Chapter 2
The Weierstrass Factorization Theorem

2.1 Mittag-Leffler's Series

A function $f(z)$ is *meromorphic* in an open set $A \subset \mathbb{C}$ if it is regular in A except for a finite or infinite sequence $z_1, z_2, \ldots \in A$ of poles of $f(z)$ (of any multiplicities). For short, in what follows a function meromorphic in the whole $\mathbb{C}$ will simply be called meromorphic.

If $f(z)$ is meromorphic with only finitely many poles $z_1, z_2, \ldots, z_N$ of respective multiplicities $\mu_1, \mu_2, \ldots, \mu_N$, for any $n = 1, 2, \ldots, N$ let the Laurent series expansion of $f(z)$ in the neighbourhood of z_n be denoted by

$$f(z) = \sum_{k=0}^{\infty} a_k^{(n)}(z - z_n)^k + \sum_{k=1}^{\mu_n} \frac{b_k^{(n)}}{(z - z_n)^k}.$$

Then

$$G(z) := f(z) - \sum_{n=1}^{N} \sum_{k=1}^{\mu_n} \frac{b_k^{(n)}}{(z - z_n)^k}$$

is plainly an entire function. Conversely, for any entire function $G(z)$, the function

$$f(z) = G(z) + \sum_{n=1}^{N} \sum_{k=1}^{\mu_n} \frac{b_k^{(n)}}{(z - z_n)^k} \tag{2.1}$$

is meromorphic, with finitely many poles $z_1, \ldots, z_N$ of respective multiplicities $\mu_1, \ldots, \mu_N$.

If a meromorphic function $f(z)$ has infinitely many poles $z_1, z_2, \ldots$, in general the decomposition similar to (2.1) with an entire function $G(z)$ does not hold, because the series

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$$\sum_{n=1}^{\infty}\sum_{k=1}^{\mu_n}\frac{b_k^{(n)}}{(z-z_n)^k} \tag{2.2}$$

is not expected to converge. However, Mittag-Leffler has shown that (2.2) can be suitably modified, so that the series thus obtained is totally convergent in any compact subset of $\mathbb{C}$ not containing the poles $z_1, z_2, \dots$.

We recall that a series of functions

$$\sum_{n=1}^{\infty} u_n(z)$$

is said to be totally convergent in a set $K \subset \mathbb{C}$, if there exists a sequence of constants $c_n > 0$ such that

$$|u_n(z)| \le c_n \text{ for all } n = 1, 2, \dots \text{ and for all } z \in K,$$

with

$$\sum_{n=1}^{\infty} c_n < +\infty.$$

By Weierstrass' test, if $\sum\limits_{n=1}^{\infty} u_n(z)$ is totally convergent in K, it is absolutely and uniformly convergent in K.

In view of subsequent applications we prove a special case of Mittag-Leffler's theorem, corresponding to the most important case where the infinitely many poles $z_1, z_2, \dots$ are all simple.

Theorem 2.1 (Mittag-Leffler) *Let $z_1, z_2, \dots \to \infty$ be a sequence of distinct complex numbers satisfying $0 < |z_1| \le |z_2| \le \dots$. Let $m_1, m_2, \dots$ be any sequence of non-zero complex numbers. Then there exists a (not unique) sequence $p_1, p_2, \dots$ of non-negative integers, depending only on the sequences (z_n) and (m_n), such that the series*

$$f(z) := \sum_{n=1}^{\infty}\left(\frac{z}{z_n}\right)^{p_n}\frac{m_n}{z-z_n} \tag{2.3}$$

is totally convergent, and hence absolutely and uniformly convergent, in any compact set $K \subset \mathbb{C} \setminus \{z_1, z_2, \dots\}$. Thus the function $f(z)$ is meromorphic, with simple poles $z_1, z_2, \dots$ having respective residues $m_1, m_2, \dots$.

