = \frac{\mathfrak{R}^{(11)}}{2\mathfrak{p}_a \beta g}, \quad (2.311)$$

$$\mathfrak{V}_{ab} = \frac{3\mathfrak{R}^{(22)}}{5\mathfrak{p}_a \beta g}. \quad (2.312)$$

Since the third and fourth equations of the set are coupled, we construct a $2r \times 2r$ square matrix $\mathfrak{S}$ consisting of $r \times r$ square submatrices $\left[\mathfrak{R}_{ab}^{(33)} / (\beta g) \right]$, $\left[\mathfrak{R}_{ab}^{(44)} / \beta g \right]$, $\left[\mathfrak{R}_{ab}^{(34)} / \beta g \right]$, and $\left[\mathfrak{R}_{ab}^{(43)} / \beta g_a^{(4)} \right]$:

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{T} & \mathfrak{H} \\ \mathfrak{K} & \mathfrak{D} \end{pmatrix}, \quad (2.313)$$

where the submatrices are defined by $r \times r$ square matrices¹¹ made up of collision bracket integrals:

$$\mathfrak{T} = \left[\frac{\mathfrak{R}_{ab}^{(33)}}{\beta g} \right], \quad (2.314)$$

$$\mathfrak{H} = \left[\frac{\mathfrak{R}_{ab}^{(34)}}{\beta g} \right], \quad (2.315)$$

$$\mathfrak{K} = \left[\frac{\mathfrak{R}_{ab}^{(43)}}{\beta g} \right], \quad (2.316)$$

$$\mathfrak{D} = \left[\frac{\mathfrak{R}_{ab}^{(44)}}{\beta g} \right]. \quad (2.317)$$

We also construct a $2r$ dimensional column vector spanned by $2r$ -dimensional thermodynamic force vectors of $T \nabla^\mu \ln T$ and $\nabla^\mu \ln c_j$:

$$\mathfrak{F}^t = (T \nabla^\mu \ln T, \dots, T \nabla^\mu \ln T, \nabla^\mu \ln c_1, \dots, \nabla^\mu \ln c_r)^t. \quad (2.318)$$

Then in the notation defined, the solution of the third and fourth equations of the set may be written in matrix form

$$\Phi = \mathfrak{S}^{-1} \mathfrak{F}, \quad (2.319)$$

where Φ denotes the $(2r - 1)$ dimensional column vector

$$\Phi = (\Phi^{(3)}, \dots, \Phi^{(4)}, \dots). \quad (2.320)$$

Explicitly written out in components, the solutions of (2.307) and (2.308) are given by

$$\widehat{\Phi}_a^{(3)\mu} = - \sum_{j=1}^r \frac{|\mathfrak{S}|_{aj}}{\det |\mathfrak{S}|} T \nabla^\mu \ln T - \sum_{j=1}^r \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} \nabla^\mu \ln c_j, \quad (2.321)$$

$$\widehat{\Phi}_a^{(4)\mu} = - \sum_{j=1}^r \frac{|\mathfrak{S}|_{a+r,j}}{\det |\mathfrak{S}|} T \nabla^\mu \ln T - \sum_{j=1}^r \frac{|\mathfrak{S}|_{a+r,r+j}}{\det |\mathfrak{S}|} \nabla^\mu \ln c_j, \quad (2.322)$$

where the subscript a runs over all species including photons and $|\mathfrak{S}|_{aj}$ is the (aj) cofactor of matrix $\mathfrak{S}$. The solutions for the linear constitutive equations

¹¹It should be noted here that we have not eliminated the dependent diffusion flux in view of the fact that the number and range of photon spectrum is indeterminate in the present model.

(2.309)–(2.322) enable us to identify the linear transport coefficients of matter and radiation species under consideration.

Linear Transport Coefficients of Matter And Radiation

The linear transport coefficients are now easily identified with matrix elements of the solutions of the constitutive equations obtained above. The shear viscosity η^0 is found to have the statistical mechanical formula

$$\eta^0 = \sum_{a=1}^r \eta_a^0 := \sum_{a,b=1}^r \frac{1}{2} (\mathfrak{B})_{ab}^{-1}; \quad (2.323)$$

the bulk viscosity ζ the formula

$$\eta_B^0 = \sum_a \eta_{Ba}^0 := \sum_{a,b=1}^r (\mathfrak{B})_{ab}^{-1}; \quad (2.324)$$

the thermal conductivity λ the formula

$$\lambda_a^0 = \sum_{j=1}^r \lambda_{aj}^0 := \sum_{j=1}^r \frac{|\mathfrak{G}|_{aj}}{\det |\mathfrak{G}|}; \quad (2.325)$$

the diffusion coefficients D_{ab} the formula

$$D_a^0 = \sum_{j=1}^r D_{aj}^0 := \sum_{j=1}^r \frac{|\mathfrak{G}|_{a,r+j}}{\det |\mathfrak{G}|}; \quad (2.326)$$

and the thermal diffusivity the formulas

$$D_{ab}^{d0} = \frac{|\mathfrak{G}|_{a,r+b}}{\det |\mathfrak{G}|}; \quad (2.327)$$

$$D_{ab}^{t0} = \frac{|\mathfrak{G}|_{a+r,b}}{\det |\mathfrak{G}|}. \quad (2.328)$$

Here the species index a runs over the material species and photons.

