Norm-Based Locality Measures of Two-Dimensional Hilbert Curves

H.K. Dai[1(✉)] and H.C. Su[2]

[1] Computer Science Department, Oklahoma State University, Stillwater, OK 74078, USA
dai@cs.okstate.edu

[2] Department of Computer Science, Arkansas State University, Jonesboro, AR 72401, USA
suh@astate.edu

Abstract. A discrete space-filling curve provides a 1-dimensional indexing or traversal of a multi-dimensional grid space. Applications of space-filling curves include multi-dimensional indexing methods, parallel computing, and image compression. Common goodness-measures for the applicability of space-filling curve families are locality and clustering. Locality reflects proximity preservation that close-by grid points are mapped to close-by indices or vice versa. We present an analytical study on the locality property of the 2-dimensional Hilbert curve family. The underlying locality measure, based on the p-normed metric d_p, is the maximum ratio of $d_p(u, v)^m$ to $d_p(\tilde{u}, \tilde{v})$ over all corresponding point-pairs (u, v) and $(\tilde{u}, \tilde{v})$ in the m-dimensional grid space and 1-dimensional index space, respectively. Our analytical results identify all candidate representative grid-point pairs (realizing the locality-measure values) for all real norm-parameters in the unit interval $[1, 2]$ and grid-orders. Together with the known results for other norm-parameter values, we have almost complete knowledge of the locality measure of 2-dimensional Hilbert curves over the entire spectrum of possible norm-parameter values.

Keywords: Space-filling curves · Hilbert curves · z-order curves · Locality

1 Preliminaries

Discrete space-filling curves have many applications in databases, parallel computation, algorithms, in which linearization techniques of multi-dimensional arrays or grids are needed. Sample applications include heuristics for Hamiltonian traversals, multi-dimensional space-filling indexing methods, image compression, and dynamic unstructured mesh partitioning.

For positive integer n, denote $[n] = \{1, 2, \ldots, n\}$. An m-dimensional (discrete) space-filling curve of length n^m is a bijective mapping $C : [n^m] \rightarrow [n]^m$, thus providing a linear indexing/traversal or total ordering of the grid points in $[n]^m$. An m-dimensional grid is said to be of order k if it has side-length $n = 2^k$;

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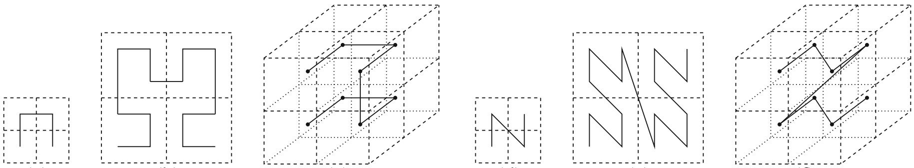
a space-filling curve has order k if its codomain is a grid of order k. The generation of a sequence of multi-dimensional space-filling curves of successive orders usually follows a recursive framework (on the dimensionality and order), which results in a few classical families, such as Gray-coded curves, Hilbert curves, Peano curves, and z-order curves.

One of the salient characteristics of space-filling curves is their "self-similarity". Denote by H_k^m and Z_k^m an m-dimensional Hilbert and z-order, respectively, space-filling curve of order k. Figure 1 illustrates the recursive constructions of H_k^m and Z_k^m for $m = 2$, and $k = 1, 2$, and $m = 3$, and $k = 1$.



(a) (b) (c) (d) (e) (f)

Fig. 1. Recursive constructions of Hilbert and z-order curves of higher order (respectively, H_k^m and Z_k^m) by interconnecting symmetric subcurves, via reflection and/or rotation, of lower order (respectively, H_{k-1}^m and Z_{k-1}^m) along an order-1 subcurve (respectively, H_1^m and Z_1^m): (a) H_1^2; (b) H_2^2; (c) H_1^3; (d) Z_1^2; (e) Z_2^2; (f) Z_1^3.

We measure the applicability of a family of space-filling curves based on: (1) their common structural characteristics that reflect locality and clustering, (2) descriptional simplicity that facilitates their construction and combinatorial analysis in arbitrary dimensions, and (3) computational complexity in the grid space-index space transformation. Locality preservation reflects proximity between the grid points of $[n]^m$, that is, close-by points in $[n]^m$ are mapped to close-by indices/numbers in $[n^m]$, or vice versa. Clustering performance measures the distribution of continuous runs of grid points (clusters) over identically shaped subspaces of $[n]^m$, which can be characterized by the average number of clusters and the average inter-cluster distance (in $[n^m]$) within a subspace.

Empirical and analytical studies of clustering performances of various low-dimensional space-filling curves have been reported in the literature (see [4] and [6] for details). These studies show that the Hilbert and z-order curve families manifest good data clustering properties according to some quality clustering measures, robust mathematical formalism, and viable indexing techniques for querying multi-dimensional data, when compared with other curve families.