Proof We choose a sequence of real numbers $0 < r_1 \le r_2 \le \ldots \to +\infty$ satisfying $r_n < |z_n|$ $(n = 1, 2, \ldots)$. In the disc $|z| \le r_n$ we have

$$\left|\frac{m_n}{z - z_n}\right| \le \frac{|m_n|}{|z_n| - |z|} \le \frac{|m_n|}{|z_n| - r_n} \tag{2.4}$$

and

$$\left|\frac{z}{z_n}\right| \le \frac{r_n}{|z_n|} < 1.$$

Let $\varepsilon_1 + \varepsilon_2 + \ldots$ be any convergent series of positive constants ε_n. For every n, let p_n be any non-negative integer such that

$$\left(\frac{r_n}{|z_n|}\right)^{p_n} < \frac{\varepsilon_n}{|m_n|}\,(|z_n| - r_n). \tag{2.5}$$

Since $r_n/|z_n| < 1$, (2.5) is satisfied for any sufficiently large p_n. Then in the disc $|z| \le r_n$ we get by (2.4) and (2.5)

$$\left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \le \left(\frac{r_n}{|z_n|}\right)^{p_n} \frac{|m_n|}{|z_n| - r_n} < \varepsilon_n. \tag{2.6}$$

Take any compact set $K \subset \mathbb{C} \setminus \{z_1, z_2, \ldots\}$, and choose an integer N such that K is contained in the disc $|z| \le r_N$. Let

$$M_n = \max_{z \in K} \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right|.$$

Then, by (2.6), for any $z \in K$ we have

$$\left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \le \begin{cases} M_n & \text{if } n < N \\ \varepsilon_n & \text{if } n \ge N. \end{cases}$$

Since the series of constants $M_1 + \ldots + M_{N-1} + \varepsilon_N + \varepsilon_{N+1} + \ldots$ converges, (2.3) is totally convergent in K. □

Remark 2.1 Fix any $z \in \mathbb{C}$, and let n be such that $|z| < |z_n|$. Then $2\,|z_n| > |z| + |z_n|$, whence

$$\left|\frac{z}{z_n}\right| \le \frac{2\,|z|}{|z| + |z_n|} \le \frac{2\,|z|}{|z - z_n|},$$

and

$$|m_n| \left|\frac{z}{z_n}\right|^{p_n + 1} \le 2\,|z| \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right|.$$

It follows that the sequence (p_n) in Theorem 2.1 is such that

$$\sum_{n=1}^{\infty} |m_n| \left|\frac{z}{z_n}\right|^{p_n+1} < +\infty \quad \text{for every } z \in \mathbb{C}. \tag{2.7}$$

Conversely, any sequence (p_n) of non-negative integers satisfying (2.7) is such that the series (2.3) is totally convergent in any compact set $K \subset \mathbb{C} \setminus \{z_1, z_2, \ldots\}$. For, if $z_* \in \mathbb{C}$ is such that $|z_*| > \max_{z \in K} |z|$, for any n satisfying $|z_n| > |z_*| + 1$ we have

$$\frac{|z_n|}{|z_n| - |z_*|} = \frac{|z_*|}{|z_n| - |z_*|} + 1 < |z_*| + 1,$$

whence

$$\frac{1}{|z_n| - |z_*|} < \frac{|z_*| + 1}{|z_n|} = \left(1 + \frac{1}{|z_*|}\right) \left|\frac{z_*}{z_n}\right|.$$

Thus, for any $z \in K$,

$$\left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \le \left|\frac{z_*}{z_n}\right|^{p_n} \frac{|m_n|}{|z_n| - |z_*|} \le \left(1 + \frac{1}{|z_*|}\right) |m_n| \left|\frac{z_*}{z_n}\right|^{p_n+1}.$$

By (2.7), the series (2.3) is totally convergent in K.

Remark 2.2 If $g(z)$ is any meromorphic function with infinitely many poles $z_1, z_2, \ldots$, all simple and with respective residues $m_1, m_2, \ldots$, then for any sequence (p_n) of non-negative integers satisfying (2.7) the function $G(z) := g(z) - f(z)$, where $f(z)$ is given by (2.3), is plainly entire. Thus $g(z)$ has the Mittag-Leffler series expansion

$$g(z) = G(z) + \sum_{n=1}^{\infty} \left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n},$$

where $G(z)$ is an entire function.