In the nonrelativistic theory, the linear transport processes calculated by the method of generalized hydrodynamic equations are found to agree completely with their equivalents calculated in the first-order Chapman–Enskog theory. Therefore the present relativistic theory results for the linear transport coefficients are also expected to give the first-order Chapman–Enskog theory results for the relativistic gases of matter and radiation (photons). It can be shown that the first-order Chapman–Enskog method of solution for the covariant kinetic equation for the system of interest indeed gives the same linear transport coefficients.

2.12.4 Quasilinear Constitutive Equations and Transport Coefficients

If the kinematic terms are linearized with respect to the thermodynamic gradients on the one hand and, on the other hand, the dissipation terms $\Lambda_a^{(q)}$ are approximated in the first-order cumulant form together with the first-order generalized potentials as in (2.294), we obtain a model for nonconserved variable evolution equations. When the model is combined with the conserved variable evolution equations, we obtain a model for generalized hydrodynamic equations, which we call quasilinear generalized hydrodynamic equations. The model for the nonconserved variable evolution equations may be written as below:

$$\rho D\widehat{\Phi}_a^{(1)\mu\nu} = -\chi_a^{(1)\mu\nu} - \frac{1}{g_a^{(1)}} \sum_{b=1}^{\{m,r\}} \mathfrak{B}_{ab} \Phi_b^{(1)\mu\nu} q_n(\kappa), \quad (2.329)$$

$$\rho D\widehat{\Phi}_a^{(2)} = -\chi_a^{(2)} - \frac{1}{g_a^{(2)}} \sum_{b=1}^{\{m,r\}} \mathfrak{B}_{ab} \Phi_b^{(2)} q_n(\kappa), \quad (2.330)$$

$$\rho D\widehat{\Phi}_a^{(3)\mu} = -\chi_a^{(3)} - \frac{1}{g_a^{(3)}} \sum_{b=1}^{\{m,r\}} \left(\mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \mathfrak{H}_{ab} \Phi_b^{(4)\mu} \right) q_n(\kappa), \quad (2.331)$$

$$\rho D\widehat{\Phi}_a^{(4)\mu} = -\chi_a^{(4)} - \frac{1}{g_a^{(4)}} \sum_{b=1}^{\{m,r\}} \left(\mathfrak{K}_{ab} \Phi_b^{(3)\mu} + \mathfrak{D}_{ab} \Phi_b^{(4)\mu} \right) q_n(\kappa), \quad (2.332)$$

where the index a runs over all species—i.e., matter and radiation. This set of equations differs from the linear evolution equations (2.297)–(2.300) by the presence of the nonlinear factor $q_n(\kappa)$. Therefore, they are still highly nonlinear and expected to describe flow processes far removed from equilibrium in fluids subjected to high thermodynamic gradients as in the case of nonrelativistic flow processes studied [5, 18, 27, 28] and discussed in Chap. 9 of Volume 1.

The quasilinear generalized hydrodynamic equations therefore include the linearized generalized hydrodynamic equations in the limit of $q_n(\kappa) \rightarrow 1$ as $\kappa \rightarrow 0$, namely, in the vicinity of equilibrium. Furthermore, they also include the classical hydrodynamic equations—the Navier–Stokes, Fourier, and Fick equations, if the linear transport processes are at steady state.

Quasilinear Transport Coefficients

The steady-state solutions of the quasilinear evolution equations (2.329)–(2.332) provide the constitutive relations for quasilinear transport processes and the associated nonlinear transport coefficients. The steady state solutions in question are as follows:

$$\Phi_a^{(1)\mu\nu} = -\eta_a [\nabla U]^{(2)\mu\nu}, \quad (2.333)$$

$$\Phi_a^{(2)} = -\eta_{Ba} \nabla_\mu U^\mu, \quad (2.334)$$

$$\Phi_a^{(3)\mu} = -\lambda_a T \nabla^\mu \ln T - \sum_{j=1}^r D_{aj}^d \nabla^\mu \ln c_j \quad (a = 1, \dots, m, r), \quad (2.335)$$

$$\Phi_a^{(4)\mu} = -D_a^t T \nabla^\mu \ln T - \sum_{j=1}^r D_{aj} \nabla^\mu \ln c_j \\ (a = r + 1, \dots, m - 1, r). \quad (2.336)$$

