Chapter 2
Magnetoquasistatic Approximation of Maxwell's Equations, Uncertainty Quantification Principles

Starting from the classical form of Maxwell's equations the magnetoquasistatic approximation will be derived and justified. Additionally, some key notions from the area of uncertainty quantification, verification and validation will be established.

2.1 Maxwell's Equations

Here, we adopt the classical 3-D Euclidean vector representation, as opposed to the 4-D space-time form of electromagnetics based on exterior calculus and differential forms. The content of this section can be found in many textbooks, see, e.g., [1, 2]. Maxwell's equations in integral form read as

$$\int_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = \int_V \rho \mathrm{d}V, \tag{2.1a}$$

$$\int_{\partial A} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_A \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathrm{d}\mathbf{A}, \tag{2.1b}$$

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0, \tag{2.1c}$$

$$\int_{\partial A} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{A}, \tag{2.1d}$$

for any surface $A \subset \mathbb{R}^3$ and volume $V \subset \mathbb{R}^3$ at rest. These relations contain the electric induction $\mathbf{D}$, the electric charge density ρ, the magnetic field $\mathbf{H}$, the magnetic induction $\mathbf{B}$, the electric field $\mathbf{E}$ and the electric current density $\mathbf{J}$. The current density is decomposed as $\mathbf{J} = \mathbf{J}_{\mathrm{src}} + \mathbf{J}_{\mathrm{con}}$, where $\mathbf{J}_{\mathrm{src}}$ and $\mathbf{J}_{\mathrm{con}}$ refer to an imposed and ohmic part, respectively. Relations (2.1) have to be supplemented by material constitutive relations. For time-invariant, isotropic media these are given by

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$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{2.2a}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \tag{2.2b}$$

$$\mathbf{J}_{\text{con}} = \sigma \mathbf{E}, \tag{2.2c}$$

where ε_0, μ_0 represent the permittivity and permeability of vacuum, σ refers to the electric conductivity and $\mathbf{P}$, $\mathbf{M}$ represent the electric polarization and the magnetization, respectively. Except for ε_0, μ_0, all quantities in (2.2) are functions of space. Moreover, $\mathbf{M}$ and $\mathbf{P}$ depend on $\mathbf{H}$ and $\mathbf{E}$, respectively. Using the theorems of Gauss and Stokes we derive from (2.1) the differential form of Maxwell's equations

$$\operatorname{div} \mathbf{D} = \rho, \tag{2.3a}$$

$$\mathbf{curl}\ \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.3b}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2.3c}$$

$$\mathbf{curl}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3d}$$

endowed with suitable boundary conditions or decay conditions at infinity if the domain is bounded or unbounded, respectively. The integral version of Maxwell's equations is often the starting point instead of (2.3) as they are formulated by means of directly measurable quantities such as voltages and fluxes. The key observation here, is that $\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{s}$ can actually be viewed as a mapping

$$S \mapsto \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{s} \tag{2.4}$$

which associates a voltage to a sufficiently smooth oriented line. This is precisely the notion of an integral form of degree one, equivalently

$$A \mapsto \int_A \mathbf{B} \cdot \mathrm{d}\mathbf{A} \tag{2.5}$$

is an integral form of degree two. The notion of integral forms will be used later on, to derive boundary and interface conditions.

2.2 Magnetoquasistatic Approximation

Although practically all phenomena of classical electromagnetics are governed by Maxwell's equations, their resolution, in particular their numerical resolution, in the most general form is often unnecessary, sometimes impracticable. In this work we are especially interested in simulating devices with dominant magnetic energy, where wave propagating effects can be neglected. As we will justify, this amounts

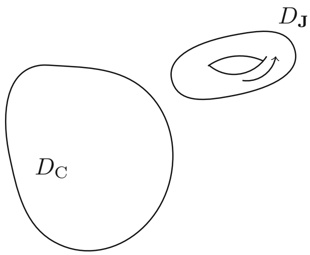

Fig. 2.1 Model geometry for the illustration of the magnetoquasistatic approximation on an unbounded domain

in neglecting the term $\partial\mathbf{D}/\partial t$ in (2.3). Typical applications are electrical machines and transformers operating at low frequencies and accelerator magnets, which are the key application in this work. The resulting set of equations is referred to as *magnetoquasistatic* model or eddy current model in the literature.