Remark 2.3 The function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}$$

in (2.3) has the Taylor series expansion around $z = 0$ given by

$$f(z) = -\sum_{k=0}^{\infty} \Bigg(\sum_{\substack{n \\ p_n \le k}} m_n \, z_n^{-(k+1)} \Bigg) z^k \tag{2.8}$$

with radius of convergence $|z_1|$. For, in the disc $|z| \le r_1 < |z_1|$ we have

$$f(z) = -\sum_{n=1}^{\infty} m_n \frac{z^{p_n}}{z_n^{p_n+1}} \frac{1}{1 - z/z_n} = -\sum_{n=1}^{\infty} m_n \sum_{k=p_n}^{\infty} \frac{z^k}{z_n^{k+1}}. \tag{2.9}$$

Since, by (2.5),

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{k=p_n}^{\infty} |m_n| \frac{|z|^k}{|z_n|^{k+1}} &= \sum_{n=1}^{\infty} |m_n| \frac{|z|^{p_n}}{|z_n|^{p_n+1}} \frac{1}{1 - |z|/|z_n|} = \sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^{p_n} \frac{|m_n|}{|z_n| - |z|} \\ &\le \sum_{n=1}^{\infty} \left(\frac{r_n}{|z_n|}\right)^{p_n} \frac{|m_n|}{|z_n| - r_n} \le \sum_{n=1}^{\infty} \varepsilon_n < +\infty,\end{aligned}$$

we may interchange the sums on the right-hand side of (2.9). Therefore

$$f(z) = -\sum_{k=0}^{\infty} z^k \sum_{\substack{n \\ p_n \le k}} m_n z_n^{-(k+1)},$$

and the radius of convergence of this Taylor series is the distance from the origin to the closest singular point of $f(z)$, i.e., to the pole z_1.

2.2 Infinite Products

Let $a_n \in \mathbb{C}$, $a_n \neq 0$ $(n = 1, 2, \ldots)$. The infinite product $\prod_n a_n$ is defined by

$$\prod_{n=1}^{\infty} a_n = \lim_{n\to\infty} \prod_{k=1}^{n} a_k = A. \tag{2.10}$$

The infinite product (2.10) converges if the limit A exists and $A \neq 0, \infty$. If $A = 0$ or $A = \infty$, the infinite product is said to be divergent to zero or, respectively, to infinity.

If (2.10) converges, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\prod_{k=1}^{n} a_k \bigg/ \prod_{k=1}^{n-1} a_k \right) = A/A = 1.$$

Let $a_n(z) \neq 0$ $(n = 1, 2, \ldots)$ be a sequence of functions defined for z in a set $K \subset \mathbb{C}$. The infinite product $\prod_n a_n(z)$ is uniformly convergent to a function $f(z) \neq 0, \infty$ in K if, for $n \to \infty$, the sequence of partial products

$$f_n(z) = \prod_{k=1}^{n} a_k(z)$$

converges uniformly to $f(z)$ in K, i.e., if for any $\varepsilon > 0$ there exists n_0 such that

$$|f(z) - f_n(z)| < \varepsilon$$

for all $n > n_0$ and for all $z \in K$.

Lemma 2.1 *Let the functions $u_n(z)$ $(n = 1, 2, \dots)$ be regular in a compact set $K \subset \mathbb{C}$, and let the series*

$$\sum_{n=1}^{\infty} u_n(z)$$

be totally convergent in K. Then the infinite product

$$\prod_{n=1}^{\infty} \exp\big(u_n(z)\big) = \exp\left(\sum_{n=1}^{\infty} u_n(z)\right)$$

is uniformly convergent in K.

Proof For any $z_1, z_2 \in \mathbb{C}$ we have

$$\begin{aligned} e^{z_1} - e^{z_2} &= (z_1 - z_2) + \frac{1}{2!}(z_1^2 - z_2^2) + \frac{1}{3!}(z_1^3 - z_2^3) + \dots \\ &= (z_1 - z_2)\Big(1 + \frac{1}{2!}(z_1 + z_2) + \frac{1}{3!}(z_1^2 + z_1 z_2 + z_2^2) + \dots\Big), \end{aligned}$$

whence

$$\begin{aligned} |e^{z_1} - e^{z_2}| &\le |z_1 - z_2|\Big(1 + \frac{1}{2!}(|z_1| + |z_2|) + \frac{1}{3!}(|z_1| + |z_2|)^2 + \dots\Big) \\ &\le |z_1 - z_2|\, e^{|z_1|+|z_2|}. \end{aligned}$$

