Chapter 2
Homogeneous Deformation without Interagent Communication

In this chapter, basics of MAS evolution as continuum deformation are presented, where an MAS is treated as particles of a continuum deforming under a homogeneous transformation. It is shown how a homogeneous deformation is acquired by the agents via no interagent communication. In this regard, agents' desired positions, defined by a homogeneous deformation in $\mathbb{R}^n$, are uniquely specified by the trajectories chosen by $n+1$ leaders. Followers acquire the desired homogeneous mapping only by knowing leaders' positions. Evolution of the followers with nonlinear constrained dynamics under a homogeneous transformation is investigated in this chapter. Homogeneous transformation of an MAS containing agents with linear dynamics is also studied.

2.1 Homogeneous Transformation

A continuum (deformable body) is a continuous region in $\mathbb{R}^n$ (n=1,2,3) containing infinite number of particles with infinitesimal size [64]. A continuum deformation is defined by the mapping $r(R,t)$, where $r \in \mathbb{R}^n$ denotes current position of a material particle which was initially placed at $R \in \mathbb{R}^n$. Notice that positions of the agents at the initial time t_0 is called *material coordinate* and denoted by R. A schematic of a continuum deformation is shown in Fig. 2.1.

It is noted that the Jacobian of the continuum deformation that is denoted by

$$Q(R,t) = \frac{\partial r}{\partial R} \in \mathbb{R}^{n\times n}. \tag{2.1}$$

is nonsingular, where $Q(R,t_0) = I_n$ ($I_n \in \mathbb{R}^{n\times n}$) is the identity matrix and t_0 denotes the initial time. If the Jacobian matrix is only a function of time, then the continuum

H. Rastgoftar, *Continuum Deformation of Multi-Agent Systems*,
DOI 10.1007/978-3-319-41594-9_2

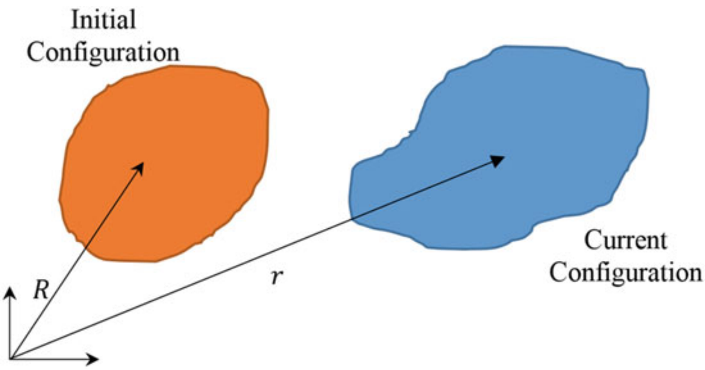

Fig. 2.1 Schematic of a continuum deformation

deformation is called *homogeneous transformation*. Therefore, a homogeneous transformation is defined by

$$r(t) = Q(t)R + D(t), \tag{2.2}$$

where $D(t) \in \mathbb{R}^n$ is called *rigid body displacement vector*.

2.1.1 Homogeneous Deformation of the Leading Polytope

Because homogeneous transformation is a linear mapping, elements of $Q(t)$ and $D(t)$ can be uniquely related to the components of the positions of $n+1$ leaders. Suppose each leader agent is identified by a number $k \in V_L$ and $V_L = \{1,\ 2, \ldots,\ n+1\}$. Leader agents are placed at the vertices of a leading polytope in $\mathbb{R}^n$, called *leading polytope*, therefore leaders' positions satisfy the following rank condition:

$$\forall t \geq t_0, Rank\left[r_2 - r_1\ \ldots\ r_{n+1} - r_1\right] = n. \tag{2.3}$$

Let R and r in Eq. (2.2) be substituted by the leaders' initial and current positions, respectively, then the following $n+1$ equations are obtained:

$$\begin{cases} r_1(t) = Q(t)R_1 + D(t) \\ r_2(t) = Q(t)R_2 + D(t) \\ \vdots \\ r_{n+1}(t) = Q(t)R_{n+1} + D(t) \end{cases}. \tag{2.4}$$

r_k and R_k ($k \in V_L$) can be expressed with respect to the Cartesian coordinate system with the unit basis ($\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\dots,\ \hat{\mathbf{e}}_n$) as follows:

$$r_k = \sum_{q=1}^{n} x_{q,k}\hat{\mathbf{e}}_q \tag{2.5}$$

$$R_k = \sum_{q=1}^{n} X_{q,k}\hat{\mathbf{e}}_q, \tag{2.6}$$

where $X_{q,k}$ and $x_{q,k}$ are the q^{th} components of the initial and current positions of the agent k. Then

$$\begin{cases} x_{q,1}(t) = \begin{bmatrix} X_{1,1} & X_{2,1} & \dots & X_{n,1} \end{bmatrix} \begin{bmatrix} Q_{q1} & Q_{q2} & \dots & Q_{qn} \end{bmatrix}^T + D_q \\ x_{q,2}(t) = \begin{bmatrix} X_{1,2} & X_{2,2} & \dots & X_{n,2} \end{bmatrix} \begin{bmatrix} Q_{q1} & Q_{q2} & \dots & Q_{qn} \end{bmatrix}^T + D_q \\ \vdots \\ x_{q,n+1}(t) = \begin{bmatrix} X_{1,n+1} & X_{2,n+1} & \dots & X_{n,n+1} \end{bmatrix} \begin{bmatrix} Q_{q1} & Q_{q2} & \dots & Q_{qn} \end{bmatrix}^T + D_q \end{cases} \tag{2.7}$$