To simplify the discussion we restrict ourselves to the time-harmonic equations in the remaining part of this subsection. A model geometry is depicted in Fig. 2.1. We consider a bounded, connected domain of homogeneous conductivity D_C in free space and an imposed divergence free current density with $\mathrm{supp}(\mathbf{J}_\mathrm{src}) =: D_\mathbf{J}$. We assume that $\overline{D_\mathbf{J}} \cap \overline{D_\mathrm{C}} = \emptyset$ and that σ is constant inside D_C and zero in free space $D_\mathrm{E} = \mathbb{R}^3 \backslash \overline{D_\mathrm{C}}$, whereas μ and ε are assumed to be piecewise constant. The restrictions of the material functions to D_C, D_E are denoted with subscripts C, E, respectively. Following [3], the time-harmonic magnetoquasistatic model is given by

$$\mathrm{div}\, \varepsilon\mathbf{E} = 0, \qquad \text{in } D_\mathrm{E}, \tag{2.6a}$$

$$\mathbf{curl}\ \mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_\mathrm{src}, \qquad \text{in } \mathbb{R}^3, \tag{2.6b}$$

$$\mathbf{curl}\ \mathbf{E} = -j\omega\mu\mathbf{H}, \qquad \text{in } \mathbb{R}^3, \tag{2.6c}$$

$$\mathbf{E}(\mathbf{x}) = \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right), \qquad \text{uniformly for } |\mathbf{x}| \to \infty, \tag{2.6d}$$

$$\mathbf{H}(\mathbf{x}) = \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right), \qquad \text{uniformly for } |\mathbf{x}| \to \infty, \tag{2.6e}$$

where j is the imaginary unit and $|\cdot|$ refers to the Euclidean norm. Uniqueness is achieved, by additionally imposing the condition $\int_{\partial D_\mathrm{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = 0$. Often, neglecting the displacement current $j\omega\mathbf{D}$ is justified as a low frequency approximation, i.e., $\omega \to 0$. In this case the modeling error, i.e., the difference between the electromagnetic fields described by Maxwell's equations $(\mathbf{E}^\mathrm{M}, \mathbf{H}^\mathrm{M})$ (M refers to full Maxwell) and the magnetoquasistatic equations $(\mathbf{E}, \mathbf{H})$, has been analyzed in [3]. In a first step it can be shown, based on a power series expansion $\mathbf{E}^\mathrm{M} = \omega\mathbf{E}_1 + \mathcal{O}(\omega^2)$, that

$$\mathbf{curl}\ \mathbf{H}^\mathrm{M} = \sigma\omega\mathbf{E}_1 + \mathbf{J}_\mathrm{src} + \mathcal{O}(\omega^2), \tag{2.7}$$

see [3], thus the neglected term is of higher order in ω. Moreover,

$$\|\mathbf{E}^{\mathrm{M}} - \mathbf{E}\|_{L^2(B_R)^3} = \mathcal{O}(\omega^2), \quad \|\mathbf{H}^{\mathrm{M}} - \mathbf{H}\|_{L^2(B_R)^3} = \mathcal{O}(\omega^2), \tag{2.8}$$

holds, see [3], where $D_{\mathbf{J}} \subset B_R \subset \mathbb{R}^3$ and B_R is a ball of radius R, provided that $\mathbf{J}_{\mathrm{src}}$ is divergence free in the limit. The asymptotic behavior (2.8) has been confirmed for a bounded domain in [4]. However, if $D_{\mathbf{J}}$ and D_{C} overlap, i.e., there exists a galvanic connection expressed as a current flow to the conductor, the asymptotic error decay reduces to $\mathcal{O}(\omega)$.

