

Chapter 2

Multimode Laser Theory for Open Cavities

In this Chapter the theory of multimode laser for open and irregular systems is presented in a form appropriate for the application of Statistical Mechanics discussed in the following Chapters. The complex structure and the extreme openness make these optical systems different from traditional cavity laser composed of lossless cavity. From a theoretical point of view, the strong coupling to the external world requires a different treatment from the standard approach of traditional laser textbook. Several attempts are present in literature to face this problem. We will briefly review the most relevant ones and present in some details the system-and-bath approach of Hackenbroich et al. [1, 2]. Using this treatment for the openness we then expose the corresponding quantum theory for the simple case of two modes laser [3]. Finally we derive the multimode theory in the semiclassical limit in Sect. 2.4.

2.1 Modes Description of Open Systems

The problem of describing quantum systems strongly interacting with the environment has large interest and it is not only relevant for the physics of lasers (see, e.g., Ref. [4]). The system is localized in space. However, there is always a natural environment into which the quantum system with discrete states is embedded. The environment consists of the continuum of extended scattering states into which the states of the system are embedded and can decay. The coupling matrix elements between the discrete states of the system and the scattering states of the continuum determine the lifetime of the states, which is, then, usually finite due to this coupling.

Several approaches are presented in literature to build a set of modes suitable for a separation of time and coordinates dependencies of various physical observables, in particular the electric and magnetic fields [5].

The difficulty originates from the non-Hermiticity of the problem as the openness becomes relevant, so that the standard methods to solve or quantize Hermitian operators do not apply in this case.

In this section some of the main alternative methods to define a basis of electromagnetic modes for open systems are reviewed. Thus, in this section only passive cavities are considered and, consequently, the dielectric constant $\epsilon(\mathbf{r})$ is taken real.

Using Gaussian units ($c = 1$), in the Coulomb gauge $\nabla \cdot [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t)] = 0$ the electric field satisfies the wave equation

$$\epsilon(\mathbf{r}) \frac{d^2}{dt^2} \mathbf{E} + \nabla \times (\nabla \times \mathbf{E}) = 0. \quad (2.1)$$

For the sake of simplicity in some approaches it is reduced to the scalar wave equation $\epsilon(\mathbf{r}) d_t^2 E + \nabla^2 E = 0$, only valid if the coupling between various polarizations is irrelevant. Numerical [6] and analytical [7] results, however, show that this assumption may be violated in three dimensional systems, since the interaction between the polarizations of the light play a fundamental role in the dynamics.

2.1.1 Fox–Li Modes

Historically the first attempt to introduce a modal description for systems with radiative losses was by Fox and Li to study the lasing properties of the unstable resonators [8]. They define a *electromagnetic mode* as a field distribution that reproduces itself after one complete round trip in the resonator and write the solution to Eq. (2.1) for $\epsilon(\mathbf{r}) = \text{const}$ as

$$E(\mathbf{r}, t) = \Re [E(\mathbf{r}) e^{i(kz - \omega t)}]$$

with $k = \omega\sqrt{\epsilon}$ and with $E(\mathbf{r})$ that changes on a scale much larger than k^{-1} . The form of the transverse field $E(\mathbf{r})$ is obtained from the solutions of the non-Hermitian problem

$$\int K(\mathbf{r}_\perp, \mathbf{r}'_\perp, z) \psi_n(\mathbf{r}'_\perp, z) d\mathbf{r}'_\perp = \lambda_n \psi_n(\mathbf{r}_\perp, z),$$

that implies the reproducibility, where $\psi_n(\mathbf{r}'_\perp, z)$ is the right eigenvector with eigenvalue λ_n of the kernel $K(\mathbf{r}_\perp, \mathbf{r}'_\perp, z)$ which depends on the properties of the resonator. Because the non-Hermiticity, the eigenmodes are not orthogonal. An analogue set of eigenmodes ϕ_n is introduced for the propagation in the opposite direction, so the two sets are biorthogonal

$$\int \phi_m^*(\mathbf{r}'_\perp, z) \psi_n(\mathbf{r}'_\perp, z) d\mathbf{r}'_\perp = \delta_{mn}.$$

The electric field $E(\mathbf{r})$ can hence be expanded along these two sets. The method allows a basic understanding of unstable resonator, however their application is rather limited and need an extension to study more general open systems.

2.1.2 Quasimodes

The idea of the quasimodes originates in quantum physics to describe resonances in atomic and molecular scattering: the Schrödinger equation is solved with boundary conditions at infinity that contain only outgoing waves [9]. The eigenvectors of the corresponding non-Hermitian problem have complex eigenvalues and it is supposed that they can give a substantial expansion of the field inside the cavity. However they diverge at infinity, so they are not a basis for the field and the quantization procedure is not possible.

The approach has been thus developed using the quasimodes only to expand the field inside, while a different set is constructed outside the resonator to avoid the divergency problem and obtain a proper basis over the whole space [10] (note however that the continuation of the external modes to the inside cannot satisfy the Dirichlet condition at the boundary [5]).

The obtained set of inner and outer modes, as a whole called “natural modes”, is complete and can then be used for the quantization procedure. Defining the annihilation operators a_n and b_n of the inner modes of the cavity and the annihilation operators $a(k)$ and $b(k)$ of the outer modes, the Hamiltonian of the system takes the form

$$\mathcal{H} = \mathcal{H}_{\text{in}}(\{a_n, b_n\}) + \mathcal{H}_{\text{out}}(\{a(k), b(k)\}) . \quad (2.2)$$

In particular there are no explicit interaction terms between the inside and outside field. In this formalism the coupling between the two regions arises from the *non commutativity* of the internal and external operator. This fact makes the use of this basis not particularly convenient. A natural question is, then, if it would be possible to build a basis of commuting operators instead. In the Sect. 2.2 we will see how using the Feshbach projection technique this is in fact possible and that it results in the presence of an explicit interaction term in the Hamiltonian.

2.1.3 Constant-Flux States

As anticipated at the end of the last subsection, a rigorous quantization of the field can be achieved by the system-and-bath approach presented in Sect. 2.2. This approach provides a clear description of the field inside the cavity. However, one would like to have to easily calculate also the field outside the resonator (that is, at positions where the detectors are, in fact, placed in the experiments). The so-called *constant-*

flux states were introduced in Ref. [11] to actually meet this request, giving the corresponding expansion of the field *outside* the resonator. For this reason they are convenient for a semiclassical theory, but it is still not clear if they are appropriate to obtain the quantization of the field.

The constant-flux modes $\tilde{\boldsymbol{\psi}}_m$ are designed in a similar way of the system-and-bath modes [11]: inside the resonator they satisfy the same eigenvalue problem (see Eq. (2.5))

$$\frac{1}{\sqrt{\epsilon(\mathbf{r})}} \nabla \times \left[\nabla \times \frac{\tilde{\boldsymbol{\psi}}_m(\omega, \mathbf{r})}{\sqrt{\epsilon(\mathbf{r})}} \right] = \Omega_m(\omega) \tilde{\boldsymbol{\psi}}_m(\omega, \mathbf{r}), \quad (2.3)$$

while outside they satisfy

$$\nabla \times \left[\nabla \times \tilde{\boldsymbol{\psi}}_m(\omega, \mathbf{r}) \right] = \omega^2 \tilde{\boldsymbol{\psi}}_m(\omega, \mathbf{r}), \quad (2.4)$$

where ω is a real parameter *different* from the eigenfrequency $\Omega_m(\omega)$. The two equations are complemented by continuity condition at the boundary and outgoing-wave boundary conditions and the infinity. In this way the modes do not diverge at the infinity while Eq. (2.4) still assures the biorthogonality of the modes. The authors of Ref. [11], then, show that these modes constitute a complete basis and describe a constant flux of energy coming out from the cavity, whence the name constant-flux modes.

2.2 Field Quantization for Open Cavities

In this section we present the field quantization of the electromagnetic field in optical cavities in presence of an arbitrary number of escape channels by the system-and-bath approach of Ref. [2]. Using the Feshbach projector technique [12] it is shown that the field Hamiltonian reduces to the system-and-bath Hamiltonian of quantum optics. Note that a quantum treatment is necessary to compute the linewidth or the photon statistics of the output radiation.

2.2.1 Normal Modes

Consider a 3D linear dielectric medium characterized by a real scalar dielectric constant $\epsilon(r)$. The case of mirrors is a special case with $\epsilon(r) \equiv 1$ and appropriate condition at the mirrors.

The quantization is conveniently performed in terms of the vector potential \mathbf{A} and scalar potential ϕ . Using Natural units ($c = \hbar = 1$), in the Coulomb gauge in absence of sources we have $\phi = 0$ and $\nabla \cdot [\epsilon(r)\mathbf{A}] = 0$ so that the electromagnetic

Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} \int d\mathbf{r} \left[\frac{\boldsymbol{\Pi}(\mathbf{r}, t)^2}{\epsilon(r)} + (\nabla \times \mathbf{A}(r, t))^2 \right]$$

with the canonical momentum field $\boldsymbol{\Pi}(r) = \epsilon(r)\mathbf{A}(r)$.

The expansion of the fields in the complete set of modes f_m , so-called *modes-of-the-universe*, take the form

$$\mathbf{A}(r, t) = \sum_m \int d\omega q_m(\omega, t) \mathbf{f}_m(\omega, \mathbf{r}), \quad \boldsymbol{\Pi}(r, t) = \sum_m \int d\omega \mathbf{f}_m^\dagger(\omega, \mathbf{r}) p_m(\omega, t),$$

where the discrete index m specifies the asymptotic boundary conditions.

The quantization is, then, achieved imposing canonical commutation relations for the coefficients of the expansion. The eigenmodes $\mathbf{f}_m(\omega, \mathbf{r})$ can be expressed as

$$\mathbf{f}_m(\omega, \mathbf{r}) = \frac{1}{\sqrt{\epsilon(r)}} \boldsymbol{\Phi}_m(\omega, \mathbf{r})$$

where $\boldsymbol{\Phi}_m(\omega, \mathbf{r})$ are taken as the solutions of the eigenvalue problem

$$L \boldsymbol{\Phi}_m(\omega, \mathbf{r}) \equiv \frac{1}{\sqrt{\epsilon(r)}} \nabla \times \left[\nabla \times \frac{\boldsymbol{\Phi}_m(\omega, \mathbf{r})}{\sqrt{\epsilon(r)}} \right] = \omega^2 \boldsymbol{\Phi}_m(\omega, \mathbf{r}), \quad (2.5)$$

and form an orthogonal basis in the subspace of L^2 defined by the transversality condition $\nabla \cdot [\sqrt{\epsilon(r)} \boldsymbol{\Phi}_m(\mathbf{r})] = 0$. The functions $\mathbf{f}_m(\omega, \mathbf{r})$ satisfy the orthonormality condition

$$\int d\mathbf{r} \epsilon(r) \mathbf{f}_m^\dagger(\omega, \mathbf{r}) \mathbf{f}_{m'}(\omega', \mathbf{r}) = \delta_{mm'} \delta(\omega - \omega').$$

Since the fields are real, we have $\mathbf{A} = \mathbf{A}^\dagger$ and $\boldsymbol{\Pi} = \boldsymbol{\Pi}^\dagger$ so that

$$q_m(\omega) = \sum_{m'} \int d\omega' \mathcal{M}_{mm'}^\dagger(\omega, \omega') q_{m'}^\dagger(\omega'), \quad p_m^\dagger(\omega) = \sum_{m'} \int d\omega' \mathcal{M}_{mm'}^\dagger(\omega, \omega') p_{m'}(\omega')$$

where $\mathcal{M}_{mm'}(\omega, \omega') \equiv \int d\mathbf{r} \epsilon(r) \mathbf{f}_m(\omega, \mathbf{r}) \cdot \mathbf{f}_{m'}(\omega', \mathbf{r})$ is an unitary and symmetric matrix. Note that mode with different frequency are orthogonal, so $\mathcal{M}(\omega, \omega') \sim \delta(\omega - \omega')$.

