A Topological Approach to Non-Archimedean Mathematics

Vieri Benci and Lorenzo Luperi Baglini

Abstract Non-Archimedean mathematics (in particular, nonstandard analysis) allows to construct some useful models to study certain phenomena arising in PDE's; for example, it allows to construct generalized solutions of differential equations and variational problems that have no classical solution. In this paper we introduce certain notions of Non-Archimedean mathematics (and of nonstandard analysis) by means of an elementary topological approach; in particular, we construct Non-Archimedean extensions of the reals as appropriate topological completions of $\mathbb{R}$. Our approach is based on the notion of Λ-limit for real functions, and it is called Λ-theory. It can be seen as a topological generalization of the α-theory presented in [6], and as an alternative topological presentation of the ultrapower construction of nonstandard extensions (in the sense of [21]). To motivate the use of Λ-theory for applications we show how to use it to solve a minimization problem of calculus of variations (that does not have classical solutions) by means of a particular family of generalized functions, called ultrafunctions.

Keywords Non-Archimedean mathematics · Nonstandard analysis · Limits of functions · Generalized functions · Ultrafunctions

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1 Introduction

In a previous series of papers [5, 9–14] we have introduced and studied a new family of generalized functions called ultrafunctions and its applications to certain problems in mathematical analysis, including some applications to PDE's in [14]. The development of a rigorous study of (a large class of) PDE's in ultrafunction theory is the object of [15], where we exemplify our approach by studying in detail Burgers' equation. Henceforth, it is our feeling that many problems in PDE's theory could be fruitfully studied by means of the theory of ultrafunctions.

However, one might have the impression that a drawback of our approach is the use of the machinery of NSA, which is not a "common working tool" for most analysts. Even if NSA has already been applied to many different fields of mathematics (such as functional analysis, probability theory, combinatorial number theory, mathematical physics and so on) to obtain important results, the original formalism of Robinson, based on model theory (see e.g. [25]), appears too technical to many researchers, and not directly usable by most mathematicians. Since Robinson's work first appeared, a simpler semantic approach (due to Robinson himself and Elias Zakon) has been developed using the purely set-theoretic notion of superstructure (see [27]); we recall also the pioneering work by Luxemburg (see [23]), where a direct use of ultrapowers was made (see [6, 8] for a complete presentation of alternative simplified approaches to NSA). However, many researcher working in NSA have the feeling that also these technical notions are not needed in order to carry out calculus with actual infinitesimals, as well as to carry out several applications of NSA. As a consequence, there have been many attempts to simplify and popularize NSA by means of simplified presentations. We recall here in particular the approaches of Henson [20], Keisler [21] and Nelson [24]; other attempts have been made by Benci, Di Nasso and Forti with algebraic (see [3, 4, 7, 17]) and topological approaches (see [8, 16]). We also suggest [22] where NSA is introduced in a simplified way suitable for many applications. In our previous papers, we tried to address the same issue by means of Λ-limits (see e.g. [11] for an axiomatic presentation of this approach to NSA). The basic idea of Λ-limits is to present nonstandard objects as limits of standard ones. However, in our previous works the word "limits" was not intended in a topological sense: the "limits" where defined axiomatically and no explicit topology was involved in the constructions.

The main aim of this paper is to show that, actually, Λ-limits can be precisely characterized as topological limits. This approach will be called Λ-theory; it allows to construct a topological approach to NSA (related to but different from the approach of Benci, Di Nasso and Forti in [8, 16]) that, in our opinion, is well-suited for researchers that are not experts in NSA and are interested to use certain Non-Archimedean arguments to study problems in analysis. In fact, it is our feeling that presenting nonstandard constructions and results by means of a topological approach might help such researchers to use them. For example, we construct extensions of the reals (in the sense of NSA) as appropriate topological completions of $\mathbb{R}$.

Λ-theory can be seen as a topological generalization of the α-theory presented in [6]. The idea behind our approach is to embed $\mathbb{R}$ in particular Hausdorff topological spaces in which it is possible to formalize the intuitive idea of hyperreals as topological limits (in a sense that we will make precise in Sect. 2.1) of real functions. From this point of view, our construction of the hyperreals starting from $\mathbb{R}$ shares some features with the construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$. We also extend our construction to define a topology on the superstructure $V(\mathbb{R})$ on $\mathbb{R}$, that we use to define Λ-limits of bounded functions defined on $V(\mathbb{R})$. Our construction is substantially equivalent to the ultrapower approach, and we will prove in Sect. 3 that within $\Lambda-$theory it is possible to construct a nonstandard universe in the sense of [21]. To motivate our feeling that Λ-theory can be fruitfully applied to study certain problems in Analysis, in Sect. 4 we apply Λ-theory to solve a minimization problem of calculus of variations that does not have classical solutions.

We want to remark that readers expert in NSA will easily recognize that Λ-theory is essentially equivalent to the ultrapower construction (we prove this fact in Sect. 3). Anyhow, in this paper, we do not assume the knowledge of NSA by the reader.

2 Λ-theory

2.1 The Λ-limit

The only technical notion that we need to develop our approach to Non-Archimedean mathematics is that of ultrafilter:

Definition 1 Let X be a set. An ultrafilter $\mathcal{U}$ on X is a family of subsets of X that has the following properties:

1. $X \in \mathcal{U}$, $\emptyset \notin \mathcal{U}$;
2. for every $A, B \subseteq X$ if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$;
3. for every $A, B \in \mathcal{U}$, $A \cap B \in \mathcal{U}$;
4. for every $A \subseteq X$ we have that $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

An ultrafiler $\mathcal{U}$ on X is principal if there exists an element $x \in X$ such that $\mathcal{U} = \{A \subseteq X \mid x \in A\}$. An ultrafilter is free if it is not principal. From now on we let $\mathfrak{L}$ be an infinite set equipped with a free ultrafilter $\mathcal{U}$. Every set $Q \in \mathcal{U}$ will be called **qualified set**. We will say that a property P is **eventually** true for the function $\varphi(\lambda)$ if it is true for every λ in a qualified set, namely if there exists $Q \in \mathcal{U}$ such that $P(\varphi(\lambda))$ holds for every $\lambda \in Q$. We let $\Lambda \notin \mathfrak{L}$ and we consider the space $\mathfrak{L} \cup \{\Lambda\}$. We equip $\mathfrak{L} \cup \{\Lambda\}$ with a topology in which the neighborhoods of Λ are of the form $\{\Lambda\} \cup Q,\ \ Q \in \mathcal{U}$. In this sense, one can imagine Λ as being a "point at infinity" for $\mathfrak{L}$ (in this sense, it plays a similar role to that of α in the Alpha-Theory, see [6]). With respect to this topology, the notion of limit of a function at Λ is specified as follows:

Definition 2 Let (X, τ) be a Hausdorff topological space, let $x_0 \in X$ and let $\varphi : \mathfrak{L} \to X$ be a function. We say that x_0 is the Λ-limit of the function φ, and we write

$$\lim_{\lambda \to \Lambda} \varphi(\lambda) = x_0, \tag{1}$$

if for every neighborhood V of x_0 the function φ is eventually in V, namely if there is a qualified set Q such that $\varphi(Q) \subset V$.