The nonlinear transport coefficients $\eta_a, \eta_{Ba}, \lambda_a, D_{aj}^d, D_a^t, D_{aj}$ —the coefficients to the thermodynamic gradients—are defined by the formulas given below:

$$\eta_a = \sum_{b=1}^r (\mathfrak{B})_{ab}^{-1} q_L(\kappa_L) := \eta_a^0 q_L(\kappa_L), \quad (2.337)$$

$$\eta_{Ba} = \sum_{b=1}^r (\mathfrak{B})_{ab}^{-1} q_L(\kappa_L) := \eta_{Ba}^0 q_L(\kappa_L), \quad (2.338)$$

$$\lambda_a = \sum_{j=1}^r \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} q_L(\kappa_L) := \lambda_a^0 q_L(\kappa_L), \quad (2.339)$$

$$D_{aj}^d = \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} q_L(\kappa_L) := D_{aj}^{d0} q_L(\kappa_L), \quad (2.340)$$

$$D_a^t = \sum_{j=1}^r \frac{|\mathfrak{S}|_{a+r,r+j}}{\det |\mathfrak{S}|} q_L(\kappa_{L\kappa}) := D_a^{t0} q_L(\kappa_{L\kappa}), \quad (2.341)$$

$$D_{aj} = \frac{|\mathfrak{S}|_{a+r,r+j}}{\det |\mathfrak{S}|} q_L(\kappa_L) := D_{aj}^0 q_L(\kappa_L), \quad (2.342)$$

where $q_L(\kappa_L) = \sinh^{-1} \kappa_L / \kappa_L$ with κ_L denoting¹² the quadratic form of thermodynamic gradients of the matter-radiation system under consideration.

The nonlinear transport coefficients $\eta_a, \dots, D_{aj}$ depends on thermodynamic gradients present in the fluid undergoing nonlinear transport processes described by the evolution equations (2.329)–(2.332). They in fact diminish to zero as $\kappa_L^{-1+\delta}$ ($\delta > 0$) as $\kappa_L \rightarrow \infty$. For example, in the case of viscosity there appears a shear thinning effect as the shear rate increases—namely, the fluid is non-Newtonian. Notice that, as their nonrelativistic counterparts do, they also break the Curie principle because the flow process in question can be affected by a thermodynamic gradient of a different spatial symmetry of the flow. In the case of nonrelativistic flow phenomena, they can adequately account for numerous flow behavior in rheology, nonlinear charge carrier transport phenomena, rarefied gas flow, etc. Therefore in the light of the nonlinear transport coefficients for nonrelativistic flows just mentioned the relativistic

¹²For the detail of getting $q_L(\kappa_L)$ from the nonlinear factor $q_n(\kappa)$ and κ_L , see Chap. 1 of this Volume.

quasilinear transport coefficients listed above and the quasilinear generalized hydrodynamic equations could be also of interest to relativistic flow phenomena in relativistic gases far removed from equilibrium.

2.13 Nonrelativistic Limits of Relativistic Generalized Hydrodynamic Equations

The nonrelativistic limits of the relativistic generalized hydrodynamic equations of matter and radiation and the related quantities presented earlier in the present chapter can be obtained by making use of the methods employed for a similar subject in the absence of radiation in the previous chapter of this Volume. Thus obtained would be the classical radiation hydrodynamic equations for flow processes in matter interacting with radiation. Of particular interest to us here is the radiation part of hydrodynamic equations, because it will provide us classical radiation hydrodynamic equations which have not been discussed in the previous chapters in this work. Since the material part of the classical hydrodynamic equations would be expected to be similar to the classical (nonrelativistic) limits studied in the previous chapter, we will focus our attention to the radiation part of the generalized hydrodynamic equations.

Since the nonrelativistic limits of various operators and quantities considered in the evolution equations in the previous chapter should also apply to the present generalized hydrodynamic equations of matter and radiation, they will be used to obtain the desired results for our goal here. It is convenient to list a few of relevant nonrelativistic limits of quantities below:

$$\begin{aligned} p_a^0 &= m_a c \left(1 + p_a^2/m_a^2 c^2\right)^{1/2} \quad (a = 1, 2, \dots, m) \\ &\rightarrow m_a c, \text{ as } u/c \rightarrow 0 \end{aligned} \quad (2.343)$$

in the case of matter particles. Since the hydrodynamic velocity U^μ may be written for the whole system as

$$U^\mu = \left(1 - \frac{u^2}{c^2}\right)^{1/2} (c, \mathbf{u}) = \gamma (c, \mathbf{u}) \rightarrow (c, \mathbf{u}) \quad (2.344)$$

in the limit of $u/c \rightarrow 0$, it is clear that the space-components of U^μ is reduced to the nonrelativistic hydrodynamic velocity in the limit. With this preparation it is straightforward to obtain the nonrelativistic limits of the relativistic generalized hydrodynamic equations. Using the aforementioned limit we find in the case of matter

$$\begin{aligned} p_a^\mu U_\mu &= \gamma (c p_a^0 - \mathbf{p}_a \cdot \mathbf{u}) \\ &= m_a c^2 + \frac{1}{2} m C_a^2 + O(c^{-2}), \end{aligned} \quad (2.345)$$

where

$$\mathbf{C}_a = \mathbf{v}_a - \mathbf{u}$$

is the nonrelativistic peculiar velocity of species a , and the $m_a c^2$ term can be neglected in the nonrelativistic theory. This means that the Jüttner function for matter has the following limit

$$f_a^e = \exp[-\beta(p_a^\mu U_\mu - \mu_a)] = \lim_{u/c \rightarrow 0} \exp\left[-\beta\left(\frac{1}{2}m\mathbf{C}_a^2 - \mu_a\right)\right]. \quad (2.346)$$

Therefore the Maxwell–Boltzmann distribution function is the nonrelativistic limit of the Jüttner distribution function.