The locality preservation of a space-filling curve family is crucial for the efficiency of many indexing schemes, data structures, and algorithms in its applications, for examples, spatial correlation in multi-dimensional indexings, compression in image processing, and communication optimization in mesh-connected parallel computing. To analyze locality, we need to rigorously define its measures that are practical – good bounds (lower and upper) on the locality measure translate into good bounds on the declustering (locality loss) in one space in the presence of locality in the other space.

A few locality measures have been proposed and analyzed for space-filling curves in the literature. Denote by d and d_p the Euclidean metric and p-normed metric (rectilinear metric ($p = 1$) and maximum metric ($p = \infty$)), respectively. Let $\mathcal{C}$ denote a family of m-dimensional curves of successive orders.

We [5] consider a locality measure conditional on a 1-normed distance of δ between points in $[n]^m$:

$$L_\delta(C) = \sum_{i,j \in [n^m] \mid i<j \text{ and } d_1(C(i),C(j))=\delta} |i-j| \text{ for } C \in \mathcal{C}.$$

They derive exact formulas for L_δ for the Hilbert curve family $\{H_k^m \mid k = 1, 2, \ldots\}$ and z-order curve family $\{Z_k^m \mid k = 1, 2, \ldots\}$ for $m = 2$ and arbitrary δ that is an integral power of 2, and $m = 3$ and $\delta = 1$ (lower-order terms collected in asymptotic form for brevity):

$$L_\delta(H_k^2) = \begin{cases} \frac{17}{2\cdot 7} \cdot 2^{3k} + O(2^{2k}) & \text{if } \delta = 1 \\ \frac{17}{2\cdot 7} \cdot 2^{3k+2\log\delta} + O(2^{2k+3\log\delta}) & \text{otherwise,} \end{cases}$$

$$L_\delta(Z_k^2) = \begin{cases} 2^{3k} + O(2^k) & \text{if } \delta = 1 \\ 2^{3k+2\log\delta} + O(2^{2k+3\log\delta}) & \text{otherwise;} \end{cases}$$

$$L_1(H_k^3) = \frac{67}{2\cdot 31} \cdot 2^{5k} + O(2^{3k}) \text{ and } L_1(Z_k^3) = 2^{5k} + O(2^{2k}).$$

With respect to the locality measure L_δ and for sufficiently large k and $\delta \ll 2^k$, the z-order curve family performs better than the Hilbert curve family for $m = 2$ and over the δ-spectrum of integral powers of 2. When $\delta = 2^k$, the domination reverses. The superiority of the z-order curve family persists but declines for $m = 3$ with unit 1-normed distance for L_δ.

For measuring the proximity preservation of close-by points in the indexing space $[n^m]$, Gotsman and Lindenbaum [7] consider the following measures: for $C \in \mathcal{C}$,

$$L_{\min}(C) = \min_{i,j\in[n^m]\mid i<j} \frac{d(C(i),C(j))^m}{|i-j|} \text{ and } L_{\max}(C) = \max_{i,j\in[n^m]\mid i<j} \frac{d(C(i),C(j))^m}{|i-j|}.$$

Alber and Niedermeier [1] generalize $L_{\max}$ to L_p by employing the p-normed metric d_p for real norm-parameter $p \geq 1$ in place of the Euclidean metric d, which is the locality measure studied in our work (and [5]). We summarize below: (1) the representative lower- and upper-bound results and exact formulas for the locality measure L_p of the 2-dimensional Hilbert curve family H_k^2 for various norm-parameter p-values and grid-order k-values, and (2) the contribution of our studies:

1. For $p = 1$: Niedermeier, Reinhardt, and Sanders [8] give a lower bound for $L_1(H_k^2)$: for all $k \geq 1$,

$$L_1(H_k^2) \geq \frac{(3 \cdot 2^{k-1} - 2)^2}{4^{k-1}},$$

and Chochia et al. [3] provide a matching upper bound for $L_1(H_k^2)$ for all $k \geq 2$. We [5] also provide the exact formula for $L_1(H_k^2)$ for all $k \geq 2$.

2. For $p = 2$: Gotsman and Lindenbaum [7] derive a lower and upper bounds for $L_2(H_k^2)$: for all $k \geq 6$,

$$\frac{(2^{k-1}-1)^2}{\frac{2}{3}\cdot 4^{k-2}+\frac{1}{3}} \leq L_2(H_k^2) \leq 6\frac{2}{3},$$

and Alber and Niedermeier [1] improves the upper bound for $L_2(H_k^2)$: for all $k \geq 1$,

$$L_2(H_k^2) \leq 6\frac{1}{2}.$$

We [5] prove that the lower bound above [7] is the exact formula for $L_2(H_k^2)$: for all $k \geq 5$,

$$L_2(H_k^2) = 6 \cdot \frac{2^{2k-3}-2^{k-1}+2^{-1}}{2^{2k-3}+1}.$$

Bauman [2] obtains a matching lower and upper bounds for $L_2(H_k^2)$ for $k = \infty$:

$$L_2(H_\infty^2) = 6.$$

3. For $2 < p \leq \infty$: Due to the monotonicity of the underlying p-normed metric: for every grid-point pair (v, u), the p-normed metric $d_p(v, u)$ is strictly decreasing in $p \in [1, \infty)$, we [5] prove the same exact formula for $L_p(H_k^2)$ as for the case when $p = 2$:

$$L_p(H_k^2) = 6 \cdot \frac{2^{2k-3}-2^{k-1}+2^{-1}}{2^{2k-3}+1} \text{ for all reals } p \geq 2.$$

