Preface

Early versions of this manuscript were developed for a course on numerical semigroups and their application to the study of planar curves, which was taught at the Lebanese University. Since the first edition, the text has been enriched with more applications that relate numerical semigroups to ongoing research in a number of fields. Nevertheless, the text is intended to be self-contained and should be accessible to beginning graduate students in mathematics.

We have included numerous examples and computational experiments to ensure that the reader develops a solid understanding of the fundamentals before moving forward. In each case it should be possible to check the examples by hand, or by plotting the code into a computer. Some of the more complicated examples can be performed with the aid of the numerical semigroups package in GAP, which is a software tool for mathematical computation available free online.

We begin with the basic notions and terminology relating to numerical semigroups. Next, we focus on the study of irreducible numerical semigroups, and in particular free numerical semigroups, which arise in the study of planar irreducible curves. Afterwards we discuss the computation of minimal presentations and how they are used to calculate nonunique factorization invariants. Factorization and division are closely related, which will become apparent in studying the Feng–Rao distance and its connection to Coding Theory.

Numerical semigroups naturally arose as the set of values of b which have nonnegative integer solutions to Diophantine equations of the form $a_1x_1 + \cdots + a_nx_n = b$, where $a_1,\ldots,a_n,b \in \mathbb{N}$ (here $\mathbb{N}$ denotes the nonnegative integers). We reduce to the case $\gcd(a_1,\ldots,a_n) = 1$. In his lectures, Frobenius asked what is the largest integer b such that a given equation has no solutions over the nonnegative integers. Sylvester and others solved the $n = 2$ case, and since then finding the largest such b has been known as the Frobenius problem. A thorough introduction to the Frobenius problem and related topics is given in [48].

An active area of study where numerical semigroups continue to play a role is within commutative algebra and algebraic geometry. Let $\mathbb{K}$ be a field, and let $\mathbf{A} = \mathbb{K}[t^{a_1},\ldots,t^{a_n}]$ be the $\mathbb{K}$-algebra of polynomials in $t^{a_1},\ldots,t^{a_n}$. The ring $\mathbf{A}$ is the coordinate ring of the curve parametrized by $t^{a_1}, \ldots, t^{a_n}$, and information from **A** can be derived from the properties of the numerical semigroup generated by the exponents $a_1,\ldots,a_n$. As a result, it is often the case that names of invariants in numerical semigroup theory are inherited from Algebraic Geometry. Similarly, Bertin and Carbonne [11], Delorme [23], Watanabe [58], and others have successfully identified properties of numerical semigroups which equate to their associated numerical semigroup ring fitting within various standard classifications in ring theory. In the monograph [10] one can find a dictionary relating much of the overlapping terminology between commutative algebra and numerical semigroup theory.

Numerical semigroups are also useful in the study of singularities over planar algebraic curves. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let $f(x,y)$ be an element of $\mathbb{K}[\![x, y]\!]$. Given another element $g \in \mathbb{K}[\![x, y]\!]$, we define the local intersection multiplicity of f with g to be the rank of the $\mathbb{K}$-vector space $\mathbb{K}[\![x, y]\!]/(f, g)$. When g runs over the set of elements of $\mathbb{K}[\![x, y]\!] \setminus (f)$, these numbers define a semigroup. If in addition f is irreducible, then the semigroup is a numerical semigroup. This leads to a classification of irreducible formal power series in terms of their associated numerical semigroups. This classification can be generalized to polynomials with one place at infinity. With regard to this topic, arithmetic properties of numerical semigroups played an essential role in the proof of the Abhyankar–Moh lemma, which says that a coordinate has a unique embedding in the plane.

Numerical semigroups associated with planar curves are free, and thus irreducible. This is why we spend some time explaining irreducible numerical semigroups and their two big subfamilies: symmetric and pseudo-symmetric numerical semigroups.

Recently, due to use of algebraic codes and Weierstrass numerical semigroups, some applications to coding theory and cryptography have arisen. The idea is to find properties of codes in terms of an associated numerical semigroup; see for instance [21] and the references therein. With this in mind we discuss Feng–Rao distances and their generalization to higher orders.

Another focus of recent interest has been the study of factorizations in monoids. Considering the equation $a_1x_1 + \cdots + a_nx_n = b$, we can think of the set of nonnegative integer solutions as the set of factorizations of b in terms of $a_1,\ldots,a_n$. It can be easily shown that no numerical semigroup other than $\mathbb{N}$ is half-factorial, or, in other words, that there are always elements with factorizations of different lengths. We will discuss some of the invariants which measure how far monoids are from being half-factorial, and how wild the sets of factorizations are. Over the last decade many algorithms for computing such invariants over numerical semigroups have been developed. As a result, studying these invariants over numerical semigroups offers a good chance to obtain families of examples, which can be used to test conjectures.

Two factorizations are expressions of the same element in terms of atoms, and one can go from one factorization to another by using a minimal presentation. Hence, minimal presentations are an important tool in the study of nonunique factorization invariants. We will show how to compute a minimal presentation of a numerical semigroup both by using graphs and combinatorics and through elimination theory.

The graphs used to compute minimal presentations can be generalized to simplicial complexes. Those having nonzero Euler characteristic are important in the expression of the generating function (Hilbert series) of the semigroup as a quotient of two polynomials.

The aim of this book is to give some basic notions related to numerical semigroups, and from these on the one hand to describe a classical application to the study of singularities of plane algebraic curves and on the other to show how numerical semigroups can be used to obtain handy examples of nonunique factorization invariants.

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