The number density tends to the nonrelativistic number density for both matter species and photons:

$$\begin{aligned} \rho_a &= c^{-2} U_\mu N_a^\mu = c^{-1} \int \frac{d^3 p_a}{p_a^0} U_\mu p_a^\mu f_a \\ &= \int d^3 p_a f_a + O(c^{-1}) \\ &= n_a^{(\text{nr})} + O(c^{-1}) \quad (a = 1, 2, \dots, m, r). \end{aligned} \quad (2.347)$$

Here $n_a^{(\text{nr})}$ denotes the number density in the nonrelativistic kinetic theory.

The covariant operators D and ∇^μ have the nonrelativistic limit as follows :

$$\begin{aligned} D &= U^\mu \partial_\mu = U^0 \partial_0 + U^k \partial_k \\ &= \partial_t + \mathbf{u} \cdot \nabla + O(c^{-1}) \\ &:= \frac{d}{dt} + O(c^{-1}). \end{aligned} \quad (2.348)$$

Namely, the relativistic substantial time derivative $\mathfrak{D} = \rho D$ tends to the nonrelativistic substantial time derivative $\rho \frac{d}{dt}$ used in classical fluid mechanics. The space-component of the operator ∇^μ is given by spatial gradient $\nabla_j = \partial/\partial x_j$ in the limit of $u/c \rightarrow 0$:

$$\begin{aligned} \nabla^j &= -\Delta^{jk} \partial_k - \Delta^{j0} \partial_0 \\ &= \frac{\partial}{\partial x_j} + O(c^{-1}) \quad (k, j = 1, 2, 3) \end{aligned} \quad (2.349)$$

Therefore there hold the limits

$$\nabla_\mu U^\nu \rightarrow \partial u_\nu / \partial x_\mu; \quad \nabla^\mu U_\nu \rightarrow -\partial u_\nu / \partial x_\mu. \quad (2.350)$$

Let us then analyze the nonrelativistic limit of the Boltzmann entropy four-flow. We have defined the Boltzmann entropy in the relativistic kinetic theory¹³

$$S^\mu(x) = -k_B c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^\mu f_\alpha \ln f_\alpha \quad (2.351)$$

and

$$\begin{aligned} S &:= \rho \mathcal{S} = c^{-2} U_\mu S^\mu(x) \\ &= -k_B \sum_{\alpha=1}^{\{m,r\}} \int d^3 p_\alpha f_\alpha \ln f_\alpha^{(\text{nr})} + O(c^{-1}) \\ &= S^{(\text{nr})} + O(c^{-1}), \end{aligned} \quad (2.352)$$

where $S^{(\text{nr})}$ is the Boltzmann entropy of a nonrelativistic gas mixture and $f_\alpha^{(\text{nr})}$ is the nonrelativistic distribution function. The space-component of Boltzmann entropy flux $J_s^\mu = S^\mu - S U^\mu$ is given by

$$\begin{aligned} J_s^k &= S^k - S U^k \\ &= -k_B \sum_{i=1}^{\{m,r\}} \int d^3 p_i \mathbf{C}_i^k f_i^{(\text{nr})} \ln f_i^{(\text{nr})} + O(c^{-1}) \\ &:= J_{sk}^{(\text{nr})} + O(c^{-1}), \end{aligned} \quad (2.353)$$

where $J_{sk}^{(\text{nr})}$ is the nonrelativistic Boltzmann entropy flux. The Boltzmann entropy production has the limit

$$\begin{aligned} \sigma_{\text{ent}} &= -k_B c \sum_{i=1}^{\{m,r\}} \sum_{j=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} \ln f_\alpha \mathfrak{R}_{\alpha\beta} [f_\alpha, f_\beta] \\ &= -k_B \sum_{i=1}^{\{m,r\}} \sum_{j=1}^{\{m,r\}} \int d^3 p_\alpha \ln f_\alpha \mathfrak{R}_{\alpha\beta} [f_\alpha^{(\text{nr})}, f_\beta^{(\text{nr})}] + O(c^{-1}) \end{aligned}$$

¹³For example, de Groot et al. [29] define the Boltzmann entropy four-flow by the formula

$$S^\mu(x) = -k_B c \sum_{a=1}^{\{m,r\}} \int \frac{d^3 p_a}{p_a^0} p_a^\mu f_a (\ln f_a - 1).$$

We define $S^\mu(x)$ by

$$S^\mu(x) = -k_B c \sum_{a=1}^{\{m,r\}} \int \frac{d^3 p_a}{p_a^0} p_a^\mu f_a \ln f_a$$

without the (-1) factor, which we find superfluous.

$$= \sigma_{\text{ent}}^{(\text{nr})} + O(c^{-1}), \quad (2.354)$$

where $\sigma_{\text{ent}}^{(\text{nr})}$ is the entropy production of nonrelativistic gas expressed in terms of nonrelativistic distribution functions.