When $p = \infty$, Alber and Niedermeier [1] establish a lower and upper bounds for $L_\infty(H_k^2)$, respectively:

$$6(1 - O(2^{-k})) \leq L_\infty(H_k^2) \leq 6\frac{2}{5}.$$

Our proofs of the exact formulas of $L_p(H_k^2)$ for $p \in \{1, 2\}$ in [5] follow a uniform approach: identifying all the representative grid-point pairs, which realize the $L_p(H_k^2)$-value, for each $p \in \{1, 2\}$. The analytical results close the gap between the current best lower and upper bounds with exact formulas for $p \in \{1, 2\}$, and extend to all reals $p \geq 2$. The identifications of candidate representative grid-point pairs rely on sequences of reduction. A reduction of a grid-point pair to another pair is based on the dominance of the underlying locality-measure values of the corresponding grid-point pairs. The geometric characteristics of the underlying p-norms (rectilinear and Euclidean metrics of $p = 1$ and $p = 2$, respectively) help distinguish candidate representative grid-point pairs and verify tedious reductions.

Our study of 2-dimensional curve family H_k^2 is focused on the exact analysis of $L_p(H_k^2)$ for all reals $p \in [1, 2]$. The intrinsic mathematical appeal in completing the computation of $L_p(H_k^2)$ for all possible norm-parameters p is our primary motivation. While the three most obviously important p-values: $\{1, 2, \infty\}$ are intimately related to intuitive concepts, in some cases the structure of applications of the Hilbert curves may suggest a different choice of p-value as the most natural setting for the underlying locality measure.

We present analytical and empirical studies on the locality measure L_p for the 2-dimensional Hilbert curve family for all reals $p \in [1, 2]$. The underlying locality measure L_p, based on the p-normed metric d_p, is the maximum ratio of $d_p(u, v)^m$ to $d_p(\tilde{u}, \tilde{v})$ over all corresponding point-pairs (u, v) and $(\tilde{u}, \tilde{v})$ in the m-dimensional grid space and (1-dimensional) index space, respectively:

1. We identify all the candidate representative grid-point pairs for all norm-parameter p-values in $[1, 2]$ and grid-order k-values. Together with the known results for other norm-parameter values, we have almost complete knowledge of $L_p(H_k^2)$ over the entire spectrum of possible norm-parameter values.
2. Our empirical study, which complements the analytical ones, shows that: (1) The analytical results are consistent with program verification over various norm-parameter p-values and sufficiently large grid-order k-values, and (2) As p increases over the real unit interval $[1, 2]$, the locations of candidate representative grid-point pairs agree with the intuitive interpolation effect over the two delimiting p-values.
3. A practical implication of our results on $L_p(H_k^2)$ is that the exact formulas provide good bounds on measuring the loss in data locality in the index space, while spatial correlation exists in the 2-dimensional grid space.

We present a high-level approach to the main results without any derivations and proofs, supplemented with an empirical study that verifies the analytical results for various p-values and sufficiently large k-values. Complete results: illustrated figures, derivations, and proofs, and verifying computer programs are available from the authors.

2 Analytical Studies of $L_p(H_k^2)$ with $p \in [1, 2]$

For 2-dimensional Hilbert curves, the self-similar structural property guides us to decompose H_k^2 into four identical H_{k-1}^2-subcurves (via reflection and rotation), which are amalgamated together by an H_1^2-curve. Following the linear order along this H_1^2-curve, we denote the four H_{k-1}^2-subcurves (quadrants) as $Q_1(H_k^2)$, $Q_2(H_k^2)$, $Q_3(H_k^2)$, and $Q_4(H_k^2)$. We extend the notion to identify all H_l^m-subcurves of a structured H_k^m for all $l \in [k]$ inductively on the order in an obvious manner.

For a space-filling curve C indexing an m-dimensional grid space, the notation "$v \in C$" refers to "grid point v indexed by C", and $C^{-1}(v)$ gives the index of v in the 1-dimensional index space. The locality measure in our study is, for all reals $p \geq 1$,

$$L_p(C) = \max_{\text{indices } i,j \in [n^m]} \frac{d_p(C(i), C(j))^m}{d_p(i, j)} = \max_{v,u \in C} \frac{d_p(v, u)^m}{|C^{-1}(v) - C^{-1}(u)|}.$$

When $m = 2$, we write $\mathcal{L}_{C,p}(u, v) = \frac{d_p(u,v)^2}{|C^{-1}(v) - C^{-1}(u)|}$.