It is noted that $q \in \{1,\ 2,\ \dots,\ n\}$, $D_q \in R$ is the q^{th} entry of the vector $D \in \mathbb{R}^n$ and Q_{qj} is the qj entry of the Jacobian matrix $Q \in \mathbb{R}^{n\times n}$. The above set of $n+1$ equations can be simplified as

$$U_q = L_0 Q_q + D_q \mathbf{1}$$

where

$$U_q = \begin{bmatrix} x_{q,1} & \dots & x_{q,n+1} \end{bmatrix}^T \in \mathbb{R}^{n+1},$$

$$Q_q = \begin{bmatrix} Q_{q1} & Q_{q2} & \dots & Q_{qn} \end{bmatrix}^T \in \mathbb{R}^{n},$$

$$\mathbf{1} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T \in \mathbb{R}^{n+1},$$

$$L_0 = \begin{bmatrix} X_{1,1} & \dots & X_{n,1} \\ \vdots & & \vdots \\ X_{1,n+1} & \dots & X_{n,n+1} \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}.$$

Therefore, entries of Q and D can be related to the components of the leaders' positions by

$$P_t = \begin{bmatrix} I_n \otimes L_0 & I_n \otimes \mathbf{1} \end{bmatrix} J_t \tag{2.8}$$

where

$$J_t = \left[Q_1{}^T \ \dots \ Q_n{}^T \ D^T\right]^T \in \mathbb{R}^{(n+1)n} \tag{2.9}$$

$$P_t = \left[U_1{}^T \ \dots \ U_n{}^T\right]^T \in \mathbb{R}^{(n+1)n}. \tag{2.10}$$

***Followers' Distribution*:** Let each follower be identified by the index $i \in V_F$, where $V_F = \{n+2, \ \dots, \ N\}$. It is assumed that followers are all placed inside the leading polytope at the initial time t_0. Because leaders' positions satisfy the rank condition (2.3), $r_i(t)$ (position of the follower $i \in V_F$ at the time t) can be uniquely expanded as a linear combination of the leaders' positions as follows:

$$\begin{aligned} r_i(t) &= r_1(t) + \sum_{k=2}^{n+1} p_{i,k}(t)(r_k(t) - r_1(t)) = (1 - \sum_{k=2}^{n+1} p_{i,k}(t))r_1(t) + \sum_{k=2}^{n+1} p_{i,k}(t)r_k(t) \\ &= \sum_{k=1}^{n+1} p_{i,k}(t)r_k(t). \end{aligned} \tag{2.11}$$

Notice that

$$\sum_{k=1}^{n+1} p_{i,k}(t) = 1. \tag{2.12}$$

By considering Eqs. (2.11) and (2.12), parameter $P_{i,k}(t)$ is uniquely obtained by solving the following set of linear algebraic equations:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n+1} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n+1} \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} p_{i,1} \\ p_{i,2} \\ \vdots \\ p_{i,n} \\ p_{i,n+1} \end{bmatrix} = \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{n,i} \\ 1 \end{bmatrix}, \tag{2.13}$$

where $x_{q,j}$ denotes the q^{th} component of the position of the agent $j \in \{i, \ 1, \ 2, \dots, \ n+1\}$. If leaders and followers deform under a homogeneous mapping, then parameters $p_{i,k}(i \in V_F$ and $k \in V_L)$ remain constant at any time $t \geq t_0$. Therefore, desired position of the followers $i \in V_F$, defined by a homogeneous deformation, is given by

$$r_{i,HT}(t) = \sum_{k=1}^{n+1} \alpha_{i,k} r_k, \tag{2.14}$$

where $\alpha_{i,k}$ is uniquely determined based on the initial positions of the follower i and $n+1$ leaders by solving the following set of linear algebraic equations:

$$\begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,n+1} \\ X_{2,1} & X_{2,2} & \dots & X_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,1} & X_{n,2} & \dots & X_{n,n+1} \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ \vdots \\ \alpha_{i,n} \\ \alpha_{i,n+1} \end{bmatrix} = \begin{bmatrix} X_{1,i} \\ X_{2,i} \\ \vdots \\ X_{n,i} \\ 1 \end{bmatrix}. \tag{2.15}$$

2.1.2 Homogeneous Deformation of the Leading Triangle

In this subsection homogeneous deformation of an MAS in a plane is considered. Let leader agents be placed at vertices of a triangle, called *leading triangle*, and followers be initially placed inside the leading triangle. Schematic of homogeneous deformation of an MAS in a plane is shown in Fig. 2.2. Because three leaders remain nonaligned, the rank condition (2.3) is simplified to

$$\forall t \geq t_0, Rank\left[r_2 - r_1 \;\; r_3 - r_1\right] = 2. \tag{2.16}$$

Additionally, the desired position of the follower i, defined by a homogeneous deformation, is given by

$$r_{i,HT}(t) = \alpha_{i,1}r_1 + \alpha_{i,2}r_2 + \alpha_{i,3}r_3, \tag{2.17}$$

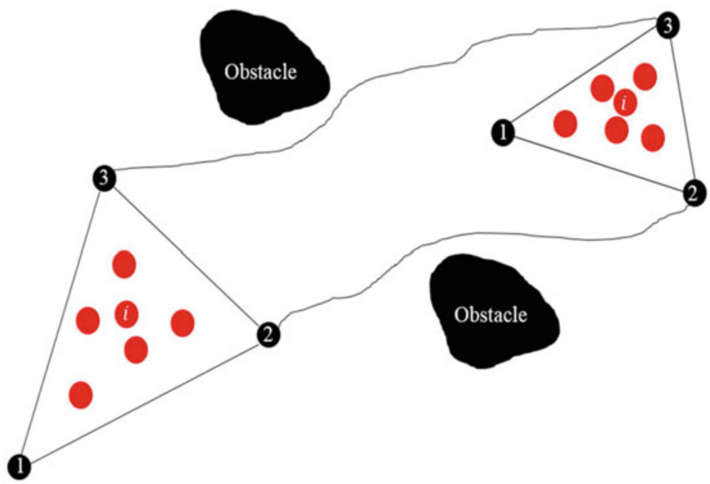

Fig. 2.2 Schematic of homogeneous deformation of an MAS in a plane

where $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ are the unique solution of

$$\begin{bmatrix} X_1 & X_1 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ \alpha_{i,3} \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix}. \tag{2.18}$$