Remark 1 We use the notation $\lim_{\lambda \to \Lambda} \varphi(\lambda)$ since, as we already noticed, one may think of $\Lambda \notin \mathfrak{L}$ as a "point at ∞" and to the sets in $\mathcal{U}$ as neighborhoods of Λ; it is conceptually similar to the point ∞ when one considers $\mathbb{R} \cup \{+\infty\}$. We prefer to use the symbol Λ rather than ∞ since one may think of Λ as a function of $\mathcal{U}$, namely $\Lambda = \Lambda(\mathcal{U})$. Thus the explicit mention of Λ is a reminder that the Λ-limit depends on $\mathcal{U}$.

Remark 2 Another way to look at the limit (1) is to consider the Stone-Čech compactification $\beta\mathfrak{L}$ of $\mathfrak{L}$ with the relative topology and to think of $\Lambda \in \beta\mathfrak{L}$ as of a nontrivial element of this compactification.

Limits as given by Eq. (1) will be called $\Lambda-$limits, and we will call Λ-theory the approach to Non-Archimedean mathematics based on the notion of Λ-limit.

Our main result is the following:

Theorem 1 *There exists a Hausdorff topological space* $(\mathbb{R}_{\mathfrak{L}}, \tau)$ *such that*

1. $\mathbb{R}_{\mathfrak{L}} = cl_\tau(\mathfrak{L} \times \mathbb{R})$;
2. $\mathbb{R} \subseteq \mathbb{R}_{\mathfrak{L}}$ *and* $\forall c \in \mathbb{R}$
$$\lim_{\lambda \to \Lambda} (\lambda, c) = c;$$
3. *for every function* $\varphi : \mathfrak{L} \to \mathbb{R}$, *the limit*
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$$
exists in $(\mathbb{R}_{\mathfrak{L}}, \tau)$;
4. *two functions* φ, ψ *are eventually equal if and only if*
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)).$$

Proof We set

$$I = \{\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \mid \varphi(x) = 0 \text{ in a qualified set}\}.$$

It is not difficult to prove that I is a maximal ideal in $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$; then

$$\mathbb{K} := \frac{\mathfrak{F}(\mathfrak{L}, \mathbb{R})}{I}$$

is a field. In the following, we shall identify a real number $c \in \mathbb{R}$ with the equivalence class of the constant function $\varphi : \mathfrak{L} \to \mathbb{R}$ such that $\varphi(\lambda) = c$ for every $\lambda \in \mathfrak{L}$.
We set

$$\mathbb{R}_{\mathfrak{L}} = (\mathfrak{L} \times \mathbb{R}) \cup \mathbb{K}.$$

We equip $\mathbb{R}_{\mathfrak{L}}$ with the following topology τ. A basis for τ is given by

$$b(\tau) = \left\{ N_{\varphi,Q} \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}), Q \in \mathscr{U} \right\} \cup \mathscr{P}(\mathfrak{L} \times \mathbb{R})$$

where

$$N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \mid \lambda \in Q\} \cup \{[\varphi]_I\}$$

is a neighborhood of $[\varphi]_I$ for every $Q \in \mathscr{U}$.

In order to show that $b(\tau)$ is a basis for a topology, we have to show that

$$\forall A, B \in b(\tau)\ \forall x \in A \cap B\ \exists C \in b(\tau) \text{ such that } x \in C \subset A \cap B.$$

Let $A, B \in b(\tau)$. Let $x \in A \cap B$. If $x \notin \mathbb{K}$ then we can just set $C = A \cap B \cap \mathfrak{L} \times \mathbb{R}$, as the topology is discrete on $\mathfrak{L} \times \mathbb{R}$. If $x \in \mathbb{K}$ then there exist $R, S \in \mathscr{U}$ such that $A = N_{\varphi,R}$ and $B = N_{\psi,S}$ with $[\varphi]_I = [\psi]_I = x$. Hence there exists $Q \in \mathscr{U}$ such that

$$\forall \lambda \in Q,\ \varphi(\lambda) = \psi(\lambda).$$

Thus if we set $C := N_{\varphi, R\cap S\cap Q}$ we have that $x \in C \subset A \cap B$.

Let us show that τ is a Hausdorff topology. Clearly it is sufficient to check it for points in $\mathbb{K}$, so let $\xi \neq \zeta \in \mathbb{K}$. Since $\xi \neq \zeta$, there exists $\varphi, \psi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}), Q \in \mathscr{U}$ such that

$$\xi = [\varphi]_I,\ \zeta = [\psi]_I \quad \text{and} \quad \forall \lambda \in Q,\ \varphi(\lambda) \neq \psi(\lambda).$$

Therefore

$$N_{\varphi,Q} \cap N_{\psi,Q} = \varnothing.$$

Let us observe that, by construction, for every function $\varphi : \mathfrak{L} \to \mathbb{R}$ we have that

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I . \tag{2}$$

In fact, given a neighborhood $N_{\varphi,Q}$ of $[\varphi]_I$, we have that $\{\varphi(\lambda) \mid \lambda \in Q\} \subseteq N_{\varphi,Q}$, so $[\varphi]_I$ is a Λ-limit of the function $(\lambda, \varphi(\lambda))$. Since the space is Hausdorff, the limit is unique, so $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I$.

Let us prove that $(\mathbb{R}_{\mathfrak{L}}, \tau)$ has the desired properties:

- property (1) follows directly by the definition of τ;
- property (2) follows by the identification of every real number $c \in \mathbb{R}$ with the equivalence class of the constant function $[c]_I$;
- properties (3) and (4) follow by Eq. (2).

Remark 3 In [8, 16], nonstandard extensions are constructed by means of similar, but different, topological considerations based on the choice of the ultrafilter $\mathcal{U}$. However the authors showed (see Theorem 4.5 in [16]) that such extensions are Hausdorff if and only if the ultrafilter $\mathcal{U}$ is Hausdorff (see again [16], Sects. 4 and 6), and in [2] Bartoszynski and Shelah proved that it is consistent with ZFC that there are no Hausdorff ultrafilters. By contrast, in our topological approach the extensions are always constructed inside Hausdorff topological spaces under the much milder request of $\mathcal{U}$ being free. This is possible because we incorporate the set of indices $\mathfrak{L}$ in the space.

Motivated by the philosophical similarity between the properties expressed in Theorem 1 and the construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$, we introduce the following definition:

Definition 3 *A Hausdorff topological space* $(\mathbb{R}_{\mathfrak{L}}, \tau)$ that satisfies conditions (1)–(4) of Theorem 1 will be called a $(\mathfrak{L}, \mathcal{U})$-**completion of** $\mathbb{R}$.

2.2 The Hyperreal Field

Let $(\mathbb{R}_{\mathfrak{L}}, \tau)$ be a $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$. Let us fix some notation: we will denote by $\mathbb{K}$ the set

$$\mathbb{K} = \left\{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \right\}.$$

The aim of this section is to study the basic properties of $\mathbb{K}$.

Proposition 1 $(\mathfrak{L} \times \mathbb{R}) \cap \mathbb{K} = \emptyset$.

