Chapter 2
Irreducible Numerical Semigroups

A numerical semigroup S is *irreducible* if it cannot be expressed as the intersection of two proper oversemigroups. The motivation of the study of these semigroups was initially to express any numerical semigroup as a finite intersection of irreducible numerical semigroups, and then derive properties of the original semigroup in terms of the irreducibles that appear in this decomposition. Historically this was not the reason to study these semigroups. It turns out that irreducible numerical semigroups are either symmetric (when their Frobenius number is odd) or pseudo-symmetric (even Frobenius number); and every symmetric or pseudo-symmetric numerical semigroup is irreducible. Kunz in [40] proved that $K[\![S]\!]$ is a Gorenstein ring if and only if S is symmetric. Consequently obtaining examples of symmetric numerical semigroups would yield examples of one dimensional Gorenstein rings. This motivated a series of tools and machinery to produce families of symmetric numerical semigroups. The name symmetric comes from the symmetry in the set of nonnegative integers less than the Frobenius number of the semigroup: there are as many gaps as elements in this interval. The closest we can get to this symmetry when the Frobenius number is even, taking into account that its half is forced to be a gap, is precisely when the semigroup is pseudo-symmetric.

Semigroups appearing in Chap. 3 are symmetric, but this is not the only reason to study them. Symmetric numerical semigroups have attracted the attention of many algebraists due to their connections to curves and their coordinate rings. This resulted in the development of a new theory and machinery for calculating examples and properties of algebraic curves.

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2.1 Characterizations of Irreducible Numerical Semigroups

Fröberg, Gottlieb and Häggkvist proved in [29] that maximal (with respect to set inclusion) numerical semigroups in the set of numerical semigroups with fixed Frobenius number correspond to symmetric (if this Frobenius number is odd) or pseudo-symmetric (Frobenius number even) numerical semigroups. We are going to see this from the unified point of view of irreducible numerical semigroups, which gather both families.

The following lemma is just a particular case of Lemma 6, taking $T = \mathbb{N}$; this fact was already pointed out in Example 11.

Lemma 7 *Let S be a numerical semigroup other than $\mathbb{N}$. Then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup.*

Example 13 The above construction allows to construct a path connecting the semigroup S and $\mathbb{N}$ in the graph of all oversemigroups of S. In Example 12, this path is $\langle 3, 7, 11\rangle$, $\langle 3, 7, 8\rangle$, $\langle 3, 5, 7\rangle$, $\langle 3, 4, 5\rangle$, $\langle 2, 3\rangle$, $\mathbb{N}$.

Theorem 1 *Let S be a numerical semigroup. The following are equivalent.*

(i) *S is irreducible.*
(ii) *S is maximal (with respect to set inclusion) in the set of numerical semigroups T such that* $\mathrm{F}(S) = \mathrm{F}(T)$.
(iii) *S is maximal (with respect to set inclusion) in the set of numerical semigroups T such that* $\mathrm{F}(S) \notin T$.

Proof *(i) implies (ii)* Let T be a numerical semigroup such that $\mathrm{F}(S) = \mathrm{F}(T)$. If $S \subseteq T$, then $S = T \cap (S \cup \{\mathrm{F}(S)\})$. Since $S \neq S \cup \{\mathrm{F}(S)\}$, we deduce $S = T$.

(ii) implies (iii) Let T be a numerical semigroup such that $\mathrm{F}(S) \notin T$ and assume that $S \subseteq T$. The set $T_1 = T \cup \{\mathrm{F}(S) + 1, \mathrm{F}(S) + 2, \ldots\}$ is a numerical semigroup with $\mathrm{F}(T_1) = \mathrm{F}(S)$. But $S \subseteq T_1$, whence $S = T_1$. Since $\mathrm{F}(S) + k \in S$ for all $k \geq 1$, it follows that $S = T$.

(iii) implies (i) Let S_1, S_2 be two numerical semigroups such that $S \subseteq S_1$, $S \subseteq S_2$, and $S = S_1 \cap S_2$. Since $\mathrm{F}(S) \notin S$, $\mathrm{F}(S) \notin S_i$ for some $i \in \{1, 2\}$. By (iii), $S_i = S$.

Example 14 In Fig. 2.1 we have drawn the Hasse diagram (ordered with respect to set inclusion) of the set of numerical semigroups with Frobenius number seven. We see that we have three maximal elements: $\langle 3, 5\rangle$, $\langle 2, 9\rangle$, and $\langle 4, 5, 6\rangle$. Thus these are the only irreducible numerical semigroups with Frobenius number seven.

Let S be a numerical semigroup. We say that S is *symmetric* if

(i) S is irreducible and
(ii) $\mathrm{F}(S)$ is odd.