The dissipation terms are given by the nonrelativistic limit

$$\begin{aligned} \Lambda_a^{(q)\mu\cdots\nu} &= c \sum_{\beta=1}^{\{m,r\}} \int \frac{d^3 p_\beta}{p_\beta^0} h_\beta^{(q)\mu\cdots\nu} \Re_\beta [f_\beta] \\ &= \sum_{\beta=1}^{\{m,r\}} \int d^3 p_\beta h_\beta^{(q)\mu\cdots\nu} \Re_\beta [f_\beta^{(\text{nr})}] + O(c^{-1}) \\ &:= \Lambda_{(\text{nr})\alpha}^{(q)\mu\cdots\nu} + O(c^{-1}). \end{aligned} \quad (2.355)$$

In order to show the nonrelativistic limit of the relativistic generalized hydrodynamic equations, it is necessary to analyze the energy-momentum tensor and other macroscopic variables. Since

$$T^{\mu\nu} = c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^\mu p_\alpha^\nu f_\alpha, \quad (2.356)$$

then its (00) component is given by the limit

$$\begin{aligned} T^{00} &= c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^0 p_\alpha^0 f_\alpha^{(\text{nr})} \\ &= nmc^2 + \mathcal{E}_k + \mathcal{E}_r + O(c^{-1}). \end{aligned} \quad (2.357)$$

On ignoring the rest energy part, it becomes the energy density of a sum of energies of matter and radiation.

Let us consider the components T^{0k} ($k = 1, 2, 3$):

$$\begin{aligned} T^{0k} &= T^{k0} = c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^0 p_\alpha^k f_\alpha \\ &= c \sum_{\alpha=1}^{\{m,r\}} \int d^3 p_\alpha p_\alpha^k f_\alpha, \end{aligned} \quad (2.358)$$

that is, T^{0k}/c becomes the momentum density of a nonrelativistic gas when $u/c \rightarrow 0$. Moreover,

$$\begin{aligned}
T^{kj} &= c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_{\alpha}}{p_{\alpha}^0} p_{\alpha}^k p_{\alpha}^j f_{\alpha} \\
&= T_{\text{m}}^{(\text{nr})kj} + T_r^{(\text{nr})kj} + O(c^{-1}) \quad (k, j = 1, 2, 3), \quad (2.359)
\end{aligned}$$

where $T_{\text{m}}^{(\text{nr})kj}$ and $T_r^{(\text{nr})kj}$ denote the nonrelativistic energy-momentum tensors of matter and radiation.

With the help of the aforementioned properties it is easy to show that the divergence term $\partial_{\tau} \left(\Delta_{\sigma a}^{\tau (q)\sigma\mu\cdots\nu} \right)$ and the kinematic term $Z_a^{(q)\mu\cdots\nu}$ have the nonrelativistic limits as follows:

$$\begin{aligned}
\partial_{\tau} \langle \Delta_{\sigma a}^{\tau (q)\sigma\mu\cdots\nu} \rangle &= -\nabla \cdot \int d^3 p_{\alpha} \mathbf{C}_{\alpha} f_{\alpha}^{(\text{nr})} h_{\alpha}^{(q)\mu\cdots\nu} + O(c^{-1}) \\
&= -\nabla \cdot \psi_{\alpha}^{(q)\mu\cdots\nu} + O(c^{-1})
\end{aligned}$$

where $\mathbf{C}_{\alpha}$ is the peculiar velocity and

$$\psi_{\alpha}^{(q)\mu\cdots\nu} = \int d^3 p_{\alpha} \mathbf{C}_{\alpha} f_{\alpha}^{(\text{nr})} h_{\alpha}^{(q)\mu\cdots\nu}.$$

The nonrelativistic limit of $Z_a^{(\alpha)\mu\cdots\nu}$ for matter species is given by

$$\begin{aligned}
Z_a^{(q)\mu\cdots\nu} &= \langle p_a^{\sigma} \partial_{\sigma} h_a^{(q)\mu\cdots\nu} f_a \rangle \\
&= \langle f_a^{(\text{nr})} (d_t + \mathbf{C}_a \cdot \nabla) h_a^{(q)\mu\cdots\nu} \rangle + O(c^{-1}), \quad (2.360)
\end{aligned}$$

for which we have used the limits given below:

$$\Delta_{\mu}^0 p_a^{\mu} = (g_{\mu}^0 - c^{-2} U^0 U_{\mu}) p_a^{\mu} = c^{-1} \mathbf{u} \cdot \mathbf{p}_a + O(c^{-1}), \quad (2.361)$$

$$\Delta_{\mu}^k p_a^{\mu} = (g_{\mu}^k - c^{-2} U^k U_{\mu}) p_a^{\mu} = -m_a C_a^k + O(c^{-1}), \quad (2.362)$$