Proof Let us suppose by contrast that there exists $\varphi : \mathfrak{L} \to X$ such that

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = (\lambda_0, r) \in \mathfrak{L} \times \mathbb{R}.$$

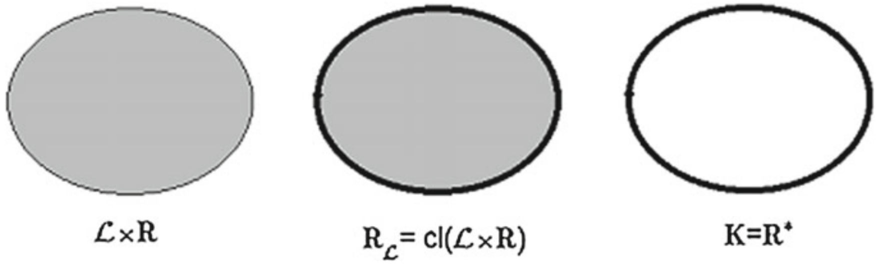
Since $\{(\lambda_0, r)\}$ is open, by definition there exists $Q \in \mathcal{U}$ such that $\forall \lambda \in Q$, $(\lambda, \varphi(\lambda)) = (\lambda_0, r)$. Therefore $Q = \{\lambda_0\}$, and this is absurd since $\mathcal{U}$ is free.

From condition (1) in Theorem 1 we know that $(\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{K} \subseteq \mathbb{R}_{\mathfrak{L}}$. In general, this inclusion might be proper; henceforth we introduce the following definition:

Definition 4 We say that $(\mathbb{R}_{\mathfrak{L}}, \tau)$ is a minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ if $\mathbb{R}_{\mathfrak{L}} = (\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{K}$.

It is immediate to see that any $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ contains a minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$, and that any minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ does not properly contain another minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ (and this is what motivates the choice of the name "minimal" for such extensions).

From now on we will be only interested in minimal $(\mathfrak{L}, \mathscr{U})$-completions.



By condition (2) in the definition of $(\mathfrak{L}, \mathscr{U})$-completions it follows that $\mathbb{R} \subseteq \mathbb{K}$. Moreover we have the following result:

Proposition 2 *For every finite subset $F \subseteq \mathbb{R}$, for every function $\varphi : \mathfrak{L} \to F$ we have that*

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \in F.$$

Proof Let $F = \{x_1, ..., x_n\}$. For every $i \leq n$ let

$$A_i = \{\lambda \in \mathfrak{L} \mid \varphi(\lambda) = x_i\}.$$

Since $\mathscr{U}$ is an ultrafilter, there exists exactly one index $i_0 \leq n$ such that $A_{i_0} \in \mathscr{U}$. Now let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let us suppose that $\xi \neq x_{i_0}$. Let O_1, O_2 be disjoint open sets such that $\xi \in O_1$, $x_{i_0} \in O_2$. Since x_{i_0} is the limit of the constant function with value x_{i_0}, there exists $B \in \mathscr{U}$ such that

$$\{(\lambda, x_{i_0}) \mid \lambda \in B\} \subseteq O_2.$$

Let $C \in \mathscr{U}$ be such that $\{(\lambda, \varphi(\lambda)) \mid \lambda \in C\} \subseteq O_1$. Then by construction we have that

$$\forall \lambda \in A_{i_0} \cap B \cap C \ (\lambda, \varphi(\lambda)) = (\lambda, x_{i_0}) \in O_1 \cap O_2,$$

and this is a contradiction since $O_1 \cap O_2 = \emptyset$. Therefore $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = x_{i_0} \in F$.

There is a natural way to define sums and products of elements of $\mathbb{K}$:

Definition 5 We set

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) + \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) + \psi(\lambda));$$
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \psi(\lambda)).$$

Theorem 2 $(\mathbb{K}, +, \cdot, 0, 1)$ *is a field which contains* $\mathbb{R}$.

Proof That $\mathbb{R} \subseteq \mathbb{K}$ follows by condition (2) of the definition of $(\mathfrak{L}, \mathscr{U})$-completion. The only non trivial property that we have to prove to show that $\mathbb{K}$ is a field is the existence of a multiplicative inverse for every $x \neq 0$. Let $x \in \mathbb{K}$, $x \neq 0$. Since the topology is Hausdorff and $x \neq 0$, there is a set $Q \in \mathscr{U}$ such that

$$\forall \lambda \in Q,\ \varphi(\lambda) \neq 0.$$

Let $\phi : \mathfrak{L} \to \mathbb{R}$ be defined as follows:

$$\phi(\lambda) = \begin{cases} 1 & \text{if } \lambda \notin Q; \\ \frac{1}{\varphi(\lambda)} & \text{if } \lambda \in Q. \end{cases}$$

Then $\varphi(\lambda) \cdot \phi(\lambda) = 1$ for every $\lambda \in Q$, thus $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \phi(\lambda)) = 1$, namely

$$x^{-1} := \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda))$$

is the inverse of x.

The ordering of $\mathbb{R}$ can be extended to $\mathbb{K}$ by setting

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) < \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \Leftrightarrow \varphi(\lambda) < \psi(\lambda) \text{ eventually,} \tag{3}$$

namely iff $\{(\lambda, \varphi(\lambda) - \psi(\lambda)) \mid \varphi(\lambda) - \psi(\lambda) \geq 0\} \cup [\varphi - \psi]$ is open (i.e. iff $\{\lambda \in \mathfrak{L} \mid \varphi(\lambda) < \psi(\lambda)\}$ is qualified). This ordering is clearly an extension of the ordering relation defined on $\mathbb{R}$ since, for every $x, y \in \mathbb{R}$, if $x \leq y$ and $\varphi_x, \varphi_y : \mathfrak{L} \to \mathbb{R}$ are the constant sequences with values resp. x, y then

$$\{\lambda \in \mathfrak{L} \mid \varphi_x(\lambda) < \varphi_y(\lambda)\} = \mathfrak{L},$$

which is qualified.

Remark 4 Usually, the inclusion $\mathbb{R} \subseteq \mathbb{K}$ is proper: e.g., let $\mathscr{U}$ be a countably incomplete ultrafilter.[1] Let $\langle A_n \mid n \in \mathbb{N} \rangle$ be a family of elements of $\mathscr{U}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, let $B_n = \bigcap_{i \leq n} A_n$ for all $n \in \mathbb{N}$ and let $\phi : \mathfrak{L} \to \mathbb{R}$ be defined as follows: for every $\lambda \in \mathfrak{L}$,

$$\phi(\lambda) = n \Leftrightarrow \lambda \in B_n \backslash B_{n+1.}$$

[1] An ultrafilter $\mathscr{U}$ is countably incomplete if there exists a family $\langle A_n \mid n \in \mathbb{N} \rangle$ of elements of $\mathscr{U}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Then $\lim_{\lambda\to\Lambda}\ (\lambda,\phi(\lambda))\notin\mathbb{R}$: in fact, $\lim_{\lambda\to\Lambda}\ (\lambda,\phi(\lambda))>n$ for every $n\in\mathbb{N}$ (and so, in particular, this limit is infinite). This holds since, for every $n\in\mathbb{N}$, by construction we have that

$$\{\lambda\in\mathfrak{L}\mid\phi(\lambda)\geq n\}=B_n\in\mathcal{U}.$$

When the inclusion $\mathbb{R}\subseteq\mathbb{K}$ is proper we have that $\mathbb{K}$ is a superreal non Archimedean field.[2] In this case, it will be called a **hyperreal field**. The terminology will be motivated by Corollary 1, where we make precise the relationship (as fields) between the hyperreal field $\mathbb{K}$ and the ultrapower $\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$. Let us recall the definition of $\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$:

Definition 6 Let $\equiv_{\mathcal{U}}$ be the equivalence relation on $\mathbb{R}^{\mathfrak{L}}$ defined as follows: for every $\varphi,\psi:\mathfrak{L}\to\mathbb{R}$

$$\varphi\equiv_{\mathcal{U}}\psi\Leftrightarrow\{\lambda\in\mathfrak{L}\mid\varphi(\lambda)=\psi(\lambda)\}\in\mathcal{U}.$$

The equivalence class of every function $\varphi:\mathfrak{L}\to\mathbb{R}$ will be denoted by $[\varphi]_{\mathcal{U}}$. The ultrapower $\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$ is the quotient $\mathbb{R}^{\mathfrak{L}}/\equiv_{\mathcal{U}}$.