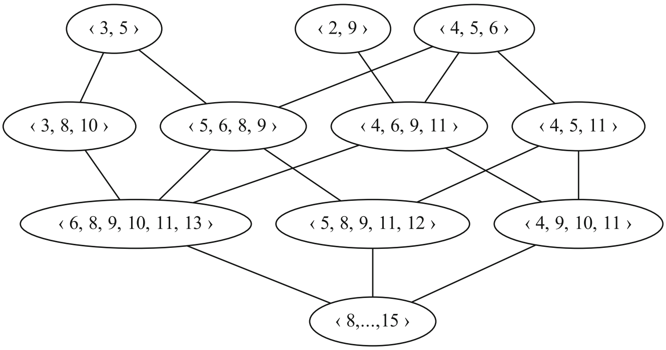

Fig. 2.1 Numerical semigroups with Frobenius number seven

We say that S is *pseudo-symmetric* if

(i) S is irreducible and
(ii) $\mathrm{F}(S)$ is even.

Next we collect some classical characterizations of symmetric and pseudo-symmetric numerical semigroups. We first prove the following. Sometimes the set H appearing in the next proposition is known as the set of holes of the semigroup.

Proposition 13 *Let S be a numerical semigroup and suppose that*

$$H = \left\{x \in \mathbb{Z} \setminus S \;\middle|\; \mathrm{F}(S) - x \notin S, x \neq \tfrac{\mathrm{F}(S)}{2}\right\}$$

is not empty. If $h = \max H$, then $S \cup \{h\}$ is a numerical semigroup.

Proof Since $S \subseteq S \cup \{h\}$, the set $\mathbb{N} \setminus (S \cup \{h\})$ has finitely many elements. Let $a, b \in S \cup \{h\}$.

- If $a, b \in S$, then $a + b \in S$.
- Let $a \in S$ and $b = h$. If $a = 0$, then $a+h = h \in S\cup\{h\}$. So assume that $a \neq 0$ and $a+h \notin S$. By the maximality of h, we deduce $\mathrm{F}(S)-a-h = \mathrm{F}(S)-(a+h) \in S$. Hence $\mathrm{F}(s) - h = a + \mathrm{F}(S) - a - h \in S$. This contradicts the definition of h.
- Finally assume that $a = b = h$. If $2h \notin S$, then the maximality of h implies that $\mathrm{F}(S) - 2h = s \in S^*$. This implies that $\mathrm{F}(S) - h = h + s$, which by the preceding paragraph is in S, contradicting the definition of h.

GAP example 7 In light of Proposition 13 and Lemma 5, if for a numerical semigroup, there exists a maximum of $\{x \in \mathbb{Z} \setminus (S \cup \{\mathrm{F}(S)/2\}) \mid \mathrm{F}(S) - x \notin S\}$, then it is a special gap.

```
gap> s:=NumericalSemigroup(7,9,11,17);
<Numerical semigroup with 4 generators>
```

```
gap> g:=GapsOfNumericalSemigroup(s);
[ 1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 15, 19 ]
gap> Filtered(g, x-> (x<>19/2) and not(19-x in s));
[ 4, 6, 13, 15 ]
gap> SpecialGapsOfNumericalSemigroup(s);
[ 13, 15, 19 ]
```

We have introduced the concepts of symmetric and pseudo-symmetric as subclasses of the set of irreducible numerical semigroups. However as we said at the beginning of this chapter, these two concepts existed before that of irreducible numerical semigroup, and thus the definitions were different than the ones we have given above. Needless to say that as in the case of irreducible numerical semigroups, there are many different characterizations of these properties. In the literature, sometimes these are chosen to be the definition of symmetric and pseudo-symmetric numerical semigroups.

Proposition 14 *Let S be a numerical semigroup.*

(i) *S is symmetric if and only if for all $x \in \mathbb{Z} \setminus S$, we have $\mathrm{F}(S) - x \in S$.*
(ii) *S is pseudo-symmetric if and only if $\mathrm{F}(S)$ is even and for all $x \in \mathbb{Z} \setminus S$, either $\mathrm{F}(S) - x \in S$ or $x = \frac{\mathrm{F}(S)}{2}$.*

Proof (i) Assume that S is symmetric. Then $\mathrm{F}(S)$ is odd, and thus $H = \{x \in \mathbb{Z} \setminus S \mid \mathrm{F}(S) - x \notin S\} = \{x \in \mathbb{Z} \setminus S \mid \mathrm{F}(S) - x \notin S, x \neq \mathrm{F}(S)/2\}$. If H is not the emptyset, then $T = S \cup \{h\}$, with $h = \max H$, is a numerical semigroup with Frobenius number $\mathrm{F}(S)$ containing properly S, which is impossible in light of Theorem 1. For the converse note that $\mathrm{F}(S)$ cannot be even, since otherwise as $\mathrm{F}(S)/2 \notin S$, we would have $\mathrm{F}(S) - \mathrm{F}(S)/2 = \mathrm{F}(S)/2 \in S$; a contradiction. So, we only need to prove that S is irreducible. Let to this end T be a numerical semigroup such that $\mathrm{F}(S) \notin T$ and suppose that $S \subset T$. Let $x \in T \setminus S$. By hypothesis $\mathrm{F}(S) - x \in S$. This implies that $\mathrm{F}(S) = (\mathrm{F}(S) - x) + x \in T$. This is a contradiction (we are using here Theorem 1 once more).
(ii) The proof is the same as the proof of (i).