$$\Delta_{\mu\nu} p_a^{\mu} p_a^{\nu} = m_a^2 C_a^2 + O(c^{-1}). \quad (2.363)$$

Therefore, the nonconserved variable evolution equation (2.169), namely, generic evolution equation, tends to the nonrelativistic limit

$$\begin{aligned}
n^{(\text{nr})} D \widehat{\Phi}_{(\text{nr})a}^{(q)\mu\cdots\nu} &= -\nabla \cdot \psi_a^{(q)\mu\cdots\nu} + \langle f_a^{(\text{nr})} (d_t + \mathbf{C}_a \cdot \nabla) h_a^{(q)\mu\cdots\nu} \rangle + \Lambda_{(\text{nr})a}^{(q)\mu\cdots\nu} \\
&\quad (a = 1, 2, \dots, m). \quad (2.364)
\end{aligned}$$

Here $d_t = \partial_t + \mathbf{u} \cdot \nabla$ is the nonrelativistic substantial time derivative. Upon explicitly working out the second term on the right of (2.364) we obtain the nonrelativistic evolution equations for nonconserved variables of matter and radiation. For notational brevity we will omit the superscript or subscript nr as well as $\mu \cdots \nu$ from the so-obtained equations with understanding all quantities involved are nonrelativistic.

They are summarized below :

$$n \frac{d}{dt} \widehat{\Phi}_a^{(1)} = -\nabla \cdot \psi_a^{(1)} - 2 [\mathbf{J}_a d_t \mathbf{u}]^{(2)} - 2 [\Pi_a \cdot \nabla \mathbf{u}]^{(2)} - \frac{2}{3} \Pi_a (\nabla \cdot \mathbf{u}) - 2 \widetilde{\Delta}_a [\nabla \mathbf{u}]^{(2)} - 2 p_a [\nabla \mathbf{u}]^{(2)} + \Lambda_a^{(1)}, \quad (2.365)$$

$$n \frac{d}{dt} \widehat{\Phi}_a^{(2)} = -\nabla \cdot \psi_a^{(2)} - \frac{2}{3} \mathbf{J}_a \cdot d_t \mathbf{u} - \frac{2}{3} \Pi_a : [\nabla \mathbf{u}]^{(2)} - \frac{2}{3} \widetilde{\Delta}_a (\nabla \cdot \mathbf{u}) - p_a d_t \ln(p_a v^{5/3}) - \nabla \cdot (\mathbf{J}_a p_a / \rho_a) + \Lambda_a^{(2)}, \quad (2.366)$$

$$n \frac{d}{dt} \widehat{\Phi}_a^{(3)} = -\nabla \cdot \psi_a^{(3)} - d_t \mathbf{u} \cdot (\mathbf{P}_a - p_a \delta) + \mathbf{Q}'_a \cdot \nabla \mathbf{u} - \mathbf{P}_a \cdot \nabla \widehat{h}_a + \phi_a^{(3)} : \nabla \mathbf{u} - \mathbf{J}_a d_t \widehat{h}_a + \Lambda_a^{(3)}, \quad (2.367)$$

$$n \frac{d}{dt} \widehat{\Phi}_a^{(4)} = -\nabla \cdot \mathbf{P}_a - p_a d_t \mathbf{u} - \mathbf{J}_a \cdot \nabla \mathbf{u} + \Lambda_a^{(4)}, \quad (2.368)$$

where the subscript a stands for matter species and the superscript (nr) on $n^{(\text{nr})}$ is dropped for brevity of notation. The number density n multiplying the substantial derivative on the left-hand side of (2.365)–(2.368) would be replaced by mass density ρ , if the density $\widehat{\Phi}_a^{(q)}$ of $\Phi_a^{(q)}$ were redefined with respect to mass density instead of the number density n , which was compelled in the relativistic theory. In the equations presented above, the following definitions of symbols are used for brevity of notation:

$$\psi_a^{(3)} = \langle f_a m_a \mathbf{C}_a \mathbf{C}_a \mathbf{C}_a \rangle. \quad (2.369)$$

In the case of radiation, we obtain the nonrelativistic limits of the evolution equations for nonconserved variables as follows:

$$n \frac{d}{dt} \widehat{\Phi}_r^{(1)} = -\nabla \cdot \psi_r^{(1)} - 2 [\mathbf{J}_r d_t \mathbf{u}]^{(2)} - 2 [\Pi_r \cdot \nabla \mathbf{u}]^{(2)} - \frac{2}{3} \Pi_r (\nabla \cdot \mathbf{u}) - 2 \widetilde{\Delta}_r [\nabla \mathbf{u}]^{(2)} - 2 p_r [\nabla \mathbf{u}]^{(2)} + \Lambda_r^{(1)}, \quad (2.370)$$

$$n \frac{d}{dt} \widehat{\Phi}_r^{(2)} = -\nabla \cdot \psi_r^{(2)} - \frac{2}{3} \mathbf{J}_r \cdot d_t \mathbf{u} - \frac{2}{3} \Pi_r : [\nabla \mathbf{u}]^{(2)} - \frac{2}{3} \widetilde{\Delta}_r (\nabla \cdot \mathbf{u}) - p_r d_t \ln(p_r v^{5/3}) - \nabla \cdot (\mathbf{J}_r p_r / \rho_r) + \Lambda_r^{(2)}, \quad (2.371)$$