Therefore

$$\begin{aligned} &\left|\exp\left(\sum_{n=1}^{\infty} u_n(z)\right) - \exp\left(\sum_{n=1}^{N} u_n(z)\right)\right| \qquad (2.11) \\ &\quad\le \left|\sum_{n=1}^{\infty} u_n(z) - \sum_{n=1}^{N} u_n(z)\right| \exp\left(\sum_{n=1}^{\infty} |u_n(z)| + \sum_{n=1}^{N} |u_n(z)|\right) \\ &\quad\le \left|\sum_{n=1}^{\infty} u_n(z) - \sum_{n=1}^{N} u_n(z)\right| \exp\left(2\sum_{n=1}^{\infty} |u_n(z)|\right). \end{aligned}$$

Let

$$\sum_{n=1}^{\infty} |u_n(z)| = U(z),$$

let

$$M = \exp\Big(2 \max_{z\in K} U(z)\Big),$$

and for any $\varepsilon > 0$ let N_0 be such that

$$\left|\sum_{n=1}^{\infty} u_n(z) - \sum_{n=1}^{N} u_n(z)\right| < \frac{\varepsilon}{M}$$

for all $N > N_0$ and all $z \in K$. Then, by (2.11),

$$\left|\exp\left(\sum_{n=1}^{\infty} u_n(z)\right) - \exp\left(\sum_{n=1}^{N} u_n(z)\right)\right| \leq M\left|\sum_{n=1}^{\infty} u_n(z) - \sum_{n=1}^{N} u_n(z)\right| < \varepsilon. \quad \square$$

2.3 Weierstrass' Products

Lemma 2.2 *Let $f(z)$ be a meromorphic function. Let $z_1, z_2, \ldots \neq 0$ be the poles of $f(z)$, all simple with respective residues $m_1, m_2, \ldots \in \mathbb{Z}$. Then the function*

$$\varphi(z) := \exp\int_0^z f(t)\,\mathrm{d}t \tag{2.12}$$

is meromorphic. The zeros (*resp. poles*) *of $\varphi(z)$ are the points z_n such that $m_n > 0$* (*resp. $m_n < 0$*)*, and the multiplicity of z_n as a zero* (*resp. pole*) *of $\varphi(z)$ is m_n* (*resp. $-m_n$*).

Proof First we remark that $\varphi(z)$ is a one-valued function, because if γ and γ' are any two paths of endpoints 0 and z not passing through the poles z_n, by the residue theorem we get

$$\int_{\gamma} f(t)\,\mathrm{d}t = \int_{\gamma'} f(t)\,\mathrm{d}t + 2\pi i R,$$

where R is the sum of residues of $f(t)$ at the poles between γ and γ', each residue being taken with $+$ or $-$ sign according to the mutual position of γ and γ' around the corresponding pole. Since $m_1, m_2, \ldots \in \mathbb{Z}$, we get $R \in \mathbb{Z}$. Therefore

$$\exp \int_\gamma f(t)\,\mathrm{d}t \;=\; e^{2\pi i R} \exp \int_{\gamma'} f(t)\,\mathrm{d}t \;=\; \exp \int_{\gamma'} f(t)\,\mathrm{d}t.$$

Plainly $\varphi(z)$ is regular and $\neq 0$ in $\mathbb{C} \setminus \{z_1, z_2, \ldots\}$. The function $f_1(z) := f(z) - m_1/(z - z_1)$ is regular in $\mathbb{C} \setminus \{z_2, z_3, \ldots\}$, whence $\exp \int_0^z f_1(t)\,\mathrm{d}t$ is regular and $\neq 0$ in $\mathbb{C} \setminus \{z_2, z_3, \ldots\}$. Thus

$$\begin{aligned}\varphi(z) \;&=\; \exp \int_0^z f(t)\,\mathrm{d}t \;=\; \exp\left(\int_0^z f_1(t)\,\mathrm{d}t + m_1 \int_0^z \frac{\mathrm{d}t}{t - z_1}\right) \\ &=\; \exp \int_0^z f_1(t)\,\mathrm{d}t \cdot \exp\left(m_1 \log\left(1 - \frac{z}{z_1}\right)\right) \;=\; (z - z_1)^{m_1} \varphi_1(z),\end{aligned}$$

where $\varphi_1(z) = (-z_1)^{-m_1} \exp \int_0^z f_1(t)\,\mathrm{d}t$ is regular and $\neq 0$ in $\mathbb{C} \setminus \{z_2, z_3, \ldots\}$. By the same argument for $z_2, z_3, \ldots$ the lemma follows. $\square$