Interpretation of parameters $\boldsymbol{\alpha_{i,k}}$ **(**$i \in V_L$**,** $k \in V_F$**):** By solving set of linear algebraic equations in (2.18), parameters $\alpha_{i,1}$,$\alpha_{i,2}$, and $\alpha_{i,3}$ are obtained as follows:

$$\begin{cases} \alpha_{i,1} = \dfrac{X_i(Y_2-Y_3)+Y_i(X_3-X_2)+X_2Y_3-X_3Y_2}{X_1(Y_3-Y_2)+X_2(Y_1-Y_3)+X_3(Y_2-Y_1)} \\ \alpha_{i,2} = \dfrac{X_i(Y_3-Y_1)+Y_i(X_1-X_3)+X_3Y_1-X_1Y_3}{X_1(Y_3-Y_2)+X_2(Y_1-Y_3)+X_3(Y_2-Y_1)} \\ \alpha_{i,3} = \dfrac{X_i(Y_1-Y_2)+Y_i(X_2-X_1)+X_1Y_2-X_2Y_1}{X_1(Y_3-Y_2)+X_2(Y_1-Y_3)+X_3(Y_2-Y_1)} \end{cases}. \tag{2.19}$$

Equation (2.19) can be rewritten as follows:

$$\begin{cases} \alpha_{i,1} = \dfrac{(X_3-X_2)(Y_i-Y_2)-(Y_3-Y_2)(X_i-X_2)}{(X_3-X_2)(Y_1-Y_2)-(Y_3-Y_2)(X_1-X_2)} \\ \alpha_{i,2} = \dfrac{(X_1-X_3)(Y_i-Y_3)-(Y_1-Y_3)(X_i-X_3)}{(X_1-X_3)(Y_2-Y_3)-(Y_1-Y_3)(X_2-X_3)} \\ \alpha_{i,3} = \dfrac{(X_2-X_1)(Y_i-Y_1)-(Y_2-Y_1)(X_i-X_1)}{(X_2-X_1)(Y_3-Y_1)-(Y_2-Y_1)(X_3-X_1)} \end{cases}. \tag{2.20}$$

Parameters $\alpha_{i,k} = c$ (c is a constant parameter.) in Eq. (2.20) define lines that are parallel to the sides of the leading triangle. Parallel lines associated with $\alpha_{i,k} = c$ are depicted in Fig. 2.3. For instance, $\alpha_{i,3} = c$ represents a line that is parallel to

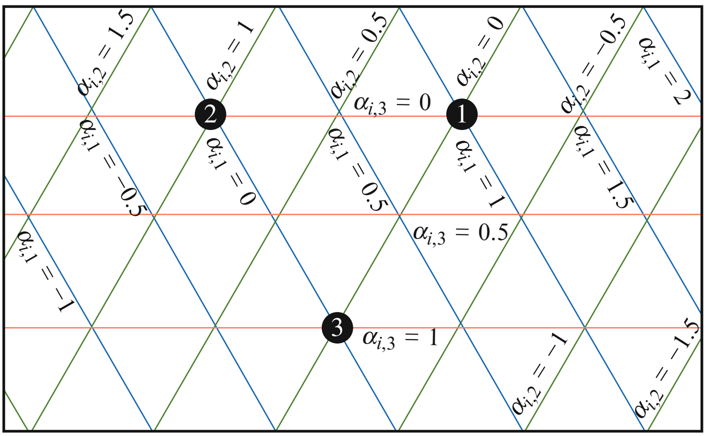

Fig. 2.3 Parameters $\alpha_{i,k}$=constant

the side s_{12}, i.e., s_{12} is the line segment connecting vertices 1 and 2 of the leading triangle. Notice that $\alpha_{i,k} = 1$ passes through the vertex k. Also, $\alpha_{i,k} = 0$ defines a line coinciding on the side of the leading triangle that is not passing through the vertex k. Moreover, $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ are all positive, if the follower i is located inside the leading triangle.

Entries of the matrix $Q \in \mathbb{R}^{2\times 2}$($Q_{11}$, Q_{12}, Q_{21}, and Q_{22}) and the vector $D \in \mathbb{R}^2$ (D_1 and D_2) can be uniquely obtained based on X and Y components of the leaders' positions as follows:

$$\begin{bmatrix} Q_{11}(t) \\ Q_{12}(t) \\ Q_{21}(t) \\ Q_{22}(t) \\ D_1(t) \\ D_2(t) \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 & 0 & 0 & 1 & 0 \\ X_2 & Y_2 & 0 & 0 & 1 & 0 \\ X_3 & Y_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & X_1 & Y_1 & 0 & 1 \\ 0 & 0 & X_2 & Y_2 & 0 & 1 \\ 0 & 0 & X_3 & Y_3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}. \tag{2.21}$$

***Spatial Description of Homogeneous Deformation*:** Material descriptions of velocity and acceleration fields of a continuum, deforming under a homogeneous transformation, are given by

$$v = \dot{r} = \dot{Q}R + \dot{D} \tag{2.22}$$

$$a = \ddot{r} = \ddot{Q}R + \ddot{D}. \tag{2.23}$$

Let R in Eqs. (2.22) and (2.23) be replaced by $Q^{-1}(r - D)$, then

$$v = \dot{Q}Q^{-1}r + (\dot{D} - \dot{Q}Q^{-1}D) \tag{2.24}$$

$$a = \ddot{Q}Q^{-1}r + (\ddot{D} - \ddot{Q}Q^{-1}D) \tag{2.25}$$

are the spatial descriptions for the velocity and acceleration fields of homogeneous deformation. Components of $v(x_1,x_2,t) = v_1(x_1,x_2,t)\hat{\mathbf{e}}_1 + v_2(x_1,x_2,t)\hat{\mathbf{e}}_2$ and $a(x_1,x_2,t) = a_1(x_1,x_2,t)\hat{\mathbf{e}}_1 + a_2(x_1,x_2,t)\hat{\mathbf{e}}_2$ are obtained as