The maximality of irreducible numerical semigroups in the set of numerical semigroups with the same Frobenius number, translates to minimality in terms of gaps. This is highlighted in the next result. Observe that for any numerical semigroup S, if $x \in S$, then $\mathrm{F}(S) - x \notin S$. In particular, $\mathrm{n}(S) = \#(S \cap [0, \mathrm{F}(S)]) \geq \mathrm{g}(S)$. As $\mathrm{F}(S) + 1 = \mathrm{n}(S) + \mathrm{g}(S)$, we deduce that $\mathrm{g}(S) \geq (\mathrm{F}(S) + 1)/2$.

Corollary 6 *Let S be a numerical semigroup.*

(i) *S is symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.*
(ii) *S is pseudo-symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.*

Hence irreducible numerical semigroups are those with the least possible genus.

Recall that the Frobenius number and genus for every embedding dimension two numerical semigroup are known; as a consequence, we get the following.

Corollary 7 *Let S be a numerical semigroup. If* $\mathrm{e}(S) = 2$*, then S is symmetric.*

The rest of the section is devoted to characterizations in terms of the Apéry sets (confirming in this way their ubiquity). First we show that Apéry sets are closed under summands.

Lemma 8 *Let S be a numerical semigroup and let $n \in S^*$. If $x, y \in S$ and $x + y \in$* $\mathrm{Ap}(S, n)$*, then $x, y \in$* $\mathrm{Ap}(S, n)$.

Proof Assume to the contrary, and without loss of generality, that $y - n \in S$. Then $x + y - n \in S$, and consequently $x + y \notin \mathrm{Ap}(S, n)$.

This in particular means that $\mathrm{Ap}(S, n)$ is fully determined by the set of maximal elements in $\mathrm{Ap}(S, n)$ with respect to $\leq_S$.

Proposition 15 (Apéry) *Let S be a numerical semigroup and let $n \in S^*$. Let* $\mathrm{Ap}(S, n) = \{0 = a_0 < a_1 < \cdots < a_{n-1}\}$*. Then S is symmetric if and only if $a_i + a_{n-1-i} = a_{n-1}$ for all $i \in \{0, \ldots, n-1\}$.*

Proof Suppose that S is symmetric. From Proposition 5, we know that $\mathrm{F}(S) = a_{n-1} - n$. Let $0 \leq i \leq n-1$. Since $a_i - n \notin S$, we get $\mathrm{F}(S) - a_i + n = a_{n-1} - a_i \in S$. Let $s \in S$ be such that $a_{n-1} - a_i = s$. Since $a_{n-1} = a_i + s \in \mathrm{Ap}(S, n)$, by Lemma 8, $s \in \mathrm{Ap}(S, n)$. Hence $s = a_j$ for some $0 \leq j \leq n-1$. As this is true for any i, we deduce that $j = n - 1 - i$.

Conversely, the hypothesis implies that $\mathrm{Maximals}_{\leq_S}\mathrm{Ap}(S, n) = a_{n-1}$. Hence $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$ (Proposition 8). Also, by Proposition 7, $\{\mathrm{F}(S)\} = \mathrm{Maximals}_{\leq_S}(\mathbb{N} \setminus S)$. If $x \notin S$, then $x \leq_S \mathrm{F}(S)$, whence $\mathrm{F}(S) - x \in S$. To prove that $\mathrm{F}(S)$ is odd, just use the same argument of the proof of Proposition 14.

As a consequence of the many invariants that can be computed using Apéry sets, we get the following characterizations of the symmetric property.

Corollary 8 *Let S be a numerical semigroup. The following conditions are equivalent.*

(i) *S is symmetric.*
(ii) $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$.
(iii) *If $n \in S$, then* $\mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, n)) = \{\mathrm{F}(S) + n\}$.
(iv) $\mathrm{t}(S) = 1$.

Example 15 Let $S = \langle 5, 6, 9 \rangle$. Figure 2.2 represents the Hasse diagram of $\mathrm{Ap}(S, 5)$ (with respect to $\leq_S$). One can see the symmetry in the shape forced by Proposition 15, and how the properties in Corollary 8 hold for this semigroup.