For any sequence $z_1, z_2, \ldots \in \mathbb{C}$, either finite or satisfying $\lim z_n = \infty$, from Theorem 2.1 and Lemma 2.2 we deduce the existence of a meromorphic function with zeros and poles at $z_1, z_2, \ldots$ with arbitrary multiplicities $m_1, m_2, \ldots$. This yields the following

Corollary 2.1 *Every meromorphic function is the quotient of two entire functions.*

Proof Let $g(z)$ be meromorphic, and let $z_1, z_2, \ldots \neq 0$ be the poles of $g(z)$ with respective multiplicities $m_1, m_2, \ldots$. Let $f(z)$ be the function (2.3) and let $\varphi(z)$ be the function (2.12). By applying Theorem 2.1 and Lemma 2.2, we see that $\varphi(z)$ is entire with zeros $z_1, z_2, \ldots$ of respective multiplicities $m_1, m_2, \ldots$. Then the product $g(z)\varphi(z)$ has no poles, and therefore is an entire function $h(z)$, whence $g(z) = h(z)/\varphi(z)$ with $h(z)$ and $\varphi(z)$ entire functions.

If $g(z)$ has a pole at $z = 0$ of multiplicity m, then $\widetilde{g}(z) := z^m g(z)$ is regular at $z = 0$, whence $\widetilde{g}(z) = h(z)/\varphi(z)$ with entire $h(z)$ and $\varphi(z)$, and $g(z) = h(z)/(z^m \varphi(z))$ with entire $h(z)$ and $z^m \varphi(z)$. $\square$

Theorem 2.2 (Weierstrass) *Let $F(z)$ be meromorphic, and regular and $\neq 0$ at $z = 0$. Let $z_1, z_2, \ldots$ be the zeros and poles of $F(z)$ with respective multiplicities $|m_1|, |m_2|, \ldots$, where $m_n > 0$ if z_n is a zero and $m_n < 0$ if z_n is a pole of $F(z)$. Then there exist integers $p_1, p_2, \ldots \geq 0$ and an entire function $G(z)$ such that*

$$F(z) \;=\; e^{G(z)} \prod_n \left(1 - \frac{z}{z_n}\right)^{m_n} \exp\left(m_n \sum_{k=1}^{p_n} \frac{1}{k}\left(\frac{z}{z_n}\right)^k\right), \tag{2.13}$$

where the product converges uniformly in any compact set $K \subset \mathbb{C} \setminus \{z_1, z_2, \dots\}$. *In* (2.13) *one can take any sequence of integers* $p_1, p_2, \dots \geq 0$ *satisfying* (2.7).

Proof Let $f(z)$ be the function (2.3) with integer exponents $p_1, p_2, \dots \geq 0$ satisfying (2.7), and let $\varphi(z)$ be the function (2.12). By Theorem 2.1 and Lemma 2.2 $\varphi(z)$ is meromorphic, with zeros z_n of multiplicities m_n if $m_n > 0$, and with poles z_n of multiplicities $|m_n|$ if $m_n < 0$. Thus $F(z)$ and $\varphi(z)$ have the same zeros and poles with the same multiplicities, whence $F(z)/\varphi(z)$ is entire and $\neq 0$. Therefore $\log\big(F(z)/\varphi(z)\big) = G(z)$ is an entire function, and

$$F(z) = e^{G(z)}\varphi(z). \tag{2.14}$$

By uniform convergence of (2.3) on a path of endpoints 0 and z not containing $z_1, z_2, \dots,$ term-by-term integration of (2.3) from 0 to z is allowed. Thus from (2.12) we get

$$\begin{aligned}
\varphi(z) &= \exp\int_0^z \sum_n \left(\frac{t}{z_n}\right)^{p_n} \frac{m_n}{t - z_n}\,\mathrm{d}t \qquad (2.15)\\
&= \prod_n \exp\int_0^z \left(\frac{m_n}{t - z_n} + \frac{m_n}{z_n}\,\frac{(t/z_n)^{p_n} - 1}{t/z_n - 1}\right)\mathrm{d}t\\
&= \prod_n \exp\int_0^z \left(\frac{m_n}{t - z_n} + \frac{m_n}{z_n}\sum_{k=1}^{p_n}\left(\frac{t}{z_n}\right)^{k-1}\right)\mathrm{d}t\\
&= \prod_n \exp\left(\log\left(1 - \frac{z}{z_n}\right)^{m_n} + m_n\sum_{k=1}^{p_n}\frac{1}{k}\left(\frac{z}{z_n}\right)^k\right)\\
&= \prod_n \left(1 - \frac{z}{z_n}\right)^{m_n}\exp\left(m_n\sum_{k=1}^{p_n}\frac{1}{k}\left(\frac{z}{z_n}\right)^k\right).
\end{aligned}$$

