Chapter 2
The Effective Field Theory of Dark Energy

When looking at alternatives to the standard ΛCDM+GR model, the simplest and most common way is to introduce an extra scalar field (see [1] for a review). It can either act as an additional dark energy fluid, or as a modification of the laws of gravity themselves. It is the easiest modification one can make and is as such the first that should be explored: there is only one additional degree of freedom to consider, making it an informative step before looking at more complicated scenarios. Even in some cases where multiple degrees of freedom are added, such as in massive [2] or bimetric gravity [3] for example, one recovers the case of a single scalar field in relevant limits.

This universality is yet more manifest for a second reason. The goal of the modifications at hand are to try and explain the current accelerated expansion of the Universe [4, 5]. Thus, in general, any field added for this purpose will have a background value that is time dependent, since the homogeneous Universe evolves in time. This explicitly breaks the time diffeomorphism invariance, that can be restored as usual with Goldstone modes, which would be a single scalar in this case (see for example [6]). Therefore, the low energy perturbations around a time dependent background will generically be described by this scalar, regardless of the fundamental origin of the theory.

These ideas were first developed in the case of inflation in [7] under the name of the Effective Field Theory of Inflation and then used for example to compute higher order correlation functions, which allow to probe non-Gaussianities [8, 9]. Later, it was applied in the context of late time acceleration in the Effective Field Theory of Dark Energy (EFT of DE) in [10, 11] and also [12].

In this section, I will present the concepts behind such an approach as well as its many advantages, based on the work I did in [13], later summarized in a review [14].

J. Gleyzes, *Dark Energy and the Formation of the Large Scale Structure of the Universe*, Springer Theses,
DOI 10.1007/978-3-319-41210-8_2

2.1 The Unitary Gauge Action

The first thing I will assume is the Weak Equivalence Principle, namely that there exists a metric that universally couples to the matter sector, even if the formalism I am going to present would apply if species coupled to different metrics (see e.g. [15, 16]).

Next, the goal is to look for a generic action that would describe cosmological perturbations around a FLRW background when looking at cosmology beyond ΛCDM. By this I mean either dark energy and/or modifications of the actual laws of gravity. For concreteness, I will consider the case of an extra scalar field, ϕ. However, the idea is to be as model independent as possible considering these assumptions.

As I mentioned before, this scalar field, in a cosmological context, is naturally expected to be spacelike, i.e. to have a gradient such that $\nabla_\mu\phi\nabla^\mu\phi < 0$. In this case, the hypersurfaces of constant ϕ define a preferred foliation of time. It is convenient to use the gauge freedom in the theory to choose this specific time: this is called the unitary gauge.

By doing so, the perturbation in the scalar field are hidden, since now we have

$$\phi(\tilde{t}, \vec{x}) = \phi_0(\tilde{t}) + \delta\phi(\tilde{t}, \vec{x}) = \phi_0(t)\,, \tag{2.1}$$

where the last equality holds because of the choice of specific time t that is made, see Fig. 2.1. Of course, the perturbation $\delta\phi$ did not disappear, it is part of the perturbations of the metric. For example, the standard kinetic term for ϕ becomes in this gauge

$$X \equiv \nabla_\mu\phi\nabla^\mu\phi = g^{00}\dot{\phi}_0^2\,, \tag{2.2}$$

so that these quantities still contribute to the perturbative expansion through $g^{00} = -1 + \delta g^{00}$. The unitary gauge has therefore the advantage of having to deal only with the metric, however it has a minor inconvenient. Since a choice of time was made, the invariance under time reparametrization is lost (while leaving the spatial one intact). This means that the theory will not be manifestly covariant, as can be seen already from Eq. (2.2). Indeed, tensors with upper indices set to 0 are allowed in this gauge (they correspond to contractions with the gradient of the scalar field, e.g. $\mathcal{P}^{00} \sim \mathcal{P}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$). This should not be worried over, as a simple redefinition of time

$$t \to t + \pi(t, \vec{x})\,, \tag{2.3}$$

allows to explicitly reintroduce the invariance under time reparametrization of the theory [6]. This is known as the Stueckelberg trick and the variable π is the field

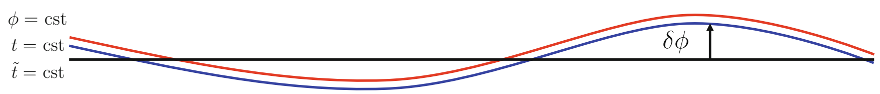

Fig. 2.1 The original time $\tilde{t}$ hypersurface in *black*. In *blue*, the new time t in unitary gauge, that is chosen so its constant hypersurfaces match the ϕ hypersurfaces (*red*).

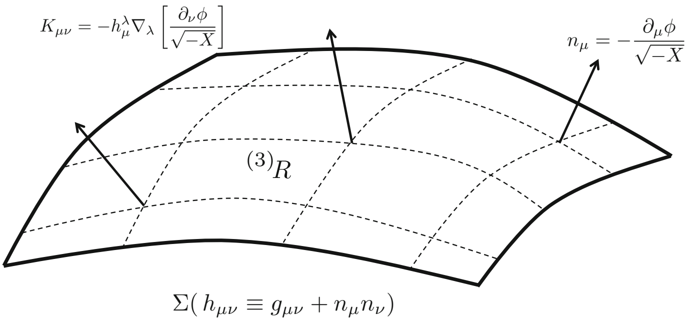

Fig. 2.2 Σ, a constant ϕ hypersurface, and its geometrical quantities: the normal vector n_μ, the projected metric $h_{\mu\nu}$, the intrinsic curvature R, as well as the extrinsic curvature $K_{\mu\nu}$.

that non linearly realizes this invariance. This will be useful to change gauge. In particular, to go to Newtonian gauge, where the equations of motion (EOM) have an easier interpretation.

Nevertheless, the unitary gauge will enable us to write the most general action for a scalar-tensor theory, without reference to a specific model. Indeed, in this gauge, all the terms that are invariant under spatial diffeomorphisms are in principle allowed. Further conditions can be imposed, such as second-order EOM for example, but the basic ingredients can be obtained from the geometry of the hypersurfaces illustrated in Fig. 2.2 and are the following:

- The normal vector orthogonal to the surfaces, $n_\mu \equiv -\frac{\nabla_\mu \phi}{\sqrt{-X}}$. This term is the one responsible for the presence of tensors with 0 as upper indices.
- The extrinsic curvature, $K_{\mu\nu}$. It quantifies the variation of the normal vector

$$K_{\mu\nu} \equiv h_{\mu\sigma} \nabla^\sigma n_\nu \,, \quad h_{\mu\sigma} \equiv g_{\mu\sigma} + n_\mu n_\sigma \,, \tag{2.4}$$

  $h_{\mu\sigma}$ being the induced metric on the hypersurface. The quantity $K_{\mu\nu}$ tells us how the hypersurfaces are embedded in the full 4-D space.
- The final ingredient is the intrinsic curvature, given be the 3-D Ricci tensor R_{ij} of the hypersurface. This is the equivalent[1] of the 4-D Riemann tensor $^{(4)}R_{\mu\nu\rho\sigma}$ for the full space. In what follows, unless specified explicitly with a (4), the Ricci tensor R_{ij} and scalar R will always be the 3-D ones.

[1]In three dimensions, there is as much information in the Ricci tensor as in the Riemann tensor since

$$R_{\mu\nu\rho\sigma} = R_{\mu\rho} h_{\nu\sigma} - R_{\nu\rho} h_{\mu\sigma} - R_{\mu\sigma} g_{\nu\rho} + R_{\nu\sigma} h_{\mu\rho} - \frac{1}{2} R (h_{\mu\rho} h_{\nu\sigma} - h_{\mu\sigma} h_{\nu\rho}) \,. \tag{2.5}$$

This is because the Weyl tensor vanishes. Another way to see it is to count the independent variables in the Riemann and Ricci tensors using their known symmetries. One obtains the same number for both in three dimensions.

The numbers of combinations of these terms is infinite. This is why in the following I will impose restrictions on the categories of action I will consider. To be more quantitative, I will discuss these restrictions in the formalism of Arnowitt-Deser-Misner (ADM) [17].

2.2 ADM Formalism and the Effective Field Theory of Dark Energy

In order to be more specific about the action, I will go one step further in the distinction between space and time. To make more explicit the 3+1 decomposition, I will use the ADM form of the metric, namely write the line element as

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) , \tag{2.6}$$

where N is the lapse, N^i the shift and h_{ij} is the spatial metric on constant time hypersurfaces, which can be decomposed into a scalar part, ζ, and a tensorial one, γ_{ij} as

$$h_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \gamma_{ij}) , \quad \partial_i \gamma_{ij} = \gamma_{ii} = 0 . \tag{2.7}$$

With this metric and in unitary gauge, the basic ingredients I mentioned above take the simpler form

$$n_\mu = -\delta^0_\mu N , \quad g^{00} = -\frac{1}{N^2} , \tag{2.8}$$

$$K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - D_i N_j - D_j N_i\right] . \tag{2.9}$$

The other components are not needed. Indeed, $K^{0i} = K^{00} = 0$ since by definition (2.4) the extrinsic curvature is orthogonal to the unit vector, $n_\mu K^{\mu\nu} = 0$. D_i is the covariant derivative associated with the spatial metric h_{ij}. The 3-D Ricci tensor R_{ij} is the standard one constructed from this metric. With this decomposition of the metric, any Lagrangian respecting the spatial diffeomorphisms invariance can be cast into the generic form

$$S_g = \int d^4x \sqrt{-g}\, L(N, K_{ij}, R_{ij}, h_{ij}, D_i, \partial^0; t) . \tag{2.10}$$

As an example, the Einstein-Hilbert action of standard GR,

$$S_{\rm GR} = \int d^4x \sqrt{-g}\, \frac{M_{\rm Pl}^2}{2}\, {}^{(4)}R , \tag{2.11}$$

can be rewritten in this form as

$$L_{\rm GR} = \frac{M_{\rm Pl}^2}{2}\left[K_{ij}K^{ij} - K^2 + R\right] , \tag{2.12}$$

using the Gauss Codazzi relation

$${}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu) . \tag{2.13}$$

Virtually all known models of dark energy involving a single field can be mapped onto a specific form of the Lagrangian (2.10). However, the real strength of this approach is that it allows to generically look at modifications of ΛCDM, without the need to specify a model.

To be quantitative, I will only look at the linearized theory, which means the action will only contain perturbations up to second order. Secondly, I will discuss the case where the three DOF of the theory (the two tensor polarizations and the additional scalar) obey second-order dynamics, to ensure stability. Moreover, I will assume that the full theory is given by an action $S_{\rm full} = S_g + S_{\rm mat}$, where $S_{\rm mat}$ is an action that describes minimally coupled matter.