$$v_1(x_1,x_2,t) = c(t)x_1 + d(t)x_2 + e(t) \tag{2.26}$$

$$v_2(x_1,x_2,t) = f(t)x_1 + g(t)x_2 + h(t) \tag{2.27}$$

$$a_1(x_1,x_2,t)) = l(t)x_1 + m(t)x_2 + o(t) \tag{2.28}$$

$$a_2(x_1,x_2,t) = p(t)x_1 + q(t)x_2 + s(t) \tag{2.29}$$

where

$$c(t) = \frac{\dot{Q}_{11}Q_{22} - \dot{Q}_{12}Q_{21}}{|Q(t)|},$$

$$
\begin{aligned}
d(t) &= \frac{\dot{Q}_{12}Q_{11} - \dot{Q}_{11}Q_{12}}{|Q(t)|}, \\
f(t) &= \frac{\dot{Q}_{21}Q_{22} - \dot{Q}_{22}Q_{21}}{|Q(t)|}, \\
g(t) &= \frac{\dot{Q}_{22}Q_{11} - \dot{Q}_{21}Q_{12}}{|Q(t)|}, \\
e(t) &= \dot{D}_1 - \frac{(\dot{Q}_{11}Q_{22} - \dot{Q}_{12}Q_{21})D_1 + (\dot{Q}_{12}Q_{11} - \dot{Q}_{11}Q_{12})D_2}{|Q(t)|}, \\
h(t) &= \dot{D}_2 - \frac{(\dot{Q}_{21}Q_{22} - \dot{Q}_{22}Q_{21})D_1 + (\dot{Q}_{22}Q_{11} - \dot{Q}_{21}Q_{12})D_2}{|Q(t)|}, \\
l(t) &= \frac{(\ddot{Q}_{11}Q_{22} - \ddot{Q}_{12}Q_{21})}{|Q(t)|} \\
m(t) &= \frac{\ddot{Q}_{12}Q_{11} - \ddot{Q}_{11}Q_{12}}{|Q(t)|} \\
p(t) &= \frac{\ddot{Q}_{21}Q_{22} - \ddot{Q}_{22}Q_{21}}{|Q(t)|} \\
q(t) &= \frac{\ddot{Q}_{22}Q_{11} - \ddot{Q}_{21}Q_{12}}{|Q(t)|} \\
s(t) &= \ddot{D}_2 - \frac{(\ddot{Q}_{21}Q_{22} - \ddot{Q}_{22}Q_{21})D_1 + (\ddot{Q}_{22}Q_{11} - \ddot{Q}_{21}Q_{12})D_2}{|Q(t)|}.
\end{aligned}
$$

2.1.3 Homogeneous Deformation of Agents with Finite Size

Consider an MAS that consists of N agents and move collectively in $\mathbb{R}^n$ ($n = 1, 2, 3$). Each agent is considered as a ball with radius ε. Because homogeneous transformation can change interagent distance among agents, it is necessary to assure that no two agents collide when the MAS deforms homogeneously.

Assume that m_1 and m_2 are index numbers of two agents having closest distance in the MAS initial configuration. To assure collision avoidance, no two agents should not get closer than 2ε ($\eta \in \mathbb{R}_+$). This requirement is assured, if

$$\forall t \geq t_0 ||r_{m_1,HT} - r_{m_2,HT}|| > 2\varepsilon. \tag{2.30}$$

By considering definition of homogeneous deformation,

$$r_{m_1,HT} - r_{m_2,HT} = Q(R_{m_1} - R_{m_2}). \tag{2.31}$$

Therefore,

$$(r_{m_1,HT} - r_{m_2,HT})^T (r_{m_1,HT} - r_{m_2,HT}) \geq (R_{m_1} - R_{m_2})^T Q^T Q (R_{m_1} - R_{m_2}). \tag{2.32}$$

and

$$||r_{m_1,HT} - r_{m_2,HT}|| \geq \lambda_{min} \left(\sqrt{Q^T Q}\right) ||R_{m_1} - R_{m_2}||. \tag{2.33}$$

By using polar decomposition, the Jacobian matrix Q can be expressed as

$$Q = R_O U_D \tag{2.34}$$

where U_D is a symmetric pure deformation matrix and R_O is an orthogonal matrix (See Appendix A). Therefore, eigenvalues of the matrix

$$Q^T Q = {U_D}^T U_D \tag{2.35}$$

are all positive, and eigenvectors are mutually orthogonal.

Let

$$\lambda_{min} = \frac{2\varepsilon}{||R_{m_1} - R_{m_2}||} \tag{2.36}$$

be the lower bound for the eigenvalues of the matrix $\sqrt{Q^T(t)Q(t)}$ $(\forall t \geq t_0)$, then the condition (2.30) is satisfied and no two agents get closer than 2ε $(\forall t \geq t_0, \|r_{m_1,HT}(t) - r_{m_2,HT}(t)\| \geq 2\varepsilon)$.

Example 2.1. Consider an MAS with five agents (three leader and two followers) with initial and final formations shown in Fig. 2.4. Leaders are initially placed

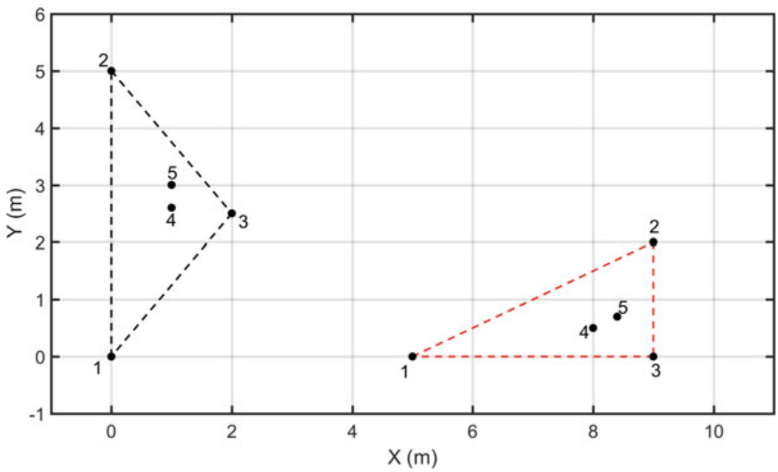

Fig. 2.4 Homogeneous transformation of agents in Example 2.2

at $(X_1,Y_1)=(0,0)$, $(X_2,Y_2)=(0,5)$, and $(X_3,Y_3)=(2,2.5)$. They are ultimately stop at $(X_{F,1},Y_{F,1})=(5,0)$, $(X_{F,2},Y_{F,2})=(9,2)$, and $(X_{F,3},Y_{F,3})=(9,0)$. By using Eq. (2.21),

$$\begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{21} \\ Q_{22} \\ D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 2 & 2.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 2 & 2.5 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \\ 9 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.8 \\ -0.5 \\ 0.4 \\ 5 \\ 0 \end{bmatrix}.$$