The Hamiltonian written in terms of q and p is, then,

$$\mathcal{H} = \frac{1}{2} \sum_m \int d\omega \left[p_m^\dagger(\omega) p_m(\omega) + \omega^2 q_m^\dagger(\omega) q_m(\omega) \right]. \quad (2.6)$$

The quantization is now achieved promoting $q(\omega)$ and $p(\omega)$ to operators with the equal time commutation relations

$$\begin{aligned} [q_m(\omega), q_{m'}(\omega')] &= [q_m(\omega), q_{m'}^\dagger(\omega')] = 0, & [p_m(\omega), p_{m'}(\omega')] &= [p_m(\omega), p_{m'}^\dagger(\omega')] = 0, \\ [q_m(\omega), p_{m'}(\omega')] &= i \delta_{mm'} \delta(\omega - \omega'), & [q_m(\omega), p_{m'}^\dagger(\omega')] &= i \mathcal{M}_{mm'}(\omega, \omega'). \end{aligned}$$

The operators $q(\omega)$ and $p(\omega)$ can be finally expressed in terms of creation and annihilation operators as

$$\begin{aligned} q_m(\omega) &= \left[\frac{1}{2\omega} \right]^{\frac{1}{2}} \left[A_m(\omega) + \sum_n \int d\omega' \mathcal{M}_{mn}^\dagger(\omega, \omega') A_n^\dagger(\omega') \right], \\ p_m(\omega) &= i \left[\frac{\omega}{2} \right]^{\frac{1}{2}} \left[A_m^\dagger(\omega) - \sum_n \int d\omega' \mathcal{M}_{mn}(\omega, \omega') A_n(\omega') \right], \end{aligned} \quad (2.7)$$

such that $[A_m(\omega), A_n(\omega')] = 0$ and $[A_m(\omega), A_n^\dagger(\omega')] = \delta_{mn} \delta(\omega - \omega')$ and the Hamiltonian takes eventually the familiar form

$$\mathcal{H} = \frac{1}{2} \sum_m \int d\omega \omega [A_m^\dagger(\omega) A_m(\omega) + A_m(\omega) A_m^\dagger(\omega)].$$

2.2.2 Resonator and Channel Modes

The previous modes-of-the-universe approach does not provide explicit information about the field inside the cavity, that is particularly relevant when an amplifying medium is present so that lasing modes may arise from the resonator modes with long lifetime. The separation is achieved using the Feshbach projector operators [12]

$$\mathcal{Q} = \int_{r \in I} |\mathbf{r}\rangle \langle \mathbf{r}|, \quad \mathcal{P} = \int_{r \notin I} |\mathbf{r}\rangle \langle \mathbf{r}|,$$

and the characteristic functions

$$\chi_-(\mathbf{r}) = \int_{\mathbf{r}' \in I} d\mathbf{r}' \delta(\mathbf{r} - \mathbf{r}'), \quad \chi_+(\mathbf{r}) = 1 - \chi_-(\mathbf{r}),$$

where I is a finite region of space such as $\epsilon(\mathbf{r}) \equiv 1$ outside it.¹ Every function Φ and its associated $\mathbf{f} = \Phi/\epsilon$ can then be decomposed as

$$\Phi(\mathbf{r}) = \chi_-(\mathbf{r})\mu(\mathbf{r}) + \chi_+\nu(\mathbf{r}), \quad \mathbf{f}(\mathbf{r}) = \chi_-(\mathbf{r})\mathbf{u}(\mathbf{r}) + \chi_+\mathbf{v}(\mathbf{r}).$$

¹It is clearly not unique but all the physical observables are independent from this choice.

The eigenvalue problem becomes

$$L\Phi_m(\omega, \mathbf{r}) = \chi_-(\mathbf{r})L\boldsymbol{\mu}(\mathbf{r}) - \int_{\partial I} d^2\mathbf{r}' K(\mathbf{r}, \mathbf{r}')\boldsymbol{\mu}(\mathbf{r}') + \chi_+L\mathbf{v}(\mathbf{r}) + \int_{\partial I} d^2\mathbf{r}' K(\mathbf{r}, \mathbf{r}')\mathbf{v}(\mathbf{r}')$$

where the operator $K(\mathbf{r}, \mathbf{r}')$ is (\mathbf{n}' being the versor normal to the boundary)

$$K(\mathbf{r}, \mathbf{r}')\boldsymbol{\mu}(\mathbf{r}') \equiv \left[\frac{\delta(\mathbf{r} - \mathbf{r}')\mathbf{n}'}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\nabla' \times \frac{\boldsymbol{\mu}(\mathbf{r}')}{\sqrt{\epsilon(\mathbf{r}')}} \right] + \left[\frac{\nabla\delta(\mathbf{r} - \mathbf{r}')}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\mathbf{n}' \times \frac{\boldsymbol{\mu}(\mathbf{r}')}{\sqrt{\epsilon(\mathbf{r}')}} \right].$$

The operator L can be decomposed into resonator ($L_{\mathcal{Q}\mathcal{Q}}$), channel ($L_{\mathcal{R}\mathcal{R}}$) and coupling contributions ($L_{\mathcal{Q}\mathcal{R}}, L_{\mathcal{R}\mathcal{Q}}$)

$$L\Phi = L_{\mathcal{Q}\mathcal{Q}}\boldsymbol{\mu} + L_{\mathcal{Q}\mathcal{R}}\mathbf{v} + L_{\mathcal{R}\mathcal{Q}}\boldsymbol{\mu} + L_{\mathcal{R}\mathcal{R}}\mathbf{v}, \quad (2.8)$$

with

$$\begin{aligned} L_{\mathcal{Q}\mathcal{Q}}\boldsymbol{\mu} &= \chi_-(\mathbf{r})L\boldsymbol{\mu}(\mathbf{r}) - \int_{\partial I} d^2\mathbf{r}'_- \left[\frac{\delta(\mathbf{r} - \mathbf{r}'_-)\mathbf{n}'}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\nabla' \times \frac{\boldsymbol{\mu}(\mathbf{r}'_-)}{\sqrt{\epsilon(\mathbf{r}'_-)}} \right], \\ L_{\mathcal{R}\mathcal{R}}\mathbf{v} &= \chi_+(\mathbf{r})L\mathbf{v}(\mathbf{r}) + \int_{\partial I} d^2\mathbf{r}'_+ \left[\frac{\nabla\delta(\mathbf{r} - \mathbf{r}'_+)}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\mathbf{n}' \times \frac{\mathbf{v}(\mathbf{r}'_+)}{\sqrt{\epsilon(\mathbf{r}'_+)}} \right], \\ L_{\mathcal{Q}\mathcal{R}}\mathbf{v} &= + \int_{\partial I} d^2\mathbf{r}'_- \left[\frac{\delta(\mathbf{r} - \mathbf{r}'_-)\mathbf{n}'}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\nabla' \times \frac{\mathbf{v}(\mathbf{r}'_+)}{\sqrt{\epsilon(\mathbf{r}'_+)}} \right], \\ L_{\mathcal{R}\mathcal{Q}}\boldsymbol{\mu} &= - \int_{\partial I} d^2\mathbf{r}'_+ \left[\frac{\nabla\delta(\mathbf{r} - \mathbf{r}'_+)}{\sqrt{\epsilon(\mathbf{r})}} \right] \times \left[\mathbf{n}' \times \frac{\boldsymbol{\mu}(\mathbf{r}'_-)}{\sqrt{\epsilon(\mathbf{r}'_-)}} \right], \end{aligned} \quad (2.9)$$

where \mathbf{r}'_- indicates that the integral must be evaluated in the limit to the boundary from the inside and \mathbf{r}'_+ that it must be evaluated in the limit to the boundary from the outside of the resonator.

The eigenvalue problem is written now

$$\begin{pmatrix} L_{\mathcal{Q}\mathcal{Q}} & L_{\mathcal{Q}\mathcal{R}} \\ L_{\mathcal{R}\mathcal{Q}} & L_{\mathcal{R}\mathcal{R}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}(\omega) \\ \mathbf{v}(\omega) \end{pmatrix} = \frac{\omega^2}{c^2} \begin{pmatrix} \boldsymbol{\mu}(\omega) \\ \mathbf{v}(\omega) \end{pmatrix}$$

with the four matching conditions

$$\begin{aligned} (1) \quad \mathbf{n} \times [\mathbf{u}(\omega) - \mathbf{v}(\omega)] &= 0, & (3) \quad \mathbf{n} \times [\nabla \times \mathbf{u}(\omega) - \nabla \times \mathbf{v}(\omega)] &= 0, \\ (2) \quad \mathbf{n} \cdot [\epsilon\mathbf{u}(\omega) - \epsilon\mathbf{v}(\omega)] &= 0, & (4) \quad \mathbf{n} \cdot [\nabla \times \mathbf{u}(\omega) - \nabla \times \mathbf{v}(\omega)] &= 0. \end{aligned}$$

The first two assure that the singular terms at the boundary vanish, the third corresponds to the gauge fixing $\nabla \cdot [\epsilon\mathbf{f}(\omega)] = 0$ and the fourth to the requirement $\nabla \cdot (\nabla \times \mathbf{f}(\omega)) = 0$. Note also as these matching conditions realize the usual boundary conditions for the electromagnetic field at an interface [13].