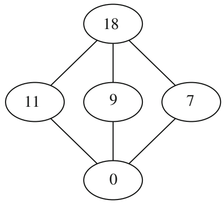

Fig. 2.2 Hasse diagram of $\mathrm{Ap}(\langle 5, 6, 9\rangle, 5)$

Example 16 Recall that for a and b integers greater than one with $\gcd(a, b) = 1$, $\mathrm{Ap}(\langle a, b\rangle, a) = \{0, b, \ldots, (a-1)b\}$. By Corollary 8, this implies that $\langle a, b\rangle$ is symmetric, recovering again Corollary 7.

Now, we are going to obtain the analogue for pseudo-symmetric numerical semigroups. The first step is to deal with one half of the Frobenius number.

Lemma 9 *Let S be a numerical semigroup and let $n \in S^*$. If S is pseudo-symmetric, then $\frac{\mathrm{F}(S)}{2} + n \in \mathrm{Ap}(S, n)$.*

Proof Clearly $\frac{\mathrm{F}(S)}{2} \notin S$. If $\frac{\mathrm{F}(S)}{2} + n \notin S$, then $\mathrm{F}(S) - \frac{\mathrm{F}(S)}{2} - n \in S$. This implies that $\frac{\mathrm{F}(S)}{2} - n \in S$ and thus $\frac{\mathrm{F}(S)}{2} \in S$, which is a contradiction.

Proposition 16 *Let S be a numerical semigroup and let $n \in S^*$. Let $\mathrm{Ap}(S, n) = \{0 = a_0 < a_1, \cdots < a_{n-2}\} \cup \left\{\frac{\mathrm{F}(S)}{2} + n\right\}$. Then S is pseudo-symmetric if and only if $a_i + a_{n-2-i} = a_{n-2}$ for all $i \in \{0, \ldots, n-2\}$.*

Proof Suppose that S is pseudo-symmetric and let $w \in \mathrm{Ap}(S, n)$. If $w \neq \frac{\mathrm{F}(S)}{2} + n$, then $w - n \notin S$ and $w - n \neq \frac{\mathrm{F}(S)}{2}$. Hence $\mathrm{F}(S) - (w - n) = \mathrm{F}(S) + n - w = \max \mathrm{Ap}(S, n) - w \in S$. Since $\mathrm{F}(S) - w \notin S$, then $\mathrm{F}(S) + n - w = \max \mathrm{Ap}(S, n) - w \in \mathrm{Ap}(S, n)$. But $\max(S, n) - w \neq \frac{\mathrm{F}(S)}{2} + n$ (otherwise $w = \frac{\mathrm{F}(S)}{2}$, a contradiction). Now we use the same argument as in the symmetric case (Proposition 15).

Conversely, let $x \neq \frac{\mathrm{F}(S)}{2}$, $x \notin S$. Take $w \in \mathrm{Ap}(S, n)$ such that $w \equiv x \pmod{n}$. There exists $k \in \mathbb{N}^*$ such that $x = w - kn$ (compare with Proposition 4).

1. If $w = \frac{\mathrm{F}(S)}{2} + n$, then $\mathrm{F}(S) - x = \frac{\mathrm{F}(S)}{2} + (k-1)n$. But $x \neq \frac{\mathrm{F}(S)}{2}$. Hence $k \geq 2$, and consequently $\mathrm{F}(S) - x = w + (k-2)n \in S$.
2. If $w \neq \frac{\mathrm{F}(S)}{2} + n$, then $\mathrm{F}(S) - x = \mathrm{F}(S) + n - w + (k-1)n = a_{n-2} - w + (k-1)n \in S$, because $a_{n-2} - w \in S$.

Again, by using the properties of the Apéry sets, we get several characterizations for pseudo-symmetric numerical semigroups.

Corollary 9 *Let S be a numerical semigroup. The following conditions are equivalent.*

(i) *S is pseudo-symmetric.*
(ii) $\mathrm{PF}(S) = \left\{\mathrm{F}(S), \frac{\mathrm{F}(S)}{2}\right\}$.

(iii) If $n \in S$, *then* $\mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S,n)) = \left\{\frac{\mathrm{F}(S)}{2} + n, \mathrm{F}(S) + n\right\}$.

Example 17 Let S be a numerical semigroup. If S is pseudo-symmetric, then $\mathrm{t}(S) = 2$. The converse is not true in general. Take for instance $S = \langle 5, 6, 8\rangle$ from Example 8. We have $\mathrm{Ap}(S, 5) = \{0, 6, 12, 8, 14\}$, $\mathrm{PF}(S) = \{7, 9\}$, and $\mathrm{t}(S) = 2$. However S is not pseudo-symmetric. The Hasse diagram depicting $\mathrm{Ap}(S, 5)$ is in Fig. 1.1.