For subcurves C_1, C_2, C_1', and C_2' of C, a grid-point pair $(v_1, v_2) \in C_1 \times C_2$ is reducible to a grid-point pair $(v_1', v_2') \in C_1' \times C_2'$ if $\mathcal{L}_{C,p}(v_1, v_2) \leq \mathcal{L}_{C,p}(v_1', v_2')$ – denoted by $(v_1, v_2) \preceq (v_1', v_2')$, and subcurve pair $C_1 \times C_2$ is reducible to subcurve pair $C_1' \times C_2'$ if for every $(v_1, v_2) \in C_1 \times C_2$, there exists $(v_1', v_2') \in C_1' \times C_2'$ such that (v_1, v_2) is reducible to (v_1', v_2') – denoted by $C_1 \times C_2 \preceq C_1' \times C_2'$. We define the strict reducibility, denoted by $\prec$, for grid-point pairs and subcurve pairs via the strict inequality of $\mathcal{L}_{C,p}$-values in an obvious manner.

A pair of grid points v and u indexed by C is representative for C with respect to L_p if $\mathcal{L}_{C,p}(v, u) = L_p(C)$, or, equivalently, for all $v', u' \in C$, $(v', u') \preceq (v, u)$. The identifications of candidate representative grid-point pairs for C often involve sequences of reductions – successive considerations of two grid-point pairs and the comparisons of their $\mathcal{L}_{C,p}$-values. Our studies of $L_p(H_k^2)$ cover all norm-parameters $p \geq 1$. However, for all reals $p \in (1, 2)$, the lack of geometric clarity for interpreting L_p-values can adversely increase the complexity: (1) of identifying candidate representative grid-point pairs, and (2) in comparing $\mathcal{L}_{H_k^2,p}$-values for reductions due to the complex interplay of the norm-parameter p-value and grid-order k-value.

2.1 Reductions of Grid-Point Pairs and Subcurve Pairs

For two grid-point pairs (v_1, v_2) and (v_1', v_2') (two subcurve pairs $C_1 \times C_2$ an $C_1' \times C_2'$) of H_k^2, the reduction $(v_1, v_2) \preceq (v_1', v_2')$ ($C_1 \times C_2 \preceq C_1' \times C_2'$, respectively) eliminates (v_1, v_2) ($C_1 \times C_2$, respectively) from the candidacy for representative grid-point pairs. We develop various sufficient conditions for reduction with an example below.

For the grid space $[2^k]^2$ of a 2-dimensional Hilbert curve H_k^2 with a referenced (x, y)-coordinate system (with origin $(1, 1)$) in a canonical orientation (see Fig. 1(a) and (b)), we denote the x- and y-coordinates of a grid point v by $x(v)$ and $y(v)$, respectively.

Lemma 1. *For all norm-parameters $p \in [1, 2]$ and three arbitrary grid points $u, v, v' \in H_k^2$ such that: (1) the sequence of three grid points: (u, v, v') is in indexing order (that is, $(H_k^2)^{-1}(u) \leq (H_k^2)^{-1}(v) \leq (H_k^2)^{-1}(v')$ or $(H_k^2)^{-1}(u) \geq (H_k^2)^{-1}(v) \geq (H_k^2)^{-1}(v')$), and (2) the two sequences of their x- and y-coordinates: $(x(u), x(v), x(v'))$ and $(y(u), y(v), y(v'))$ have the same monotone property (both increasing or both decreasing), if $|(H_k^2)^{-1}(u) - (H_k^2)^{-1}(v)|(2|x(u) - x(v)||x(v) - x(v')| + |x(v) - x(v')|^2 + 2|y(u) - y(v)||y(v) - y(v')| + |y(v) - y(v')|^2) - |(H_k^2)^{-1}(v) - (H_k^2)^{-1}(v')|(|x(u) - x(v)| + |y(u) - y(v)|)^2 \geq 0$ (> 0), then $(u, v) \preceq (u, v')$ $((u, v) \prec (u, v'))$ via $\mathcal{L}_{H_k^2,p}(u, v) \leq \mathcal{L}_{H_k^2,p}(u, v')$ $(\mathcal{L}_{H_k^2,p}(u, v) < \mathcal{L}_{H_k^2,p}(u, v')$, respectively).*

Note that the sufficient condition for the reduction is independent of the p-value for $\mathcal{L}_{H_k^2,p}$.

For reductions of grid-point pairs, we mostly use various p-independence sufficient conditions as the one in Lemma 1. For reductions of subcurve pairs, simple ones are realized by symmetry arguments with regard to relative subcurve-orientations or succinct geometric interpretations of the $\mathcal{L}_{H_k^2,p}$-computation if possible.