The problem can then be solved by standard methods [14] obtaining

$$|\Phi(\omega)\rangle = \sum_{\lambda} \alpha_{\lambda}(\omega) |\mu_{\lambda}\rangle + \int d\omega' \beta(\omega, \omega') |\nu(\omega')\rangle ,$$

where μ_{λ} and $\nu(\omega)$ are the solutions of the uncoupled problems $L_{\mathcal{Q}\mathcal{Q}}$ and $L_{\mathcal{P}\mathcal{P}}$ (with the appropriate boundary conditions). The expansion coefficients are

$$\begin{aligned} \alpha_{\lambda}(\omega) &= \langle \mu_{\lambda} | G_{\mathcal{Q}\mathcal{Q}} L_{\mathcal{Q}\mathcal{P}} | \nu(\omega) \rangle , \\ \beta(\omega, \omega') &= \langle \nu(\omega') | \left[1 + \frac{1}{\omega^2 + i\epsilon - L_{\mathcal{P}\mathcal{P}}} L_{\mathcal{P}\mathcal{Q}} G_{\mathcal{Q}\mathcal{Q}} L_{\mathcal{Q}\mathcal{P}} \right] | \nu(\omega) \rangle , \end{aligned}$$

and $G_{\mathcal{Q}\mathcal{Q}}$ is the Green function of the resonator in presence of the coupling with the channel

$$G_{\mathcal{Q}\mathcal{Q}}(\omega^2) = \frac{1}{\omega^2 - L_{\text{eff}}(\omega)} \quad \text{with} \quad L_{\text{eff}} \equiv L_{\mathcal{Q}\mathcal{Q}} + L_{\mathcal{Q}\mathcal{P}} \frac{1}{\omega^2 + i\epsilon - L_{\mathcal{P}\mathcal{P}}} L_{\mathcal{P}\mathcal{Q}} . \quad (2.10)$$

2.2.3 System-and-Bath Hamiltonian

The separation in resonator and channel modes suggests a quantization based on these modes instead of the mode-of-the-universe ones. The vector potential and the canonical momentum can be extracted in terms of these modes as

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sum_{\lambda} Q_{\lambda} \mathbf{u}_{\lambda}(\mathbf{r}) + \int d\omega Q(\omega) \mathbf{v}(\omega, \mathbf{r}) , \\ \mathbf{\Pi}(\mathbf{r}, t) &= \sum_{\lambda} \mathbf{u}_{\lambda}^*(\mathbf{r}) P_{\lambda} + \int d\omega \mathbf{v}^{\dagger}(\omega, \mathbf{r}) P(\omega) , \end{aligned}$$

where Q_{λ} and P_{λ} ($Q(\omega)$ and $P(\omega)$) are time-dependent operators that represent the amplitude associated with the resonator (channel) fields. Their relations with the operator $q(\omega)$ and $p(\omega)$ of the total system are

$$\begin{aligned} q(\omega) &= \sum_{\lambda} \alpha_{\lambda}^{\dagger}(\omega) Q_{\lambda} + \int d\omega' \beta^{\dagger}(\omega, \omega') Q(\omega') , \\ p(\omega) &= \sum_{\lambda} P_{\lambda} \alpha_{\lambda}(\omega) + \int d\omega' P(\omega') \beta(\omega', \omega) , \end{aligned}$$

so that the Hamiltonian takes the form

$$\begin{aligned} \mathcal{H} = & \sum_{\lambda} \left[P_{\lambda}^{\dagger} P_{\lambda} + \omega_{\lambda}^2 Q_{\lambda}^{\dagger} Q_{\lambda} \right] + \sum_m \int d\omega \left[P_m^{\dagger}(\omega) P_m(\omega) + \omega_{\lambda}^2 Q_m^{\dagger}(\omega) Q_m(\omega) \right] + \\ & + \sum_{\lambda} \sum_m \int d\omega \left[W_{\lambda m}(\omega) Q_{\lambda}^{\dagger} Q_m(\omega) + \text{h.c.} \right], \end{aligned}$$

with

$$2W_{\lambda m}(\omega) = \langle \mathbf{v}_{\lambda} | L | \mathbf{v}_m(\omega) \rangle.$$

Also, introducing as usual the creation and annihilation operators a_{λ}^{\dagger} and a_{λ} ($b_m^{\dagger}(\omega)$ and $b_m(\omega)$) for the resonator (channel) fields (cf., e.g., Eqs. 2.7), we have

$$\begin{aligned} \mathcal{H} = & \sum_{\lambda} \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \sum_m \int d\omega \omega b_m^{\dagger}(\omega) b_m(\omega) + \\ & + \sum_{\lambda} \sum_m \int d\omega \left[\mathcal{W}_{\lambda m}(\omega) a_{\lambda}^{\dagger} b_m(\omega) + \mathcal{V}_{\lambda m}(\omega) a_{\lambda} b_m(\omega) + \text{h.c.} \right] \end{aligned}$$

$$\text{with } \mathcal{W}_{\lambda m}(\omega) = \frac{1}{2\sqrt{\omega_{\lambda}\omega}} \langle \mu_{\lambda} | L_{\mathcal{D}\mathcal{D}} | \mathbf{v}_m(\omega) \rangle, \quad \mathcal{V}_{\lambda m}(\omega) = \frac{1}{2\sqrt{\omega_{\lambda}\omega}} \langle \mu_{\lambda}^* | L_{\mathcal{D}\mathcal{D}} | \mathbf{v}_m(\omega) \rangle,$$

where $\langle \mu_{\lambda}^* |$ means $\langle \mu_{\lambda}^* | \mathbf{r} \rangle = \boldsymbol{\mu}(\mathbf{r})$. Finally, the corresponding expressions for the intracavity fields are given by

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sum_{\lambda} \left[\frac{1}{2\omega_{\lambda}} \right]^{\frac{1}{2}} \left[a_{\lambda} \mathbf{u}_{\lambda}(\mathbf{r}) + a_{\lambda}^{\dagger} \mathbf{u}_{\lambda}^*(\mathbf{r}) \right], \\ \boldsymbol{\Pi}(\mathbf{r}, t) &= -i \sum_{\lambda} \left[\frac{\omega_{\lambda}}{2} \right]^{\frac{1}{2}} \left[a_{\lambda} \mathbf{u}_{\lambda}(\mathbf{r}) - a_{\lambda}^{\dagger} \mathbf{u}_{\lambda}^*(\mathbf{r}) \right]. \end{aligned}$$

Note that the resonator modes are now coupled to the external ones via both resonant ($a^{\dagger}b$, $b^{\dagger}a$) and non resonant (ab , $b^{\dagger}a^{\dagger}$) terms. In Ref. [15] the explicit form for some simple case is reported.

2.2.4 Langevin Equations

In most cases of interest, the *rotating wave approximation* is valid: the frequencies are spread over a range much smaller than the typical frequency ($\Delta\omega \ll \omega$), so that the nonresonant terms can be neglected.

In this case the equations of motion are then given by

$$\begin{aligned}\dot{a}_\lambda &= -i\omega_\lambda a_\lambda - i \sum_m \int d\omega \mathcal{W}_{\lambda m}(\omega) b_m(\omega), \\ \dot{b}_m(\omega) &= -i\omega b_m(\omega) - i \sum_\lambda \mathcal{W}_{\lambda m}^*(\omega) a_\lambda.\end{aligned}$$

Integrating the equation for $b_m(\omega)$ from some initial time $t_0 < t$ and assuming that the coupling amplitudes $\mathcal{W}_{\lambda m}(\omega)$ are independent from ω (consistently with the rotating-wave approximation), we finally obtain the Langevin equations for the internal modes

$$\dot{a}_\lambda(t) = -i\omega_\lambda a_\lambda(t) - \pi \sum_{\lambda'} [\mathcal{W} \mathcal{W}^\dagger]_{\lambda\lambda'} a_{\lambda'}(t) + F_\lambda(t) \quad (2.11)$$

where the noise operator $F_\lambda(t)$ is

$$F_\lambda(t) = -i \int d\omega e^{-i\omega(t-t_0)} \sum_m \mathcal{W}_{\lambda m} b_m(\omega, t_0).$$

This result differs from the equations of standard laser theory in two aspects:

- the mode operators a_λ are coupled by the damping matrix

$$\gamma_{\lambda\mu} \equiv \pi [\mathcal{W} \mathcal{W}^\dagger]_{\lambda\mu}; \quad (2.12)$$

- the noise operators are correlated

$$\langle F_\lambda^\dagger(t) F_\mu(t') \rangle \propto 2\gamma_{\lambda\mu} \delta(t - t') \neq \delta_{\lambda\mu} \delta(t - t'). \quad (2.13)$$

Note as the rotating-wave approximation assures that the noise is Markovian, equivalently to assuming a time scale separation so that the typical inner modes lifetimes are much bigger than the “bath correlation time” [16].

2.2.5 Comparison with the Constant Flux States

It is interesting to compare the system-and-bath and the constant flux modes. When using the eigenfunctions $\tilde{\psi}_m(\omega, \mathbf{r})$ and their biorthogonal adjoint $\tilde{\phi}_m(\omega, \mathbf{r})$, it is possible to write the spectral representation of the Green function operator in the form

$$[G_{\mathcal{Q}\mathcal{Q}}(\omega, \mathbf{r}, \mathbf{r}')]_{\alpha\alpha'} = \sum_m \frac{[\tilde{\psi}_m(\omega, \mathbf{r})]_{\alpha} [\tilde{\phi}_m^*(\omega, \mathbf{r}')]_{\alpha'}}{\omega^2 - \Omega_m^2(\omega)}, \quad (2.14)$$

with $\alpha, \alpha' = x, y, z$ indicates the polarization of the field.

The same form is indeed obtained in the system-and-bath approach as $G_{\mathcal{Q}\mathcal{Q}}(\omega) = [\omega^2 - L_{\text{eff}}(\omega)]^{-1}$, with (cf. Eq. (2.10))

$$L_{\text{eff}}(\omega) = L_{\mathcal{Q}\mathcal{Q}} + L_{\mathcal{Q}\mathcal{P}} (\omega^2 - L_{\mathcal{P}\mathcal{P}} + i\epsilon)^{-1} L_{\mathcal{P}\mathcal{Q}}, \quad (2.15)$$

where the first term concerns the inside of the resonator, while the second one the inside-outside coupling (see Eq. (2.9)).

The modes $\tilde{\psi}_m(\omega, \mathbf{r})$ are then the eigenfunctions of $L_{\text{eff}}(\omega)$ with eigenvalue $\Omega_m^2(\omega)$. In fact for the one-dimensional case it was shown in Ref. [15] that the second term in Eq. (2.15), corresponding to the boundary, disappears for outgoing boundary condition, consistently with the constant-flux result.

However the scattering resonances, that correspond to the poles of the Green function, are found differently: for the system-and-bath case the Green function is analytically continued in the complex plane in ω so that the condition $\omega^2 = \Omega_m^2(\omega)$ is imposed; for the constant flux case ω is always a real parameter instead (it corresponds to the physical frequency outside the cavity), so the respective condition would be $\omega = \Re[\Omega_m(\omega)]$.

A comparison between the two approaches can be obtained writing L_{eff} in the eigenbasis of $L_{\mathcal{Q}\mathcal{Q}}$ as [5]

$$L_{\text{eff}}(\omega) \simeq \Omega_0^2 - 2\sqrt{\Omega_0} [i\pi \mathcal{W} \mathcal{W}^\dagger + \Delta(\omega)] \sqrt{\Omega_0},$$

where Ω_0 is the real diagonal matrix of ω_λ . The difference between $\Omega^2(\omega)$ and this matrix is then

$$\Omega^2(\omega) - L_{\text{eff}} = [\Delta\Omega(\omega), \Omega_0] + \Delta\Omega^2(\omega), \quad \Delta\Omega(\omega) \equiv \Omega(\omega) - \Omega_0.$$

The difference is then of second order in the parameters $|\Delta\Omega_{\lambda\lambda'}|/|\omega_\lambda|$ and $|\omega_{\lambda'} - \omega_\lambda|/|\omega_\lambda|$; when these parameters are small (consistently with *the rotating-wave approximation*) the constant-flux eigenmodes and eigenfrequencies are thus expected to become close to those corresponding obtained in the system-and-bath approach.

2.3 Quantum Theory of a Two-Mode Laser in an Open-Cavity

In the previous section the system-and-bath approach based on the Feshbach projectors is employed to obtain a proper quantization of open systems. To complete the theory for multimode open-cavity laser also the effect of the gain must be considered,

i.e. the interaction of the field with an ensemble of atoms continuously pumped in an excited state.

In this section the complete theory for the simple case of a two modes system is exposed, as proposed in the work of Eremeev et al. [3], with particular focus on the novelties with respect to the standard theory of multimode lasers (where the leakages are considered negligible).