Although appropriate in many cases the notion of low-frequency approximation is not general enough, as it is well known that in several circumstances the magnetoquasistatic model can be used for moderate up to high frequencies, too. Let us therefore consider the more general, and widely used condition, that

$$\omega\varepsilon_{\mathrm{C}} \ll \sigma_{\mathrm{C}} \tag{2.9}$$

has to be small. For a bounded domain D, we supplement this condition by

$$\mathrm{diam}(D) \ll \lambda, \tag{2.10}$$

where diam refers to the diameter of D. This means that the dimensions of the domain D have to be small compared to the wavelength $\lambda = 2\pi/(\sqrt{\mu\epsilon}\omega)$. Again, these conditions can be mathematically justified, as done in [4], where the relation

$$\frac{\|\mathbf{E}^{\mathrm{M}} - \mathbf{E}\|_{L^2(D)^3}}{\|\mathbf{E}^{\mathrm{M}}\|_{L^2(D)^3}} \leq C_1 \frac{\mathrm{diam}(D)^2}{\lambda_{\mathrm{C}}^2} + C_2 \frac{\omega\varepsilon_{\mathrm{C}}}{\sigma_{\mathrm{C}}} \tag{2.11}$$

was derived, with positive constants $C_1, C_2 > 0$. However, one should keep in mind, that even if (2.9) and (2.10) are satisfied the magnetoquasistatic approximation might still be unjustified as the constants C_1, C_2 in the previous expressions can become quite large in some situations: the conductor geometry has to be such that capacitive effects are negligible [4, 5].

2.3 Magnetoquasistatic Model

Although the magnetoquasistatic model was justified in a time-harmonic setting in the previous section, we will choose a time-domain setting from now on, due to the following reasons. The magnetic material law is typically nonlinear and higher order harmonics of the fields need to be taken into account, even if the current is excited at a single frequency. This can be done, and actually is a popular choice for the simulations of electrical machines [6]. It should also be noted that usually a few higher order harmonics are sufficient and the truncation error is well understood [7]. However, as a more serious drawback, a time-harmonic setting is inappropriate for modeling strongly time transient phenomena we want to include into our setting, such as the ramping of a magnet. We are now going to introduce in some detail the model problem that, under simplifications from time to time, will be the basis for the remaining part of this work. For many applications it is acceptable to consider a bounded computational domain D. This reflects the fact, that the fields decay as given in (2.6d), (2.6e) and the energy stored in a region far away from the current excitation is close to zero. In this context, the shape of D has no physical significance and we assume that D is a simply connected polyhedral Lipschitz domain. Lipschitz boundaries are needed to apply many results on Sobolev spaces, whereas polyhedral domains, i.e., domains with boundaries consisting of plain faces, straight edges and corner points, can be exactly covered by conventional tetrahedral finite element meshes. For a precise definition of these terms, see, e.g., [8, 9]. The model geometry is depicted in Fig. 2.2, where D (strictly) contains a conducting, ferromagnetic region D_{C} and an air region filled with coil parts, defined as D_{E} such that $D_{\mathrm{E}} = D\backslash\overline{D_{\mathrm{C}}}$. We do not assume that D_{C} is simply connected, as this would exclude many applications. Concerning the constitutive relations the effects of isotropy and hysteresis and permanent magnetization are neglected here. Although this might oversimplify many practical setups, it allows for a more thorough modeling of uncertainties, which is a main issue of this work. More precisely, we assume the following:

Assumption 2.1 (*Conductivity*) The electric conductivity satisfies

$$\sigma(\mathbf{x}) = \begin{cases} \sigma_{\mathrm{C}}, & \text{in } D_{\mathrm{C}}, \\ 0, & \text{in } D_{\mathrm{E}}, \end{cases} \tag{2.12}$$

where $\sigma_{\mathrm{C}} > 0$ is supposed to be constant.