Therefore,

$$U_D^2 = Q^T Q = \begin{bmatrix} 1.2500 & 0.6000 \\ 0.6000 & 0.8000 \end{bmatrix},$$

$\lambda_1(Q^TQ)=0.3842$ and $\lambda_2(Q^TQ)=1.6658$. Moreover, $\mathbf{n}_1=0.5696\hat{\mathbf{e}}_x-0.8219\hat{\mathbf{e}}_y$, $\mathbf{n}_2=-0.8219\hat{\mathbf{e}}_x-0.5696\hat{\mathbf{e}}_y$ are the eigenvectors of both matrices Q^TQ and $U_D=\sqrt{Q^TQ}$. Thus,

$$U_D = \begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{bmatrix} \begin{bmatrix} \sqrt{0.3842} & 0 \\ 0 & \sqrt{1.6658} \end{bmatrix} \begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{bmatrix}^T = \begin{bmatrix} 1.0730 & 0.3141 \\ 0.3141 & 0.8375 \end{bmatrix}$$

$\lambda_1(U_D)=0.6198$ and $\lambda_2(Q^TQ)=1.2907$. Given U_D and Q,

$$R_O = QU_D{}^{-1} = \begin{bmatrix} 0.7328 & 0.6805 \\ -0.6805 & 0.7328 \end{bmatrix}.$$

Follower 4 and 5 are considered as disks with radius $10cm$ with initial separation $||R_4-R_5||=40cm$. Hence, $2\varepsilon=20cm$ and

$$\lambda_{min} = \frac{20}{40} = 0.5.$$

Because $\lambda_{min}\le\lambda_1(U_D)=0.6198$, followers 4 and 5 do not collide in the final configuration.

2.1.4 Force Analysis

Let followers all have the same mass m, and they are all distributed inside the leading polytope at the initial time t_0. Then, the parameter $\alpha_{i,k}$ ($k=1,\ 2,\dots,\ n+1,\ i=\ n+2,\dots,\ N$) is positive. Under a homogeneous transformation, the q^{th} component of acceleration of the follower agent i ($i=n+2,\dots,\ N$) becomes

$$\ddot{x}_{q,i}(t) = \sum_{k=1}^{n+1} \alpha_{i,k}\ddot{x}_{q,k}(t). \tag{2.37}$$

If $a_q(t)$ denotes the maximum for the magnitude of the q^{th} component of the leaders' acceleration at the time t, i.e.

$$||\ddot{x}_{q,k}(t)|| \leq a_q(t),\ k = 1,2,\dots,n+1, \tag{2.38}$$

then the q^{th} component of the acceleration of the follower i satisfies the following inequality:

$$||\ddot{x}_{q,i}(t)|| = ||\sum_{k=1}^{n+1} \alpha_{i,k}\ddot{x}_{q,k}(t)|| \leq a_q(t)\sum_{k=1}^{n+1} \alpha_{i,k} = a_q(t). \tag{2.39}$$

Equation (2.39) yields an upper bound for maximum force required for the motion of the follower i. Let

$$f_i(t) = m\sum_{q=1}^{n} \ddot{x}_{q,k}(t)\hat{\mathbf{e}}_q \tag{2.40}$$

be the force required for the motion of follower i at time t, then

$$||f_i(t)|| = m\sqrt{\sum_{q=1}^{n} \ddot{x}^2_{q,i}(t)} \leq m\sqrt{\sum_{q=1}^{n} {a_q}^2(t)}. \tag{2.41}$$

Example 2.2. Consider an MAS with the initial formation shown in Fig. 2.5. As it is seen, 3 leaders are placed at the vertices of the leading triangle and 17 followers are distributed inside the leading triangle. Therefore, the parameter $\alpha_{i,k}$ ($i = 4,\ 5,\dots,\ 20$ and $k = 1,2,3$) is positive. In Fig. 2.6, paths of the leaders in the $X-Y$ plane are shown. Leaders initiate their motion from rest at $t = 0s$ and settle at $T = 60s$. Configurations of the leading triangle are depicted at the initial and final times, where both have the same area of $50m^2$.

Each follower has the mass $m = 1kg$. In Fig. 2.7, magnitudes of forces acting on the followers are shown by black curves. Also, the upper limit for these acting forces is obtained by Eq. (2.41) and shown by the red curve. As it is seen in Fig. 2.7, magnitudes of the acting forces do not exceed the upper limit $\sqrt{{a_1}^2 + {a_2}^2}$, where $a_1(t)$ and $a_2(t)$ are the supremum for the X and Y components of the leaders' accelerations over $t \in [0,60]$, respectively.