The operations on $\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$ are defined componentwise: for every $\varphi,\psi:\mathfrak{L}\to\mathbb{R}$ we set

$$[\varphi]_{\mathcal{U}}+[\psi]_{\mathcal{U}}:=[\varphi+\psi]_{\mathcal{U}};\ [\varphi]_{\mathcal{U}}+[\psi]_{\mathcal{U}}:=[\varphi\cdot\psi]_{\mathcal{U}}.$$

A well-known result (see e.g. [21]) is that $(\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}},[0]_{\mathcal{U}},[1]_{\mathcal{U}},+,\cdot)$ is a field. Moreover, we have the following:

Corollary 1 *$\mathbb{K}$ and $\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$ are isomorphic as fields.*

Proof The isomorphism is given by the map $\Psi:\mathbb{K}\to\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$ such that, for every $\varphi:\mathfrak{L}\to\mathbb{R}$,

$$\Psi\left(\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))\right)=[\varphi]_{\mathcal{U}}.$$

Condition (4) in the definition of $(\mathfrak{L},\mathcal{U})$-completion entails that Ψ is injective, whereas the definition of $\mathbb{K}$ as the set of all possible Λ-limits entails that Ψ is surjective. Since it is immediate to see that Ψ also preserves the operations, we have that it is an isomorphism.

We will strengthen Corollary 1 in Theorem 4. By Corollary 1 it clearly follows that, if the $(\mathfrak{L},\mathcal{U})$ -completion is minimal, as sets $\mathbb{R}_{\mathfrak{L}}\cong(\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}}$.

Remark 5 Let us note that $((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}^{\mathfrak{L}}_{\mathcal{U}},\tau)$ is a $(\mathfrak{L},\mathcal{U})$-completion of $\mathbb{R}$ for different choices of τ. One such choice is the topology $\tau_{\mathcal{U}}$ introduced in the proof of Theorem 1; a different topology can be constructed as follows: let us fix a function φ with $\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))\notin\mathbb{R}$, a nonempty infinite set $B\notin\mathcal{U}$, a free filter $\mathcal{F}$ on B and

[2] A superreal non Archimedean field is an ordered field that properly contains $\mathbb{R}$.

let us consider the following topology $\widetilde{\tau}$ on $(\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}$: if $\xi \neq \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ then a family of open neighborhoods of ξ is

$$\left\{ O_{\psi,Q} \mid Q \in \mathscr{U}, \psi \text{ function with } \xi = \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \right\};$$

if $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ then a family of open neighborhoods of ξ is

$$\{O_{F,Q} \mid F \in \mathscr{F}, Q \in \mathscr{U}\}$$

where, for every $F \in \mathscr{F}$, $Q \in \mathscr{U}$ we set

$$O_{F,Q} = O_{\varphi,Q} \cup \{(\lambda, x) \mid \lambda \in F, x \in \mathbb{R}\}.$$

By construction, $\left((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}, \widetilde{\tau}\right)$ is a $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$.

A consequence of Remark 5 is that there are infinitely many topologies τ that make $((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}, \tau)$ a $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$. However, the topology introduced in the proof of Theorem 1 plays a central role in our approach. For this reason, we introduce the following definition.

Definition 7 Let $(\mathbb{R}_{\mathfrak{L}}, \tau)$ be a $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$. We call **slim topology**, and we denote by $\tau_{\mathscr{U}}$, the topology on $\mathbb{R}_{\mathfrak{L}}$ generated by the family of open sets

$$\left\{N_{\varphi,Q} \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}), Q \in \mathscr{U}\right\} \cup \mathscr{P}(\mathfrak{L} \times \mathbb{R})$$

where, for every $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R})$, $Q \in \mathscr{U}$ we set

$$N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \mid \lambda \in Q\} \cup \left\{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \right\}.$$

Proposition 3 *The slim topology $\tau_{\mathscr{U}}$ is finer than any topology τ that makes $\left((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}, \tau\right)$ a $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$.*

Proof Let τ be given, let O be an open set in τ and let $x \in O$. If $x \in \mathfrak{L} \times \mathbb{R}$ then $\{x\}$ is an open neighborhood of x in $\tau_{\mathscr{U}}$ contained in O; if $x = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ for some function $\varphi : \mathfrak{L} \to \mathbb{R}$ then let $B \in \mathscr{U}$ be such that $\{(\lambda, \varphi(\lambda)) \mid \lambda \in B\} \subseteq O$; therefore, by construction, $O_{\varphi,B}$ is an open neighborhood of x in $\tau_{\mathscr{U}}$ entirely contained in O. This proves that O is an open set in $\tau_{\mathscr{U}}$, therefore $\tau_{\mathscr{U}}$ is finer than τ.

The slim topology can also be characterized in terms of closure of subsets of $(\mathfrak{L} \times \mathbb{R})$:

Proposition 4 *Let* $\left((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}, \tau\right)$ *be a* $(\mathfrak{L}, \mathscr{U})$*-completion of* $\mathbb{R}$*. The following facts are equivalent:*

1. $\tau = \tau_{\mathscr{U}}$;
2. *for every set* $B \subseteq (\mathfrak{L} \times \mathbb{R})$ *we have that*

$$cl_\tau(B) = B \cup \left\{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \mid \exists A \in \mathscr{U}\ \forall \lambda \in A\ (\lambda, \varphi(\lambda)) \in B \right\}.$$

Proof (1) $\Rightarrow$ (2) Let $\varphi : \mathfrak{L} \to \mathbb{R}$, let $B \subseteq (\mathfrak{L} \times \mathbb{R})$ and let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let $A = \{\lambda \in \mathfrak{L} \mid (\lambda, \varphi(\lambda)) \in B\}$. If $A \in \mathscr{U}$ then for every open neighborhood O of ξ we have that $O \cap B \neq \emptyset$ by construction, so $\xi \in cl_{\tau_{\mathscr{U}}}(B)$; if $A \notin \mathscr{U}$ then $O_{\varphi,A}$ is a neighborhood of ξ such that $O_{\varphi,A} \cap B = \emptyset$, therefore $\xi \notin cl_{\tau_{\mathscr{U}}}(B)$.

(2) $\Rightarrow$ (1) Let $A \in \mathscr{U}$, let $\varphi : \mathfrak{L} \to \mathbb{R}$ and let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let us consider $B = (\mathfrak{L} \times \mathbb{R}) \setminus O_{A,\varphi}$. By hypothesis and construction

$$cl_\tau(B) = \left[(\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}\right] \setminus O_{A,\varphi}.$$

Therefore $O_{A,\varphi}$ is open for every $A \in \mathscr{U}$, $\varphi : \mathfrak{L} \to \mathbb{R}$, so τ is finer than $\tau_{\mathscr{U}}$ which, as a consequence of Proposition 3, entails that $\tau = \tau_{\mathscr{U}}$.