2.2.1 Background Evolution

As I said in the Introduction, it is fairly simple in general to reproduce the same background as that of ΛCDM, even if the perturbations might be different. Nevertheless, I will firstly discuss the background equations by considering a spatially flat FLRW spacetime, whose metric reads

$${\rm d}s^2 = -\bar{N}^2(t){\rm d}t^2 + a^2(t)\delta_{ij}{\rm d}x^i{\rm d}x^j . \tag{2.14}$$

In this spacetime, the intrinsic curvature tensor of the constant time hypersurfaces vanishes, i.e. $R_{ij} = 0$, and the components of the extrinsic curvature tensor are given by

$$K^i_j = \frac{\dot{a}}{\bar{N}a}\delta^i_j \equiv H\delta^i_j , \tag{2.15}$$

where H is the Hubble parameter. Note that since this is the background level, only a time dependence can appear. Substituting into the Lagrangian L of (2.10), one thus obtains an homogeneous Lagrangian, which is a function of $\bar{N}(t)$, $a(t)$ and of time:

$$\bar{L}(a, \dot{a}, \bar{N}) \equiv L\left[K^i_j = \frac{\dot{a}}{\bar{N}a}\delta^i_j, R^i_j = 0, N = \bar{N}(t)\right] . \tag{2.16}$$

The variation of the homogeneous action,

$$\bar{S}_g = \int dt\, d^3x \bar{N} a^3 \bar{L}, \tag{2.17}$$

leads to

$$\delta \bar{S}_g = \int dt d^3x \left\{ a^3 \left(\bar{L} + \bar{N} L_N - 3H\mathcal{F} \right) \delta \bar{N} + 3a^2 \bar{N} \left(\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} \right) \delta a \right\} , \tag{2.18}$$

where L_N denotes the partial derivative $\partial L/\partial N|_{\text{bgd}}$, evaluated on the homogeneous background. We have also introduced the coefficient $\mathcal{F}$, which is defined from the derivative of the Lagrangian with respect to the extrinsic curvature, evaluated on the background

$$\left(\frac{\partial L}{\partial K_{ij}} \right)_{\text{bgd}} \equiv \mathcal{F} \bar{g}^{ij} , \tag{2.19}$$

where $\bar{g}^{ij} = a^{-2} \delta^{ij}$ are the spatial components of the inverse background metric.

If we add some matter minimally coupled to the metric $g_{\mu\nu}$, the variation of the corresponding action with respect to the metric defines the energy-momentum tensor,

$$\delta S_{\text{m}} = \frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu} . \tag{2.20}$$

In a FLRW spacetime, this reduces to

$$\delta \bar{S}_{\text{m}} = \int d^4x \bar{N} a^3 \left(-\rho_{\text{m}} \frac{\delta \bar{N}}{\bar{N}} + 3 p_{\text{m}} \frac{\delta a}{a} \right) . \tag{2.21}$$

Consequently, variation of the total homogeneous action $\bar{S} = \bar{S}_g + \bar{S}_{\text{m}}$ with respect to N and a yields, respectively, the first and second Friedmann equations in a very unusual form:

$$\bar{L} + \bar{N} L_N - 3H\mathcal{F} = \rho_{\text{m}} \tag{2.22}$$

and

$$\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_{\text{m}} . \tag{2.23}$$

These two equations, which are the generalization of the Friedmann equations, also imply

$$\frac{\dot{\mathcal{F}}}{\bar{N}} + \bar{N} L_N = \rho_{\text{m}} + p_{\text{m}} . \tag{2.24}$$

It is easy to check that one recovers the usual Eqs. (1.8)–(1.9) when gravity is described by general relativity only. Indeed, in this case,

$$\frac{\partial L_{\rm GR}}{\partial K^i_j} = M_{\rm Pl}^2 \left(K_i^j - K\delta_i^j \right) , \tag{2.25}$$

which, after substituting $K^i_j = H\delta^i_j$, yields,

$$\mathcal{F}_{\rm GR} = -2M_{\rm Pl}^2 H , \tag{2.26}$$

whereas $\bar{L}_{\rm GR} = -3M_{\rm Pl}^2 H^2$ and $L_N = 0$. In the rest of this thesis, I will set $\bar{N} = 1$, which can always be achieved through at redefinition of time.

2.2.2 The Quadratic Action

To obtain the quadratic action, that will yield the linear equations of motion, one expands Eq. (2.10) in terms of the perturbative quantities

$$\delta N \equiv N - 1 , \quad \delta K^i{}_j \equiv K^i{}_j - H\delta^i{}_j , \quad R^i{}_j . \tag{2.27}$$

Then, the expansion of the Lagrangian L up to quadratic order yields

$$L(N, K^i_j, R^i_j, \dots) = \bar{L} + L_N\delta N + \frac{\partial L}{\partial K^i_j}\delta K^i_j + \frac{\partial L}{\partial R^i_j}\delta R^i_j + L^{(2)} + \cdots , \tag{2.28}$$

with the quadratic part given by

$$\begin{aligned} L^{(2)} =& \frac{1}{2}L_{NN}\delta N^2 + \frac{1}{2}\frac{\partial^2 L}{\partial K^i_j \partial K^k_l}\delta K^i_j\delta K^k_l + \frac{1}{2}\frac{\partial^2 L}{\partial R^i_j \partial R^k_l}\delta R^i_j\delta R^k_l + \\ &+ \frac{\partial^2 L}{\partial K^i_j\,\partial R^k_l}\delta K^i_j\delta R^k_l + \frac{\partial^2 L}{\partial N\partial K^i_j}\delta N\delta K^i_j + \frac{\partial^2 L}{\partial N\partial R^i_j}\delta N\delta R^i_j + \cdots , \end{aligned} \tag{2.29}$$

where all the partial derivatives are evaluated on the FLRW background (without explicit notation, as will be the case in the rest of this Chapter). The coefficient L_{NN} denotes the second derivative of the Lagrangian with respect to N. The dots in the two above equations correspond to other possible terms which are not indicated explicitly to avoid too lengthy equations, but can be treated exactly in the same way.

The third term on the right hand side of (2.28) can be simplified as follows. Rewriting it as

$$\frac{\partial L}{\partial K^i_j}\delta K^i_j = \mathcal{F}\delta K = \mathcal{F}(K - 3H) , \tag{2.30}$$

and noting that $K = \nabla_\mu n^\mu$, one can use the integration by parts

$$\int d^4x\sqrt{-g}\,\mathcal{F}K = -\int d^4x\sqrt{-g}\,n^\mu\nabla_\mu\mathcal{F} = -\int d^4x\sqrt{-g}\,\frac{\dot{\mathcal{F}}}{N}\,. \tag{2.31}$$

This implies that the Lagrangian (2.28) can be replaced by the equivalent Lagrangian

$$L^{\text{new}} = \bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{N} + L_N\delta N + L^{(2)}\,. \tag{2.32}$$

Let us now consider the quadratic part (2.29). Because of the symmetries of background, the coefficient of the second term is necessarily of the form[2]

$$\frac{\partial^2 L}{\partial K^j_i\,\partial K^l_k} = \hat{\mathcal{A}}_K\,\delta^i_j\,\delta^k_l + \mathcal{A}_K\left(\delta^i_l\,\delta^k_j + \delta^{ik}\delta_{jl}\right)\,, \tag{2.33}$$

where we have introduced the (a priori time-dependent) coefficients $\hat{\mathcal{A}}_K$ and $\mathcal{A}_K$. Similarly, one can write

$$\frac{\partial^2 L}{\partial R^j_i\,\partial R^l_k} = \hat{\mathcal{A}}_R\,\delta^i_j\,\delta^k_l + \mathcal{A}_R\left(\delta^i_l\,\delta^k_j + \delta^{ik}\delta_{jl}\right)\,, \tag{2.34}$$

and

$$\frac{\partial^2 L}{\partial K^j_i\,\partial R^l_k} = \hat{\mathcal{C}}\,\delta^i_j\,\delta^k_l + \mathcal{C}\left(\delta^i_l\,\delta^k_j + \delta^{ik}\delta_{jl}\right)\,. \tag{2.35}$$

The mixed coefficients that appear on the second line are proportional to δ^j_i and can be written as

$$\frac{\partial^2 L}{\partial N\partial K^i_j} = \mathcal{B}\,\delta^j_i\,, \qquad \frac{\partial^2 L}{\partial N\partial R^i_j} = \mathcal{B}_R\,\delta^j_i\,. \tag{2.36}$$

Taking into account the term $\sqrt{-g} = N\sqrt{h}$, it is straightforward to derive the quadratic part of the full Lagrangian $\mathcal{L} \equiv \sqrt{-g}\,L$, which is relevant to study linear perturbations. After some cancellations due to the background equations of motion,[3] one finds

[2] This is equivalent to the definition below, expressed with covariant indices for the extrinsic curvature tensors, which makes the symmetry under exchange of the indices more manifest:

$$\frac{\partial^2 L}{\partial K_{ij}\,\partial K_{kl}} \equiv \hat{\mathcal{A}}_K\,\bar{g}^{ij}\,\bar{g}^{kl} + \mathcal{A}_K\left(\bar{g}^{ik}\,\bar{g}^{jl} + \bar{g}^{il}\,\bar{g}^{jk}\right)\,.$$

[3] If matter is present, one must also include in the quadratic Lagrangian the terms from the expansion of the matter action with respect to the metric perturbations.

$$
\begin{aligned}
\mathcal{L}_2 = {} & \bar{N}\mathcal{G}\,\delta_1 R\,\delta\sqrt{h} + a^3\left(L_N + \frac{1}{2}\bar{N}L_{NN}\right)\delta N^2 \\
& + \bar{N}a^3\left[\mathcal{G}\delta_2 R + \frac{1}{2}\hat{\mathcal{A}}_K\,\delta K^2 + \mathcal{B}\,\delta K\delta N + \hat{\mathcal{C}}\,\delta K\delta R + \mathcal{C}\,\delta K^i_j\,\delta R^j_i \right. \\
& \left. + \mathcal{A}_K\,\delta K^i_j\,\delta K^j_i + \mathcal{A}_R\,\delta R^i_j\,\delta R^j_i + \frac{1}{2}\hat{\mathcal{A}}_R\,\delta R^2 + \left(\frac{\mathcal{G}}{\bar{N}} + \mathcal{B}_R\right)\delta N\delta R\right] + \cdots\,,
\end{aligned}
\tag{2.37}
$$

where, in analogy with the definition (2.19) of $\mathcal{F}$, we have introduced the coefficient $\mathcal{G}$ defined by

$$
\frac{\partial L}{\partial R^i_j} = \mathcal{G}\,\delta^j_i\,. \tag{2.38}
$$

We have also denoted as $\delta_1 R$ and $\delta_2 R$, respectively, the first and second order terms of the curvature R expressed in terms of the metric perturbations.