For subcurves in the form of nested subquadrants of H_k^2, we may prove the reduction between subcurve pairs $C_1 \times C_2 \preceq C_1' \times C_2'$ with a divide-and-conquer approach by considering all possible reductions between quadrant-subcurve pairs $Q_{i_1}(C_1) \times Q_{i_2}(C_2)$ (for all $i_1, i_2 \in [4]$) to $Q_{j_1}(C_1') \times Q_{j_2}(C_2')$ (for some $j_1, j_2 \in [4]$). Some reductions of quadrant-subcurve pairs may be resolved by simple symmetry/geometric arguments, while others may entail further reductions of subquadrant-subcurve pairs. These nested reductions generally arrive at some forms of recursive patterns, and mathematical induction is applied to resolve the reductions.

2.2 Identification of Candidate Representative Grid-Point Pairs

The upper-bound argument [5] in establishing the exact formulas for $L_p(H_k^2)$ for $p \in \{1, 2\}$ does not translate into a viable application for $p \in (1, 2)$. For identifying all possible candidate representative grid-point pairs in H_k^2, we consider all grid-point pairs in $Q_i(H_k^2) \times Q_j(H_k^2)$ with $1 \leq i < j \leq 4$ and their possible systematic reductions. Due to a simple reduction ($Q_1(H_k^2) \times Q_4(H_k^2) \preceq Q_2(H_k^2) \times Q_3(H_k^2)$) and geometric symmetry ($Q_2(H_k^2) \times Q_4(H_k^2)$ to $Q_1(H_k^2) \times Q_3(H_k^2)$ and $Q_3(H_k^2) \times Q_4(H_k^2)$ to $Q_1(H_k^2) \times Q_2(H_k^2)$), three cases remain: $Q_1(H_k^2) \times Q_2(H_k^2)$, $Q_1(H_k^2) \times Q_3(H_k^2)$, and $Q_2(H_k^2) \times Q_3(H_k^2)$. An involved analysis of $Q_1(H_k^2) \times Q_3(H_k^2)$ reveals that the quadrant-subcurve pair is void of any candidate representative grid-point pairs.

We summarize the findings below in Theorem 1, in which the sources of (candidate) representative grid-point pairs (named A, B, and C) are illustrated in Fig. 2 and elaborated with (local) (x, y)-coordinates and $\mathcal{L}_{H_k^2,p}$-values in Table 1. For brevity we omit the symmetry ones.

Theorem 1. *Consider the following cases determined by the interplay of the grid-order $k \geq 1$ and norm-parameter $p \in [1, 2]$ of H_k^2:*

1. *Case when $k = 1$:*
   For all $p \in [1, 2)$: One representative grid-point pair with coordinates $((1, 1), (2^k, 2^k))$ and its symmetry.
   For $p = 2$: Three representative grid-point pairs with coordinates $((1, 1), (1, 2^k))$, $((1, 1), (2^k, 2^k))$, and $((1, 2^k), (2^k, 2^k)$, and their symmetries.
2. *Case when $k \in \{2, 3\}$:*
   For all $p \in [1, 2]$: One representative grid-point pair B and its symmetry.
3. *Case when $k = 4$: The p-interval $[1, 2]$ is decomposed into two p-subintervals: $[1, \rho)$ and $(\rho, 2]$, where $\rho \approx 1.825$.*
   For all $p \in [1, \rho)$: One representative grid-point pair B and its symmetry.
   For all $p \in (\rho, 2]$: One representative grid-point pair A and its symmetry.
   For $p = \rho$: Two representative grid-point pairs B and A, and their symmetries.

4. *Case when $k \geq 5$: For all $p \in [1,2]$: $1+(k-2)+(k-4) = 2k-5$ candidate representative grid-point pairs $B, C_1, D_1, C_2, \ldots, C_{k-5}, D_{k-5}, C_{k-4}, D_{k-4}, C_{k-3}, C_{k-2}$, and their symmetries.*

Refined analysis with further reductions eliminates $D_1, \ldots, D_{k-5}$ and D_{k-4}, and their symmetries from the candidacy for representative grid-point pairs.

Theorem 2. *For all grid-orders $k \geq 5$ and norm-parameters $p \in [1,2]$ of H_k^2, the candidate representative grid-point pairs are $B, C_1, C_2, \ldots, C_{k-5}, C_{k-4}, D_{k-4}, C_{k-3}, C_{k-2}$, and their symmetries.*

For all norm-parameters $p \in [1,2]$, there exists a sufficiently large grid-order $k_0 \geq 5$ such that for all grid-orders $k \geq k_0$ of H_k^2, the candidate representative grid-point pairs are $B, C_1, C_2, \ldots, C_{k-5}, C_{k-4}, C_{k-3}, C_{k-2}$, and their symmetries.

Our future work will be focused on establishing analytically the association of representative grid-point pairs in $\{B, C_1, C_2, \ldots, C_{k-5}, C_{k-4}, C_{k-3}, C_{k-2}\}$ with their $\mathcal{L}_{H_k^2,p}$-dominance p-subintervals and relevant grid-orders.