GAP example 8 Let us see how many numerical semigroups with Frobenius number 16 and type 2 are not pseudo-symmetric.

```
gap> l:=NumericalSemigroupsWithFrobeniusNumber(16);;
gap> Length(l);
205
gap> Filtered(l, s->TypeOfNumericalSemigroup(s)=2);
[ <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup> ]
gap> Filtered(last,IsPseudoSymmetricNumericalSemigroup);
[ <Numerical semigroup>, <Numerical semigroup>,
 <Numerical semigroup>,  <Numerical semigroup>,
  <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup> ]
gap> Difference(last2,last);
[ <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 3 generators>,
  <Numerical semigroup with 4 generators>,
  <Numerical semigroup with 5 generators> ]
gap> List(last, MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 3, 14, 19 ], [ 3, 17, 19 ], [ 5, 7, 18 ], [ 5, 9, 12 ],
  [ 6, 7, 11 ], [ 6, 9, 11, 13 ], [ 7, 10, 11, 12, 13 ] ]
```

We have seen in this section that if S is irreducible, then its type is either one (symmetric case) or two (pseudo-symmetric case). Hence $\mathrm{t}(S) + 1 \leq \mathrm{e}(S)$ (we do not have pseudo-symmetric numerical semigroups with embedding dimension two, since they are all symmetric). This, together with Corollary 5 proves that Wilf's conjecture holds for irreducible numerical semigroups (as mentioned in the last chapter).

2.2 Decomposition of a Numerical Semigroup into Irreducible Semigroups

Recall that a numerical semigroup S is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. We show in this section that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups.

Theorem 2 *Let S be a numerical semigroup. There exists a finite set of irreducible numerical semigroups $\{S_1, \dots, S_r\}$ such that $S = S_1 \cap \cdots \cap S_r$.*

Proof If S is not irreducible, then there exist two numerical semigroups S^1 and S^2 such $S = S^1 \cap S^2$ and $S \subset S^1$ and $S \subset S^2$. If S^1 is not irreducible, then we restart with S^1, and so on. We construct this way a sequence of oversemigroups of S. This process will stop, because $\mathcal{O}(S)$ has finitely many elements.

Example 18 Figure 2.3 represents the Hasse diagram of irreducible oversemigroups of $\langle 5, 7, 9\rangle$ with respect to set inclusion (we have included $\langle 5, 7, 9\rangle$ in the diagram). Since the minimal irreducible oversemigroups of $\langle 5, 7, 9\rangle$ are $\langle 5, 7, 9, 11\rangle$ and $\langle 5, 7, 8, 9\rangle$, we have that $\langle 5, 7, 9\rangle = \langle 5, 7, 9, 11\rangle \cap \langle 5, 7, 8, 9\rangle$ is a decomposition of $\langle 5, 7, 9\rangle$ into irreducibles.

The next step is to find a way to compute an "irredundant" decomposition into irreducible numerical semigroups. The key result to accomplish this task is the following proposition.

Proposition 17 *Let S be a numerical semigroup and let $S_1, \dots, S_r \in \mathcal{O}(S)$. The following conditions are equivalent.*

(i) $S = S_1 \cap \cdots \cap S_r$.
(ii) *For all $h \in \mathrm{SG}(S)$, there is $i \in \{1, \dots, r\}$ such that $h \notin S_i$.*

Proof (i) implies (ii) Let $h \in \mathrm{SG}(S)$. Then $h \notin S$, which implies that $h \notin S_i$ for some $i \in \{1, \dots, r\}$.

(ii) implies (i) Suppose that $S \subset S_1 \cap \cdots \cap S_r$, and let $h = \max(S_1 \cap \cdots \cap S_r \setminus S)$. In light of Lemma 6, $h \in \mathrm{SG}(S)$, and for all $i \in \{1, \dots, r\}$, $h \in S_i$, contradicting the hypothesis.

Remark 5 Let $\mathcal{I}(S)$ be the set of irreducible numerical semigroups of $\mathcal{O}(S)$, and let $\mathrm{Min}_{\subseteq}(I(S))$ be the set of minimal elements of $\mathcal{I}(S)$ with respect to set inclusion. Assume that $\mathrm{Min}_{\leq_{\subseteq}}(I(S)) = \{S_1, \dots, S_r\}$. Define $\mathrm{C}(S_i) = \{h \in \mathrm{SG}(S) : h \notin S_i\}$. We have $S = S_1 \cap \cdots \cap S_r$ if and only if $\mathrm{SG}(S) = \mathrm{C}(S_1) \cup \cdots \cup \mathrm{C}(S_r)$. This gives a procedure to compute a (nonredundant) decomposition of S into irreducibles. This decomposition might not be unique, and not all might have the same number of irreducibles involved.