Then (2.13) follows from (2.14). Moreover, for any $z \in K$ we can choose the integration path from 0 to z of length bounded by a constant depending only on K, whence, by Theorem 2.1, the series

$$\sum_n \int_0^z \left(\frac{t}{z_n}\right)^{p_n} \frac{m_n}{t - z_n}\,\mathrm{d}t$$

is totally convergent in K. Hence, by Lemma 2.1, the product in (2.13) is uniformly convergent in K. □

If the meromorphic function $F(z)$ has a zero of multiplicity m or a pole of multiplicity $-m$ at $z = 0$, then $z^{-m}F(z)$ is regular and $\neq 0$ at $z = 0$, and therefore can be represented by (2.13). Thus in this case the Weierstrass factorization formula (2.13) becomes

$$F(z) \;=\; z^m e^{G(z)} \prod_n \left(1 - \frac{z}{z_n}\right)^{m_n} \exp\left(m_n \sum_{k=1}^{p_n} \frac{1}{k}\left(\frac{z}{z_n}\right)^k\right). \tag{2.16}$$

Corollary 2.2 *The Taylor series expansion with centre $z = 0$ of the entire function $G(z)$ in (2.13) is*

$$G(z) \;=\; \log F(0) + \sum_{k=1}^{\infty}\left(\frac{1}{(k-1)!}\left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\,\frac{F'(z)}{F(z)}\right]_{z=0} + \sum_{\substack{n\\ p_n<k}} m_n\, z_n^{-k}\right)\frac{z^k}{k}. \tag{2.17}$$

Proof From (2.8), (2.12), (2.13) and (2.15) we get, for $|z| < |z_1|$,

$$\begin{aligned}
G(z) \;&=\; \log F(z) - \log\varphi(z) \;=\; \log F(z) - \int_0^z f(t)\,\mathrm{d}t \\
&=\; \log F(z) + \int_0^z \sum_{k=0}^{\infty}\left(\sum_{\substack{n\\ p_n\le k}} m_n\, z_n^{-(k+1)}\right) t^k\,\mathrm{d}t \\
&=\; \log F(0) + \sum_{k=1}^{\infty}\frac{z^k}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\log F(z)\right]_{z=0} + \sum_{k=0}^{\infty}\left(\sum_{\substack{n\\ p_n\le k}} m_n\, z_n^{-(k+1)}\right)\frac{z^{k+1}}{k+1} \\
&=\; \log F(0) + \sum_{k=1}^{\infty}\frac{z^k}{k!}\left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\,\frac{F'(z)}{F(z)}\right]_{z=0} + \sum_{k=1}^{\infty}\left(\sum_{\substack{n\\ p_n< k}} m_n\, z_n^{-k}\right)\frac{z^{k}}{k},
\end{aligned}$$

and (2.17) follows. □

By Corollary 2.1, the Weierstrass factorization (2.13) or (2.16) of a meromorphic function $F(z)$ can be reduced to the Weierstrass factorizations of two entire functions $F_1(z)$ and $F_2(z)$ such that $F(z) = F_1(z)/F_2(z)$. Thus, in what follows, we will employ (2.13) or (2.16) only for entire functions $F(z)$, i.e., when $m_n > 0$ ($n = 1, 2, \ldots$) and $m \geq 0$. Moreover, with no loss of generality, in the product on the right-hand sides of (2.13) and (2.16) we can put $m_n = 1$, with the convention that if a zero z_n has multiplicity m_n, the corresponding factor

$$\left(1 - \frac{z}{z_n}\right)\exp\sum_{k=1}^{p_n}\frac{1}{k}\left(\frac{z}{z_n}\right)^k$$

is repeated m_n times in the product. Accordingly, condition (2.7) for the sequence (p_n) becomes

$$\sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right|^{p_n+1} < +\infty \quad \text{for every } z \in \mathbb{C}, \tag{2.18}$$

where if z_n is a zero of multiplicity m_n, the term $|z/z_n|^{p_n+1}$ is repeated m_n times in the series.

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