3 Empirical Study on $L_p(H_k^2)$ with $p \in [1,2]$

To complement the analytical results for $L_p(H_k^2)$ for all reals $p \in [1,2]$, we conduct an empirical study on $L_p(H_k^2)$ for all $k \in \{2,3,\ldots,12\}$ and some reals $p \in [1,2]$. We cover the grid space $[2^k]^2$ of a 2-dimensional Hilbert curve H_k^2 in a canonical orientation with Cartesian coordinates: 2^k columns (respectively, rows) indexed by x-coordinates (respectively, y-coordinates) $1, 2, \ldots, 2^k$. For every grid-order $k \in \{2,3,\ldots,12\}$ and real $p \in [1,2]$ with granularity of 0.01 (for $2 \leq k \leq 12$), we locate with computer programs all representative grid-point pairs for H_k^2 with respect to L_p. Figure 2(a) illustrates the three sources $\{A, B, C\}$ of candidate representative grid-point pairs for $k \geq 2$.

Source A identifies the grid-point pair $(u_A, v_A) = ((1, \frac{1}{4} \cdot 2^k + 1), (1, 2^k))$ and its symmetry. The pair (u_A, v_A) serves as the representative grid-point pair "briefly" – for $k = 4$ and $1.83 \leq p \leq 2.00$.

Source B identifies the grid-point pair $(u_B, v_B) = ((2^{k-1}, 1), (1, 2^k))$ and its symmetry. The pair (u_B, v_B) serves as the representative grid-point pair for every $k \in \{2,3,\ldots,12\}$ and all reals p of a (shrinking) prefix-interval $[1, \rho_k) \subseteq [1,2]$ – with ρ_k decreasing as k increases.

Source C identifies a sequence $(C_1, C_2, \ldots, C_{k-2})$ of grid-point pairs:

$$C_t = (u_{C_t}, v_{C_t}) = ((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-2-t})),$$

for $t = 1, 2, \ldots, k-2$, and their symmetries, with:

$$x(v_{C_{t+1}}) = x(v_{C_t}) \text{ and } y(v_{C_{t+1}}) - 2^{k-1} = \frac{y(v_{C_t}) - 2^{k-1}}{2},$$

and eventually v_{C_t} converges to $v_{C_{k-2}}$. Note that, for $t = 0$, the grid-point pair $C_0 = (u_{C_0}, v_{C_0}) = ((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-2}))$ is not included in C since C_0 can not be a candidate representative grid-point pair (for any k and real $p \in [1, 2]$):

$$\mathcal{L}_{H_k^2,p}(u_B, v_B) = \frac{((2^{k-1} - 1)^p + (2^k - 1)^p)^{\frac{2}{p}}}{2^{2k-2}}$$
$$> \mathcal{L}_{H_k^2,p}(u_{C_0}, v_{C_0}) = \frac{((2^{k-1} - 1)^p + (2^{k-2} - 1)^p)^{\frac{2}{p}}}{\frac{1}{3} \cdot 2^{2k-3} + \frac{1}{3} \cdot 2^{2k-4}}.$$

Empirically, for all $k \in \{5, 6, \ldots, 12\}$ and all reals p of the (growing) suffix-interval $(\rho_k, 2] \subseteq [1, 2]$, all the representative grid-point pairs form a subsequence C' of C composed of: (1) a prefix of C and (2) $(u_{C_{k-2}}, v_{C_{k-2}})$. The suffix-interval $(\rho_k, 2]$ is partitioned into disjoint successive p-subintervals, each of which supports a grid-point pair in the subsequence C' as the representative grid-point pair for H_k^2 (for all reals p of the subinterval). The length of C' (number of all representative grid-point pairs from the source C) should depend on k in general, and on the p-granularity in our empirical setting. Figure 2(b) depicts the sequence of candidate representative grid-point pairs from the source C.

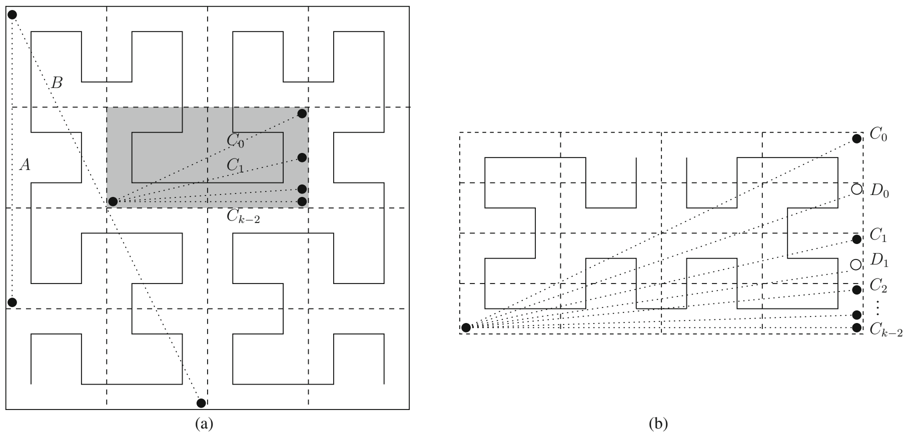


Fig. 2. Candidate representative grid-point pairs for H_k^2 with respect to L_p for $k \geq 2$: (a) three sources $\{A, B, C\}$ of candidate representative grid-point pairs; (b) detailed view of the source C.