The openness is addressed using the approach of the previous section, so that the modes of the electromagnetic field are separated in the system “atoms+field” and the “bath”. Using the density operator description [17], the field is obtained tracing over the atomic degrees of freedom $\rho_F = \text{Tr}_A \rho$ and its evolution can be expressed as

$$\dot{\rho}_F = L^{(\text{gain})} \rho_F + L^{(\text{loss})} \rho_F .$$

From the previous section the loss term takes the form [16]

$$L^{(\text{loss})} \rho_F = \sum_{\lambda\mu} \gamma_{\lambda\mu} \left(2a_\mu \rho_F a_\lambda^\dagger - \rho_F a_\lambda^\dagger a_\mu - a_\lambda^\dagger a_\mu \rho_F \right) ,$$

where $\lambda, \mu = 1, 2$, a_λ and a_λ^\dagger are the annihilation and creation operators and $\gamma_{\lambda\mu}$ is defined in Eq. (2.12), so that the openness of the cavity is associated to nonzero off-diagonal terms. For the sake of simplicity in the following we assume that \mathcal{W} is Hermitian, so that the matrix γ is symmetric: $\gamma_{21} = \gamma_{12}$.

2.3.1 Atom-Field Interaction

The $L^{(\text{gain})}$ term is instead standard and it can be expressed by the Jaynes-Cumming Hamiltonian in the case of a two-level atom as [18, 19]

$$\mathcal{H}_{\text{AF}} = \frac{\omega_a}{2} \sigma_z + \sum_{\lambda} \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \sum_{\lambda} \left(g_{\lambda} \sigma_{-}^{\dagger} a_{\lambda} + \text{h.c.} \right) , \quad (2.16)$$

where ω_a is the frequency of the atomic transition, $\sigma_{-}^{\dagger} = |e\rangle \langle g|$ and $\sigma_z = |e\rangle \langle e| - |g\rangle \langle g|$, $|g\rangle$ and $|e\rangle$ being the ground and excited states, are the atomic raising and inversion operator and g_{λ} describes the coupling between the atom and the field. Defining the parameters $\delta = \omega_a - \bar{\omega}$ and $\Delta_{\lambda} = \omega_{\lambda} - \bar{\omega}$,

$$\mathcal{H}_{\text{AF}} = \frac{\bar{\omega}}{2} \sigma_z + \bar{\omega} \sum_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \frac{\delta}{2} \sigma_z + \sum_{\lambda} \Delta_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \sum_{\lambda} (g_{\lambda} \sigma_{-}^{\dagger} a_{\lambda} + \text{h.c.}) \equiv H_0 + V ,$$

so that in the interaction picture then the time evolution of the density operator is governed by the operator $U(t) = \exp(-iVt)$. In particular, in the limit $\Delta_{\lambda} \ll g_{\lambda}$ (an approximation compatible with the rotating-wave approximation but stronger) the

expression of V is simply

$$V = \begin{pmatrix} \delta/2 & g\alpha \\ g\alpha & \delta/2 \end{pmatrix}, \quad \text{with } g \text{ and } \alpha \text{ given by } \begin{matrix} g = \sqrt{g_1^2 + g_2^2} \\ g\alpha = g_1 a_1 + g_2 a_2 \end{matrix}. \quad (2.17)$$

In this case the explicit expression for ρ_F can be obtained as [3]

$$\begin{aligned} \rho_F(t) &= \Phi_- \rho_F(0) \Phi_+ + g^2 \alpha^\dagger \frac{\sin(\varphi t)}{\varphi} \rho_F(0) \frac{\sin(\varphi t)}{\varphi} \alpha \equiv \\ &\equiv \Lambda(t) \rho_F(0), \end{aligned} \quad (2.18)$$

where the auxiliary operators φ and Φ_\pm are defined as

$$\varphi = g \left[\alpha \alpha^\dagger + \left(\frac{\delta}{2g} \right)^2 \right]^{\frac{1}{2}}, \quad \Phi_\pm = \cos(\varphi t) \pm i \frac{\delta \sin(\varphi t)}{2\varphi},$$

and it is assumed that at $t = 0$ the system is in the state $\rho(0) = \rho_F(0) \otimes \rho_A(0)$ with $\rho_A(0) = |e\rangle \langle e|$.

The generalization to the interaction with an ensemble of many atoms is achieved assuming that only one atom at a time interacts with the field and that the time τ of the interaction is much shorter than the time on which the field evolves. Under these assumptions the field at time t , during which it interacts with k atoms, is $\rho_F(t) = \Lambda^k(\tau) \rho_F(0)$ [20].

The last basic element to add is the external pumping. The number of atoms that are excited during the time t is taken Poissonian with average $\langle k \rangle = \mathcal{P} t$, corresponding to the realistic case of random pumping [21].

Eventually also the fact that interaction time τ is a random variable must be taken into account. Assuming a finite lifetime Γ^{-1} for the excited state (due, e.g., to the decay to another excited state not involved in the lasing process) the statistical distribution of τ is $P(\tau) = \Gamma \exp(-\Gamma \tau)$. The final expression for the evolution of the field due to the gain is then

$$\begin{aligned} L^{(\text{gain})} \rho_F &= \mathcal{P} \int_0^\infty d\tau \Gamma \exp(-\Gamma \tau) \cdot \\ &\cdot \left\{ \Phi_- \rho_F(t) \Phi_+ + g^2 \alpha^\dagger \frac{\sin(\varphi \tau)}{\varphi} \rho_F(t) \frac{\sin(\varphi \tau)}{\varphi} \alpha - \rho_F(t) \right\}. \end{aligned} \quad (2.19)$$

2.3.2 Properties of the Lasing Mode

The resolution of the master equation Eq. (2.19) is easier in the orthogonal basis of the composite modes $\alpha = (g_1 a_1 + g_2 a_2)/g$ and $\beta = (g_2 a_1 - g_1 a_2)/g$; assuming real $g_{1,2}$ we have then $[\beta, \beta^\dagger] = [\alpha, \alpha^\dagger] = 1$ and $[\alpha, \beta] = [\alpha, \beta^\dagger] = 0$.

In particular, defining the distribution of the photons in the α mode as

$$\rho(n_\alpha) = \sum_{n_\beta} \langle n_\alpha, n_\beta | \rho_F | n_\alpha, n_\beta \rangle$$

where $|n_\alpha, n_\beta\rangle$ are the Fock states of the field, at the steady-state regime $\dot{\rho}(n_\alpha) = \dot{\rho}(n_\beta) = 0$ the resulting equations are (for $n_\alpha, n_\beta > 1$)

$$p(n_\alpha) \left\{ C_1 - 2C_3^2 \left[(\bar{n}_\beta + 1) K_{\bar{n}_\alpha, \bar{n}_\beta + 1}^{-1} - 2\bar{n}_\beta K_{\bar{n}_\alpha, \bar{n}_\beta}^{-1} \right] \right\} + \\ -p(n_\alpha - 1) \left[\frac{A}{1 + (\delta/\Gamma)^2 + (B/A)n_\alpha} + 2C_3^2 \bar{n}_\beta K_{\bar{n}_\alpha, \bar{n}_\beta}^{-1} \right] = 0, \quad (2.20)$$

$$p(n_\beta) \left\{ C_2 - 2C_3^2 \left[(\bar{n}_\alpha + 1) K_{\bar{n}_\alpha + 1, \bar{n}_\beta}^{-1} - 2\bar{n}_\alpha K_{\bar{n}_\alpha, \bar{n}_\beta}^{-1} \right] \right\} + \\ -p(n_\beta - 1) \times 2C_3^2 \bar{n}_\alpha K_{\bar{n}_\alpha, \bar{n}_\beta}^{-1} = 0, \quad (2.21)$$

where \bar{n}_α and \bar{n}_β are the average values of the corresponding distributions and the parameters shown are

$$\begin{aligned} A &= 2(g/\Gamma)^2 \mathcal{P}, \\ B &= 8(g/\Gamma)^4 \mathcal{P} = 4(g/\Gamma)^2 A, \\ C_1 &= 2g^{-2}(\gamma_{11}g_1^2 + 2\gamma_{12}g_1g_2 + \gamma_{22}g_2^2), \\ C_2 &= 2g^{-2}(\gamma_{11}g_2^2 - 2\gamma_{12}g_1g_2 + \gamma_{22}g_1^2), \\ C_3 &= g^{-2}[(\gamma_{11} - \gamma_{22})g_1g_2 + 2\gamma_{12}(g_2^2 - g_1^2)], \\ M_{n_\alpha, n_\beta} &= [(n_\alpha + 1/2)A + B/4][1 + (\delta/\Gamma)^2 + (n_\alpha + 1/2)B/A + (B/4A)^2]^{-1} + \\ &\quad + (n_\alpha - 1/2)C_1 + (n_\beta - 1/2)C_2, \\ K_{n_\alpha, n_\beta} &= M_{n_\alpha, n_\beta} + (\delta A/2\Gamma)^2[1 + (\delta/\Gamma)^2 + (B/A)(n_\alpha + 1/2) + (B/4A)^2]^{-2} M_{n_\alpha, n_\beta}^{-1}. \end{aligned}$$

In particular, the coupling between the modes is proportional to the parameter C_3 . When $\gamma_{12} \rightarrow 0$, the master equation reduces to the case of a lossless cavity [3] and, then, the coupling between the modes arises uniquely from the coupling to the same atomic transition.

From Eq. (2.21) it is quite clear that $p(n_\beta) = 0$ is the only solution, since there are no pumping terms in the equation. Instead, Eq. (2.20) has a nontrivial solution above a threshold in \mathcal{P} . In particular for low pump the photon distribution approaches the thermal distribution

$$p(n_\alpha) \simeq \left(1 - \frac{A}{\tilde{C}_1}\right) \left(\frac{A}{\tilde{C}_1}\right)_\alpha^n, \quad \mathcal{P} \rightarrow 0, \quad (2.22)$$

where $\tilde{C}_1 = C_1(1 + (\delta/\Gamma)^2)$.

For strong pump instead $\bar{n}_\alpha \gg (1 + (\delta/\Gamma)^2)A/B$ and the solution tends to the Poissonian distribution

$$p(n_\alpha) \simeq p(0) \frac{(\tilde{A}/B)!(A^2/BC_1)^{n_\alpha}}{(n_\alpha + \tilde{A}/B)!}, \quad \mathcal{P} \rightarrow \infty, \quad (2.23)$$

where $\tilde{A} = A(1 + (\delta/\Gamma)^2)$ and $p(0)$ is fixed by the normalization. Above the threshold the distribution is very peaked around \bar{n}_α , so that

$$\bar{n}_\alpha \simeq \frac{\tilde{A}}{B} \left(\frac{A}{\tilde{C}_1} - 1 \right)$$

and the pumping threshold in \mathcal{P} is consequently approximately given by

$$\frac{A}{C_1} = 1 + \left(\frac{\delta}{\Gamma}\right)^2 \rightarrow \mathcal{P} \frac{g^4}{\Gamma^2 + \delta^2} = \gamma_{11}g_1^2 + 2\gamma_{12}g_1g_2 + \gamma_{22}g_2^2.$$

Note, in particular, that the threshold in \mathcal{P} increases linearly in the dumping parameter γ_{12} , that is associated with the openness of the cavity. Incidentally, we note also as the inhomogeneity of the gain results in a reduction of the threshold, as observed by Deych [22]. Taking for simplicity $\gamma_{12} = 0$, $\gamma_{1,2} \equiv \gamma$ and defining $g_{1,2} = g_0 \pm \Delta g$ the threshold becomes

$$\mathcal{P} \simeq \gamma \frac{\Gamma^2 + \delta^2}{2g_0^2} \left(1 - \frac{(\Delta g)^2}{g_0^2} \right).$$

It is also known that the openness of the cavity results in a broadening of the emission line by the so-called Petermann factor [23]. In the two mode system, the linewidth $\delta\omega_\alpha$ can be obtained from the evolution of the complete density matrix: using the ansatz $\dot{\rho}_{n_\alpha, n_\beta; n_\alpha+k_1, n_\beta+k_2} = -\mu(k_1, k_2)\rho_{n_\alpha, n_\beta; n_\alpha+k_1, n_\beta+k_2}$, it indeed coincides with the real part of $\mu(0, 1)$, hence obtaining [3]

$$\delta\omega_\alpha = \frac{1}{4} \left[\frac{A/(\bar{n}_\alpha + 1) + 2B}{1 + (\delta/\Gamma)^2 + (B/A)(\bar{n}_\alpha + 3/2) + (B/4A)^2} + \frac{C_1}{\bar{n}_\alpha} \right]. \quad (2.24)$$

In particular, γ_{12} enters in the expression by C_1 and \bar{n}_α . Far above the threshold $A \gg C_1$, \bar{n}_α becomes independent from γ_{12} and thus the linewidth increases linearly with γ_{12} .