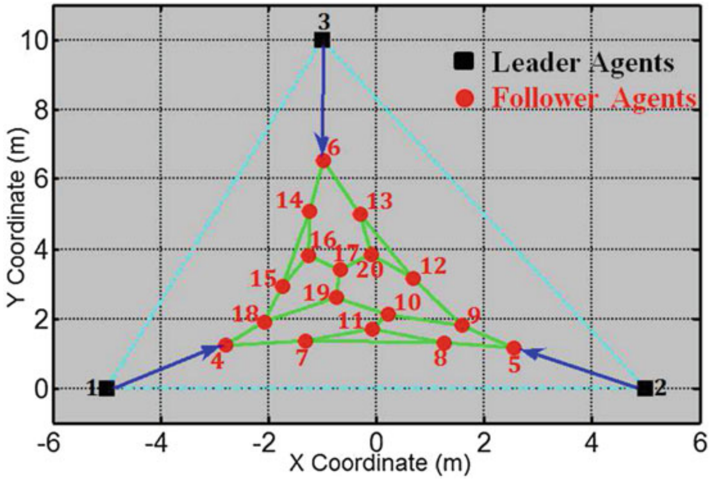

Fig. 2.5 Initial distribution of the agents in Example 2.2

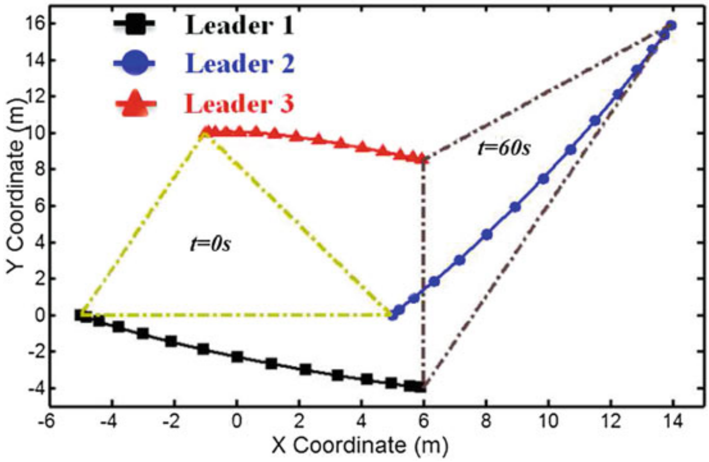

Fig. 2.6 Paths of the leaders

2.1.5 *Path Planning: MAS As Particles of Newtonian Viscous Flow*

A continuum can be considered as a portion of the incompressible Newtonian viscous flow, if the deformation mapping satisfies both the continuity and Navier-Stokes constitutive equations. Let $v(x_1,x_2,x_3,t) = v_1(x_1,x_2,x_3,t)\hat{\mathbf{e}}_1 + v_2(x_1,x_2,x_3,t)\hat{\mathbf{e}}_2 + v_3(x_1,x_2,x_3,t)\hat{\mathbf{e}}_3$ be the spatial description of the velocity field of a continuum, then continuity equation is expressed by

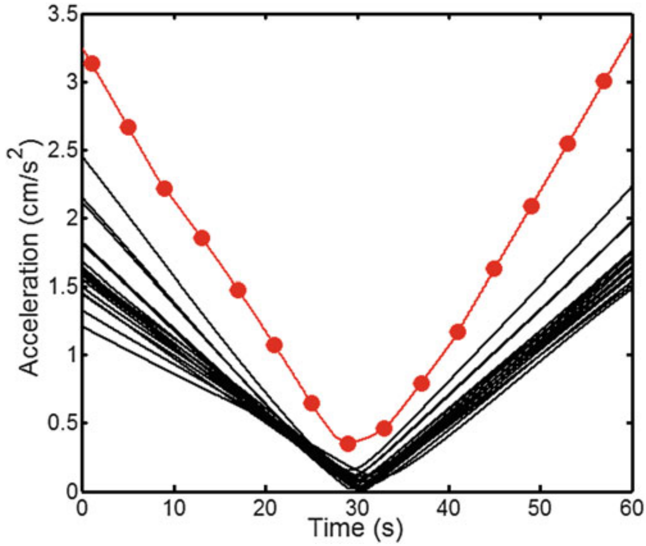

Fig. 2.7 Acting forces required for followers to deform as homogeneous transformation

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0. \tag{2.42}$$

Also, the Navier-Stokes equations are expressed as follows:

$$\rho a_1 = -\frac{\partial p}{\partial x_1} + \mu\left(\frac{\partial^2 v_1}{\partial {x_1}^2} + \frac{\partial^2 v_1}{\partial {x_2}^2} + \frac{\partial^2 v_1}{\partial {x_3}^2}\right) \tag{2.43}$$

$$\rho a_2 = -\frac{\partial p}{\partial x_2} + \mu\left(\frac{\partial^2 v_2}{\partial {x_1}^2} + \frac{\partial^2 v_2}{\partial {x_2}^2} + \frac{\partial^2 v_2}{\partial {x_3}^2}\right) \tag{2.44}$$

$$\rho a_3 = -\frac{\partial p}{\partial x_3} + \mu\left(\frac{\partial^2 v_3}{\partial {x_1}^2} + \frac{\partial^2 v_3}{\partial {x_2}^2} + \frac{\partial^2 v_3}{\partial {x_3}^2}\right). \tag{2.45}$$

It is noted that p is the pressure filed, ρ and μ are density and viscosity of the fluid flow, respectively. Additionally,

$$a_1(x_1, x_2, x_3, t) = \frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3} \tag{2.46}$$

$$a_2(x_1, x_2, x_3, t) = \frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3} \tag{2.47}$$

$$a_3(x_1, x_2, x_3, t) = \frac{\partial v_3}{\partial t} + v_1\frac{\partial v_3}{\partial x_1} + v_2\frac{\partial v_3}{\partial x_2} + v_3\frac{\partial v_3}{\partial x_3} \tag{2.48}$$

are the components of the acceleration $a(x_1,x_2,x_3,t) = a_1(x_1,x_2,x_3,t)\hat{\mathbf{e}}_1 + a_2(x_1,x_2,x_3,t)\hat{\mathbf{e}}_2 + a_3(x_1,x_2,x_3,t)\hat{\mathbf{e}}_3$.

***Holonomic Constraints Imposed on the Leaders' Motion*:** Continuity and Navier-Stokes equations impose two holonomic constraints on the leaders' motion, if homogeneous deformation of the MAS is treated as incompressible fluid flow. Before these two holonomic constraints are obtained, the following lemma is stated:

Lemma. *Let $Q \in \mathbb{R}^{n\times n}$ be a non-singular second order tensor, then*

$$\frac{d|Q|}{dt} = |Q| trace(\dot{Q}Q^{-1}). \tag{2.49}$$

Note that $|Q|$ is the determinant of the matrix Q.