$$n \frac{d}{dt} \widehat{\Phi}_r^{(3)} = -\nabla \cdot \psi_r^{(3)} - d_t \mathbf{u} \cdot (\mathbf{P}_r - p_r \delta) + \mathbf{Q}'_r \cdot \nabla \mathbf{u} - \mathbf{P}_r \cdot \nabla h_r - \mathbf{J}_r d_t h_r + \Lambda_r^{(3)}, \quad (2.372)$$

$$n \frac{d}{dt} \widehat{\Phi}_r^{(4)} = -\nabla \cdot \mathbf{P}_r - p_r d_t \mathbf{u} - \mathbf{J}_r \cdot \nabla \mathbf{u} + \Lambda_r^{(4)}, \quad (2.373)$$

In this sense, the relativistic generalized hydrodynamic equations we have derived are reduced to their nonrelativistic counterparts—namely, the nonrelativistic radiation hydrodynamic equations—under the condition $u/c \rightarrow 0$, which we compactly

summarize as below:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{u}, \quad (2.374)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot \mathbf{P}, \quad (2.375)$$

$$\rho \frac{\partial \mathcal{E}}{\partial t} = -\nabla \cdot \mathbf{Q} - \mathbf{P} : \nabla \mathbf{u}, \quad (2.376)$$

$$\rho \frac{\partial}{\partial t} \Phi_{\alpha}^{(q)} = -\nabla \cdot \psi_{\alpha}^{(q)} + \mathcal{Z}_{\alpha}^{(q)} + \Lambda_{\alpha}^{(q)} \quad (q = 1, 2, \dots, 4, \text{ etc.}; \alpha = 1, 2, \dots, m, r). \quad (2.377)$$

The matter part of the nonrelativistic generalized hydrodynamic equations are well known results in the scheme of the modified moment method for nonrelativistic gases [5, 18]. The radiation part of the generalized hydrodynamic equations is new and one of important objectives of the present chapter on radiation and matter.

2.14 Relative Boltzmann Entropy Density and Fluctuations

The relative Boltzmann entropy density and its balance equation can be calculated similarly to the calortropy density and its balance equation obtained in the previous section. However, the relative Boltzmann entropy density balance equation is expected to be not integrable in the thermodynamic manifold.

For the task we face here for the relative Boltzmann entropy it is necessary to have the distribution function f_a explicitly as a function of momentum four flow, which of course would be forthcoming if the covariant kinetic equation were solved explicitly, but it is not available at this point in the development of theory. Nevertheless, we may propose a way to determine f_a and therewith deduce possible nature and properties of the relative Boltzmann entropy. Therefore we look for f_a in a form isomorphic mathematically to the nonequilibrium canonical form f_a^c , but the parameters therein are not the same as β , μ_a , and $X_a^{(q)}$ appearing in f_a^c . Therefore we postulate f_a in the form

$$f_a = \left\{ \exp \left[\beta \left(p_a^{\mu} U_{\mu} + \sum_{q \geq 1} X_a^{(q) \mu \dots \sigma} h_{a \nu \dots \omega}^{(q)} - \mu_a \right) - \epsilon_a \right] \right\}^{-1} \quad (a = 1, \dots, m, r), \quad (2.378)$$

which should be contrasted to the nonequilibrium canonical form

$$f_a^c = \left\{ \exp \left[\beta^t \left(p_a^\mu U_\mu + \sum_{q \geq 1} X_a^{t(q)\mu \cdots \sigma} h_{a\nu \cdots \omega}^{(q)} - \mu_a^t \right) - \epsilon_a \right] \right\}^{-1} \quad (a = 1, \dots, m, r), \quad (2.379)$$

in which the superscript t is affixed on the parameters β^t , μ_a^t , and $X_a^{t(q)\mu \cdots \sigma}$ to distinguish them from β , μ_a , and $X_a^{(q)}$. The superscript t stands for thermodynamics and the meanings of β^t , μ_a^t , and $X_a^{t(q)\mu \cdots \sigma}$ are fixed according to the thermodynamic correspondence discussed in connection with the calortropy. Therefore the burden of difference between the exact solution f_a of the covariant kinetic equation and the nonequilibrium canonical form f_a^c falls on the parameters $(\beta, \mu_a, X_a^{(q)})$, and $\Delta f_a := f_a - f_a^c$ may be regarded as a functional of fluctuations of the parameters from the thermodynamic parameters $(\beta^t, \mu_a^t, X_a^{t(q)})$. This viewpoint therefore motivates us to define the fluctuations of parameters:

$$\delta\beta = \beta - \beta^t, \quad \delta(\beta\mu_a) = \beta\mu_a - \beta^t\mu_a^t, \quad \delta(\beta X_a^{(q)}) = \beta X_a^{(q)} - \beta^t X_a^{t(q)} \quad (2.380)$$

to which we add the fluctuation in $\beta\mathbf{p}$:

$$\delta(\beta\mathbf{p}) = \beta\mathbf{p} - \beta^t\mathbf{p}^t, \quad (2.381)$$

which will be found necessary.