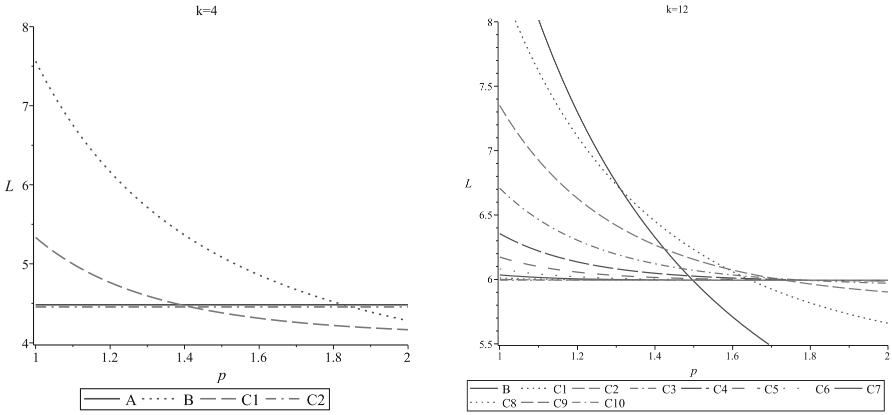
Table 1 tabulates: (1) for each $k \in \{2, 3, \ldots, 12\}$, the partitioning p-subintervals of $[1, 2]$, and the corresponding representative grid-point pair and its source; and (2) $\mathcal{L}_{H_k^2,p}(u, v)$ $(= L_p(H_k^2))$ for a representative grid-point pair (u, v) in the three sources A, B, and C:

Table 1. Representative grid-point pairs for H_k^2 with respect to L_p for $k \in \{2, 3, \ldots, 12\}$ and $p \in [1.00, 2.00]$ with granularity of 0.01

k	p	(x, y)-coordinates	representative grid-point pair coordinates in terms of k	source
2	$[1.00, 2.00]$	$((2, 1), (1, 4))$	$((2^{k-1}, 1), (1, 2^k))$	B
3	$[1.00, 2.00]$	$((4, 1), (1, 8))$	$((2^{k-1}, 1), (1, 2^k))$	B
4	$[1.00, 1.82]$	$((8, 1), (1, 16))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.83, 2.00]$	$((1, 5), (1, 16))$	$((1, \frac{1}{4} \cdot 2^k + 1), (1, 2^k))$	A
5	$[1.00, 1.61]$	$((16, 1), (1, 32))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.62, 2.00]$	$((9, 17), (24, 17))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_3
6	$[1.00, 1.51]$	$((32, 1), (1, 64))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.52, 1.55]$	$((17, 33), (48, 40))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.56, 1.60]$	$((17, 33), (48, 36))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.61, 2.00]$	$((17, 33), (48, 33))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_4
7	$[1.00, 1.41]$	$((64, 1), (1, 128))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.42, 1.57]$	$((33, 65), (96, 80))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.58, 1.66]$	$((33, 65), (96, 72))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.67, 1.67]$	$((33, 65), (96, 68))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.68, 2.00]$	$((33, 65), (96, 65))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_5
8	$[1.00, 1.36]$	$((128, 1), (1, 256))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.37, 1.57]$	$((65, 129), (192, 160))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.58, 1.68]$	$((65, 129), (192, 144))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.69, 1.72]$	$((65, 129), (192, 136))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.73, 2.00]$	$((65, 129), (192, 129))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_6
9	$[1.00, 1.33]$	$((256, 1), (1, 512))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.34, 1.58]$	$((129, 257), (384, 320))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.59, 1.69]$	$((129, 257), (384, 288))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.70, 1.75]$	$((129, 257), (384, 272))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.76, 1.77]$	$((129, 257), (384, 264))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	$[1.78, 2.00]$	$((129, 257), (384, 257)$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_7
10	$[1.00, 1.32]$	$((512, 1), (1, 1024))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.33, 1.58]$	$((257, 513), (768, 640))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.59, 1.70]$	$((257, 513), (768, 576))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.71, 1.76]$	$((257, 513), (768, 544))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.77, 1.79]$	$((257, 513), (768, 528))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	$[1.80, 1.80]$	$((257, 513), (768, 520))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-7}))$	C_5
	$[1.81, 2.00]$	$((257, 513), (768, 513))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_8
11	$[1.00, 1.31]$	$((1024, 1), (1, 2048))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.32, 1.58]$	$((513, 1025), (1536, 1280))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.59, 1.70]$	$((513, 1025), (1536, 1152))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.71, 1.76]$	$((513, 1025), (1536, 1088))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.77, 1.80]$	$((513, 1025), (1536, 1056))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	$[1.81, 1.82]$	$((513, 1025), (1536, 1040))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-7}))$	C_5
	$[1.83, 2.00]$	$((513, 1025), (1536, 1025))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_9
12	$[1.00, 1.31]$	$((2048, 1), (1, 4096))$	$((2^{k-1}, 1), (1, 2^k))$	B
	$[1.32, 1.58]$	$((1025, 2049), (3072, 2560))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-3}))$	C_1
	$[1.59, 1.70]$	$((1025, 2049), (3072, 2304))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-4}))$	C_2
	$[1.71, 1.77]$	$((1025, 2049), (3072, 2176))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-5}))$	C_3
	$[1.78, 1.81]$	$((1025, 2049), (3072, 2112))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-6}))$	C_4
	$[1.82, 1.83]$	$((1025, 2049), (3072, 2080))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-7}))$	C_5
	$[1.84, 1.84]$	$((1025, 2049), (3072, 2064))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 2^{k-8}))$	C_6
	$[1.85, 2.00]$	$((1025, 2049), (3072, 2049))$	$((\frac{1}{4} \cdot 2^k + 1, 2^{k-1} + 1), (\frac{3}{4} \cdot 2^k, 2^{k-1} + 1))$	C_{10}