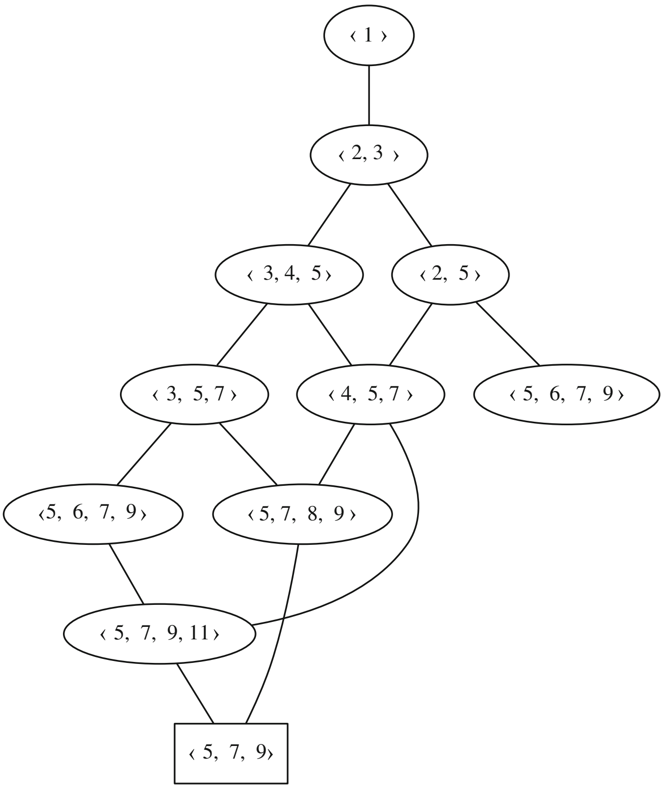

Fig. 2.3 The Hasse diagram of the irreducible oversemigroups of $\langle 5, 7, 9\rangle$

GAP example 9 Let us decompose the semigroup $S = \langle 7, 9, 11, 17\rangle$ into irreducibles.

```
gap> s:=NumericalSemigroup(7,9,11,17);;
gap> DecomposeIntoIrreducibles(s);
[ <Numerical semigroup>, <Numerical semigroup>,
  <Numerical semigroup> ]
gap> List(last,MinimalGeneratingSystemOfNumericalSemigroup);
[ [ 7, 8, 9, 10, 11, 12 ], [ 7, 9, 10, 11, 12, 13 ],
  [ 7, 9, 11, 13, 15, 17 ] ]
```

There are exactly 17 irreducible oversemigroups of S.

```
gap> Length(Filtered(OverSemigroupsNumericalSemigroup(s),
> IsIrreducibleNumericalSemigroup));
17
```

There are some (inefficient) bounds for the number of irreducible numerical semigroups appearing in a minimal decomposition of a numerical semigroup into irreducibles. Actually, a numerical semigroup might have different minimal decompositions (in the sense that they cannot be refined to other decompositions) with different cardinalities. So it is an open problem to know the minimal cardinality among all possible minimal decompositions.

Also it is not clear how this decomposition translates to $\mathbb{K}[\![S]\!]$, or if the study of the curve associated to S can benefit from this decomposition.

2.3 Free Numerical Semigroups

We present in this section a way to construct easily symmetric numerical semigroups. This idea was originally exploited by Bertin, Carbonne, and Watanabe among others (see [11, 23, 59]) and goes back to the 70's. As mentioned above this is a way to produce one dimensional local Gorenstein rings. The semigroups appearing in Chap. 3 are of this form, and this is why we pay special attention to them.

Let S be a numerical semigroup and let $\{r_0, \ldots, r_h\}$ be its minimal set of generators. Let $d_1 = r_0$ and for all $k \in \{2, \ldots, h+1\}$, set $d_k = \gcd(d_{k-1}, r_{k-1})$. Define $e_k = \frac{d_k}{d_{k+1}}$, for $k \in \{1, \ldots, h\}$.

We say that S is *free* for the arrangement $(r_0, \ldots, r_h)$ if for all $k \in \{1, \ldots, h\}$:

(i) $e_k > 1$.
(ii) $e_k r_k$ belongs to the semigroup generated by $\{r_0, \ldots, r_{k-1}\}$.

We say that S is *telescopic* if $r_0 < r_1 < \cdots < r_h$ and S is free for the arrangement $(r_0, \ldots, r_h)$.

Example 19 Let $S = \langle 4, 6, 9\rangle$. Then $h = 2$, and $d_1 = 4$, $d_2 = 2$ and $d_3 = 1$. Hence, $e_1 = 2 = e_2$. As $2 \times 6 \in \langle 4\rangle$ and $2 \times 9 \in \langle 4, 6\rangle$, we have that S is telescopic.

Free numerical semigroups were named in this way by Bertin and Carbonne [11]; the name telescopic was used by Kirfel and Pellikaan in [39]. The motivation of Bertin and Carbonne was finding families of complete intersection numerical semigroups (that are always symmetric); while Kirfel and Pellikaan interest was determining numerical semigroups with associated algebraic–geometric codes with nice properties.