For nonlinear materials, the magnetic properties at each point are expressed through the so called $B - H$ curve $f_{BH} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, given by

$$|\mathbf{B}| = f_{BH}(|\mathbf{H}|) := \mu_0(|\mathbf{H}| + |\mathbf{M}|). \tag{2.13}$$

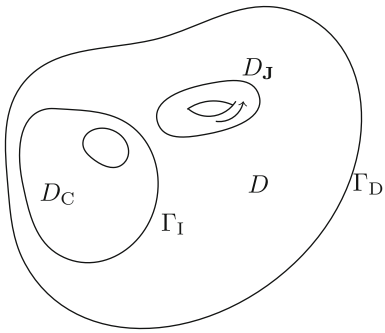

Fig. 2.2 General model geometry for the magnetoquasistatic model on a bounded domain

Let $\mathcal{C}^1(\mathbb{R}^+)$ denote the space of continuously differentiable functions on $\mathbb{R}^+$. Well-known physical properties of $f_{BH} \in \mathcal{C}^1(\mathbb{R}^+)$ are expressed as

$$f_{BH}(0) = 0, \tag{2.14a}$$

$$\partial_s f_{BH}(s) \geq \mu_0, \quad \forall s \geq 0, \tag{2.14b}$$

$$\lim_{s\to\infty} \partial_s f_{BH}(s) = \mu_0, \tag{2.14c}$$

see [10, 11]. From (2.14) it can be deduced that f_{BH} is a bijective function and the inverse $f_{HB} := f_{BH}^{-1}$ satisfies properties similar to (2.14). This motivates the following assumption for the magnetic reluctivity, defined point-wise as $\nu(s) := f_{HB}(s)/s, \ \forall s \in \mathbb{R}^+$ ($\nu(0)$ is defined by taking the limit $s \to 0$):

Assumption 2.2 (*Reluctivity*) The magnetic reluctivity $\nu : D \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies

$$\nu(\mathbf{x}, \cdot) = \begin{cases} \nu_{\mathrm{C}}(\cdot), & \text{in } D_{\mathrm{C}}, \\ \nu_0, & \text{in } D_{\mathrm{E}}, \end{cases} \tag{2.15}$$

with $\mathbf{x} \in D$ and value $\nu_0 = 1/\mu_0$, the reluctivity of vacuum. Moreover, $\nu_{\mathrm{C}} : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function such that for all $s \in \mathbb{R}_0^+$,

$$0 < \nu_{\min} \leq \nu_{\mathrm{C}}(s) \leq \nu_0 < \infty, \tag{2.16a}$$

$$\nu_{\mathrm{C}}(\cdot)\cdot, \text{ is Lipschitz continuous with constant } \nu_0, \tag{2.16b}$$

$$\nu_{\mathrm{C}}(\cdot)\cdot, \text{ is strongly monotone with constant } \nu_{\min}, \tag{2.16c}$$

holds.

More precisely, Lipschitz continuity and strong monotonicity for $\nu_{\mathrm{C}}(\cdot)\cdot$ or f_{HB} are expressed as

$$|f_{HB}(s) - f_{HB}(t)| \leq \nu_0 |s - t|, \tag{2.17}$$

$$(f_{HB}(s) - f_{HB}(t))\,(s - t) \geq \nu_{\min}(s - t)^2, \tag{2.18}$$

for all $s, t \in \mathbb{R}^+$, respectively. In a more general setting equations (2.16) would hold for each $\mathbf{x} \in D$ and additionally $\nu(\cdot, s)$ would be assumed to be measurable for all $s \in \mathbb{R}_0^+$, see [12]. The assumption $\nu = \nu_{\mathrm{E}}$ in D_{E} is justified as the magnetic properties of coil parts, such as copper, can be well approximated by the respective vacuum properties. Working with the magnetic reluctivity and the inverse function f_{HB}, is particularly appropriate for the magnetic vector potential formulation. There exists a vast literature on the magnetoquasistatic model, see, e.g., [7, 13, 14]. Among the different formulations, the vector potential formulation is a rather general approach, well suited for the applications covered in this treatise and derived formally as follows.