$$\mathcal{L}_{H_k^2,p}(u,v) = \begin{cases} \frac{(3\cdot 2^{k-2}-1)^2}{\frac{5}{3}\cdot 2^{2k-4}+\frac{1}{3}} & \text{if } (u,v) \text{ is in } A \\ \frac{((2^{k-1}-1)^p+(2^k-1)^p)^{\frac{2}{p}}}{2^{2k-2}} & \text{if } (u,v) \text{ is in } B \\ \frac{((2^{k-1}-1)^p+(2^{k-2-t}-1)^p)^{\frac{2}{p}}}{\frac{1}{3}\cdot 2^{2k-3}+\frac{1}{3}\cdot 2^{2k-4-2t}} & \text{if } (u,v) = (u_{C_t}, v_{C_t}) \text{ in } C, \\ & \quad \text{where } t = 1, 2, \ldots, k-2. \end{cases}$$

Figure 3(a) and (b) show the graphs, using the mathematical software Maple, of the locality measure $\mathcal{L}_{H_k^2,p}(u,v)$ for $k = 4$ and 12, respectively, for all reals $p \in [1,2]$ and all (u,v) in the three sources A, B, and C. Our future work will involve determining, for each k, the dominant functions/measures over successive subintervals of $[1,2]$, whose piece-wise combination yields the (overall) locality measure $L_p(H_k^2)$ for all reals $p \in [1,2]$.



(a) (b)

Fig. 3. Locality measures corresponding to the grid-point pairs in: (a) A, B, and $C = \{C_2\}$ for $k = 4$ and p-granularity of 0.01; (b) B and $C = \{C_t \mid 1 \leq t \leq k-2\}$ for $k = 12$ and p-granularity of 0.01. (Color figure online)

For the extreme case of $k = 4$ with p-granularity of 0.01, two representative grid-point pairs emerge from the sources B and A over the partitioning subintervals $[1.00, 1.82]$ and $[1.83, 2.00]$, respectively.

For a more general case of $k = 12$ with p-granularity of 0.01, the representative grid-point pairs are from the sources B and C over the partitioning subintervals $[1.00, 1.31]$ and $[1.32, 2.00]$, respectively. Observe that the subsequence C' of all representative grid-point pairs (from the source $C = \{C_t \mid 1 \leq t \leq 10\}$) is $\{C_1, C_2, C_3, C_4, C_5, C_6, C_{10}\}$.

4 Conclusion

Our analytical study of the locality properties of the Hilbert curve family, $\{H_k^2 \mid k = 1, 2, \ldots\}$, is based on the locality measure L_p, which is the maximum ratio of $d_p(u, v)^m$ to $d_p(\tilde{u}, \tilde{v})$ over all corresponding point-pairs (u, v) and $(\tilde{u}, \tilde{v})$ in the m-dimensional grid space and index space, respectively. Our analytical results identify all the candidate representative grid-point pairs of H_k^2 from the three sources A, B, and C (which realize $L_p(H_k^2)$-values) for all norm-parameters $p \in [1, 2]$ and grid-orders k, which enable us to have almost complete knowledge of $L_p(H_k^2)$ for all $p \geq 1$ – except for the relation between the candidate grid-point pairs and their dominance p-subintervals. For all real norm-parameters $p \in [1, 2]$ with sufficiently small granularity and grid-orders $k \in \{2, 3, \ldots, 12\}$, our empirical study reveals the three major sources (A, B, and C) of representative grid-point pairs (v, u) that give $\mathcal{L}_{H_k^2, p}(v, u) = L_p(H_k^2)$. The results also suggest that all the representative grid-point pairs of B and C are from B and C', which is a prefix-subsequence of C together with C_{k-2} for some sufficiently large grid-orders $k \in \{5, 6, \ldots, 12\}$. The study has shed some light on a continuing study of determining the interplay pattern between the norm-parameter p and grid-order k for emerging representative grid-point pairs.

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