The above quadratic expression can be further simplified by reexpressing $\delta K^i_j\,\delta R^j_i$ in terms of the other terms, thanks to the identity

$$
\int d^4x\sqrt{-g}\,\lambda(t)R_{ij}K^{ij} = \int d^4x\sqrt{-g}\left[\frac{\lambda(t)}{2}R\;K + \frac{\dot{\lambda}(t)}{2N}\;R\right]. \tag{2.39}
$$

This implies the following replacement at quadratic order:

$$
\bar{N}a^3\mathcal{C}\,\delta K^i_j\,\delta R^j_i \quad\to\quad \frac{\bar{N}a^3}{2}\left[\left(\frac{\dot{\mathcal{C}}}{\bar{N}} + H\mathcal{C}\right)\left(\delta_2 R + \frac{\delta\sqrt{h}}{a^3}\delta R\right) + \mathcal{C}\,\delta R\delta K + \frac{H\mathcal{C}}{\bar{N}}\delta N\delta R\right]. \tag{2.40}
$$

Consequently, the quadratic Lagrangian (2.37) is equivalent to the new one

$$
\begin{aligned}
\mathcal{L}_2^{\rm new} = {} & \bar{N}\mathcal{G}^*\,\delta_1 R\,\delta\sqrt{h} + a^3\left(L_N + \frac{1}{2}\bar{N}L_{NN}\right)\delta N^2 \\
& + \bar{N}a^3\left[\mathcal{G}^*\delta_2 R + \frac{1}{2}\hat{\mathcal{A}}_K\,\delta K^2 + \mathcal{B}\,\delta K\delta N + \mathcal{C}^*\,\delta K\delta R \right. \\
& \left. + \mathcal{A}_K\,\delta K^i_j\,\delta K^j_i + \mathcal{A}_R\,\delta R^i_j\,\delta R^j_i + \frac{1}{2}\hat{\mathcal{A}}_R\,\delta R^2 + \left(\frac{\mathcal{G}^*}{\bar{N}} + \mathcal{B}^*_R\right)\delta N\delta R\right] + \cdots\,,
\end{aligned}
\tag{2.41}
$$

with the “renormalized” coefficients

$$
\begin{aligned}
\mathcal{G}^* &= \mathcal{G} + \frac{\dot{\mathcal{C}}}{2\bar{N}} + H\mathcal{C}\,, \\
\mathcal{C}^* &= \hat{\mathcal{C}} + \frac{1}{2}\mathcal{C}\,, \\
\mathcal{B}^*_R &= \mathcal{B}_R - \frac{\dot{\mathcal{C}}}{2\bar{N}^2}\,.
\end{aligned}
\tag{2.42}
$$

Let me concentrate more particularly on the scalar sector, since this is where restrictions need to be imposed in order to keep second-order dynamics. I will use the further parametrization

$$N^i = \delta^{ij}\partial_j\psi\,, \tag{2.43}$$

for the scalar part of g^{0i}. Together with the form of the metric (2.7), the perturbations of the geometrical quantities read

$$\delta\sqrt{h} = 3a^3\zeta\,, \qquad \delta K^i{}_j = \left(\dot{\zeta} - H\delta N\right)\delta^i_j - \frac{1}{a^2}\delta^{ik}\partial_k\partial_j\psi\,, \tag{2.44}$$

and

$$\delta_1 R_{ij} = -\delta_{ij}\partial^2\zeta - \partial_i\partial_j\zeta\,, \qquad \delta_2 R = -\frac{2}{a^2}\left[(\partial\zeta)^2 - 4\zeta\partial^2\zeta\right]. \tag{2.45}$$

I will restrict to the case where no time derivatives ∂^0 appear explicitly in the Lagrangian, since it leads in general to extra DOF (see [14] for a discussion on including such derivatives). In this case, the variation with respect to δN and ψ gives constraint equations. They allow to express δN and ψ in terms of ζ and its derivatives, yielding an action only for this variable. It is on this action that conditions need to be imposed to get second-order dynamics.[4] They read

$$\hat{\mathcal{A}}_K + 2\mathcal{A}_K = 0\,, \qquad \mathcal{C}^* = 0\,, \qquad 4\hat{\mathcal{A}}_R + 3\mathcal{A}_R = 0\,, \tag{2.46}$$

Then the most general action that abides by these criteria can be written as

$$\boxed{\begin{aligned} S_g = \int d^4x a^3 \frac{M^2}{2}\Bigg[&\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2 + (1+\alpha_T)\left(\delta_{(2)}R + \frac{\delta\sqrt{h}}{a^3}R\right) + H^2\alpha_K\delta N^2 \\ &+ 4H\alpha_B\delta N\delta K + (1+\alpha_H)R\,\delta N\Bigg] + \cdots \end{aligned}}\,, \tag{2.47}$$

where $h = \det h_{ij}$ and the $\cdots$ denotes terms that vanish when the background equations are enforced. The functions M and α_i are all in principle dependent on time, which is allowed by the presence of the extra scalar field. Additionally, one can define

$$\alpha_M \equiv \frac{2\dot{M}}{HM}\,, \tag{2.48}$$

which parametrizes the potential time dependence of the Planck mass. These coefficients, originally introduced in [18], are defined so that the standard case of ΛCDM+GR would correspond to setting all of them to zero.

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[4] It is too restrictive to impose no higher derivatives in all of the equations before the constraint are solved. Indeed, such constraints might remove these higher derivatives so that the actual propagating DOF still obeys a second-order EOM. See Sect. 3.2 for more details.

Table 2.1 In the first row, the parameters α_i in the Lagrangian of Eq. (2.47)

Equation (2.47)	M^2	α_M	α_K	α_B	α_T	α_H
Equation (2.41)	$2\mathcal{A}_K$	$\frac{1}{H}\frac{d}{dt}\ln\mathcal{A}_K$	$\frac{2L_N + L_{NN}}{2H^2\mathcal{A}_K}$	$\frac{\mathcal{B}}{4H\mathcal{A}_K}$	$\frac{\mathcal{G}^*}{\mathcal{A}_K} - 1$	$\frac{\mathcal{G}^* + \mathcal{B}_R^*}{\mathcal{A}_K} - 1$
Parameters of [13]	$M_{\rm Pl}^2 f + 2m_4^2$	$\frac{M_{\rm Pl}^2\dot{f} + 2(m_4^2)^{\cdot}}{M^2 H}$	$\frac{2c + 4M_2^4}{M^2H^2}$	$\frac{M_{\rm Pl}^2\dot{f} - m_3^3}{2M^2H}$	$-\frac{2m_4^2}{M^2}$	$\frac{2(\tilde{m}_4^2 - m_4^2)}{M^2}$

These parameters are written in terms of the Lagrangian coefficients of Eq. (2.41), defined in Eqs. (2.33)–(2.36) (second row), and in terms of the parameter of the EFT Lagrangian in [13] (third row). All these quantities are understood to be evaluated on the background, with $\bar{N} = 1$

They can be related to the original Lagrangian (2.10) and its derivatives with respect to the various quantities N, K_{ij}, ... The starting point is to define the equivalent of the Planck mass, M, which is associated with the normalization of the tensor kinetic term, $\dot{\gamma}_{ij}^2$. Since $\dot{\gamma}_{ij}$ only appears in K_{ij}, the M is going to be given by the derivative of the Lagrangian with respect to the extrinsic curvature, Eq. (2.33). More precisely,

$$M^2 \equiv 2\mathcal{A}_K \,. \tag{2.49}$$

Then, all the coefficients α_i follow almost algorithmically (Table 2.1)

In the next section, for concreteness, I will give examples on how to get these parameters in the case of specific models.

2.3 Going from Models to the EFT of DE

Once a model is decomposed in 3+1 quantities, computing its parameters is completely automatic, making the link with possible constraints straightforward. Let me go through the functions α_a one at a time, increasing the complexity of the model needed to illustrate the parameter.

- α_K

  Taking the simplest case of GR plus quintessence [19], i.e.

$$L = \frac{M_{\rm Pl}^2}{2}\left[K_{\mu\nu}K^{\mu\nu} - K^2 + R\right] - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\,. \tag{2.50}$$

  After going to unitary gauge, one finds

$$M = M_{\rm Pl}\,, \quad \alpha_K = \frac{\dot{\phi}_0^2}{H^2 M_{\rm Pl}^2}\,, \tag{2.51}$$

  while all the others coefficients and $\dot{\phi}_0$ is the background value of the scalar field. One can indeed check that ΛCDM corresponds to all the α_i being zero: one recovers the cosmological constant for $\dot{\phi}_0 = 0$, which would set $\alpha_K = 0$.

  As a side note, it might seem odd that the potential V does not appear in Eq. (2.51). The reason is that this parametrization is specifically designed to look at linear perturbations, while V is a background quantity in unitary gauge. More precisely, the Friedmann equations impose

$$V = \frac{M_{\rm Pl}^2}{2}\left[2\dot{H} + 3H^2\left(2 - \Omega_{\rm m}\right)\right]\,. \tag{2.52}$$

  Therefore, if the history of H and the matter content are known, V is fixed.

- α_B

This example requires a more complicated model: kinetic braiding [20]. This theory is characterized by a Lagrangian of the form

$$L_3 = L_{\rm GR} + G_3(X)\,\Box\phi = L_{\rm GR} - \int G_{3X}\sqrt{-X}\,{\rm d}X K\,. \qquad (2.53)$$

Since the $\Box$ operator is made with covariant derivatives, $\Box\phi$ contains derivative couplings $(\partial g)(\partial\phi)$ between gravity and the scalar, hence its name kinetic gravity braiding.

The last term is going to give a nonzero α_B in the EFT Lagrangian (2.47), and the whole set of coefficients is given by

$$M = M_{\rm Pl} \quad \alpha_K = 12\dot\phi_0^3\,\frac{G_{3X} - \dot\phi_0^2 G_{3XX}}{HM_{\rm Pl}^2}\,, \quad \alpha_B = -\frac{G_{3X}\dot\phi_0^3}{HM_{\rm Pl}^2}\,, \qquad (2.54)$$

where I have used the fact that in unitary gauge $X = -\dot\phi_0^2/N^2$, so that a dependence on X can be seen as a dependence on N and vice versa.