While the restriction to just two cavity modes makes the application to random lasers quite limited (though some comparison with experiments may be possible,

see, e.g., Ref. [24]), this analysis shows some general properties of high-open cavity lasers. In particular: the lasing modes are different from the cold cavity ones; the lasing threshold and the emission linewidth increase with the openness of the cavity.

2.4 Semiclassical Multimode Theory

The generalization of the quantum theory of the previous section to the general multimode case is still an open problem. In this section we limit ourselves to the derivation of the multimode laser theory for open resonators in the semiclassical limit [25, 26].

The evolution of the atom-field operators given by the cavity loss and the Jaynes-Cumming Hamiltonian Eq. (2.16) can be expressed through the equations in the Heisenberg representation

$$\begin{aligned}\dot{a}_\lambda &= -i\omega_\lambda a_\lambda - \sum_\mu \gamma_{\lambda\mu} a_\mu + \int d\mathbf{r} g_\lambda^\dagger(\mathbf{r}) \sigma_-(\mathbf{r}) + F_\lambda, \\ \dot{\sigma}_-(\mathbf{r}) &= -(\gamma_\perp + i\omega_a) \sigma_-(\mathbf{r}) + 2 \sum_\mu g_\mu(\mathbf{r}) \sigma_z(\mathbf{r}) a_\mu + F_-(\mathbf{r}), \\ \dot{\sigma}_z(\mathbf{r}) &= +\gamma_\parallel (S\rho(\mathbf{r}) - \sigma_z(\mathbf{r})) - \sum_\mu (g_\mu^\dagger(\mathbf{r}) a_\mu^\dagger \sigma_-(\mathbf{r}) + \text{h.c.}) + F_z(\mathbf{r}),\end{aligned}\tag{2.25}$$

where $\gamma_{\lambda\mu}$ is the dumping matrix Eq. (2.12) associated to the openness of the cavity, $\rho(\mathbf{r})$ is the density of atoms, γ_\perp (γ_\parallel) is the polarization (population-inversion) decay rate, S is the pump intensity as resulting from the interaction between atoms and external baths. The interaction also gives rise to the noises $F_-(\mathbf{r})$ and $F_z(\mathbf{r})$, due, for example, to the finite lifetime of the excited states for the decay to states non involved in the lasing process. The noise F_λ is due to the coupling with the bath (cf. Eq. (2.11)). The field-atoms coupling constants are

$$g_\lambda(\mathbf{r}) \equiv \frac{\omega_a p}{\sqrt{2\hbar\epsilon_0\omega_\lambda}} \mu_\lambda(\mathbf{r}),\tag{2.26}$$

where p is the atomic dipole matrix element and the $\mu_\lambda(\mathbf{r})$ are the orthogonal set of the resonator eigenstates (cf. Sect. 2.2).

The semiclassical theory consists in neglecting all the noise terms in the evolution and replacing the operators with their expectation values.

As in the previous section, it is supposed that the lifetimes of the modes are much longer than the characteristic times of pump and loss: in this way the atomic variables can be adiabatically removed to obtain the nonlinear equations for the field alone.

2.4.1 Linear Regime

Consider first the case of weak pumping, such that it is possible to assume $\sigma_z(\mathbf{r}) = S\rho(\mathbf{r})$ and the unique stationary solution is $a_\lambda = 0$ for all the modes. Supposing that the deviations from the stationary state relax to zero with complex frequency ω_k , so that $\delta a_\lambda(t) = \delta a_\lambda \exp(-i\omega_k t)$, the evolution Eqs. 2.25 give

$$\sum_{\mu} (\omega_k \delta_{\mu\lambda} - H_{\lambda\mu}) \delta a_\mu = 0, \quad (2.27)$$

and, then, the frequencies ω_k are the eigenvalues of the non-Hermitian matrix [25]

$$H_{\lambda\mu} = \omega_\lambda \delta_{\lambda\mu} - i\gamma_{\lambda\mu} + iG_{\lambda\mu}^{(2)}(\omega_k), \quad (2.28)$$

$$\text{with } G_{\lambda\mu}^{(2)}(\omega) \equiv 2S \int d\mathbf{r} \rho(\mathbf{r}) \frac{g_\mu^*(\mathbf{r}) g_\lambda(\mathbf{r})}{i(\omega_a - \omega) + \gamma_\perp}.$$

The three terms account respectively for the linear gain, the escape losses and the internal resonator dynamics. In general, if the atoms are not uniformly distributed in the resonator, the matrix $G_{\lambda\mu}^{(2)}(\omega)$ is not diagonal: the eigenvalues and eigenvectors are, hence, different from the cold cavity ones and depend parametrically on the pump strength S . In particular, increasing S the eigenvalues move up in the complex plane. The lasing threshold is reached when one eigenvalue takes a positive imaginary part. In this case the gain exceeds the loss and the solution $a_\lambda = 0$ becomes unstable.

2.4.2 Lasing Regime

To obtain the expressions in the lasing regime it is useful to write the time evolution in the Fourier space:

$$a_\lambda(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega a_\lambda(\omega) e^{-i\omega t},$$

and similarly for $\sigma_-(t)$ and $\sigma_z(t)$, with $\sigma_z^*(\omega) = \sigma_z(-\omega)$. In particular it is useful, then, to define $f(\omega)$ such that $\sigma_z(\omega) = f(\omega) + f^*(-\omega)$.

The atoms-field evolution of Eqs. 2.25 becomes²

$$-i\omega a_\lambda(\omega) = -i\omega_\lambda a_\lambda(\omega) - \sum_{\mu} \gamma_{\lambda\mu} a_\mu(\omega) + \int d\mathbf{r} g_\lambda^* \sigma_-(\omega), \quad (2.29)$$

²The space dependence is not explicitly written for the sake of simplicity.

$$-i\omega\sigma_-(\omega) = -(\gamma_\perp + i\omega_a)\sigma_-(\omega) + \sum_\mu \frac{g_\mu}{\pi} \int d\omega' a_\mu(\omega - \omega') (f(\omega') + f^*(-\omega')) , \quad (2.30)$$

$$-i\omega f(\omega) = +\gamma_\parallel \left(\frac{S\rho}{2} \delta(\omega) - f(\omega) \right) - \sum_\mu \left(\frac{g_\mu^*}{2\pi} \int d\omega' a_\mu^*(-\omega + \omega') \sigma_-(\omega') \right) . \quad (2.31)$$

The previous system of equations can be reduced to a systems of equations of the field modes alone using a perturbation theory in the modes amplitude.

For later convenience we define here

$$D_\parallel(\omega) \equiv \left(1 - i \frac{\omega}{\gamma_\parallel} \right)^{-1} , \quad D(\omega) \equiv \left(1 - i \frac{\omega - \omega_a}{\gamma_\perp} \right)^{-1} , \quad A \equiv -\frac{1}{\pi \gamma_\perp \gamma_\parallel} . \quad (2.32)$$

One starts neglecting the quadratic term in Eq. (2.31) so that the zeroth-order it is (note that $D_\parallel(\omega)\delta(\omega) = \delta(\omega)$)

$$f^{(0)}(\omega) = \frac{1}{2} S\rho \delta(\omega) . \quad (2.33)$$

Replacing this expression in Eq. (2.30) we have to the first order

$$\sigma_-^{(1)}(\omega) = -\gamma_\parallel A S\rho D(\omega) \sum_\mu g_\mu a_\mu(\omega) , \quad (2.34)$$

this expression can then be replaced back in Eq. (2.31) and so obtain the second order

$$f^{(2)}(\omega) = \frac{A S\rho}{2\pi} D_\parallel(\omega) \sum_{\mu_1 \mu_2} g_{\mu_1}^* g_{\mu_2} \int d\omega_1 a_{\mu_1}^*(-\omega + \omega_1) D(\omega_1) a_{\mu_2}(\omega_1) . \quad (2.35)$$

Replacing back in the equation for σ_- , one finally gets the third order term

$$\begin{aligned} \sigma_-^{(3)}(\omega) = & -\frac{\gamma_\parallel A^2}{2\pi} S\rho D(\omega) \sum_{\mu_1 \mu_2 \mu_3} g_{\mu_1} \cdot \\ & \left[g_{\mu_2}^* g_{\mu_3} \int d\omega_1 d\omega_2 a_{\mu_1}(\omega - \omega_1) D_\parallel(\omega_1) a_{\mu_2}^*(-\omega_1 + \omega_2) D(\omega_2) a_{\mu_3}(\omega_2) + \right. \\ & \left. + g_{\mu_2} g_{\mu_3}^* \int d\omega_1 d\omega_2 a_{\mu_1}(\omega - \omega_1) D_\parallel(\omega_1) a_{\mu_2}(\omega_1 + \omega_2) D^*(\omega_2) a_{\mu_3}^*(\omega_2) \right] \Rightarrow \end{aligned} \quad (2.36)$$

$$\Rightarrow \sigma_-^{(3)}(\omega) = - \frac{\gamma_{\parallel} A^2}{2\pi} S\rho D(\omega) \sum_{\mu_1 \mu_2 \mu_3} g_{\mu_1} g_{\mu_2}^* g_{\mu_3} \int d\omega_1 d\omega_2 \cdot \quad (2.37)$$

$$a_{\mu_1}(\omega - \omega_1 + \omega_2) D_{\parallel}(\omega_1 - \omega_2) a_{\mu_2}^*(\omega_2) a_{\mu_3}(\omega_1) [D(\omega_1) + D^*(\omega_2)]$$

Proceeding recursively in this way all the terms of the expansions for $\sigma_z(\omega)$ and $\sigma_-(\omega)$ can be obtained. Then, replacing them in Eq. (2.29) finally gives the equation for the modes alone.

In the particular case of the *free-running approximation* [25], that is assuming that the different lasing modes oscillate independently from each other (so that the phases are uncorrelated and the interaction concerns the intensity alone), it is possible to resum the equation and obtain an expression for the mode intensities valid to all the orders in the perturbation theory (cf. Ref. [26]).