There is an alternative way of introducing free semigroups with the use of gluings (more modern notation), see for instance [53, Chap. 8].

Example 20 The easiest example of free numerical semigroup (apart from $\mathbb{N}$) is $\langle a, b\rangle$ (and as this is a numerical semigroup other than $\mathbb{N}$, a and b coprime integers greater than one).

Example 21 We know that $\langle 4, 6, 9\rangle$ is telescopic. Let us see how to construct another telescopic numerical semigroup from this one. Take any positive integer, for this example, we choose 2, and multiply the sequence (4, 6, 9) by 2, obtaining (8, 12, 18). In this way the old d_i's are multiplied by 2 as well. Now we need another positive integer coprime with 2 so that the whole sequence has greatest common divisor one. Also 2 times this integer must be in $\langle 8, 12, 18\rangle$, or equivalently, our integer must be in $\langle 4, 6, 9\rangle$. Since we are looking for a telescopic numerical semigroup, we need an integer greater than 18. We can for instance choose 19, which is in $\langle 4, 6, 9\rangle$ and it is coprime with 2. Thus $\langle 8, 12, 18, 19\rangle$ is a telescopic numerical semigroup. We can repeat this process as many times as desired.

Any telescopic numerical semigroup can be obtained in this way.

One of the advantages of dealing with free numerical semigroups is that every integer admits a unique representation in terms of its minimal generators if we impose some bounds on the coefficients.

Lemma 10 *Assume that S is free for the arrangement $(r_0, \ldots, r_h)$, and let $x \in \mathbb{Z}$. There exist unique $\lambda_0, \ldots, \lambda_h \in \mathbb{Z}$ such that the following holds*

(i) $x = \sum_{k=0}^{h} \lambda_k r_k$.
(ii) *For all $h \in \{1, \ldots, h\}$, $0 \le \lambda_k < e_k$.*

Proof *Existence* The group generated by S is $\mathbb{Z}$, and so there exist $\alpha_0, \ldots, \alpha_h \in \mathbb{Z}$ such that $x = \sum_{k=0}^{h} \alpha_k r_k$. Write $\alpha_h = q_h e_h + \lambda_h$, with $0 \le \lambda_h < e_h$. Now we use that $e_h r_h = \sum_{i=0}^{h-1} \beta_i r_i$, with $\beta_i \in \mathbb{N}$ for all $i \in \{1, \ldots, h-1\}$. Hence

$$x = \sum_{k=0}^{h-1} (\lambda_k + q_h \beta_k) r_k + \lambda_h r_h,$$

and $0 \le \lambda_h < e_h$. Now the result follows by an easy induction on h.

Uniqueness. Let $x = \sum_{k=0}^{h} \alpha_k r_k = \sum_{k=0}^{h} \beta_k r_k$ be two distinct such representations, and let $j \ge 1$ be the greatest integer such that $\alpha_j \ne \beta_j$. We have

$$(\alpha_j - \beta_j) r_j = \sum_{k=0}^{j-1} (\beta_k - \alpha_k) r_k.$$

In particular, d_j divides $(\alpha_j - \beta_j) r_j$. But $\gcd(d_j, r_j) = d_{j+1}$, whence $\frac{d_j}{d_{j+1}}$ divides $(\alpha_j - \beta_j)\frac{r_j}{d_{j+1}}$. As $\gcd(d_j/d_{j+1}, r_j/d_{j+1}) = 1$, this implies that $\frac{d_j}{d_{j+1}}$ divides $\alpha_j - \beta_j$. However $|\alpha_j - \beta_j| < e_j = \frac{d_j}{d_{j+1}}$, yielding a contradiction.

An expression of x like in the preceding lemma is called a *standard representation*.

Example 22 Let $S = \langle 4, 6, 9\rangle$, and let us consider the integer 30. There are several ways to represent 30 as a linear combination of $\{4, 6, 9\}$ with nonnegative integer coefficients:

```
gap> FactorizationsIntegerWRTList(30,[4,6,9]);
[ [ 6, 1, 0 ], [ 3, 3, 0 ], [ 0, 5, 0 ], [ 3, 0, 2 ],
  [ 0, 2, 2 ] ]
```

The first one in the list corresponds with the standard representation of 30 with respect to S (recall that in this example $e_1 = 2 = e_2$).

As a consequence of this representation we obtain the following characterization for membership to a free numerical semigroup.