- α_T

To get a non zero α_T, one needs a model that does not preserve the relation between the intrinsic and the extrinsic curvatures in Eq. (2.12). Since the extrinsic curvatures give terms in $\dot\gamma_{ij}^2$ while the intrinsic one gives $\left(\partial_k\gamma_{ij}\right)^2$, changing the relation between them brings a change in the speed of sound of tensors. This happens for example for what is known as the quartic galileon [21], whose Lagrangian is

$$L_4 = G_4(X)\,{}^{(4)}\!R - 2G_{4X}(X)\left[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right]. \qquad (2.55)$$

The covariant second derivatives of the scalar field introduce first derivatives for the metric through the Christoffel symbols, which modifies the kinetic terms for gravity and gives a non zero α_T. In unitary gauge this Lagrangian reads

$$L_4 = G_4 R + (2XG_{4X} - G_4)(K^2 - K^{ij}K_{ij})\,, \qquad (2.56)$$

so that the EFT coefficients are

$$M^2 = 2\left(G_4 + G_{4X}\dot\phi_0^2\right)\,, \quad \alpha_K = -12\dot\phi_0^2\,\frac{G_{4X} - 8\dot\phi_0^2 G_{4XX} + 4\dot\phi_0^4 G_{4XXX}}{M^2}\,, \qquad (2.57)$$

$$\alpha_B = 4\dot\phi_0^2\,\frac{G_{4X} - 2\dot\phi_0^2 G_{4XX}}{M^2}\,, \quad \alpha_T = -4\dot\phi_0^2\,\frac{G_{4X}}{M^2}\,, \qquad (2.58)$$

I will not discuss here the case of α_H, which parametrizes deviations from Horndeski theories, since the next chapter is specifically focused on theories beyond Horndeski. In particular, the effect of α_H will be explored in Sect. 3.6.

The theoretical origin of the parameters α_a of Eq. (2.47) is summarized in Table 2.2.

Table 2.2 In the first row, the parameters α_i introduced in Eq. (2.47)

	M^2	α_M	α_K	α_B	α_T	α_H
Interpretation	Normalization of the tensor quadratic action $\equiv$ Planck mass	Planck mass rate of change	Kinetic term for the scalar	Kinetic braiding between gravity and scalar	Modification of tensor sound speed	Theories beyond Horndeski
Example	GR (when constant)	$f(R)$ [22] Brans-Dicke [23]	k-essence [24]	Cubic Galileon [20]	Quartic Galileon [21]	G^3 theories (see Chap. 3)

2.4 Stability and Theoretical Consistency

Even if the terms in Eq. (2.47) passed the first condition of yielding second-order dynamics (which guarantees the absence of extra, ghost-like DOF), further restrictions need to be imposed on the EFT parameters. Indeed, before thinking about comparing the predictions of a theory to observations, stringent constraints must be imposed in order for the theory to be stable. This is where using a parametrization at the level of the action and not of the EOM has a clear advantage, since these stability conditions can in principle be read off directly from the action. The idea can be simplified thusly: in the case of two scalar fields[5] $\psi_1(t,\vec{x}),\ \psi_2(t,\vec{x})$ their quadratic Lagrangian is generically of the form:

$$L = \xi\dot{\psi}_1^2 - c_1\partial_i\psi_1^2 + \dot{\psi}_2^2 - c_2\partial_i\psi_2^2 + V_{\rm int}(\psi_1,\psi_2). \tag{2.59}$$

In this illustrative case, the stability of the theory requires the coefficient ξ to be positive. When this is not the case, the field ψ_1 is called a ghost and in general violent instabilities are present in the theory.

Let me give some intuition on why that is, by thinking of the Lagrangian as $L = T - V$, where T is the kinetic energy and V the potential one. If the two signs are not the same in T, kinetic energy can flow without limits from one field to the other without changing the total energy $E = T + V$, meaning that the ground state of the theory is not stable (see [25] for a discussion on classical and quantum ghosts).

On top of this, one needs to impose that the coefficients c_1 and c_2 (which represent the squared sound speeds) are positive, to avoid gradient instabilities. These instabilities can be understood very easily from the EOM: when varying (2.59) with respect to ψ_1 for example, one gets

$$\ddot{\psi}_1 - c_1\Delta\psi_1 = \frac{1}{2}\frac{\partial V_{\rm int}}{\partial\psi_1}\,. \tag{2.60}$$

If c_1 is negative, this equation admits in Fourier space a solution $\psi_{\vec{k}}$ proportional to $e^{\sqrt{|c_1|}kt}$, which is divergent.

The analysis in the case of the action (2.47) is more involved, since tensor modes are present on top of the scalar. Moreover, other non dynamical variables are present (scalar and vector), so that at first glance the form of the quadratic action is not as simple as (2.59). If we parametrize the unitary gauge metric as before

$$N = 1 + \delta N\,, \quad N^i = \partial_i\psi + N_V^i\,, \quad h_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + \gamma_{ij}\right)\,, \tag{2.61}$$

[5] I will not treat the case of one field, as it present less interests. In particular, one cannot have a ghost field in this case: the sign of the kinetic term does not matter when there is nothing to compare it to. Moreover, in cosmology, the scalar field is always coupled to gravity.

with $\partial_i N_V^i = 0$ and $\gamma_{ii} = \partial_i \gamma_{ij} = 0$, only ζ and γ_{ij} are dynamical.[6] Once the constraints are solved, the quadratic part of the action can be rewritten in terms of dynamical DOF only, in a manner very similar to Eq. (2.59):

$$S = \int d^4x \, \frac{M^2 a^3}{2} \left\{ \frac{\alpha}{(1+\alpha_B)^2} \left[\dot{\zeta}^2 - c_s^2 \frac{\partial_i \zeta^2}{a^2} \right] + \frac{\dot{\gamma}_{ij}^2}{4} - (1+\alpha_T) \frac{\partial_k \gamma_{ij}^2}{4a^2} + \frac{(\partial_i N_j^V + \partial_j N_i^V)^2}{4a^4} \right\} . \tag{2.62}$$

I have used the following definitions

$$\alpha \equiv \alpha_K + 6\alpha_B^2 \, , \tag{2.63}$$

and

$$c_s^2 \equiv 2 \left\{ 1 + \alpha_T - \frac{1+\alpha_H}{1+\alpha_B} \left(1 + \alpha_M - \frac{\dot{H}}{H^2} \right) - \frac{1}{H} \frac{d}{dt} \left(\frac{1+\alpha_H}{1+\alpha_B} \right) \right\} , \tag{2.64}$$

the latter being valid only in the absence of matter. The stability conditions discussed above can be stated as

$$\begin{aligned} &M^2 > 0 \, , \quad \alpha_K + 6\alpha_B^2 > 0 \, , \\ &c_T^2 \equiv (1+\alpha_T) > 0 \, , \quad c_s^2 > 0 \, , \end{aligned} \tag{2.65}$$

which defines the tensor sound speed.

The presence of matter, both at the background and perturbative levels, slightly complicates the situation. In the case $\alpha_H = 0$, one finds

$$c_s^2 = 2 \frac{(1+\alpha_B)^2}{\alpha} \left\{ \frac{1}{1+\alpha_B} \left(1 + \alpha_M - \frac{\dot{H}}{H^2} \right) - (1+\alpha_T) - \frac{\dot{\alpha_B}}{H(1+\alpha_B)^2} \right\} - \frac{\rho_\mathrm{m} + p_\mathrm{m}}{\alpha \, M^2 H^2} \, , \tag{2.66}$$

while the speed of sound for matter and tensors are unchanged. In the case $\alpha_H \neq 0$, which will be treated in more details in Chap. 3, both the sound speed of matter and the extra scalar field are affected.

Of course, the conditions (2.65) can be translated into conditions on parameters of models, using for example Sect. 2.3. However, the advantage of the EFT of DE is that those conditions are really imposed on deviations from ΛCDM, not just on a specific model. It might well be that the regions of the parameter space they allow are not fully explored by any of the known theories (which led us to the theories beyond Horndeski of Chap. 3). As we will see, the same kind of reasoning applies to the comparison with observations.

[6] In general, the spatial metric contains also a (non-dynamical) vectorial part, which can be set to zero by using the spatial gauge freedom.

2.5 Evolution of Cosmological Perturbations

In this section I will discuss the effects of the deviations from ΛCDM on the evolution of perturbations, in the vector, tensor and scalar sectors, the latter being the richest–and most complicated–in term of phenomenology. The matter sector will be parametrized by its total stress energy tensor, decomposed at linear order as

$$T^0_{\ 0} \equiv -(\rho_\mathrm{m} + \delta\rho_\mathrm{m}) \ , \tag{2.67}$$

$$T^0_{\ i} \equiv \partial_i q_\mathrm{m} + \left(T^0_{\ i}\right)^T \equiv (\rho_\mathrm{m} + p_\mathrm{m})\partial_i v_\mathrm{m} + \left(T^0_{\ i}\right)^V \ , \tag{2.68}$$

$$T^i_{\ j} \equiv (p_\mathrm{m} + \delta p_\mathrm{m})\delta^i_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\sigma_\mathrm{m} + \left(\partial^i C_j + \partial_j C^i\right)^V + \left(T^i_{\ j}\right)^{TT} \ , \tag{2.69}$$

where $\delta\rho_\mathrm{m}$ and δp_m are the energy density and pressure perturbations, q_m and v_m are respectively the 3-momentum and the 3-velocity potentials; σ_m is the anisotropic stress potential. $(T^0_{\ i})^V$ is the transverse part of the matter energy flux, $\left(\partial^i C_j + \partial_j C^i\right)^V$ and $(T_{ij})^{TT}$ are respectively the transverse and the transverse-traceless parts of the spatial matter stress tensor.

2.5.1 Vector Sector

As we have seen from Eq. (2.62), the vector sector is the simplest one as it does not contain propagating DOF. However, the presence of a time varying Planck mass, characterized by $\alpha_M \neq 0$ still affects the perturbations. Indeed, when considering the full action supplemented by matter, the vector equation reads:

$$\frac{1}{2}\nabla^2 N_i^V = \frac{a^2}{M^2}\left(T^0_{\ i}\right)^V \ . \tag{2.70}$$

For a perfect fluid where $C_i^V = 0$, the conservation of the matter stress-energy tensor implies that $(T^0_{\ i})^T \propto 1/a^3$ [26]. Thus, the metric vector perturbations scale as

$$N_V^i \propto \frac{1}{aM^2} = \frac{1}{a^{1+\alpha_M}} \ , \tag{2.71}$$

where the last equality holds for a constant α_M. It is therefore interesting to see that the evolution of the vector sector only depends on a single parameter.

Since they typically decay, vector modes are very difficult to observe. This very fact already signals that α_M cannot be too negative, i.e. the Planck mass cannot have been growing too strongly in time, otherwise they would not necessarily be negligible today. If vectors mode were to be detected, this would allow to constrain α_M without having to treat the other parameters.

2.5.2 Tensor Sector

The tensor sector, slightly more complicated, leads to the evolution equation

$$\ddot{\gamma}_{ij} + H(3+\alpha_M)\dot{\gamma}_{ij} - (1+\alpha_T)\frac{\nabla^2}{a^2}\gamma_{ij} = \frac{2}{M^2}\left(T_{ij}\right)^{TT} \, . \tag{2.72}$$

Thus, even for a perfect fluid where the anisotropic stress is zero, the propagation of tensor modes is affected both by an additional friction term proportional to α_M, as well as a different speed of propagation. In principle, the combined observation of vector and tensor modes could therefore provide constraints on α_M and α_T independently of each other and of the other α_i.