This approximation may be valid for the so-called nonresonant or incoherent feedback “random lasers” [27], where the interference effects are neglected. In this case the emission is due solely to amplified spontaneous emission, and, then, the spectrum is determined only by the gain curve of the active material. This simple approach can explain some properties of random laser [28, 29]. In random lasers, however, the multiple-scattering process defines optical modes, with a certain frequency, bandwidth and rich spatial profile [30]. A complete model of random lasers should, hence, include the mode structure, as was clear after the observations of Cao et al. [31], which revealed narrow spikes in the emission spectrum on top of a global narrowing, that can not be explained by amplified spontaneous emission alone.

We do not use, hence, the free-running approximation and limit ourselves to the third order theory. The subsequent orders may become relevant far above the threshold. From a statistical mechanics point of view the orders beyond the third are not expected to change the universality class for a large class of models (see for example [32]). Being specific, we consider that $g^2|a|^2 \ll \gamma_{\perp}\gamma_{\parallel}$, where $|a|^2$ is the typical intensity in the lasing regime, so that the third order theory is valid.

A *slow amplitude mode* with index l is a solution such that it has a harmonic form for $t \gg 1$ and, therefore, its Fourier transform is proportional to $\delta(\omega - \omega_l)$. By definition, a *lasing mode* is a slow amplitude mode with a positive intensity at the solution. In general the lasing modes are different from the cold cavity ones (the steady-state solutions are different already in the linear regime as $G_{\lambda\mu}^{(2)}$ is not diagonal, cf. Eq. (2.28)). We express these modes in the form

$$a_{\lambda}(t) = \sum_k a_{\lambda k} \alpha_k(t), \quad \alpha_k(t) = \bar{a}_k(t) e^{-i\omega_k t} \quad (2.38)$$

with $\bar{a}_k(t)$ evolving on time scales much larger than ω_k^{-1} , so that

$$\alpha_k(\omega) \simeq \delta(\omega - \omega_k),$$

where $a_{\lambda k}$, hence, expresses the component of the cold cavity mode λ at the lasing frequency ω_k .

In general the number of lasing modes will be smaller than the dimension of the space and it will increase with the pumping. Nevertheless, we can always define a *slow amplitude modes basis* of which the lasing modes are a subset. For example, in the two modes laser of Sect. 2.3 such basis may be composed by the modes α and β , such that only α lases above the threshold. Note that the modes in the subspace orthogonal to the lasing modes will have a zero intensity at the solution by definition, therefore, the components along these modes will be irrelevant in the dynamics. However the slow amplitude modes basis is useful to expand any mode and, in particular, invert the relation Eq. (2.38)

$$\alpha_k(t) = \sum_{\lambda} b_{k\lambda}^* a_{\lambda}(t), \quad (b_{k\lambda}^*) = (a_{k\lambda})^{-1}. \quad (2.39)$$

Here and in the following we use Greek letters for the indices of cold cavity modes and Latin letters for the indices of the slow amplitude modes.

Using the slow amplitude modes expansion Eq. (2.38) in the evolution given by Eq. (2.29) we obtain at the third order

$$\begin{aligned} \sum_k i(\omega_{\lambda} - \omega_k) a_{\lambda k} \delta(\omega - \omega_k) = & + \sum_{\mu k} \left[\gamma_{\lambda\mu} - G_{\lambda\mu; k}^{(2)} \right] a_{\mu k} \delta(\omega - \omega_k) + \\ & - \sum_{\mu} \sum_{k_1 k_2 k_3} G_{\lambda\mu; \mathbf{k}}^{(4)} a_{\mu_1 k_1} a_{\mu_2 k_2}^* a_{\mu_3 k_3} \delta(\omega - \omega_{k_3} + \omega_{k_2} - \omega_{k_1}), \end{aligned} \quad (2.40)$$

where $G_{\lambda\mu; k}^{(2)}$ and $G_{\lambda\mu_1\mu_2\mu_3; \mathbf{k}}^{(4)}$ are the functions of the frequencies ω_k given by

$$G_{\lambda\mu; k}^{(2)} = \gamma_{\parallel} A S D(\omega_k) \int d\mathbf{r} \rho(\mathbf{r}) g_{\lambda}^*(\mathbf{r}) g_{\mu}(\mathbf{r}), \quad (2.41)$$

$$\begin{aligned} G_{\lambda\mu; \mathbf{k}}^{(4)} = & \frac{\gamma_{\parallel} A^2}{2\pi} S D(\omega_{k_3} - \omega_{k_2} + \omega_{k_1}) D_{\parallel}(\omega_{k_3} - \omega_{k_2}) [D(\omega_{k_3}) + D^*(\omega_{k_2})] \cdot \\ & \int d\mathbf{r} \rho(\mathbf{r}) g_{\lambda}^*(\mathbf{r}) g_{\mu_1}(\mathbf{r}) g_{\mu_2}^*(\mathbf{r}) g_{\mu_3}(\mathbf{r}). \end{aligned} \quad (2.42)$$

Taking the terms at the frequency ω_l , summing over λ and using orthonormality of the cold cavity modes the expression becomes

$$\begin{aligned} \left[i(\omega_{\mu} - \omega_l) - \sum_{\lambda} \left(\gamma_{\lambda\mu} - G_{\lambda\mu; l}^{(2)} \right) \right] a_{\mu l} = & \sum_k \left[- \sum_{\lambda \mu_2 \mu_3} \sum_{k_2 k_3} G_{\lambda\mu; \mathbf{k}}^{(4)} a_{\mu_2 k_2}^* a_{\mu_3 k_3} \right] a_{\mu k} \rightarrow \\ \rightarrow & [-i\omega_l + \Omega_{\mu}(\omega_l)] a_{\mu l} = \sum_k v_{\mu k}^{(l)} a_{\mu k}, \end{aligned} \quad (2.43)$$

and note that the matrix $V_{\mu k}^{(l)}$ itself depends on all the amplitudes and frequencies of the lasing modes, so Eq. (2.43) is not a standard eigenvalue problem.

The derivation can be repeated including a slow time dependence of $\bar{a}_l(t)$ of Eq. (2.38), so that $a_l(\omega)$ is not exactly a Dirac delta though it is still sharply peaked in ω_l . In this case, expanding the linear coupling as $\Omega_\mu(\omega) \simeq \Omega_\mu(\omega_l) + (\omega - \omega_l)\Omega'_\mu(\omega_l)$, and considering the time dependence of $a_{\mu l}(t)$ in the nonlinear term only, the equations become

$$\left[i\Omega'_\mu(\omega_l) \frac{d}{dt} - i\omega_l + \Omega_\mu(\omega_l) \right] a_{\mu l}(t) = \sum_k V_{\mu k}^{(l)}(t) a_{\mu k}(t). \quad (2.44)$$

In the stationary limit the previous equation is recovered. The steady-state solutions are thus not changed when the slow time dependence is included, although the term $\Omega'_\mu(\omega_l)$ may change their stability, so it cannot be in general neglected in a complete analysis [26].

2.4.2.1 Langevin Equation for the Lasing Modes

Eventually, we write explicitly the Langevin equation for the lasing modes $\alpha_l(t)$. Using the expansion in the slow amplitude modes $\alpha_k(\omega)$, Eq. (2.38) in the mode evolution Eq. (2.29), multiplying by $b_{\lambda l}^*$ (defined in Eq. (2.39)) and summing over λ we obtain

$$\begin{aligned} \sum_k \sum_\lambda i(\omega_\lambda - \omega) b_{\lambda l}^* a_{\lambda k} \alpha_k(\omega) &= \sum_k \sum_{\mu\lambda} \left[\gamma_{\lambda\mu} - G_{\lambda\mu,k}^{(2)} \right] b_{\lambda l}^* a_{\mu k} \alpha_k(\omega) + \\ &- S \frac{\gamma_{\parallel} A^2}{2\pi} D(\omega) \sum_{\mathbf{k}} \sum_{\lambda\mu} \int d\mathbf{r} \rho g_{\lambda}^* g_{\mu_1} g_{\mu_2}^* g_{\mu_3} b_{\lambda l}^* a_{\mu_1 k_1} a_{\mu_2 k_2}^* a_{\mu_3 k_3} \int d\omega_1 d\omega_2 \times \\ &\times D_{\parallel}(\omega_1 - \omega_2) \left[D(\omega_1) + D^*(\omega_2) \right] \alpha_{k_1}(\omega - \omega_1 + \omega_2) \alpha_{k_2}^*(\omega_2) \alpha_{k_3}(\omega_1), \end{aligned} \quad (2.45)$$

and then (making also use of the explicit expression of $G^{(2)}$)

$$\begin{aligned} -i\omega \alpha_l(\omega) &= \sum_k \left[-i\omega_l \delta_{lk} + \tilde{\gamma}_{lk} - S M_k^{(2)} \int d\mathbf{r} \rho(\mathbf{r}) g_l^{L*}(\mathbf{r}) g_k^R(\mathbf{r}) \right] \alpha_k(\omega) + \\ &- S \sum_{\mathbf{k}} M_{\mathbf{k}}^{(4)} \int d\mathbf{r} \rho(\mathbf{r}) g_l^{L*}(\mathbf{r}) g_{k_1}^R(\mathbf{r}) g_{k_2}^{R*}(\mathbf{r}) g_{k_3}^R(\mathbf{r}) \times \\ &\times \int d\omega_1 d\omega_2 \alpha_{k_1}(\omega - \omega_1 + \omega_2) \alpha_{k_2}^*(\omega_2) \alpha_{k_3}(\omega_1), \end{aligned} \quad (2.46)$$

where the tilde matrix $\tilde{\gamma}_{lk}$ is

$$\tilde{\gamma}_{lk} \equiv \sum_{\lambda\mu} b_{\lambda l}^* \gamma_{\lambda\mu} a_{\mu k}, \quad (2.47)$$

the left and right coupling constants for the slow amplitude modes are given by

$$g_k^L = \sum_{\mu} b_{\mu k} g_{\mu}, \quad g_k^R = \sum_{\mu} a_{\mu k} g_{\mu}, \quad (2.48)$$

and the coefficients $M_k^{(2)}$ and $M_{\mathbf{k}}^{(4)}$ are defined by

$$\begin{aligned} M_k^{(2)} &\equiv \gamma_{\parallel} A D(\omega_k), \\ M_{\mathbf{k}}^{(4)} &\equiv \frac{\gamma_{\parallel} A^2}{2\pi} D(\omega_{k_1} - \omega_{k_2} + \omega_{k_3}) \cdot D_{\parallel}(\omega_{k_3} - \omega_{k_2}) \cdot [D(\omega_{k_3}) + D^*(\omega_{k_2})]. \end{aligned} \quad (2.49)$$

In the time domain we have

$$\begin{aligned} \frac{d}{dt} \alpha_l(t) &= \sum_{k|FM(l,k)} \left[-i\omega_l \delta_{lk} + \tilde{\gamma}_{lk} - S M_k^{(2)} \int d\mathbf{r} \rho(\mathbf{r}) g_{k_1}^{L*}(\mathbf{r}) g_{k_2}^R(\mathbf{r}) \right] \alpha_k(t) + \\ &\quad - S \sum_{\mathbf{k}|FM(l,\mathbf{k})} M_{\mathbf{k}}^{(4)} \int d\mathbf{r} \rho(\mathbf{r}) g_l^{L*}(\mathbf{r}) g_{k_1}^R(\mathbf{r}) g_{k_2}^{R*}(\mathbf{r}) g_{k_3}^R(\mathbf{r}) \cdot \alpha_{k_1}(t) \cdot \alpha_{k_2}^*(t) \cdot \alpha_{k_3}(t), \end{aligned} \quad (2.50)$$

where we have stressed that, by definition, $\alpha_k(\omega) \simeq \delta(\omega - \omega_k)$, so that the relevant terms in the sums are only those that meet the *frequency matching conditions*

$$FM(k_1, \dots, k_{2n}) : \quad |\omega_{k_1} - \omega_{k_2} + \dots + \omega_{k_{2n-1}} - \omega_{k_{2n}}| \lesssim \delta\omega. \quad (2.51)$$

The finite linewidth $\delta\omega$ of the modes can be thoroughly derived only in the complete quantistic theory, in particular including the noise factors in Eqs. (2.29)–(2.31) in the approach, resulting in a weak time dependence of $\bar{a}_k(t)$ of Eq. (2.38). Here we include it in an effective way, as a parameter to suitably conform to different experimental physical situations.