Proof. It is known that

$$\frac{\partial|Q|}{\partial Q_{ij}} = Q_{ij}{}^{C} = M_{ji} \tag{2.50}$$

where $Q_{ij}{}^{C}$ is the cofactor of the entry ij of the matrix Q, and

$$Q^{-1} = \frac{M^T}{|Q|}. \tag{2.51}$$

Thus,

$$\frac{d|Q|}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial|Q|}{\partial Q_{ij}}\dot{Q}_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ji}\dot{Q}_{ij} = |Q| trace(Q\dot{Q}^{-1}). \tag{2.52}$$

Theorem 2.1. *Homogeneous transformation of a continuum can be considered as deformation of incompressible Newtonian viscous fluid flow, if*

$$\forall t \geq t_0, |Q(t)| = 1 \tag{2.53}$$

$$curl\ a = \mathbf{0} \tag{2.54}$$

Proof. Let the velocity of a homogeneous deformation satisfy the continuity Eq. (2.42), then

$$\nabla .v = trace(\dot{Q}Q^{-1}) = 0. \tag{2.55}$$

Therefore, the right-hand side of Eq. (2.52) vanishes and $|Q(t)|$ remains constant. Because $Q(t_0) = I_n \in \mathbb{R}^{n\times n}$, $|Q(t)| = 1$ at any time during deformation.

Under homogeneous deformation,

$$\nabla^2 v_i = 0,\ i = 1,2,3,$$

therefore, the Navier-Stokes equations simplify to

$$\rho a = -\nabla p. \tag{2.56}$$

Note that pressure field is smooth, therefore,

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1}$$
$$\frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2}$$
$$\frac{\partial^2 p}{\partial x_3 \partial x_1} = \frac{\partial^2 p}{\partial x_1 \partial x_3}$$

and *curl* $a = \mathbf{0}$.

Corollary 1. *If homogeneous deformation of a continuum in* $\mathbb{R}^2$ *satisfies the continuity equation, then the area of the leading triangle is preserved during deformation. Therefore, the following holonomic constraint can be considered for motion of the leaders:*

$$\begin{aligned} C_1 : \frac{1}{2}\begin{vmatrix} x_1(t) & x_2(t) & x_3(t) \\ y_1(t) & y_2(t) & y_3(t) \\ 1 & 1 & 1 \end{vmatrix} = \\ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) - 2a_0 = 0. \end{aligned} \tag{2.57}$$

Note that

$$a_0 = \frac{1}{2}\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{vmatrix}. \tag{2.58}$$

is the area of the leading triangle at the initial time t_0.

Corollary 2. *If homogeneous deformation satisfies the Navier-Stokes equations, then motion of the leaders should satisfy the following constraint:*

$$C_2 = \ddot{x}_1(x_2 - x_3) + \ddot{x}_2(x_3 - x_1) + \ddot{x}_3(x_1 - x_2) + \ddot{y}_1(y_3 - y_2) + \ddot{y}_2(y_1 - y_3) + \ddot{y}_3(y_2 - y_1) = 0. \tag{2.59}$$

***Minimum Acceleration Norm of Homogeneous Deformation*: ** In this section, optimal trajectories of the leaders minimizing acceleration norm of homogeneous transformation are determined, where leaders' positions satisfy the holonomic constraints (2.57) and (2.59) at any time during MAS evolution, and initial and final positions and velocities of leaders are known. It is assumed that leaders' positions are updated by

$$\ddot{x}_i = q_i,\ i = 1,\ 2,\ 3, \tag{2.60}$$

$$\ddot{y}_i = q_i,\ i = 4,\ 5,\ 6. \tag{2.61}$$

The dynamics of the leaders can be converted into a state space representation,

$$\dot{P} = A_p P + B_p U_p, \tag{2.62}$$

where $P = [p_i] \in \mathbb{R}^{12}$, $p_1 = x_1, p_2 = x_2, p_3 = x_3, p_4 = y_1, p_5 = y_2, p_6 = y_3, p_7 = \dot{x}_1$, $p_8 = \dot{x}_2, p_9 = \dot{x}_3, p_{10} = \dot{y}_1, p_{11} = \dot{y}_2, p_{12} = \dot{y}_3$,

$$U_p = \begin{bmatrix} q_1 & \dots & q_6 \end{bmatrix}^T$$

$$A_p = \begin{bmatrix} \mathbf{0} & I_6 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T$$

$$B_p = \begin{bmatrix} \mathbf{0} \\ I_6 \end{bmatrix}^T .$$

It is noted that $\mathbf{0}, I_6 \in \mathbb{R}^{6\times 6}$ are zero-entry and identity matrices, respectively.

The constraints (2.57) and (2.59) can be rewritten as

$$\begin{aligned} C_1' :\ & q_1(p_5 - p_6) + q_2(p_6 - p_4) + q_3(p_4 - p_5) + q_4(p_3 - p_2) + \\ & q_5(p_1 - p_3) + q_6(p_2 - p_1) + p_7(p_{11} - p_{12}) + p_8(p_{12} - p_{10}) + \\ & p_9(p_{10} - p_{11}) + p_{10}(p_9 - p_8) + p_{11}(p_7 - p_9) + p_{12}(p_8 - p_7) = 0. \end{aligned} \tag{2.63}$$

$$\begin{aligned} C_2 :\ & q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2) + q_4(p_5 - p_6) \\ & + q_6(p_6 - p_4) + q_6(p_4 - p_5) = 0. \end{aligned} \tag{2.64}$$

It is noted that the constraint Eq. (2.63) is obtained by taking the second time derivative from Eq. (2.57).

Leaders start their motion from the rest at the initial time $t = 0$ and they stop in a finite horizon of time T, where the leaders' initial and final positions are known.