2.5.3 Scalar Sector

2.5.3.1 Obtaining the Equations

In principle, five (non independent) scalar equations can be derived from the action (2.47). Four are the Einstein scalar equations (00, $0i$, ii and ij traceless), where one needs to further introduce the scalar part of the traceless component of the spatial metric, χ

$$h_{ij} = a^2(1+2\zeta)\left[\delta_{ij} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\,\partial^2\right)\chi\right] . \tag{2.73}$$

Then, the action needs to be varied with respect to ζ, δN, ψ and χ, giving the four Einstein equations.

The fifth equation is the one for the scalar field ϕ. However, in unitary gauge this field is not explicit. One can still derive what would be the unitary gauge version of this equation (that will depend only on metric quantities) by imposing the invariance under time reparametrization of the action. Indeed, by definition of the unitary gauge,

$$\left.\frac{\delta S[\phi, g_{\mu\nu}]}{\delta\phi(x)}\right|_{\phi=t} = \frac{\delta S_{\text{u.g.}}[t, g_{\mu\nu}]}{\delta t}, \tag{2.74}$$

where the time derivative is understood as a partial one (that is to say, not taking into account the time dependence of the metric).

For a general infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$, the metric changes as $\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$. Therefore,

$$\delta S_{\text{u.g.}} = \int d^4x \frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}(x)}(\nabla_\mu\xi_\nu(x) + \nabla_\nu\xi_\mu(x)) + \frac{\delta S_{\text{u.g.}}}{\delta t}\xi^0 = 0 \, . \tag{2.75}$$

After integrating by parts and combining this with Eq. (2.74), one obtains that the equation of the scalar field in unitary gauge is simply the zero component of the divergence of Einstein's equations,[7]

$$\left.\frac{\delta S[\phi, g_{\mu\nu}]}{\delta\phi(x)}\right|_{\phi=t} = \frac{\delta S_{\text{u.g.}}}{\delta t} = 2g^{0\nu}\nabla_\mu \frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}} = 0\,, \tag{2.76}$$

where the last equality holds when Einstein's equations $\frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}} = 0$ are inforced. Hence, this yields the fifth scalar equation, which is not independent from the others.

These five equations are for the scalar variables of the metric, namely ζ, δN, ψ and χ. To describe scalar perturbations and their physics, the Newtonian gauge is more adapted than the unitary gauge. In order to go from one to the other, a time diffeomorphism is performed

$$t \to t + \pi(t, \vec{x})\ , \tag{2.77}$$

where π describes the fluctuations of the scalar field

$$\phi = t + \pi\ . \tag{2.78}$$

In Newtonian gauge the scalar part of the metric is parametrized as

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j\ . \tag{2.79}$$

One can relate the metric perturbations in unitary gauge defined in Eq. (2.61) to he metric perturbations Φ and Ψ, as well as the scalar fluctuation π by[8]

$$\delta N = \Phi - \dot{\pi}\ , \qquad \zeta = -\Psi + H\pi\ , \qquad \psi = a^{-2}\pi\ , \quad \chi = 0\ . \tag{2.80}$$

Then, the five equations can be put in the following form (in Fourier space):

- The Hamiltonian constraint ((00) component of Einstein's equation) is

$$\begin{aligned} &6(1+\alpha_B)H\dot{\Psi} + (6 - \alpha_K + 12\alpha_B)H^2\Phi + 2(1+\alpha_H)\frac{k^2}{a^2}\Psi + (\alpha_K - 6\alpha_B)\,H^2\dot{\pi} \\ &+ 6\left[(1+\alpha_B)\dot{H} + \frac{\rho_{\text{m}} + p_{\text{m}}}{2M^2} + \frac{1}{3}\frac{k^2}{a^2}(\alpha_H - \alpha_B)\right]H\pi = -\frac{\delta\rho_{\text{m}}}{M^2}\ , \end{aligned} \tag{2.81}$$

[7] Since we assumed the presence of a Jordan frame, where matter is minimally coupled, its stress energy tensor is conserved independently.

[8] More precisely, to remove also the variable χ one needs a spatial diffeomorphism $x^i \to x^i + \partial_i\beta$.

- The momentum constraint ((0*i*) components of Einstein's equation) reads

$$2\dot{\Psi}+2(1+\alpha_B)H\Phi-2H\alpha_B\dot{\pi}+\left(2\dot{H}+\frac{\rho_{\rm m}+p_{\rm m}}{M^2}\right)\pi=-\frac{(\rho_{\rm m}+p_{\rm m})v_{\rm m}}{M^2}\ . \quad (2.82)$$

- The traceless part of the ij components of Einstein's equation gives

$$(1+\alpha_H)\Phi-(1+\alpha_T)\Psi+(\alpha_M-\alpha_T)H\pi-\alpha_H\dot{\pi}=-\frac{\sigma_{\rm m}}{M^2}\ , \quad (2.83)$$

- The trace of the same components gives, using the equation above,

$$\begin{aligned}&2\ddot{\Psi}+2(3+\alpha_M)H\dot{\Psi}+2(1+\alpha_B)H\dot{\Phi}\\&+2\left[\dot{H}-\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}+(\alpha_B H)^{\cdot}+(3+\alpha_M)(1+\alpha_B)H^2\right]\Phi\\&-2H\alpha_B\,\ddot{\pi}+2\left[\dot{H}+\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}-(\alpha_B H)^{\cdot}-(3+\alpha_M)\alpha_B H^2\right]\dot{\pi}\\&+2\left[(3+\alpha_{\rm m})H\dot{H}+\frac{\dot{p}_{\rm m}}{2M^2}+\ddot{H}\right]\pi=\frac{1}{M^2}\left(\delta p_{\rm m}-\frac{2}{3}\frac{k^2}{a^2}\sigma_{\rm m}\right)\ .\end{aligned} \quad (2.84)$$

- Finally, the evolution equation for π reads

$$\begin{aligned}&H^2\alpha_K\ddot{\pi}+\left\{\left[H^2(3+\alpha_M)+\dot{H}\right]\alpha_K+(H\alpha_K)^{\cdot}\right\}H\dot{\pi}\\&+6\left\{\left(\dot{H}+\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right)\dot{H}+\dot{H}\alpha_B\left[H^2(3+\alpha_M)+\dot{H}\right]+H(\dot{H}\alpha_B)^{\cdot}\right\}\pi-2\frac{k^2}{a^2}\dot{H}\pi\\&-2\frac{k^2}{a^2}\left\{\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}+H^2\left[1+\alpha_B(1+\alpha_M)+\alpha_T-(1+\alpha_H)(1+\alpha_M)\right]+(H(\alpha_B-\alpha_H))^{\cdot}\right\}\pi\\&+6H\alpha_B\ddot{\Psi}+H^2(6\alpha_B-\alpha_K)\dot{\Phi}+6\left[\dot{H}+\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}+H^2\alpha_B(3+\alpha_M)+(\alpha_B H)^{\cdot}\right]\dot{\Psi}\\&+\left[6\left(\dot{H}+\frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right)+H^2(6\alpha_B-\alpha_K)(3+\alpha_M)+2(9\alpha_B-\alpha_K)\dot{H}+H(6\dot{\alpha}_B-\dot{\alpha}_K)\right]H\Phi\\&+2\frac{k^2}{a^2}\left\{\alpha_H\dot{\Psi}+\left[H(\alpha_M+\alpha_H(1+\alpha_M)-\alpha_T)-\dot{\alpha}_H\right]\Psi+(\alpha_H-\alpha_B)H\Phi\right\}=0\ .\end{aligned} \quad (2.85)$$

These equations are much more involved than in the two other sectors and as such are not readily useful. Nevertheless, one has to remember that there is only one propagating degree of freedom, which means that 4 of these equations are just constraints. Therefore, the five equations can be combined into a single equation for a single variable, e.g.

$$\boxed{\begin{aligned}&\ddot{\Psi}+\frac{\beta_1\beta_2+\beta_3\alpha_B^2\tilde{k}^2}{\beta_1+\alpha_B^2\tilde{k}^2}H\dot{\Psi}+\frac{\beta_1\beta_4+\beta_1\beta_5\tilde{k}^2+c_s^2\alpha_B^2\tilde{k}^4}{\beta_1+\alpha_B^2\tilde{k}^2}H^2\Psi=\\&-\frac{1}{2M^2}\left[\frac{\beta_1\beta_6+\beta_7\alpha_B^2\tilde{k}^2}{\beta_1+\alpha_B^2\tilde{k}^2}\delta\rho_\mathrm{m}+\frac{\beta_1\beta_8+\beta_9\alpha_B^2\tilde{k}^2}{\beta_1+\alpha_B^2\tilde{k}^2}H(\rho_\mathrm{m}+p_\mathrm{m})v_\mathrm{m}-\frac{\alpha_K}{\alpha}\delta p_\mathrm{m}\right]\end{aligned}}\,, \tag{2.86}$$

where $\tilde{k}\equiv k/(aH)$, α is defined in Eq. (2.63) and for simplicity, I have assumed that the anisotropic stress of matter is zero. The β_i are functions of the coefficients α_j, whose–rather cumbersome–expressions are given in the Appendix in the case $\alpha_H=0$. Although this equation is enough to describe the dynamics of the scalar sector, it is useful to have the relation between the two metric potentials Φ and Ψ to connect with observations (in particular lensing, as explained in Sect. 1.2.3). This relation takes the form

$$\begin{aligned}&\alpha_B^2\tilde{k}^2\left[\Phi-\Psi\left(1+\alpha_T+\frac{\alpha_T-\alpha_M}{\alpha_B}\right)\right]+\beta_1\left[\Phi-\Psi(1+\alpha_T)\left(1+\alpha\frac{\alpha_T-\alpha_M}{2\beta_1}\right)\right]=\\&\frac{\alpha_T-\alpha_M}{2H^2M^2}\left\{\alpha_B\left[\delta\rho_\mathrm{m}-3H(\rho_\mathrm{m}+p_\mathrm{m})v_\mathrm{m}\right]+HM^2\alpha\,\dot{\Psi}+H\frac{\alpha_K}{2}(\rho_\mathrm{m}+p_\mathrm{m})v_\mathrm{m}\right\}\,.\end{aligned} \tag{2.87}$$

To complete the system of equations, one needs to provide the evolution equations for the matter sector. Since it is assumed to be minimally coupled, these equations come from the conservation of the stress energy tensor. At linear order in the perturbations, treating one species of matter only for simplicity, they read

$$\dot{\delta}_\mathrm{m}-3H(w_\mathrm{m}\delta_\mathrm{m}-\delta p_\mathrm{m})-(1+w_\mathrm{m})\left(\frac{k^2}{a^2}v_\mathrm{m}+3\dot{\Psi}\right)=0\,, \tag{2.88}$$

$$\dot{v}_\mathrm{m}-\left[3Hw_\mathrm{m}-\frac{\dot{w}_\mathrm{m}}{1+w_\mathrm{m}}\right]v_\mathrm{m}+\frac{\delta p_\mathrm{m}}{1+w_\mathrm{m}}+\Phi=0\,, \tag{2.89}$$

with the definitions

$$w_\mathrm{m}\equiv\frac{p_\mathrm{m}}{\rho_\mathrm{m}}\,,\quad \delta_\mathrm{m}\equiv\frac{\delta\rho_\mathrm{m}}{\rho_\mathrm{m}}\,, \tag{2.90}$$

where w_m is the usual equation of state parameter and δ_m the density contrast. Note that in general, when the fluid is not at rest, the relation between the pressure perturbation and the density contrast involves more than just the speed of sound (see for example [27]) which is why I kept explicitly δp_m in these equations.