We stress as, in general, the linear term of the Langevin equation Eq. (2.50) may have nonzero off-diagonal terms. They are all zero when the frequencies are all different and well spaced, $\Delta\omega \gg \delta\omega$, so that the frequencies matching condition of the linear term is never satisfied. While this is generally true for standard high quality-factor lasers,³ for random lasers in general there can be a significant frequency overlap between the lasing modes.

The actual values of the couplings are in principle, and in some simple case, entirely computable in the cold cavity basis (cf. Eqs. (2.41)–(2.42)). The main prob-

³The quality factor (or Q factor) of a resonator is the ratio between the resonance frequency and the full width at half-maximum bandwidth of the resonance. It is then also equals to the finesse times the optical frequency divided by the free spectral range.

lem remain to express the interactions in the slow amplitude mode basis actually used in the dynamics. In some cases the solution can be found using some self-consistent procedures that progress iteratively starting from the solution obtained neglecting the nonlinear coupling [33–35]. In particular, when the nonlinear term is entirely neglected, a possible (not unique) solution is the one that diagonalizes the linear interaction. Nonetheless, when the lasing threshold is exceeded, the nonlinear interaction term becomes relevant and then the diagonalization of the linear term does not in general correspond to a slow amplitude basis anymore. In addition, for hyper-connected modes and, in particular, in the mean-field approximation, the nonlinear term is not abruptly relevant at the transition: the optical power is approximately equipartited also above the transition, as we will show in the rest of Thesis. In the following, then, we consider the possible case of a non diagonal linear term in all regimes.

The fulfillment of the frequency matching condition on the nonlinear term gives rise to the interaction associated to the *mode locking*,⁴ cf. next section. In the mode locking regime, then, the frequency dependence of the interaction makes the phases in linear relation with the frequencies $\phi_k = \omega_k \Delta + \phi_0$, with Δ in good approximation independent from k (cf. Chap. 4). This relation itself, in particular, is the origin of the term “mode locking”.

2.5 Hamiltonian Formulation

The main result of the previous section is in the Langevin equations of the lasing modes Eq. (2.50). This shows that in presence of an open resonator the lasing modes are coupled by both a linear and a nonlinear term. In particular, not too far from the threshold ($g^2|a|^2 \ll \gamma_\perp \gamma_\parallel$) the nonlinear term is approximately given by the third order with the coupling expressed by Eq. (2.49).

As usual, in the previous semiclassical derivation we neglected all the noise sources. From a statistical point of view this means that the model is at zero temperature, so that the entropy is completely neglected. However, to obtain a complete statistical description of model, the noise, hence, the temperature, must be taken in account. As we will see in the next Chapter the role of the entropy becomes, in particular, crucial in the case of disordered multimode lasers, where a random first order transition [36] is expected when the mean-field approximation is valid (cf. Chap. 3).

Some insight in the physical meaning of the different terms of the Langevin equation can be obtained comparing with the master equation of standard mode locking lasers [37]:

⁴The term mode locking indicates a whole group of methods to obtain ultrashort pulses from lasers. In general, a mode locked laser resonator contains either an active element (an optical modulator) or a nonlinear passive element (a saturable absorber), which causes the formation of an ultrashort pulse circulating in the laser resonator.

$$\frac{d}{dt}\alpha_l(t) = (G_l + iD_l)\alpha_l(t) + (\Gamma - i\Delta) \sum_{\mathbf{k}|\text{FM}(l,\mathbf{k})} \alpha_{k_1}(t) \cdot \alpha_{k_2}^*(t) \cdot \alpha_{k_3}(t) + F_l(t); \quad (2.52)$$

this can be seen as a special case of Eq. (2.50) for regular and high quality-factor cavities. In particular, $\tilde{\gamma}_{lk}$ is then very low and the frequencies are equispaced with $\Delta\omega \gg \delta\omega$,⁵ while the coupling of the nonlinear term is approximately the same for every mode. In particular note that here the lasing modes are the same of the cold cavity modes. The approach based on the master equation has provided a deep understanding of passive mode-locking lasers: the pulse shape, the stability analysis, the threshold behavior and much more [37].

In the standard master equation the real parameter G_l represents the difference between the gain and loss of the mode l in a complete round-trip through the cavity, D_l is the group velocity dispersion of the wave packet, Γ is the nonlinear self-amplitude modulation coefficient associated to a saturable absorber and, then, to the passive mode-locking, and Δ is the self-phase modulation coefficient (responsible of the Kerr lens effect). The noise $F_l(t)$ is generally assumed Gaussian, white and uncorrelated:

$$\begin{aligned} \langle F_{k_1}^*(t_1) F_{k_2}(t_2) \rangle &= 2T \delta_{k_1 k_2} \delta(t_1 - t_2), \\ \langle F_{k_1}(t_1) F_{k_2}(t_2) \rangle &= 0, \end{aligned} \quad (2.53)$$

where T is the spectral power of the noise and it is related to the effective temperature.

In general, there are different sources of noises (see F_λ , F_- and F_z in Eqs. 2.25). In the case of open cavities, it is known that the noise due to the coupling to the external bath F_λ may be correlated in general in the cold cavity modes basis, cf. Sect. 2.2. Note as a not-unitary change of basis can also affect the noise correlation:

$$\begin{aligned} \alpha_l &= \sum_{\lambda} a_{l\lambda}^{-1} \alpha_{\lambda} \rightarrow F_l = \sum_{\lambda} a_{l\lambda}^{-1} F_{\lambda} \rightarrow \\ \rightarrow \langle F_{k_1}^*(t_1) F_{k_2}(t_2) \rangle &= \sum_{\lambda_1 \lambda_2} a_{l_1 \lambda_1}^{-1*} \langle F_{\lambda_1}^*(t_1) F_{\lambda_2}(t_2) \rangle a_{l_2 \lambda_2}^{-1}. \end{aligned} \quad (2.54)$$

The decomposition in the slow amplitude modes, Eq. (2.38), is by no means unique. This freedom may be used to try to build a mode basis for which the noise is approximately uncorrelated. In the following we assume that the various independent noise sources act so that such basis construction is possible and, then, the noise can be assumed white and uncorrelated also in the general case of open and irregular cavities. Note as this request may result in a further source of non diagonality for the linear coupling, even when the nonlinear interaction is neglected.

⁵A notable example is that of an optical frequency comb, where the an optical spectrum consists of well-defined equidistant lines. A frequency comb can thus be used as an optical ruler: If the comb frequencies are known, the frequency comb can be used, e.g., to measure unknown frequencies by measuring beat notes, which reveal the difference in frequency between the unknown frequency and the comb frequencies [38].

Defining $\mathcal{H} = \mathcal{H}_R + i\mathcal{H}_I$ [39]

$$\begin{aligned}\mathcal{H}_R &\equiv - \sum_k G_k \alpha_k^* \alpha_k - \frac{\Gamma}{2} \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} \alpha_{k_1}(t) \cdot \alpha_{k_2}^*(t) \cdot \alpha_{k_3}(t) \cdot \alpha_{k_4}^*(t), \\ \mathcal{H}_I &\equiv - \sum_k D_k \alpha_k^* \alpha_k + \frac{\Delta}{2} \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} \alpha_{k_1}(t) \cdot \alpha_{k_2}^*(t) \cdot \alpha_{k_3}(t) \cdot \alpha_{k_4}^*(t),\end{aligned}$$

the master equation can be expressed as

$$\frac{d}{dt} \alpha_l(t) = - \frac{\partial \mathcal{H}}{\partial \alpha_l^*} + F_l = - \frac{\partial (\mathcal{H}_R + i\mathcal{H}_I)}{\partial \alpha_l^*} + F_l, \quad (2.55)$$

and switching to the real and imaginary parts of the mode amplitudes $\alpha_l \equiv \sigma_l + i\tau_l$

$$\begin{aligned}\frac{\partial \sigma_l}{\partial t} &= - \frac{1}{2} \frac{\partial \mathcal{H}_R}{\partial \sigma_l} + \frac{1}{2} \frac{\partial \mathcal{H}_I}{\partial \tau_l} + F_l^R, \\ \frac{\partial \tau_l}{\partial t} &= - \frac{1}{2} \frac{\partial \mathcal{H}_R}{\partial \tau_l} - \frac{1}{2} \frac{\partial \mathcal{H}_I}{\partial \sigma_l} + F_l^I.\end{aligned} \quad (2.56)$$

The corresponding expression of the Hamiltonian for general open and irregular cavities is obtained as

$$\begin{aligned}\mathcal{H} &= - \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} \left[-i\omega_{k_1} \delta_{k_1 k_2} + \tilde{\gamma}_{k_1 k_2} - S M_{k_1}^{(2)} \int d\mathbf{r} \rho(\mathbf{r}) g_{k_1}^{L*}(\mathbf{r}) g_{k_2}^R(\mathbf{r}) \right] \cdot \alpha_{k_1}^* \alpha_{k_2} + \\ &+ \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} S \left[M_{\mathbf{k}}^{(4)} \int d\mathbf{r} \rho(\mathbf{r}) g_{k_1}^{L*}(\mathbf{r}) g_{k_2}^R(\mathbf{r}) g_{k_3}^{R*}(\mathbf{r}) g_{k_4}^R(\mathbf{r}) \right] \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^* \equiv \\ &\equiv \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} g_{k_1 k_2}^{(2)} \cdot \alpha_{k_1} \alpha_{k_2}^* + \frac{1}{2} \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} g_{k_1 k_2 k_3 k_4}^{(4)} \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^*\end{aligned} \quad (2.57)$$

\mathcal{H}_R and \mathcal{H}_I being the real and imaginary part of the previous expression:

$$\begin{aligned}\mathcal{H}_R &= \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} G_{k_1 k_2} \cdot \alpha_{k_1} \alpha_{k_2}^* + \frac{1}{2} \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} \Gamma_{k_1 k_2 k_3 k_4} \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^*, \\ \mathcal{H}_I &= \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} D_{k_1 k_2} \cdot \alpha_{k_1} \alpha_{k_2}^* + \frac{1}{2} \sum_{\mathbf{k}|\text{FM}(\mathbf{k})} \Delta_{k_1 k_2 k_3 k_4} \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^*,\end{aligned}$$

where now

$$\begin{aligned} G_{k_1 k_2} &\equiv \frac{1}{2} \left(g_{k_1 k_2}^{(2)} + g_{k_2 k_1}^{(2)*} \right), & \Gamma_{k_1 k_2 k_3 k_4} &\equiv \frac{1}{2} \left(g_{k_1 k_2 k_3 k_4}^{(4)} + g_{k_2 k_1 k_4 k_3}^{(4)*} \right) \\ iD_{k_1 k_2} &\equiv \frac{1}{2} \left(g_{k_1 k_2}^{(2)} - g_{k_2 k_1}^{(2)*} \right), & i\Delta_{k_1 k_2 k_3 k_4} &\equiv \frac{1}{2} \left(g_{k_1 k_2 k_3 k_4}^{(4)} - g_{k_2 k_1 k_4 k_3}^{(4)*} \right). \end{aligned} \quad (2.58)$$

From Eqs. (2.55)–(2.56) it is clear that \mathcal{H}_R is associated with a purely dissipative motion (a gradient flow in the $2N$ dimensional space $\sigma_1, \dots, \sigma_N, \tau_1, \dots, \tau_N$), while \mathcal{H}_I generates an Hamiltonian motion for the N conjugated variables (σ_l, τ_l) .