Definition 8 We will call $\left((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}, \tau_{\mathscr{U}}\right)$ the **canonical** $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$.

From the next section on we will work only with the minimal canonical $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$.

2.3 Natural Extension of Sets and Functions

From now on, $\overline{(\cdot)}$ will denote the closure operator in the canonical $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$.

Definition 9 For every $E \subseteq \mathbb{R}$ we set

$$E_{\mathfrak{L}} := \overline{\mathfrak{L} \times E}.$$

A different and related (as we will show in Proposition 5) extension of E is the following:

Definition 10 Given *a set* $E \subset \mathbb{R}$, we set

$$E^* := \left\{ \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \,|\psi(\lambda) \in E \right\};$$

E^* is called the **natural extension** of E.

Let us observe that by property (2) of the definition of $(\mathfrak{L}, \mathscr{U})$-completions it follows that $E \subseteq E^*$. Following the notation introduced in Definition 10, from now on we will denote $\mathbb{K}$ by $\mathbb{R}^*$.

It is easy to modify the proof of Proposition 1 to obtain the following result:

Proposition 5 *For every $E \subseteq \mathbb{R}$ we have that $E_{\mathfrak{L}} = (\mathfrak{L} \times E) \uplus E^*$.*

It is also possible to extend functions to $\mathbb{R}_{\mathfrak{L}}$. To this aim, given a function

$$f : A \to B$$

we will denote by

$$f_{\mathfrak{L}} : \mathfrak{L} \times A \to \mathfrak{L} \times B$$

the function defined as follows:

$$f_{\mathfrak{L}}(\lambda, x) = (\lambda, f(x)).$$

Lemma 1 *For every $A, B \subseteq \mathbb{R}$, for every function $f : A \to B$, f can be extended to a continuous function*

$$\overline{f_{\mathfrak{L}}} : A_{\mathfrak{L}} \to B_{\mathfrak{L}}.$$

Moreover, the restriction of $\overline{f_{\mathfrak{L}}}$ to A coincides with f.

Proof The extension of f to $\mathfrak{L} \times A$ is given by $f_{\mathfrak{L}}$. Therefore to get the desired extension to $A_{\mathfrak{L}}$ it is sufficient to extend $f_{\mathfrak{L}}$ on A^*. For every $\varphi \in A^{\mathfrak{L}}$ we set

$$\overline{f_{\mathfrak{L}}}\left(\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))\right) = \lim_{\lambda \to \Lambda} (\lambda, f(\varphi(\lambda))).$$

Let us note that the definition is well posed and that $\overline{f_{\mathfrak{L}}}(\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))) \in B^*$ since, for every $\varphi \in A^{\mathfrak{L}}$, the function $f \circ \varphi \in B^{\mathfrak{L}}$. This extension is continuous: let Ω be a basis open subset of $B_{\mathfrak{L}}$. If $\Omega = \{(\lambda, x)\}$ then

$$\overline{f_{\mathfrak{L}}}^{-1}(\Omega) = \bigcup_{y \in f^{-1}(x)} (\lambda, y),$$

which is open. If $\Omega = N_{\varphi, Q}$ for some $\varphi : \mathfrak{L} \to \mathbb{R}$, $Q \in \mathscr{U}$ then let $\xi \in \overline{f_{\mathfrak{L}}}^{-1}(\Omega)$. If $\xi = (\lambda, x)$ for some $x \in A$ then $\{(\lambda, x)\}$ is a neighborhood of (λ, x) included in $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$; if $\xi = \lim_{\lambda \to \Lambda}(\lambda, \psi(\lambda))$ then $\overline{f_{\mathfrak{L}}}(\xi) = \lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$, therefore there exists $Q_1 \in \mathscr{U}$ such that $f(\psi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in Q_1$, hence if we set $Q_2 = Q \cap Q_1$ we have that N_{ψ, Q_2} is a neighborhood of ξ included in $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$, thus $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$ is open, and this proves that $\overline{f_{\mathfrak{L}}}$ is continuous.

Finally, $\overline{f_{\mathfrak{L}}}$ restricted to A coincides with f since, for every $a \in A$, by definition

$$\overline{f_{\mathfrak{L}}}(a) = \overline{f_{\mathfrak{L}}}\left(\lim_{\lambda \to \Lambda} (\lambda, a)\right) = \lim_{\lambda \to \Lambda} (\lambda, f(a)) = f(a).$$

Lemma 1 entails that the following definition is well posed:

Definition 11 Given a function

$$f : A \to B$$

the restriction of $\overline{f_{\mathfrak{L}}}$ to A^* is called the **natural extension** of f and it will be denoted by

$$f^* : A^* \to B^*.$$

In particular, $f^*(a) = f(a)$ for every $a \in A$.

2.4 *The Λ-limit in $V_\infty(\mathbb{R})$*

In this section we want to extend the notion of Λ-limit to a wider family of functions. To do that, we have to introduce the notion of superstructure on a set (see also [21]):

Definition 12 Let E be an infinite set. The superstructure on E is the set

$$V_\infty(E) = \bigcup_{n \in \mathbb{N}} V_n(E),$$

where the sets $V_n(E)$ are defined by induction by setting

$$V_0(E) = E$$

and, for every $n \in \mathbb{N}$,

$$V_{n+1}(E) = V_n(E) \cup \mathscr{P}(V_n(E)).$$

Here $\mathscr{P}(E)$ denotes the power set of E. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $V_\infty(E)$ contains almost every usual mathematical object that can be constructed starting with E; in particular, $V_\infty(\mathbb{R})$ contains almost every usual mathematical object of analysis.

Sometimes, following e.g. [21], we will refer to

$$\mathbb{U} := V_\infty(\mathbb{R})$$

as to the **standard universe**. A mathematical entity (number, set, function or relation) is said to be **standard** if it belongs to $\mathbb{U}$.

Now we want to formally define the Λ-limit of $(\lambda, \varphi(\lambda))$ where $\varphi(\lambda)$ is any bounded function of mathematical objects in $V_\infty(\mathbb{R})$ (a function $\varphi : \mathfrak{L} \to V_\infty(\mathbb{R})$ is called bounded if there exists n such that $\forall\lambda \in \mathfrak{L},\ \varphi(\lambda) \in V_n(\mathbb{R})$). To this aim, let us consider a function

$$\varphi : \mathfrak{L} \to V_n(\mathbb{R}). \tag{4}$$

We will define $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ by induction on n.

Definition 13 For $n = 0$, $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ exists by Theorem 1; so by induction we may assume that the limit is defined for $n - 1$ and we define it for the function (4) as follows:

$$\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda)) = \left\{ \lim_{\lambda\to\Lambda}(\lambda, \psi(\lambda)) \mid \psi : \mathfrak{L} \to V_{n-1}(\mathbb{R}) \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

Clearly $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ is a well defined set in $V_\infty(\mathbb{R}^*)$.

Definition 14 A mathematical entity (number, set, function or relation) which is the Λ-limit of a function is called **internal**.

Notice that $V_\infty(\mathbb{R}^*)$ contains sets which are not internal.