As $\mathbf{B}$ is divergence free (2.1c), it can be represented by a vector potential $\mathbf{B} = \mathbf{curl}\ \mathbf{A}$. Then from (2.1b) and the magnetoquasistatic approximation we obtain

$$\sigma\mathbf{E} + \mathbf{curl}\ (\nu\,\mathbf{curl}\ \mathbf{A}) = \mathbf{J}_{\mathrm{src}}. \tag{2.19}$$

From Eq. (2.1d) in turn we infer $\partial\mathbf{A}/\partial t = \mathbf{E}$ (up to a gradient field) and hence the differential equation

$$\sigma\frac{\partial\mathbf{A}}{\partial t} + \mathbf{curl}\ (\nu\,\mathbf{curl}\ \mathbf{A}) = \mathbf{J}_{\mathrm{src}}, \tag{2.20}$$

which is valid, separately in each subdomain. To achieve uniqueness in the vector potential representation a gauging procedure is required. In particular, any gradient field can be added to $\mathbf{A}$ without changing $\mathbf{B}$, as $\mathbf{curl}\ \mathbf{grad}\ = 0$. To this end

$$\mathrm{div}\ \mathbf{A} = 0, \text{ in } D_{\mathrm{E}}, \quad \int_{\Gamma_{\mathrm{E},i}} \mathbf{A}\cdot\mathbf{n}\ \mathrm{d}\mathbf{x} = 0, \tag{2.21}$$

with exterior unit normal $\mathbf{n}$, is imposed for all connected components $\Gamma_{\mathrm{E},i} \subset \partial D_{\mathrm{E}}$. Equation (2.21) is an extension of the Coulomb gauge to multiply connected domains, see [15, 16] and also [17] for a rigorous treatment of vector potentials in a more general context. In D_{C}, $\mathbf{A}$ will automatically satisfy these conditions. This can be seen by taking the divergence of (2.19), provided that additionally both the current excitation and the initial condition for the vector potential are divergence free. To close the setting, the interface conditions at Γ_{I} have to be specified. The patch condition [18] implies that, in order for $\mathbf{A}$ and $\mathbf{H}$ to give rise to a valid integral 1-form, for all $S \subset D$,

$$\mathbf{n} \times (\mathbf{A}^+ - \mathbf{A}^-) = 0, \tag{2.22a}$$

$$\mathbf{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{n} \times ((\nu\,\mathbf{curl}\ \mathbf{A})^+ - (\nu\,\mathbf{curl}\ \mathbf{A})^-) = 0. \tag{2.22b}$$

Here, we denote with $\mathbf{H}^+$ and $\mathbf{H}^-$, the restriction of $\mathbf{H}$ to S, from the exterior and interior (with respect to $\mathbf{n}$), respectively. In this setting we exclude the presence of surface currents, that can be incorporated in a more general, distributional setting [19]. As interface conditions, such as (2.22), play an important role in this work we also introduce the operator

$$(\mathbf{A})_S := \mathbf{n} \times (\mathbf{A}^+ - \mathbf{A}^-). \tag{2.23}$$