If $\mathcal{H}_R = 0$ the total optical intensity $\mathcal{E} \equiv \sum_k |\alpha_k|^2$ is a constant of motion under the previous Langevin equations (like \mathcal{H} itself):

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t} \sum_k \alpha_k \alpha_k^* = \sum_k \left(-i \frac{\partial \mathcal{H}_I}{\partial \alpha_k^*} \alpha_k^* + i \frac{\partial \mathcal{H}_I}{\partial \alpha_k} \alpha_k \right) = 0. \quad (2.59)$$

When $\mathcal{H}_R \neq 0$ this is no longer true. In this case the system is stable because the gain decreases as the optical intensity increases [40]. For standard lasers this is usually modeled assuming that the gain in the master equation Eq. (2.52) is given by $G_I = G_0 / (1 + \mathcal{E} / E_{\text{sat}})$. To study the equilibrium properties of the model, it is possible to consider a simpler model: at any instant the gain is supposed to assume the value that exactly keeps \mathcal{E} constant in the motion, as Gordon and Fisher have proposed in Ref. [39]. In this way the system evolves over the hypersphere $\mathcal{E} \equiv E_0$. The relation between the thermodynamics in the fixed-power ensemble and a variable-power ensemble may be seen as similar to the relation between the canonical and grand canonical ensembles in statistical mechanics [41]. The constraint $\mathcal{E} \equiv E_0$ will induce a correlation of order N^{-1} in the noise F_I . However, as far as we are interested in the limit $N \gg 1$, such correlation can be neglected and the noise is considered white.

More explicitly we require that

$$\frac{\partial \mathcal{E}}{\partial t} \propto \sum_{\mathbf{k} | \text{FM}(\mathbf{k})} G_{k_1 k_2} \cdot \alpha_{k_1} \alpha_{k_2}^* + \sum_{\mathbf{k} | \text{FM}(\mathbf{k})} \Gamma_{k_1 k_2 k_3 k_4} \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^* = 0;$$

so exploiting the central gain G_0 as $G_{kk} \equiv G_0 + G_{kk}^\delta$, we obtain that the required value for G_0 is given by

$$G_0 = -\frac{1}{\mathcal{E}} \left\{ \sum_{\mathbf{k} | \text{FM}(\mathbf{k})} G_{k_1 k_2}^\delta \cdot \alpha_{k_1} \alpha_{k_2}^* + \sum_{\mathbf{k} | \text{FM}(\mathbf{k})} \Gamma_{k_1 k_2 k_3 k_4} \cdot \alpha_{k_1} \alpha_{k_2}^* \alpha_{k_3} \alpha_{k_4}^* \right\}.$$

In particular for $G_{k_1 k_2}^\delta = G_{k_1}^\delta \delta_{k_1 k_2}$ and $\Gamma_{k_1 k_2 k_3 k_4} = \Gamma$, the known result for the standard mode-locking case of Eq. (2.52) is recovered [39].

Replacing the expression for G_0 in Eq. (2.55) it becomes

$$\frac{d\alpha_l}{dt} = G_0\alpha_l + \sum_{k|FM(l,k)} (G_{kl}^\delta + iD_{kl})\alpha_k + \sum_{\mathbf{k}|FM(l,\mathbf{k})} (\Gamma_{lk_1k_2k_3} + i\Delta_{lk_1k_2k_3})\alpha_{k_1}\alpha_{k_2}^*\alpha_{k_3},$$

so that the corresponding \mathcal{H}_R is eventually

$$\mathcal{H}_R \equiv -\frac{E_0}{\mathcal{E}} \sum_{\mathbf{k}|FM(\mathbf{k})} G_{k_1k_2}^\delta \cdot \alpha_{k_1}\alpha_{k_2}^* - \frac{E_0^2}{2\mathcal{E}^2} \sum_{\mathbf{k}|FM(\mathbf{k})} \Gamma_{k_1k_2k_3k_4} \cdot \alpha_{k_1}\alpha_{k_2}^*\alpha_{k_3}\alpha_{k_4}^*. \quad (2.60)$$

In such a way the total optical intensity \mathcal{E} is a constant of the motion, unlike the functional \mathcal{H} .

2.5.1 Purely Dissipative Case ($\mathcal{H}_I = 0$)

In the case $\mathcal{H}_R \gg \mathcal{H}_I$ the functional \mathcal{H} is approximately real. For the standard mode-locking lasers this corresponds to the physical situation where the group velocity dispersion and the Kerr effect can be neglected. The purely dissipative case does also apply to the important case of soliton lasers [42]. In the general case of Eq. (2.57) the situation is more complex. First note that it is not necessary that the coefficients the single couplings $g_{k_1k_2}^{(2)}$ and $g_{k_1k_2k_3k_4}^{(4)}$, that appear in the Hamiltonian Eq. (2.57), are real (cf. Eqs. (2.58)), i.e. that $D_{k_1k_2} \ll G_{k_1k_2}$ and $\Delta_{k_1k_2k_3k_4} \ll \Gamma_{k_1k_2k_3k_4}$. A more generic sufficient condition is that, with the definitions of Sect. 2.4,

$$\omega_l - \omega_a \ll \gamma_\perp, \quad \Delta\omega \ll \gamma_\parallel \quad (2.61)$$

and the matrix $a_{\lambda l}$, that connects the cold cavity and the lasing mode basis (cf. Eq. (2.38)), is Hermitian (as obtained for example in the two modes model of Sect. 2.3). This is consistent with, but stronger than, the rotating-wave approximation.

The case with a real functional \mathcal{H} is of particular interest because it can be studied using the standard methods of the equilibrium statistical physics. In fact, when the functional \mathcal{H} is real, the Eqs. (2.56) reduce to

$$\begin{aligned} \frac{\partial \sigma_l}{\partial t} &= -\frac{1}{2} \frac{\partial \mathcal{H}_R}{\partial \sigma_l} + F_l^R, \\ \frac{\partial \tau_l}{\partial t} &= -\frac{1}{2} \frac{\partial \mathcal{H}_R}{\partial \tau_l} + F_l^I, \end{aligned} \quad (2.62)$$

so they have the familiar “potential form”: the evolution is the derivative of a “potential” respect to the considered variable plus white Gaussian noise. Hence, the steady-state solution of the associated Fokker-Plank equation

$$\dot{\rho} = - \sum_k \frac{\partial}{\partial \sigma_k} \left\{ \frac{\partial \mathcal{H}_R}{\partial \sigma_k} \rho \right\} - \sum_k \frac{\partial}{\partial \tau_k} \left\{ \frac{\partial \mathcal{H}_R}{\partial \tau_k} \rho \right\} + T \sum_k \left(\frac{\partial^2}{\partial \sigma_k^2} + \frac{\partial^2}{\partial \tau_k^2} \right) \rho \quad (2.63)$$

is given by the familiar Gibbs distribution

$$\rho(\sigma_1, \tau_1 \dots \sigma_n, \tau_N) = \frac{e^{-\mathcal{H}_R/T}}{\int e^{-\mathcal{H}_R/T} d\sigma_1 d\tau_1 \dots d\sigma_n d\tau_N}. \quad (2.64)$$

This case, then, is the most interesting for the application of statistical mechanics and it will thoroughly analyzed in this thesis.

2.5.2 General Case ($\mathcal{H}_I \neq 0$)

In the case in which the functional \mathcal{H}_I cannot be neglected, the Fokker-Plank equation for statistical distribution of the modes is given by

$$\begin{aligned} \dot{\rho} = & - \sum_k \frac{\partial}{\partial \sigma_k} \left\{ \left(\frac{\partial \mathcal{H}_R}{\partial \sigma_k} - \frac{\partial \mathcal{H}_I}{\partial \tau_k} \right) \rho \right\} - \sum_k \frac{\partial}{\partial \tau_k} \left\{ \left(\frac{\partial \mathcal{H}_R}{\partial \tau_k} + \frac{\partial \mathcal{H}_I}{\partial \sigma_k} \right) \rho \right\} + \\ & + T \sum_k \left(\frac{\partial^2}{\partial \sigma_k^2} + \frac{\partial^2}{\partial \tau_k^2} \right) \rho. \end{aligned} \quad (2.65)$$

Replacing the Gibbs distribution for \mathcal{H}_R (cf. Eq. (2.64)) in the steady-state equation $\dot{\rho} = 0$, one finds that the term of order $\mathcal{O}(T^0)$ is always satisfied (so the Gibbs distribution is always the solution for $T \gg 1$), while the term of order $\mathcal{O}(T^{-1})$ imposes

$$\sum_k \left(\frac{\partial \mathcal{H}_R}{\partial \sigma_k} \frac{\partial \mathcal{H}_I}{\partial \tau_k} - \frac{\partial \mathcal{H}_R}{\partial \tau_k} \frac{\partial \mathcal{H}_I}{\partial \sigma_k} \right) = 0. \quad (2.66)$$

That is, \mathcal{H}_I must be a constant of the motion under the evolution given by \mathcal{H}_R alone. This is true in particular for the case of the solitons in standard lasers [37, 42].

Otherwise, looking for a solution in the form $\rho = \exp(-f(\mathcal{H}_R, \mathcal{H}_I)/T)$, one finds that at the first order in $\mathcal{H}_I/\mathcal{H}_R$ the steady-state distribution is

$$\rho(\sigma_1, \tau_1 \dots \sigma_n, \tau_N) = \frac{1}{\mathcal{Z}} \exp \left[-\frac{1}{T} \left(\mathcal{H}_R + \frac{\partial_\sigma \mathcal{H}_R \partial_\tau \mathcal{H}_I - \partial_\tau \mathcal{H}_R \partial_\sigma \mathcal{H}_I}{\partial_\sigma \mathcal{H}_R \partial_\sigma \mathcal{H}_I + \partial_\tau \mathcal{H}_R \partial_\tau \mathcal{H}_I} \mathcal{H}_I \right) \right], \quad (2.67)$$

where \mathcal{Z} is the normalization factor, and the sum over the modes in the numerator and the denominator of the factor in front of \mathcal{H}_I is implicit.

In this case it becomes very hard to study the solution analytically. Numerical simulations for the case of standard lasers (cf. Eq. (2.52)) show that the presence of a nonzero \mathcal{H}_1 term does not change the qualitative scenario in the discontinuous transition for the continuous wave to the passive mode locking phase, while it moves the transition at lower values of the “temperature” T [42–44]. In general, it is expected that transitions of the first order are not removed by slight modification of the dynamics.

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