2.5.3.2 Fluid Description

Similarly to the case of the cosmological constant Λ which can be seen either as a modification of gravity (belonging to the $G_{\mu\nu}$ side of Einstein's equation) or as a new fluid with a density ρ_Λ (see Sect. 1.1.3), one can describe the dark energy, both in the background and perturbative equations, as an effective fluid. The idea is to regroup under a effective stress energy tensor $T^\Lambda_{\mu\nu}$ everything that is not either $G_{\mu\nu}$ nor $T^{\rm matter}_{\mu\nu}$. At the background level this gives

$$\rho_D \equiv 3M^2H^2 - \rho_{\rm m} \,, \qquad p_D \equiv -M^2(2\dot{H} + 3H^2) - p_{\rm m} \,. \tag{2.91}$$

With these definitions, and using the conservation of the background matter stress-energy tensor,

$$\dot{\rho}_{\rm m} + 3H(\rho_{\rm m} + p_{\rm m}) = 0 \,, \tag{2.92}$$

the conservation of the background $T^\Lambda_{\mu\nu}$ reads

$$\dot{\rho}_D = -3H(\rho_D + p_D) + 3\alpha_M M^2 H^3 = 3H(\rho_{\rm m} + p_{\rm m}) + 6M^2H(\dot{H} + \alpha_M H^2) \,. \tag{2.93}$$

Another useful relation that one can use to express $\dot{p}_D$ in terms of matter and geometry is

$$\dot{p}_D = -\dot{p}_{\rm m} - M^2\big[2\ddot{H} + 2H\dot{H}(3 + \alpha_M) + 3\alpha_M H^3\big] \,, \tag{2.94}$$

which can be derived from the equations above.

Equations (2.81)–(2.84) can be then rewritten in the usual form,

$$\frac{k^2}{a^2}\Psi + 3H\big(\dot{\Psi} + H\Phi\big) = -\frac{1}{2M^2}\sum_I \delta\rho_I \,, \tag{2.95}$$

$$\dot{\Psi} + H\Phi = -\frac{1}{2M^2}\sum_I q_I \,, \tag{2.96}$$

$$\Psi - \Phi = \frac{1}{M^2}\sum_I \sigma_I \,, \tag{2.97}$$

$$\ddot{\Psi} + H\dot{\Phi} + 2\dot{H}\Phi + 3H\big(\dot{\Psi} + H\Phi\big) = \frac{1}{2M^2}\sum_I\left(\delta p_I - \frac{2}{3}\frac{k^2}{a^2}\sigma_I\right) \,, \tag{2.98}$$

where the sum is over the matter and the dark energy components. These equations implicitly define the quantities $\delta\rho_D$, q_D, δp_D and σ_D as the energy density perturbation, momentum, pressure perturbation and anisotropic stress of the dark energy fluid. An explicit definition is given in the Appendix.

With these definitions, one can verify that the evolution equation for π, Eq. (2.85), is equivalent to a conservation equation of the dark energy fluid quantities,

$$\delta\dot{\rho}_D + 3H(\delta\rho_D + \delta p_D) - 3(\rho_D + p_D)\dot{\Psi} - \frac{k^2}{a^2}q_D = \alpha_M H\sum_I \delta\rho_I \,. \tag{2.99}$$

The Euler equation,

$$\dot{q}_D + 3Hq_D + (\rho_D + p_D)\Phi + \delta p_D - \frac{2}{3}\frac{k^2}{a^2}\sigma_D = \alpha_M H \sum_I q_I \,, \qquad (2.100)$$

is identically satisfied by the definitions of q_D, δp_D and σ_D.

To close the system, one needs to provide a relation between δp_D and σ_D in terms of $\delta\rho_D$, q_D and the other matter variables. In order to do so in the simpler case where $\alpha_H = 0$, we follow a procedure similar to the previous section. First, we solve Eqs. (2.81)–(2.83) for Ψ, $\dot{\Psi}$ and $\dot{\pi}$ and then we plug these solutions in Eqs. (2.95) and (2.96) to express π and Φ in terms of $\delta\rho_{\rm m}$, $q_{\rm m}$, $\sigma_{\rm m}$, $\delta\rho_D$ and q_D. $\dot{\Phi}$ is obtained from the first derivative of (2.83). To obtain $\ddot{\Psi}$ and $\ddot{\pi}$ we use Eqs. (2.84) and (2.85). Combining all these solutions we can finally express σ_D and δp_D in terms of the other fluid variables. We obtain

$$\begin{aligned}\delta p_D =& \frac{\gamma_1\gamma_2 + \gamma_3\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_1\gamma_4 + \gamma_5\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D \\ &+ \gamma_7(\delta\rho_{\rm m} - 3Hq_{\rm m}) + \frac{\gamma_1\gamma_6 + 3\gamma_7\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_{\rm m} - \frac{6\alpha_B^2}{\alpha}\delta p_{\rm m} \,, \qquad (2.101)\\ \sigma_D =& \frac{a^2}{2k^2}\left[\frac{\gamma_1\alpha_T + \gamma_8\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_9\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D\right. \\ &\left. + \alpha_T(\delta\rho_{\rm m} - 3Hq_{\rm m}) + \frac{\gamma_{10}\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_{\rm m}\right], \qquad (2.102)\end{aligned}$$

where we use the notation $\tilde{k} \equiv k/(aH)$ and we have defined dimensionless coefficients γ_a, whose expressions are explicitly given in the Appendix. These relations for δp_D and σ_D are the equivalent of the Eqs. (2.86)–(2.87).

2.5.3.3 Interpretation

The system of Equations (2.86)–(2.89) is complete (provided $\delta p_{\rm m}$ and $w_{\rm m}$ are specified) and can in principle be solved to get the evolution of the matter perturbations and gravitational potentials. To do so without approximations would require a numerical implementation. However, the physics can be discussed analytically in specific cases, that give an idea of the effects expected. In particular, I will focus on the role played by kinetic braiding. Indeed, one can see appearing in Eq. (2.86) a new scale when $\alpha_B \neq 0$:

$$k_B = \frac{aH\beta_1^{1/2}}{\alpha_B} \,, \qquad (2.103)$$

which has been called braiding scale [18]. We shall explore two examples that show it is associated with noticeable modifications of gravity.

- $\alpha_B = 0$:
  It can be seen as the extreme limit where $k_B \to \infty$, meaning that all modes are outside of the braiding length, $k \ll k_B$. In this case most of the scale dependences go away. We are left with the simpler expression

$$\ddot{\Psi} + (4 + 2\alpha_M + 3\Upsilon)\, H\dot{\Psi} + \left(\beta_4 H^2 + c_s^2 \frac{k^2}{a^2}\right)\Psi =$$
$$- \frac{1}{2M^2}\left\{c_s^2\,[\delta\rho_{\rm m} - 3H(\rho_{\rm m} + p_{\rm m})v_{\rm m}] + (\alpha_M - \alpha_T + 3\Upsilon)H(\rho_{\rm m} + p_{\rm m})v_{\rm m} - \delta p_{\rm m}\right\}\,, \tag{2.104}$$

  where Υ is defined in the Appendix. Although both α_M and α_T can be nonzero here, the form of this equation is very similar to that obtained in the standard k-essence case [24]. One recovers in the quasistatic limit (i.e. by neglecting time derivatives and taking $k \gg aH/c_s$)

$$-\frac{k^2}{a^2}\Psi = \frac{1}{2M^2}\delta\rho_{\rm m}\,, \quad \Phi = (1 + \alpha_T)\left[1 + \alpha_K \frac{\alpha_T - \alpha_M}{2\beta_1}\right]\Psi\,. \tag{2.105}$$

  This means that no scale dependence is introduced in the effective Newton constant defined as

$$-\frac{k^2}{a^2}\Phi \equiv 4\pi G_{\rm eff}\,\delta\rho_{\rm m}\,. \tag{2.106}$$

  As we will see, this no longer necessarily holds when $\alpha_B \neq 0$.
- $\alpha_B^2 \gg \alpha_K$:
  This case corresponds to having most of the kinetic energy of the scalar field coming from kinetic braiding. Indeed, one can see in this case that the kinetic energy (the term in $\dot{\zeta}^2$ in Eq. (2.62)) is dominated by the contribution of α_B.
  For simplicity we consider only the case $\alpha_T = 0$. Moreover, to avoid gradient instabilities the following relation is required (see Eq. (2.66))

$$\alpha_B \lesssim \mathcal{O}(\alpha_M)\,. \tag{2.107}$$

  However, no restrictions are imposed on α_M, whose value can affect the braiding scale. Indeed, when $\alpha_B^2 \gg \alpha_K$, this is given by

$$\frac{k_B^2}{a^2} \simeq 3(H^2\alpha_M - \dot{H})\,, \tag{2.108}$$

which can be inside the Hubble horizon. In this case, considering modes with $k \gg k_B$, Eq. (2.86) simplifies to

$$\ddot{\Psi} + (3+\alpha_M)H\dot{\Psi} + \left(\frac{k_B^2\beta_5}{a^2} + c_s^2\frac{k^2}{a^2}\right)\Psi \simeq -\frac{1}{2M^2}\left(\frac{k_B^2\beta_6}{k^2} + c_s^2 + \frac{1}{3} - \frac{\alpha_M}{3\alpha_B}\right)\delta\rho_{\rm m}\ , \tag{2.109}$$

where we have neglected relativistic terms on the right hand side of (2.86). If the ratio β_5/c_s^2 is larger than one, the scale dependence cannot be neglected even in the case $k \gg k_B$. Therefore, a non vanishing α_B, or the fact that $k_B < \infty$, brings a transition scale in the effective Newton constant,[9] which is a strong signal that gravity is modified.
Another interpretation would be that dark energy clusters: one can write Einstein equations as

$$G^{\mu\nu} = \frac{T_{\rm m}^{\mu\nu} + T_D^{\mu\nu}}{M^2}\ , \tag{2.110}$$

which defines effective fluid variables for dark energy/modified gravity. Thus, for subhorizon scales, the Poisson equation has the form

$$-\frac{k^2}{a^2}\Phi = \frac{1}{M^2}\left(\delta\rho_{\rm m} + \delta\rho_D\right)\ . \tag{2.111}$$

For a cosmological constant, there are no perturbation in the dark energy fluid, $\delta\rho_D = 0$, and the standard behavior is recovered. However, as soon as dark energy clusters, i.e. $\delta\rho_D \sim \mathcal{O}(\delta\rho_{\rm m})$, the relation between the gravitational potential and matter is no longer as simple, leading to a different (and potentially scale dependent) effective Newton constant.