Example 1 Each real number is standard and internal. However the set of real numbers $\mathbb{R} \in V_\infty(\mathbb{R}^*)$ is standard, but not internal. In order to see this let us suppose that there is a function $\varphi : \mathfrak{L} \to V_1(\mathbb{R})$ such that $\mathbb{R} = \lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$. Therefore, by definition, we would have

$$\mathbb{R} = \left\{ \lim_{\lambda\to\Lambda}(\lambda, \psi(\lambda)) \mid \psi : \mathfrak{L} \to \mathbb{R} \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

In particular, for every constant $c \in \mathbb{R}$ we have that $c \in \varphi(\lambda)$; therefore, $\varphi(\lambda) = \mathbb{R}$ for every $\lambda \in \mathfrak{L}$, and this is absurd because

$$\lim_{\lambda\to\Lambda}(\lambda, \mathbb{R}) = \mathbb{R}^*,$$

and (except trivial cases) $\mathbb{R}^*$ properly includes $\mathbb{R}$. Let us explicitly observe that (except trivial cases), while for every $c \in \mathbb{R}$ the function $\lambda \to (\lambda, c)$ converges to c, given $A \in V_n(\mathbb{R})$, for $n \geq 1$ the function $\lambda \to (\lambda, A)$ converges to a proper superset of A.

Definition 15 A mathematical entity (number, set, function or relation) which is not internal is called **external.**

As it is given, the definition of limit given by Definition 13 is not related to any topology. Thus a question arises naturally: is there a topological Hausdorff space such that the limit given by Definition 13 is the topological limit of a function?

The answer is affirmative, and it is a consequence of the possibility to topologize the set

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \uplus V_\infty(\mathbb{R}^*).$$

To topologize $\mathbb{U}_{\mathfrak{L}}$ we take as open sets:

- every subset of $\mathfrak{L} \times V_\infty(\mathbb{R})$;
- $\{x\}$ for every $x \in V_\infty(\mathbb{R}^*)$ that is external;
- $N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \,|\, \lambda \in Q\} \cup \{x\}$ for every x internal such that φ is a bounded sequence with
$$x = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) .$$

We let $\sigma_{\mathscr{U}}$ be the topology on $\mathbb{U}_{\mathfrak{L}}$ generated by these open sets. It is clear that this topology is Hausdorff and that the Λ-limit is a limit in this topology.

The set

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \cup V_\infty(\mathbb{R}^*)$$

will be called the **expanded universe**. Let us note that, by construction, $\mathbb{U}_{\mathfrak{L}} \subseteq V_\infty(\mathbb{R}_{\mathfrak{L}})$.

The results about extensions of subsets of $\mathbb{R}$ and of functions $f : A \to B$, $A, B \subseteq \mathbb{R}$, can be generalized to our new general setting. Since a function f can be identified with its graph then the natural extension of a function is defined by the above definition. Moreover we have the following result, that can be proved as Lemma 1:

Theorem 3 *For every sets $E, F \in V_\infty(\mathbb{R})$ and for every function $f : E \to F$ the* ***natural extension*** *of f is a continuous function*

$$f^* : E^* \to F^*,$$

and for every function $\varphi : \mathfrak{L} \to E$ we have that

$$\lim_{\lambda \to \Lambda} f(\lambda, \varphi(\lambda)) = f^* \left(\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \right).$$

3 Comparison Between Λ-theory and Ultrapowers

3.1 Λ-theory and Nonstandard Universes

It should be evident to any reader with a background in NSA that Λ -theory (when restricted to minimal canonical extensions) is closely related to ultrapowers (which, from a purely logical point of view, are even easier to define). In this section we want to detail the relationship between Λ-theory and NSA. We will show that $\mathbb{U}_{\mathfrak{L}}$ contains a nonstandard universe in the sense of Keisler [21]. We recall the main definitions of [21].

Definition 16 A **superstructure embedding** is a one to one mapping $*$ of $V_\infty(\mathbb{R})$ into another superstructure $V_\infty(\mathbb{S})$ such that

1. $\mathbb{R}$ is a proper subset of $\mathbb{S}$, $r^* = r$ for all $r \in \mathbb{R}$, and $\mathbb{R}^* = \mathbb{S}$;
2. for $x, y \in V_\infty(\mathbb{R})$, $x \in y$ if and only if $x^* \in y^*$.

To avoid confusion, in this section we will use the letter $\mathbb{K}$ to denote the Non-Archimedean field constructed in Sect. 2.2, while $\mathbb{R}^*$ will be used as in Definition 16.

Let us denote by $\mathscr{L}$ a formal language relative to a first order predicate logic with the equality symbol, a binary relation symbol $\in$, and a constant symbol for each element in $V_\infty(\mathbb{R})$. We recall that a sentence $p \in \mathscr{L}$ is bounded if every quantifier in p is bounded (see e.g. [21]). The notion of bounded sequence allows to define the notion of nonstandard universe.

Definition 17 A **nonstandard universe** is a superstructure embedding $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{R}^*)$ which satisfies Leibniz' Principle, which is the property that states that for each bounded sentence $p \in \mathscr{L}$, p is true in $V_\infty(\mathbb{R})$ if and only if p^* is true[3] in $V_\infty(\mathbb{R}^*)$.

Definition 18 We let $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ be the map defined as follows: for every element $x \in V_\infty(\mathbb{R})$ we set

$$x^* = \lim_{\lambda \to \Lambda} (\lambda, x).$$

Remark 6 Following Keisler (see [21]), in Definition 17 we have called nonstandard universe just the superstructure embedding; however, in our approach, probably, it would be more appropriate to call nonstandard universe the set $V_\infty(\mathbb{K})$; in this case the global picture would be the following one: the extended universe

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \uplus V_\infty(\mathbb{K})$$

contains pairs (λ, x) and elements of the nonstandard universe $V_\infty(\mathbb{K})$; the latter contains the following objects:

- standard elements, namely objects $x \in V_\infty(\mathbb{R}) \subset V_\infty(\mathbb{K})$;
- nonstandard elements, namely objects $x \in V_\infty(\mathbb{K}) \backslash V_\infty(\mathbb{R})$;
- hyperimages, namely objects x such that there exists $y \in V_\infty(\mathbb{R})$ with $x = y^*$;
- internal objects, namely Λ-limits of bounded functions;
- external objects.

To give some examples: 7, $\mathbb{R}$, $\mathscr{P}(\mathbb{R} \times \mathscr{P}(\mathbb{R}))$ are all standard elements; 7 is also an hyperimage, while $\mathbb{R}$, $\mathscr{P}(\mathbb{R} \times \mathscr{P}(\mathbb{R}))$ are not; $\mathbb{K}$, $\mathscr{P}(\mathbb{R})^*$ and $\lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$

[3] p^* is the bounded sentence obtained by changing every constant symbol $c \in V_\infty(\mathbb{R})$ that appears in p with c^*.

for every $\varphi : \mathfrak{L} \to \mathbb{R}$ which is not eventually constant are nonstandard elements, and they are all internal; $\mathbb{R}$ and $\mathbb{K}\backslash\mathbb{R}$ are external objects.

An interesting class of internal objects, particularly important for our applications to PDEs, is that of hyperfinite objects[4]:

Definition 19 An object $\xi \in V_\infty(\mathbb{K})$ is hyperfinite if there exists a natural number n and a bounded function $\varphi : \mathfrak{L} \to \mathscr{P}_{fin}(V_n(\mathbb{R}))$ such that $\xi = \lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$.

Hyperfinite objects are the analogue, in the universe $V_\infty(\mathbb{K})$, of finite objects in $V_\infty(\mathbb{R})$. The notion of hyperfinite object will be used in Sect. 4 to show some applications of Λ-theory.