Let $I_T = (0, T]$ denote the time interval of interest. In summary, we want to determine the magnetic vector potential $\mathbf{A}(t, \mathbf{x})$ subject to

$$\sigma(\cdot)\frac{\partial\mathbf{A}}{\partial t} + \mathbf{curl}\ (\nu(\cdot, |\,\mathbf{curl}\ \mathbf{A}|)\,\mathbf{curl}\ \mathbf{A}) = \mathbf{J}_{\mathrm{src}}, \qquad \text{in } I_T \times (D_{\mathrm{C}} \cup D_{\mathrm{E}}), \tag{2.24a}$$

$$\mathbf{n} \times (\mathbf{A}_\mathrm{C} - \mathbf{A}_\mathrm{E}) = 0, \qquad \text{on } I_T \times \Gamma_\mathrm{I}, \tag{2.24b}$$

$$\mathbf{n} \times (\nu_\mathrm{C}(|\,\mathbf{curl}\ \mathbf{A}_\mathrm{C}|)\,\mathbf{curl}\ \mathbf{A}_\mathrm{C} - \nu_0\,\mathbf{curl}\ \mathbf{A}_\mathrm{E}) = 0, \qquad \text{on } I_T \times \Gamma_\mathrm{I}, \tag{2.24c}$$

$$\mathbf{A} \times \mathbf{n} = 0, \qquad \text{on } I_T \times \Gamma_\mathrm{D}, \tag{2.24d}$$

$$\mathbf{A}(0) = \mathbf{A}_\mathrm{init}, \qquad \text{on } \{0\} \times D, \tag{2.24e}$$

$$\operatorname{div} \mathbf{A} = 0, \qquad \text{in } I_T \times D_\mathrm{E}, \tag{2.24f}$$

$$\int_{\Gamma_{\mathrm{E},i}} \mathbf{A} \cdot \mathbf{n}\ \mathrm{d}\mathbf{x} = 0, \qquad \text{on } I_T, \tag{2.24g}$$

where $\mathbf{A}_E = \mathbf{A}|_{D_\mathrm{E}}$ and $\mathbf{A}_C = \mathbf{A}|_{D_\mathrm{C}}$ and Γ_I is oriented such that $\mathbf{n}$ is the exterior unit normal with respect to the domain D_C. To simplify notation the spatial dependency is omitted when possible, i.e., $\mathbf{A}(t) := \mathbf{A}(t, \mathbf{x})$. We observe that equation (2.24) is an initial boundary value problem of parabolic-elliptic type, parabolic in D_C and elliptic in D_E. Let us note that gauging is rather simple in our case, due the fact that σ_C is constant. For the general case of varying σ_C we refer, e.g., to [20] and for a more detailed discussion to [21]. Also, a similar formulation holds true for the electric field. From now on, for simplicity, the electric current density $\mathbf{J}$ always refers to the imposed part, i.e., we set $\mathbf{J} = \mathbf{J}_\mathrm{src}$.

2.4 Uncertainty Quantification, Verification and Validation

The aim of this section is to give a brief introduction to uncertainty quantification as well as verification and validation and to specify the required terminology. The exposition will be based mainly on work from mechanical engineering [22–25] and the monograph [26], that we consider to be the most appropriate for our purposes.

- Following [22], a (mathematical) *model* is defined as a "collection of mathematical constructions that provide abstractions of a physical event consistent with a scientific theory proposed to cover that event". In the context of this work, this refers to the magnetoquasistatic model, i.e., the system of nonlinear partial differential equations given by (2.24). Model input data is identified with "data from the description of the surroundings", see [23]. In general this refers to initial and boundary conditions, shapes and material constitutive laws. A *model output* in the general case is given by the solution of the underlying system of differential equations. However, in many practical situation not the field itself is used within the design process but some derived *quantities of interest*. Mathematically, these are given as functionals, i.e., maps from the solution to the real numbers. Important examples, are the magnetic energy, the inductance, power losses and most notably in our case multipole coefficients, i.e., Fourier coefficients of the magnetic field.
- Despite the fact that the magnetoquasistatic model may provide a very accurate mathematical description of reality, it is always a simplification of the real underlying physics. Assessing this discrepancy is commonly denoted as *validation* [22, 24, 26]. This comprises both the comparison with measurements and the quantitative

estimation of modeling errors by means of a posteriori error estimation, as outlined, e.g., in [27]. We also refer to [28] for the important case of a posteriori error estimation of the linearization error. Also the arguments given in Sect. 2.2 to justify the magnetoquasistatic approximation can be assigned to the process of validation.