The Eqs. (2.86) and (2.87) can be seen as the generalization to arbitrary scales of the usual parametrization in term of $G_{\rm eff}$ (defined in Eq. (2.106)) and the slip parameter

$$\gamma \equiv \frac{\Psi}{\Phi}\ , \tag{2.112}$$

that are employed in the quasistatic limit. However, if this limit is clearly defined in GR where it means focusing on subhorizon scales $k \gg aH$, its definition in the presence of an extra scalar field is more ambiguous. Indeed, in general, new scales (see [28] for a general discussion concerning new scales in modified gravity) and time dependences appear and it is not always clear how this limit would translate, although in general it is expected to hold well inside the sound horizon of the scalar perturbations, $kc_s \gg aH$.

[9] Although the standard relation defining $G_{\rm eff}$ involves Φ and not Ψ, it easy to convince oneself that the relation between them set by Eq. (2.87) does not remove this transition.

To alleviate this uncertainty, one can look at what is called the extreme quasistatic limit [18] corresponding to wavenumber k much bigger than any scale in the problem, i.e. taking $k \to \infty$ in Eqs. (2.86)–(2.87). This yields the following expressions

$$8\pi G_{\rm eff} = \frac{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2}{\alpha\, c_s^2} M^{-2}\,, \tag{2.113}$$

$$\gamma = \frac{\alpha\, c_s^2 + 2\alpha_B\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]}{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2}\,, \tag{2.114}$$

where I have expressed both quantities directly in terms of the functions α_a (recall that $\alpha = \alpha_K + 6\alpha_B^2$ and α_H is here set to zero). These two quantities are observable since the first affects directly the growth of structures and therefore affects the power spectrum of the large scale structure. The second is related to the gravitational potential felt by photon, $\Phi + \Psi$, and thus can be probed in weak lensing experiments (see for example [29]).

In this Section, I have shown that by looking at the evolution of cosmological perturbations, one can relate the parametrization of the action in Eq. (2.47) to observable quantities. The simplest cases from the theoretical side are the vector and tensor sectors. They only depend on the time variation of the Planck mass, α_M, and on the deviation from unity of the tensor sound speed, α_T. However, these sectors are precisely the fields of observations where the signals are the weakest.

The more experimentally accessible scalar sector corresponds to the most complicated domain, where all five functions α_i play a role. Although their effects are understood from a theoretical point of view (see Table 2.2), they appear in a non trivial way when going to observable quantities such as the growth of structures or weak lensing. This can be seen analytically in the quasistatic limit with the modifications of the way matter sources the gravitational potential (through $G_{\rm eff}$) or the way the two potentials are related to each other (through γ). This is why, to break the degeneracies that remain, one may need to go beyond the quasistatic limit, starting for example from Eq. (2.86).

One idea would be to solve perturbatively Eqs. (2.86)–(2.89) around $k \to \infty$ without necessarily making assumptions on the time derivatives. This would be a way to see the range of validity of the quasistatic approximation (see also [30]). We have actually started looking into this, but taking care of the time dependence is rather subtle and requires more work.

2.6 Conclusions

In this chapter, I presented a method called the Effective Field Theory for Dark Energy, that allows to explore the vast landscape beyond the standard model of cosmology, ΛCDM. It is based on the parametrization of an action, describing scalar-tensor theories in a very broad sense. I used the preferred time foliation that the scalar field offers, along with its $3+1$ geometry, to construct a very generic Lagrangian that describes linear perturbations with second-order dynamics. This Lagrangian depends only on five functions of time, provided the expansion of the Universe and its matter content are known.

This has many advantages, both theoretically and observationally. The stability conditions that one needs to impose for a theory to be sensible can be easily read from this action. Moreover, this reduces to a single channel of analysis the comparison to experiments. The straigthforward links that we developped between wide classes of models and the parameters make it particularly convenient to use, since constraints on the five parameters easily translate to constraints on models.

However, this point of view is somewhat limiting the potential of this approach. The action (2.47) explores domains beyond the models currently known, potentially leading to new models, as we shall see in the next chapter. Indeed, it is solely based on the fact that in general, the background solution of an additional field in a cosmological setting explicitly breaks time reparametrization invariance. This opens the possibility of new terms in the action beside the standard Ricci scalar. It really is deviations from ΛCDM +GR that are captured by this formalism.

Because of its minimal number of parameters, the EFT of DE has started to be used by the community. It first started with people developing codes, in particular [31], that is based on the popular CMB code CAMB [32] and others doing forecasts for galaxy surveys [33]. Now, the parametrization, conveniently optimized by [18], is being used in the analysis of the Planck collaboration [34]. Hopefully, future surveys such as EUCLID [35] and LSST [36] will also use it, and the constraints on the α_a will improve.

From a theoretical point of view, there is still work to be done. As I mentioned above, there is a yet untamed wealth of information contained in Eq. (2.86), which includes for example relativistic effects that become important when looking at increasingly large surveys. It would be interesting to see how much of this information can be extracted using numerical solutions, or analytical method generalizing the quasistatic limit.

Another point I have been working on recently consists of extending this formalism to the case where the Weak Equivalence Principle (WEP) is violated, i.e. species couple to different metrics. This has been studied for ΛCDM under the name of interacting dark energy (see for example [37–40]). The idea is to investigate the interplay between these two properties, namely modifications of gravity and violation of the WEP. In particular, we generalized the stability conditions (2.65) in [15] to include the different couplings of the matter fields. In a subsequent publication [16], we looked at the effect of deviation from ΛCDM and the WEP on various observables using Fisher matrices (in the quasistatic limit). The bottom line is that, even with future surveys, there is not enough information to disentangle the effects of the various α. It might be possible when looking beyond the quasistatic regime, where more scale and time dependence arise, that can potentially break degeneracies.

Looking back at the motivations presented in the Introduction, it will appear to the careful reader that the analysis presented here does not address what I referred to as the old cosmological constant problem. This is a good remark. At the moment, there are no outstanding candidate theory that gives a hint at a resolution of this problem. The hope is that, by looking at the simplest deviation from ΛCDM, namely adding a scalar field, one might be then guided by the data towards the beginning of an answer. An extra scalar field might well be the manifestation of a more complex theory in certain limits (as it is for massive gravity for example [2]).

Appendix

In this Appendix, I compiled a few complicated expressions that were omitted from the main text of this chapter. The following shorthand notations for the variable Φ, Ψ and π

$$\mathcal{P} \equiv M^2(\dot{\pi}-\Phi)\ , \qquad \mathcal{Q} \equiv M^2(\dot{\Psi}+H\Phi+\dot{H}\pi)\ , \qquad \mathcal{R} \equiv M^2(\Psi+H\pi)\ , \tag{2.115}$$

allow to express the fluid quantities of Sect. 2.5.3.2 in a compact manner

$$\delta\rho_D \equiv 2\frac{k^2}{a^2}\left(\alpha_H\mathcal{R}-\alpha_B M^2 H\pi\right) - 3H\left[(\rho_D+p_D)\pi - 2\alpha_B\mathcal{Q}\right] + H^2(\alpha_K - 6\alpha_B)\mathcal{P}\ , \tag{2.116}$$

$$q_D \equiv -2\alpha_B H\mathcal{P} - (\rho_D + p_D)\pi\ , \tag{2.117}$$

$$\sigma_D \equiv \alpha_M M^2 H\pi - \alpha_T\mathcal{R} - \alpha_H\mathcal{P}\ , \tag{2.118}$$

$$\begin{aligned}\delta p_D \equiv& \left[\dot{p}_D + \alpha_M H M^2(2\dot{H}+3H^2)\right]\pi - 2\alpha_M H\mathcal{Q}\\ &+\left(\frac{\rho_D+p_D}{M^2}+6\alpha_B H^2\right)\mathcal{P} + 2\left(\alpha_B H\mathcal{P}\right)^{\cdot} + \frac{2}{3}\frac{k^2}{a^2}\sigma_D\ .\end{aligned} \tag{2.119}$$

Moreover, the parameters β in Eq. (2.86) can be related to the initial α parameters of the action (2.47) through (in the case $\alpha_H = 0$)

$$\beta_1 \equiv -\alpha_K\frac{\rho_\mathrm{m}+p_\mathrm{m}}{4H^2M^2} - \frac{1}{2}\alpha\left(\frac{\dot{H}}{H^2}+\alpha_T-\alpha_M\right)\ , \tag{2.120}$$

$$\beta_2 \equiv 2(2+\alpha_M) + 3\Upsilon\ , \tag{2.121}$$

$$\beta_3 \equiv 3 + \alpha_M + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot}\ , \tag{2.122}$$

$$\beta_4 \equiv (1+\alpha_T)\left[2\dot{H}/H^2 + 3(1+\Upsilon) + \alpha_M\right] + \dot{\alpha}_T/H\ , \tag{2.123}$$

$$\beta_5 \equiv c_s^2 - \frac{2\alpha_B(\beta_3-\beta_2)}{\alpha} + \frac{\alpha_B^2}{\beta_1}(1+\alpha_T)(\beta_3-\beta_2) + \frac{\alpha_B^2\beta_4}{\beta_1}\ , \tag{2.124}$$

$$\beta_6 \equiv \beta_7 - 2\frac{\alpha_B(\beta_3-\beta_2)}{\alpha}\ , \tag{2.125}$$

$$\beta_7 \equiv c_s^2 + 2\frac{\alpha_B^2(1+\alpha_T)+\alpha_B(\alpha_T-\alpha_M)}{\alpha} \ , \tag{2.126}$$

$$\beta_8 \equiv \beta_9 - \frac{(\alpha_K - 6\alpha_B)(\beta_3-\beta_2)}{\alpha} \ , \tag{2.127}$$

$$\beta_9 \equiv -(1+3c_s^2+\alpha_T) + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \ , \tag{2.128}$$

$$\beta_{10} \equiv -6(1+\Upsilon) - 4\dot{H}/H^2 \ , \tag{2.129}$$

$$\beta_{11} \equiv \frac{2}{3} - 2\frac{\alpha_B^2}{\beta_1}\left[(2-\alpha_M)+2\dot{H}/H^2\right] - 2\frac{\alpha_B^4}{\beta_1 H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \ , \tag{2.130}$$

with

$$\begin{aligned} 12\beta_1 H^3 M^2 \Upsilon \equiv {} & 2\alpha M^2\Big\{\left[\dot{H}+(\alpha_T-\alpha_M)H^2\right]^{\cdot} + (3+\alpha_M)H\left[\dot{H}+(\alpha_T-\alpha_M)H^2\right]\Big\} \\ & + \alpha_K \dot{p}_\mathrm{m} - (\rho_\mathrm{m}+p_\mathrm{m})H(\alpha_K-6\alpha_B)(\alpha_T-\alpha_M) + 6(\rho_\mathrm{m}+p_\mathrm{m})\frac{\alpha_B^4}{\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \ . \end{aligned} \tag{2.131}$$