To detail the relationship between Λ-theory and nonstandard universes in the sense of Keisler we need to specify how we interpret formulas in $V_\infty(\mathbb{K})$[5]:

Definition 20 Let $p(x_1, \ldots, x_n) \in \mathscr{L}$ be a bounded formula having $x_1, \ldots, x_n$ as its only free variables. Let $\xi_1 = \lim_{\lambda\to\Lambda}(\lambda, \varphi_1(\lambda)), \ldots, \xi_n = \lim_{\lambda\to\Lambda}(\lambda, \varphi_n(\lambda))$. We say that $p^*(\xi_1, \ldots, \xi_n)$ holds in $V_\infty(\mathbb{K})$ iff $p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda))$ is eventually true in $V_\infty(\mathbb{R})$, namely iff

$$\{(\lambda, (\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \mid p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \cup \{(\xi_1, \ldots, \xi_n)\}$$

is open in $\sigma_{\mathscr{U}}$.

Theorem 4 *Let $*$ be defined as in Definition 18; then*

$$(V_\infty(\mathbb{R}), V_\infty(\mathbb{K}), *)$$

is a nonstandard universe.

Proof That $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ is a superstructure embedding follows clearly from the definitions.

Moreover, for every bounded formula $p(x_1, \ldots, x_n) \in \mathscr{L}$ having $x_1, \ldots, x_n$ as its only free variables, for every $\xi_1 = \lim_{\lambda\to\Lambda}(\lambda, \varphi_1(\lambda)), \ldots, \xi_n = \lim_{\lambda\to\Lambda}(\lambda, \varphi_n(\lambda))$, we have that

$$\begin{gathered} p(\xi_1, \ldots, \xi_n) \text{ holds in } V_\infty(\mathbb{K}) \Leftrightarrow \\ \{(\lambda, (\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \mid p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \cup \{(\xi_1, \ldots, \xi_n)\} \\ \text{is open in } \sigma_{\mathscr{U}} \Leftrightarrow \\ \{\lambda \in \mathscr{L} \mid p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \in \mathscr{U} \Leftrightarrow \\ p([\varphi_1], \ldots, [\varphi_n]) \text{ holds in } \mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}. \end{gathered}$$

This equivalence can be used to easily prove the transfer property for $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ by induction on the complexity of formulas.

[4]See e.g. [1], where many different applications of hyperfinite objects and other nonstandard tools are developed.

[5]Once again, it should be evident to readers expert in NSA that our definition is precisely analogous to the one that is given for ultrapowers.

3.2 General Remarks

Theorem 4 makes precise the intuition that the topological approach to Non-Archimedean mathematics given by Λ-theory is closely related with NSA as presented by Keisler in [21]. As we said in the introduction, we think of Λ-theory as a way to present to a non-expert reader many basic ideas of NSA in a more familiar language. Nevertheless, we think that from a philosophical point of view point there are some differences between Λ-theory and the ultrapower approach:

1. in Λ-theory we assume the existence of a unique mathematical universe $\mathbb{U}_{\mathfrak{L}} \subset V_{\infty}(\mathfrak{L} \cup \mathbb{K})$. Inside this universe there are entities that do not appear in traditional mathematics but that can be obtained as limits of traditional objects, namely the internal elements. Moreover, there are also external objects, and some of them are objects of traditional mathematics (e.g., $\mathbb{R}$);
2. in NSA the primitive concept is that of hyperimage, the other concepts (e.g., the concept of internal object) are derived by that one; in Λ -theory, the primitive concept is that of Λ-limit, while the concept of hyperimage is derived by the limit. So, within Λ-theory the notion of internal object (being defined as a Λ-limit) is more primitive than that of hyperimage;
3. the construction of the hyperreal field in our approach has a topological "flavour" which is similar to other constructions in traditional mathematics. In fact, e.g. whitin our approach the construction of $\mathbb{R}^*$ as "set of limits of functions with values in $\mathfrak{L} \times \mathbb{R}$" has some similarities with the construction of $\mathbb{R}$ as set of limits of Cauchy sequences with values in $\mathbb{Q}$.

4 Generalized Solutions

In many circumstances, the notion of function is not sufficient to the needs of a theory and it is necessary to extend it. Many different constructions have been considered in the literature to deal with this problem, both with standard (for example, Colombeau's Theory, see e.g. [19] and references therein for a complete presentation of the theory and [18] and reference therein for some new developments of the theory with applications to generalized ODE's) and nonstandard techniques (see e.g. [26]). In this section we want to apply Λ-theory to construct spaces of generalized functions called ultrafunctions (see also [5, 9–14], and to use them to study a simple class of problems in calculus of variations. As we are going to show, ultrafunctions are constructed by means of a particular version of the hyperfinite approach which can be naturally introduced by means of Λ-theory.

In this section we will use the following shorthand notation: for every bounded function $\varphi : \mathfrak{L} \to V_{\infty}(\mathbb{R})$ we let

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) .$$

4.1 Ultrafunctions

Let N be a natural number, let Ω be a set in $\mathbb{R}^N$ and let $V(\Omega)$ be a function vector space. We want to define the space of ultrafunctions generated by $V(\Omega)$. We assume that

$$\mathfrak{L} = \mathscr{P}_{fin}(V(\Omega)),$$

and we let $\mathscr{U}$ be a fine ultrafilter[6] on $\mathfrak{L}$. For any $\lambda \in \mathfrak{L}$, we set

$$V_\lambda(\Omega) = Span\{\lambda \cap V(\Omega)\}.$$

Let us note that, by construction, $V_\lambda(\Omega)$ is a finite dimensional vector subspace of $V(\Omega)$.

Definition 21 Given the function space $V(\Omega)$ we set

$$V_\Lambda(\Omega) := \lim_{\lambda \uparrow \Lambda} V_\lambda(\Omega) = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V_\lambda(\Omega) \right\}.$$

$V_\Lambda(\Omega)$ will be called the **space of ultrafunctions** generated by $V(\Omega)$.

Given any vector space of functions $V(\Omega)$, we have the following three properties:

1. the ultrafunctions in $V_\Lambda(\Omega)$ are Λ-limits of functions valued in $V(\Omega)$, so they are all internal functions;
2. the space of ultrafunctions $V_\Lambda(\Omega)$ is a vector space of hyperfinite dimension;
3. if we identify a function f with its natural extension f^* then $V_\Lambda(\Omega)$ includes $V(\Omega)$, hence we have that

$$V(\Omega) \subset V_\Lambda(\Omega) \subset V(\Omega)^*.$$

Remark 7 Notice that the natural extension f^* of a function f is an ultrafunction if and only if $f \in V(\Omega)$.