Then inserting (2.378) and (2.379) into the relative Boltzmann entropy density balance equation (2.188) we obtain the equation for $\widehat{\mathcal{S}}_r[f|f^c]$:

$$D\widehat{\mathcal{S}}_r[f|f^c] = -k_B \left[\delta\beta D\mathcal{E} + \delta(\beta\mathbf{p}) D\mathbf{v} - \sum_{a=1}^r \delta(\beta\mu_a) D\mathfrak{c}_a + \sum_{a=1}^r \sum_{q \geq 1} (\beta X_a^{(q)}) \delta \odot D\widehat{\Phi}_a^{(q)} \right] + \rho^{-1} \Delta\Xi_r, \quad (2.382)$$

where $\Delta\Xi_r$ is given by

$$\begin{aligned} \Delta\Xi_r = & -k_B \beta \left(c^{-2} Q^\mu D U_\mu + P^{\mu\nu} \nabla_\nu U_\mu - \mathbf{p} \nabla_\nu U^\nu + \sum_{a=1}^r \mu_a \nabla_\nu J_a^\nu \right) \\ & - k_B \sum_{a=1}^r \sum_{q \geq 1} \beta X_a^{(q)} \odot \langle f_a \partial_\mu p_a^\mu h_a^{(q)} \rangle \\ & - k_B \sum_{a=1}^r \partial_\mu \left(\Delta_\nu^\mu \langle \epsilon_a p_a^\nu \ln(1 + \epsilon_a f_a) \rangle \right). \end{aligned} \quad (2.383)$$

This balance equation for $\widehat{\mathcal{S}}_r[f|f^c]$ is similar to the Pfaffian differential form for calortropy but for $\rho^{-1}\Delta\Xi_r$, which does not vanish. Because of the term $\rho^{-1}\Delta\Xi_r$ the balance equation for $\widehat{\mathcal{S}}_r[f|f^c]$ is not integrable in the thermodynamic manifold, but the fluctuations in parameters $\left\{\delta\beta, \delta(\beta\mu_a), \delta(\beta X_a^{(q)}), \delta(\beta\mathbf{p})\right\}$ are determining factors of the relative Boltzmann entropy density and ultimately the distribution function f_a . In the limits of

$$\delta\beta, \delta(\beta\mu_a), \delta(\beta X_a^{(q)}), \delta(\beta\mathbf{p}) \rightarrow 0$$

Equation (2.382) reduces to the form

$$D\widehat{\mathcal{S}}_r[f|f^c] = \rho^{-1}\Delta\Xi_r, \quad (2.384)$$

which may be integrated along the path of the irreversible process of under consideration, but its value would be path-dependent. A stochastic theory may be pursued along the line suggested in the previous chapter of this volume. However, we will not pursue such a study here, leaving to future study on the subject.

2.15 Concluding Remarks

We have formulated a covariant kinetic theory for a mixture consisting of matter and radiation by putting the material particles and photons on an equal footing. This kinetic theory has an attendant theory of irreversible processes in a system of radiation and material gases, which is consistent with the laws of thermodynamics. The present formulation is achieved by treating the system as a dilute gas mixture of photons and material particles which interact with each other according to the dynamical laws of mechanics and, especially, at the collisional level, by quantum mechanics. The covariant kinetic equations used are the Boltzmann equations suitably generalized to accommodate the quantum nature of radiation, at least, with regard to their collisions. By applying the modified moment method, thermodynamically consistent solutions for the kinetic equations are obtained and a theory of irreversible thermodynamics is formulated therewith for the system. As it was the case for nonradiative systems, the Boltzmann entropy differential is found to be nonexact—namely, not integrable—in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ if the system is away from equilibrium, but the Pfaffian differential form for calortropy is exact in manifold $\mathfrak{P} \cup \mathfrak{T}$ and therefore serves for formulation of irreversible thermodynamics and hydrodynamics consistent with a local form of the thermodynamic laws. Therefore we now have fundamental equations for thermodynamics of irreversible processes in a system of radiation and matter undergoing nonequilibrium processes under the aegis of thermodynamic principles.

A theory of radiative transport processes can be developed by the means of the flux evolution equations presented and study of hydrodynamics can be made. The

present covariant formulation removes the weakness inherent to the nonrelativistic kinetic theory reported in Ref. [1] in which photons are treated nonrelativistically. The covariant generalized hydrodynamic equations derived from the relativistic Boltzmann equation have better balanced structures, although more difficult to solve in practice. The generalized hydrodynamic equations, namely, the conservation equations and the flux evolution equations, for the system of radiation and matter can be used to describe irreversible thermodynamic and hydrodynamic processes occurring far removed from equilibrium. The quasilinear generalized hydrodynamic equations presented appear to be applicable to and practicable for nonlinear systems. The theory of transport processes obtained is applied to study radiation transport phenomena in the next chapter.

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