On the other hand, the γ in Eq. (2.102) read

$$\gamma_1 \equiv \alpha_K \frac{\rho_D + p_D}{4H^2M^2} - 3\alpha_B^2\frac{\dot{H}}{H^2} \ , \tag{2.132}$$

$$\gamma_2 \equiv c_s^2 + \frac{\alpha_T}{3} - 2\frac{\alpha_B(2+\Gamma)+(1+\alpha_B)(\alpha_M-\alpha_T)}{\alpha} \ , \tag{2.133}$$

$$\gamma_3 \equiv c_s^2 + \frac{\gamma_8}{3} \ , \tag{2.134}$$

$$\begin{aligned} \gamma_4 \equiv {} & \frac{1}{\rho_D+p_D}\Big\{-\dot{p}_D/H + \alpha_M\left[\rho_D+p_D-3H^2M^2\right] \\ & + 6\frac{\alpha_B^2}{\alpha}\Big[(3+\alpha_M+\Gamma)(\rho_\mathrm{m}+p_\mathrm{m}) - \dot{p}_\mathrm{m}/H\Big]\Big\} \ , \end{aligned} \tag{2.135}$$

$$\gamma_5 \equiv -1 - \frac{(6\alpha_B-\alpha_K)(\alpha_T-\alpha_M)}{6\alpha_B^2} + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \tag{2.136}$$

$$\gamma_6 \equiv -6\alpha_B^2\frac{2+\Gamma}{\alpha} + \frac{\alpha_K\alpha_M - 6\alpha_B^2}{\alpha} \ , \tag{2.137}$$

$$\gamma_7 \equiv \frac{\alpha_K\alpha_M - 6\alpha_B^2}{3\alpha} - \frac{(6\alpha_B-\alpha_K)(\alpha_T-\alpha_M)}{3\alpha} \ , \tag{2.138}$$

$$\gamma_8 \equiv \alpha_T + \frac{\alpha_T-\alpha_M}{\alpha_B} \ , \tag{2.139}$$

$$\gamma_9 \equiv \alpha\frac{\alpha_T-\alpha_M}{2} \ , \tag{2.140}$$

$$\gamma_{10} \equiv 3\alpha_B^2(\alpha_T-\alpha_M) \ , \tag{2.141}$$

where

$$\gamma_1 \Gamma \equiv \frac{\alpha_K}{4H^2M^2}\left[(3+\alpha_M)(\rho_{\rm m}+p_{\rm m}) - \dot{p}_{\rm m}/H - \frac{\alpha_B^2(\rho_D+p_D)}{\alpha_K H}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot}\right] - \alpha\frac{\ddot{H}}{2H^3}\,, \tag{2.142}$$

and

$$c_s^2 = -\frac{2(1+\alpha_B)\Big[\dot{H} - (\alpha_M-\alpha_T)H^2 + H^2\alpha_B(1+\alpha_T)\Big] + 2H\dot{\alpha}_B + (\rho_{\rm m}+p_{\rm m})/M^2}{H^2\alpha}\,. \tag{2.143}$$

References

1. A. Joyce, B. Jain, J. Khoury, M. Trodden, Beyond the cosmological standard model. Phys. Rept. **568**, 1–98 (2015). 1407.0059
2. C. de Rham, Massive gravity. Living Rev. Rel. **17**, 7 (2014). 1401.4173
3. S. Hassan, R.A. Rosen, Bimetric gravity from ghost-free massive gravity. JHEP **1202**, 126 (2012). 1109.3515
4. **Supernova Search Team** Collaboration, A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J. **116**, 1009–1038 (1998). astro-ph/9805201
5. **Supernova Cosmology Project** Collaboration, S. Perlmutter et al., Measurements of Omega and Lambda from 42 high redshift supernovae. Astrophys. J. **517**, 565–586 (1999). astro-ph/9812133
6. N. Arkani-Hamed, H.-C. Cheng, M.A. Luty, S. Mukohyama, Ghost condensation and a consistent infrared modification of gravity. JHEP **0405**, 074 (2004). hep-th/0312099
7. C. Cheung, P. Creminelli, A.L. Fitzpatrick, J. Kaplan, L. Senatore, The effective field theory of inflation. JHEP **0803**, 014 (2008). 0709.0293
8. L. Senatore, K.M. Smith, M. Zaldarriaga, Non-Gaussianities in single field inflation and their optimal limits from the WMAP 5-year data. JCAP **1001**, 028 (2010). 0905.3746
9. P. Creminelli, G. D'Amico, M. Musso, J. Noreña, E. Trincherini, Galilean symmetry in the effective theory of inflation: new shapes of non-Gaussianity. JCAP **1102**, 006 (2011). 1011.3004
10. P. Creminelli, G. D'Amico, J. Noreña, F. Vernizzi, The effective theory of quintessence: the $w < -1$ side unveiled. JCAP **0902**, 018 (2009). 0811.0827
11. G. Gubitosi, F. Piazza, F. Vernizzi, The effective field theory of dark energy. JCAP **1302**, 032 (2013). 1210.0201
12. J.K. Bloomfield, E.E. Flanagan, M. Park, S. Watson, Dark energy or modified gravity? An effective field theory approach. JCAP **1308**, 010 (2013). 1211.7054
13. J. Gleyzes, D. Langlois, F. Piazza, F. Vernizzi, Essential building blocks of dark energy. JCAP **1308**, 025 (2013). 1304.4840
14. J. Gleyzes, D. Langlois, F. Vernizzi, A unifying description of dark energy. Int. J. Mod. Phys. D **23**, 3010 (2014). 1411.3712
15. J. Gleyzes, D. Langlois, M. Mancarella, F. Vernizzi, Effective theory of interacting dark energy. JCAP **1508**(08), 054 (2015). 1504.05481
16. J. Gleyzes, D. Langlois, M. Mancarella, F. Vernizzi, Effective theory of dark energy at redshift survey scales. JCAP **1602**(02) 056 (2016). 1509.02191
17. R.L. Arnowitt, S. Deser, C.W. Misner, The dynamics of general relativity. Gen.Rel.Grav. **40**, 1997–2027 (2008). gr-qc/0405109

18. E. Bellini, I. Sawicki, Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity. JCAP **1407**, 050 (2014). 1404.3713
19. R. Caldwell, R. Dave, P.J. Steinhardt, Cosmological imprint of an energy component with general equation of state. Phys. Rev. Lett. **80**, 1582–1585 (1998). astro-ph/9708069
20. C. Deffayet, O. Pujolas, I. Sawicki, A. Vikman, Imperfect dark energy from kinetic gravity braiding. JCAP **1010**, 026 (2010). 1008.0048
21. C. Deffayet, X. Gao, D. Steer, G. Zahariade, From k-essence to generalised Galileons. Phys. Rev. D **84**, 064039 (2011). 1103.3260
22. H.A. Buchdahl, Non-linear Lagrangians and cosmological theory. Mon. Not. R. Astron. Soc. **150**, 1 (1970)
23. C. Brans, R. Dicke, Mach's principle and a relativistic theory of gravitation. Phys. Rev. **124**, 925–935 (1961)
24. C. Armendariz-Picon, V.F. Mukhanov, P.J. Steinhardt, A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration. Phys. Rev. Lett. **85**, 4438–4441 (2000). astro-ph/0004134
25. F. Sbisà, Classical and quantum ghosts. Eur. J. Phys. **36**, 015009 (2015). 1406.4550
26. S. Weinberg, *Cosmology* (Oxford University Press, 2008)
27. R. Bean, O. Doré, Probing dark energy perturbations: the dark energy equation of state and speed of sound as measured by WMAP. Phys. Rev. D **69**, 083503 (2004). astro-ph/0307100
28. T. Baker, P.G. Ferreira, C.D. Leonard, M. Motta, New gravitational scales in cosmological surveys. Phys. Rev. D **90**(12), 124030, (2014). 1409.8284
29. G.-B. Zhao, L. Pogosian, A. Silvestri, J. Zylberberg, Searching for modified growth patterns with tomographic surveys. Phys. Rev. D **79**, 083513 (2009). 0809.3791
30. I. Sawicki, E. Bellini, Limits of quasistatic approximation in modified-gravity cosmologies. Phys. Rev. D **92**(8), 084061 (2015). 1503.06831
31. B. Hu, M. Raveri, N. Frusciante, A. Silvestri, Effective field theory of cosmic acceleration: an implementation in CAMB. Phys. Rev. D **89**(10) 103530 (2014). 1312.5742
32. A. Lewis, A. Challinor, A. Lasenby, Efficient computation of CMB anisotropies in closed FRW models. Astrophys. J. **538**, 473–476 (2000). astro-ph/9911177
33. F. Piazza, H. Steigerwald, C. Marinoni, Phenomenology of dark energy: exploring the space of theories with future redshift surveys. JCAP **1405**, 043 (2014). 1312.6111
34. **XXX Collaboration** Collaboration, P. Ade et. al., Planck 2015 results. XIV. Dark energy and modified gravity. 1502.01590
35. **EUCLID Collaboration** Collaboration, R. Laureijs et. al., Euclid Definition Study Report. 1110.3193
36. **LSST Science, LSST Project** Collaboration, P.A. Abell et. al., LSST Science Book, Version 2.0. 0912.0201
37. L. Amendola, Coupled quintessence. Phys. Rev. D **62**, 043511 (2000). astro-ph/9908023
38. J. Valiviita, R. Maartens, E. Majerotto, Observational constraints on an interacting dark energy model. Mon. Not. R. Astron. Soc. **402**, 2355–2368 (2010). 0907.4987
39. M. Baldi, V. Pettorino, G. Robbers, V. Springel, Hydrodynamical N-body simulations of coupled dark energy cosmologies. Mon. Not. R. Astron. Soc. **403** 1684–1702 (2010). 0812.3901
40. M. Zumalacárregui, T. Koivisto, D. Mota, P. Ruiz-Lapuente, Disformal scalar fields and the dark sector of the universe. JCAP **1005**, 038 (2010). 1004.2684

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Dark Energy and the Formation of the Large Scale
Structure of the Universe
Gleyzes, J.
2016, XV, 113 p. 19 illus., Hardcover
ISBN: 978-3-319-41209-2