Proof The proof of this result is trivial.[7]

Ultrafunctions can be used to give generalized solutions to some problems in the calculus of variations (see e.g. [11]). Usually this kind of problems have a "natural space" where to look for solutions: the appropriate function space has to be a space in which the problem is well posed and (relatively) easy to solve. For a very large class of problems the natural space is a Sobolev space. However, many times even the best candidates to be natural spaces are inadequate to study the problem, since there is no solution in them. So the choice of the appropriate function space is part of the problem

[6] Let us recall that an ultrafilter $\mathscr{U}$ on $\mathfrak{L}$ is fine if for every $\lambda \in \mathfrak{L}$ the set $\{\mu \in \mathfrak{L} \mid \mu \subseteq \lambda\} \in \mathscr{U}$. We also point out that, for more complicated applications, it would be better to take $\mathfrak{L} = \mathscr{P}_{fin}(V_\infty(\mathbb{R}))$.
[7] Any interested reader can find it in [10].

itself; this choice is somewhat arbitrary and it might depend on the final goals. In the framework of ultrafunctions this situation persists. The general rule is: choose the "natural space" $V(\Omega)$ and look for a generalized solution in $V_\Lambda(\Omega)$. For many applications, an hypothesis[8] that we need to assume is that $D(\Omega) \subseteq V(\Omega) \subseteq L^2(\Omega)$. In this case, since $V_\Lambda(\Omega) \subset \left[L^2(\Omega)\right]^*$, we can equip $V_\Lambda(\Omega)$ with the following scalar product:

$$(u, v) = \int^* u(x)v(x)\, dx, \tag{5}$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \to \mathbb{R}.$$

The norm[9] of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2\, dx\right)^{\frac{1}{2}}.$$

Moreover, using the inner product (5), we can identify $L^2(\Omega)$ with a subset of $V'(\Omega)$ and hence $\left[L^2(\Omega)\right]^*$ with a subset of $\left[V'(\Omega)\right]^*$; in this case, $\forall f \in \left[L^2(\Omega)\right]^*$, we let $\widetilde{f}$ be the unique ultrafunction such that, $\forall v \in V_\Lambda(\Omega)$,

$$\int^* \widetilde{f}(x)v(x)\, dx = \int^* f(x)v(x)\, dx,$$

namely we associate to every $f \in L^2(\Omega)^*$ the function $\widetilde{f} = P_\Lambda(f)$, where

$$P_\Lambda : \left[L^2(\Omega)\right]^* \to V_\Lambda(\Omega)$$

is the orthogonal projection.

Remark 8 There are a few different ways to prove the existence of an orthogonal projection of $L^2(\Omega)^*$ on $V_\Lambda(\Omega)$. For example, consider, for every $\lambda \in \mathfrak{L}$, the orthogonal projection $P_\lambda : L^2(\Omega) \to V_\lambda(\Omega)$. Let $F := \lim_{\lambda \uparrow \Lambda} P_\lambda$. It is immediate to see that $F : L^2(\Omega)^* \to V_\Lambda(\Omega)$ is an orthogonal projection.

Let us note that the key property to associate an ultrafunction to every function in $\left[L^2(\Omega)\right]^*$ is that $\left[L^2(\Omega)\right]^*$ can be identified with a subset of $\left[V'(\Omega)\right]^*$. Therefore, using a similar idea, it is also possible to extend a large class of operators:

[8] E.g., in [12] a (slightly modified) version of this hypothesis is used to construct an embedding of the space of distributions in a particular algebra of functions constructed by means of ultrafunctions.

[9] Let us observe that both the scalar product and the norm take values in $\mathbb{R}^*$.

Definition 22 Given an operator

$$\mathscr{A} : V(\Omega) \to V'(\Omega),$$

we can extend it to an operator

$$\widetilde{\mathscr{A}} : V_\Lambda(\Omega) \to V_\Lambda(\Omega)$$

in the following way: given an ultrafunction u, $\mathscr{A}_\Lambda(u)$ is the unique ultrafunction such that

$$\forall v \in V_\Lambda(\Omega), \int^* \widetilde{\mathscr{A}}(u)v dx = \int^* \mathscr{A}^*(u)v dx;$$

namely

$$\widetilde{\mathscr{A}} = P_\Lambda \circ \mathscr{A}^*,$$

where P_Λ is the canonical projection.

This association can be used, e.g., to define the derivative of an ultrafunction, by setting

$$Du := \widetilde{\partial} u = P_\Lambda(\partial^* u)$$

for every ultrafunction $u \in V_\Lambda(\Omega) \cap \mathscr{C}^1(\Omega)^*$.

4.2 Applications to Calculus of Variations

To give an example of application of ultrafunctions to calculus of variations, we will show the ultrafunction interpretation of the Lavrentiev phenomenon. Let us consider the following problem: minimize the functional

$$J_0(u) = \int_0^1 \left[\left(|\nabla u|^2 - 1 \right)^2 + |u|^2 \right] dx$$

in the function space $\mathscr{C}_0^1(\Omega) = \mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$. We assume Ω to be bounded to avoid problems of summability.[10]

It is not difficult to realize that any minimizing sequence u_n converges uniformly to 0 and that $J_0(u_n) \to 0$, but $J_0(0) > 0$ for any $u \in \mathscr{C}_0^1(0, 1)$. Hence there is no minimizer in $\mathscr{C}_0^1(\Omega)$.

[10]This example has already been studied in greater detail in [11].

On the contrary, it is possible to show that this problem has a minimizer in the space of ultrafunctions

$$V_0^1(\Omega) = \left[\mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})\right]_\Lambda .$$

In $V_0^1(\Omega)$ our problem becomes

$$\text{find } v \in V_0^1(\Omega) \text{ s.t. } \widetilde{J}_0(v) = \min_{u \in V_0^1(\Omega)} \widetilde{J}_0(u). \qquad (P)$$

To solve (P), let us prove the following "ultrafunction version" of an existence result for minimizers of coercive continuous operators; the proof is based on a variant of Faedo-Galerkin method.

Theorem 5 *Let $V(\Omega) \subseteq L^2(\Omega)$ be a vector space and let*

$$J : V(\Omega) \to \mathbb{R}$$

be an operator continuous and coercive on finite dimensional spaces. Then the operator

$$\widetilde{J} : V_\Lambda(\Omega) \to \mathbb{R}^*$$

has a minimum point. If J itself has a minimizer u, then u^* is a minimizer of $\widetilde{J}$.

Proof Take $\lambda \in \mathfrak{L}$; since the operator

$$J|_{V_\lambda} : V_\lambda(\Omega) \longrightarrow \mathbb{R}$$

is continuous and coercive, it has a minimizer; namely

$$\exists u_\lambda \in V_\lambda \; \forall v \in V_\lambda \; J(u_\lambda) \leq J(v).$$

We set

$$u_\Lambda = \lim_{\lambda \uparrow \Lambda} u_\lambda .$$

We show that u_Λ is a minimizer of $\widetilde{J}$. Let $v \in V_\Lambda(\Omega)$. Let us suppose that $v = \lim_{\lambda \uparrow \Lambda} v_\lambda$; then by construction

$$\forall \lambda \in \mathfrak{L} \; J(u_\lambda) \leq J(v_\lambda),$$

therefore

$$\widetilde{J}(u_\Lambda) \leq \widetilde{J}(v).$$

If J itself has a minimizer $\overline{u}$, then u_λ is eventually equal to $\overline{u}$ and hence $u_\Lambda = \overline{u}^*$.

As a consequence, problem (P) has a solution, since the functional J_0 satisfies the hypothesis of Theorem 5. So there exists an ultrafunction $u \in V_0^1(\Omega)$ that minimizes $\widetilde{J}_0$. Moreover, it can be represented as the Λ-limit of a function of minimizers of the approximate problems on the spaces $\left[\mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})\right]_\lambda$. By using this characterization, it is also possible to derive some qualitative properties of u, e.g. it is not difficult to show that, $\forall x \in (0,1)^*$, the minimizer $u_\Lambda(x) \sim 0$ and that $\widetilde{J}_0(u_\Lambda)$ is a positive infinitesimal.

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