- A set of partial differential equations posed on complicated domains, cannot be resolved directly in general and approximations have to be introduced. By means of numerical approximations, a *computational model*, i.e., a linear system of equations solved by a computer, is derived [22]. Several different types of approximation errors occur at this stage. These are discretization errors, round-off errors as well as errors from the numerical resolution of the linear system of equations. Considerable progress has been made in controlling most of them by means of a posteriori error analysis. We refer to [29] and the references therein for an overview of discretization error estimation in a finite element context. This error contribution, as well as the linearization error, will be of central importance in this work. Note that several errors such as coding errors, might not even be known. The general process of evaluating whether an implemented computational model can be used to accurately represent the mathematical model is referred to as *verification*.
- The setting introduced so far is completely deterministic, in the sense that the input data is considered to be known exactly and to each input is associated a solution by the computational model. This view has several shortcomings with respect to depicting real life devices and machines. Indeed, in practice uncertainties arise and should be incorporated in several different parts of the model. Every single part of a device, as produced from chain production, has a different material composition and shape. Furthermore, it is often unknown whether the chosen form of the model is appropriate to describe the underlying physics. Consequently, model inputs as well as the model form exhibit uncertainties and any reliable design demands for their quantification. In the literature two types of uncertainties are widely acknowledged: *aleatory* and *epistemic* uncertainty [23]. Aleatory uncertainty is defined as "the inherent variation associated with the physical system or the environment under consideration", see [26]. It originates usually from manufacturing imperfections and is considered irreducible for the system under consideration [23]. Epistemic uncertainty is defined as "any lack of knowledge of information in any phase or activity of the modeling process", [26]. In contrast to aleatory uncertainty, epistemic uncertainty might be considered reducible as, e.g., by measurements the belief in a specific model might be increased. Mathematically, aleatory uncertainty is described in the most general form by a probability density function, whereas non-probabilistic quantities subject to epistemic uncertainty, will belong to an admissible set with equal probability of occurrence for each element. Note that in practice distinguishing between both types may be difficult, sometimes rather subjective.
- If a mathematical description of the input uncertainties is at hand, another important step of uncertainty quantification consists in propagating them through the model. Thereby, even more errors, consisting of both modeling and numerical errors occur

as, e.g., solving stochastic equations might quickly become very costly and simplifications are required. As soon as the output uncertainties are quantified, strong sensitivities may be taken into account to increase the robustness of the design.

Remark 2.1 There is an ambiguity in the literature, whether numerical error should be considered as epistemic uncertainty. In [25, 26], numerical error is considered to be a "recognizable deficiency in any phase or activity of modeling that is not due to the lack of knowledge", whereas in [23] it is argued that for complex systems this might be not practicable and error cannot always be identified and reduced. We adapt the latter view, as it might be useful to compare uncertainties and errors in the context of error balancing.

Another important aspect is that uncertainty quantification is not feasible for any kind of mathematical model. Following [26] we identify as minimal requirements for the mathematical model, the existence of a unique solution as well as a continuous dependence of the solution, or output, on the input data. Models that feature these properties are denoted *well-posed* after Hadamard. Indeed, especially the continuity can be seen as essential for the purpose of uncertainty quantification, as it assures that small changes in the input produce small changes in the output. Further, highly desirable, model features are the existence of a numerical solution as well as the differentiability of the model outputs with respect to the model inputs [26]. In the latter case, sensitivity analysis can be used to efficiently propagate uncertainties.

2.5 Conclusion

So far, the magnetoquasistatic approximation to Maxwell's equations was derived and justified. Key assumptions on data, such as reluctivity, conductivity and geometry were specified. Finally, notions from verification and validation, in particular concerning uncertainty quantification, were introduced and discussed.

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