Lemma 11 *Suppose that S is free for the arrangement $(r_0, \ldots, r_h)$ and let $x \in \mathbb{N}$. Let $x = \sum_{k=0}^{h} \lambda_k r_k$ be the standard representation of x. We have $x \in S$ if and only if $\lambda_0 \geq 0$.*

Proof If $\lambda_0 \geq 0$ then clearly $x \in S$. Suppose that $x \in S$ and write $x = \sum_{k=0}^{h} \alpha_k r_k$ with $\alpha_0, \ldots, \alpha_h \in \mathbb{N}$. As in Lemma 10, whenever $\alpha_i \geq e_i$, with $i > 0$, we can replace $e_i r_i$ with its expression in terms of $r_0, \ldots, r_{i-1}$. At the end we will have the standard representation of x, and by construction the coefficient of r_0 will be nonnegative (will be greater than or equal to α_0).

We will come back to the rewriting procedure used in the above lemmas in Sect. 4.2.

With all this information, it is easy to describe the Apéry set of the first generator in the arrangement that makes the semigroup free.

Corollary 10 *Suppose that S is free for the arrangement $(r_0, \ldots, r_h)$. Then*

$$\mathrm{Ap}(S, r_0) = \left\{ \sum_{k=1}^{h} \lambda_k r_k \;\middle|\; 0 \leq \lambda_k < e_k \text{ for all } k \in \{1, \ldots, h\} \right\}.$$

Proof Let $x \in S$ and let $x = \sum_{k=0}^{h} \lambda_k r_k$ be the standard representation of x. Clearly $x - r_0 = (\lambda_0 - 1)r_0 + \sum_{k=1}^{h} \lambda_k r_k$ is the standard representation of $x - r_0$. Hence $x - r_0 \notin S$ if and only if $\lambda_0 = 0$. This proves our assertion.

As we have seen in this last result, the shape of the Apéry set of a free numerical semigroup is rectangular. D'Anna, Micale and Sammartano have studied recently a generalization of these semigroups by considering Apéry sets with these shapes [17].

As usual, once we know an Apéry set, we can derive many properties of the semigroup.

Proposition 18 *Let S be free for the arrangement $(r_0, \ldots, r_h)$.*

(i) $\mathrm{F}(S) = \sum_{k=1}^{h} (e_k - 1) r_k - r_0$.
(ii) *S is symmetric.*
(iii) $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.
(iv) $r_0 = \prod_{i=1}^{h} e_i$.

Proof We have $\mathrm{F}(S) = \max \mathrm{Ap}(S, r_0) - r_0$, by Proposition 5. As $\max \mathrm{Ap}(S, r_0) = \sum_{k=1}^{h}(e_k - 1)r_k$, (i) follows easily.

Assertion (ii) is a consequence of Corollary 10 and Proposition 15.

Assertion (iii) is a consequence of (ii) and Corollary 6.

We know from Lemma 1 that $\#\mathrm{Ap}(S, r_0) = r_0$. In light Lemma 10, $\#\mathrm{Ap}(S, r_0) = e_1 \times \cdots \times e_h$. This proves (iv).

Example 23 Let us revisit the semigroup $S = \langle 8, 12, 18, 19\rangle$, which we know it is a telescopic numerical semigroup. In this setting, $(d_1, d_2, d_3, d_4) = (8, 4, 2, 1)$, whence $(e_1, e_2, e_3) = (2, 2, 2)$. By Corollary 10, this means that

$$\mathrm{Ap}(S, 8) = \left\{a \times 12 + b \times 18 + c \times 19 \mid (a, b, c) \in \{0, 1\}^3\right\}.$$

Hence $\mathrm{Ap}(S, 8) = \{0, 12, 18, 19, 30, 31, 37, 49\}$. Also, we have that $\mathrm{F}(S) = 49 - 8 = \sum_{k=1}^{3}(e_i - 1)r_i - r_0 = 12 + 18 + 19 - 8 = 41$, and $\mathrm{g}(S) = (41 + 1)/2 = 21$.

GAP example 10 The proportion of free numerical semigroup compared with symmetric numerical semigroups with fixed Frobenius number (or genus) is small.

```
gap> List([1,3..51], i ->
> [Length(FreeNumericalSemigroupsWithFrobeniusNumber(i)),
> Length(IrreducibleNumericalSemigroupsWithFrobeniusNumber(i))]);
[ [ 1, 1 ], [ 1, 1 ], [ 2, 2 ], [ 3, 3 ], [ 2, 3 ], [ 4, 6 ],
  [ 5, 8 ], [ 3, 7 ], [ 7, 15 ], [ 8, 20 ], [ 5, 18 ],
  [ 11, 36 ], [ 11, 44 ], [ 9, 45 ], [ 14, 83 ], [ 17, 109 ],
  [ 12, 101 ], [ 18, 174 ], [ 24, 246 ], [ 16, 227 ], [ 27, 420 ],
  [ 31, 546 ], [ 21, 498 ], [ 35, 926 ], [ 38, 1182 ], [ 27, 1121 ] ]
```

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