***Objective Function*:** The objective is to minimize the acceleration norm of homogeneous transformation of the MAS, where leaders satisfy the constraints C_1' and C_2 (Eqs. (2.63) and (2.64)). Therefore,

$$J = \int_0^T \{ U_p{}^T U_p + \lambda^T (A_p P + B_p U_p - \dot{P}) + \gamma_1 C_1' + \gamma_2 C_2 \} dt \tag{2.65}$$

is the cost function that is desired to be minimized, where $\lambda \in \mathbb{R}^{12}$ is the costate vector, γ_1 and γ_2 are Lagrange multipliers. This is a well-known two point boundary value problem that can be solved by using calculus of variation. The cost J is minimized, if

$$\begin{aligned}\delta J = \int_0^T \Bigg\{ \sum_{i=1}^{6} (2q_i\delta q_i + \delta\lambda_i(p_{i+6} - \dot{p}_i) + \\ \lambda_i(\delta p_{i+6} - \delta\dot{p}_i) + \delta\lambda_{i+6}(q_i - \dot{p}_{i+6}) + \\ \lambda_{i+6}(\delta q_i - \delta\dot{p}_{i+6})) + \delta\gamma_1 C_1' + \gamma_1\delta C_1' + \delta\gamma_2 C_2 + \gamma_2\delta C_2 \Bigg\} dt = 0.\end{aligned} \tag{2.66}$$

Therefore, X and Y components of the leaders' positions are determined by solving the following 24^{th} order dynamics:

$$\dot{S} = A_s S, \tag{2.67}$$

where

$$S = \begin{bmatrix} P \\ \lambda \end{bmatrix} \tag{2.68}$$

$$A_s = \begin{bmatrix} \mathbf{0} & I_6 & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}(\gamma_1 K_1 + \gamma_2 K_3) & \mathbf{0} & \mathbf{0} & -\frac{1}{2}I_6 \\ \frac{1}{2}(\gamma_1^2 + \gamma_2^2)K_2 & \mathbf{0} & \mathbf{0} & \frac{1}{2}(\gamma_1 K_1 - \gamma_2 K_3) \\ \mathbf{0} & -2\gamma_1 K_1 & -I_6 & \mathbf{0} \end{bmatrix} \tag{2.69}$$

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.70}$$

$$K_2 = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \tag{2.71}$$

$$K_3 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}. \tag{2.72}$$

It is noted that γ_1 and γ_2 in Eq. (2.69) are equal to

$$\gamma_1 = \frac{2\tau - \sigma}{\rho} \tag{2.73}$$

$$\gamma_2 = -\frac{\varphi}{\rho}, \tag{2.74}$$

where

$$\rho = (p_5 - p_6)^2 + (p_6 - p_4)^2 + (p_4 - p_5)^2 + (p_3 - p_2)^2 + (p_1 - p_3)^2 + (p_2 - p_1)^2 \tag{2.75}$$

$$\tau = p_7(p_{11} - p_{12}) + p_8(p_{12} - p_{10}) + p_9(p_{10} - p_{11}) + p_{10}(p_9 - p_8) + p_{11}(p_7 - p_9) + p_{12}(p_8 - p_7) \tag{2.76}$$

$$\sigma = \lambda_7(p_5 - p_6) + \lambda_8(p_6 - p_4) + \lambda_9(p_4 - p_5) + \lambda_{10}(p_3 - p_2) + \lambda_{11}(p_1 - p_3) + \lambda_{12}(p_2 - p_1) \tag{2.77}$$

$$\varphi = \lambda_7(p_3 - p_2) + \lambda_8(p_1 - p_3) + \lambda_9(p_2 - p_1) + \lambda_{10}(p_6 - p_5) + \lambda_{11}(p_4 - p_6) + \lambda_{12}(p_5 - p_4). \tag{2.78}$$

Furthermore, optimal control inputs become

$$q_1 = -\frac{1}{2}(\lambda_7 + \gamma_1(p_5 - p_6) + \gamma_2(p_2 - p_3)), \tag{2.79}$$

$$q_2 = -\frac{1}{2}(\lambda_8 + \gamma_1(p_6 - p_4) + \gamma_2(p_3 - p_1)), \tag{2.80}$$

$$q_3 = -\frac{1}{2}\lambda_9 + \gamma_1(p_4 - p_5) + \gamma_2(p_1 - p_2)), \tag{2.81}$$

$$q_4 = -\frac{1}{2}(\lambda_{10} + \gamma_1(p_3 - p_2) + \gamma_2(p_5 - p_6)), \tag{2.82}$$

$$q_5 = -\frac{1}{2}(\lambda_{11} + \gamma_1(p_1 - p_3) + \gamma_2(p_6 - p_4)), \tag{2.83}$$

$$q_6 = -\frac{1}{2}(\lambda_{12} + \gamma_1(p_5 - p_6) + \gamma_2(p_4 - p_5)). \tag{2.84}$$

***Numerical Solution*:** It is difficult to obtain an analytic solution for the optimal trajectories minimizing acceleration norm of MAS evolution. This is because the dynamics of Eq. (2.67) is nonlinear and 24^{th} order, where the initial values for the co-estate vector λ are not defined. Therefore, trajectories of the leaders are found by using a distributed gradient algorithm.

First, $\gamma_1(t)$ and $\gamma_2(t)$ are estimated by $\gamma_{1_1}(t)$ and $\gamma_{2_1}(t)$, over the time interval $t \in [t_0, T]$. Then $\gamma_1(t)$ and $\gamma_2(t)$ are kept updating until the solution of the optimal trajectories are obtained.

Let $\Phi_k(t,t_0)$, P_k, and λ_k denote state transition matrix, control state, and costate at the attempt ($k = 1,\ 2,\ 3,\ \ldots$), then

$$\forall t \geq t_0, \begin{bmatrix} P_k(t) \\ \lambda_k(t) \end{bmatrix} = \phi_k(t,t_0) \begin{bmatrix} P_k(t_0) \\ \lambda_k(t_0) \end{bmatrix} = \begin{bmatrix} \phi_{11_k}(t,t_0) & \phi_{12_k}(t,t_0) \\ \phi_{21_k}(t,t_0) & \phi_{22_k}(t,t_0) \end{bmatrix} \begin{bmatrix} P(t_0) \\ \lambda(t_0) \end{bmatrix}. \tag{2.85}$$

Thus,

$$P(T) = P_k(T) = \phi_{11_k}(T,t_0)P(t_0) + \phi_{12_k}\lambda_k(t_0). \tag{2.86}$$

Notice that $P(t_0)$ and $P(T)$ are both known because positions and velocities of the leaders are given at the initial time t_0 and final time T. Hence, $\lambda_k(t_0)$ is obtained as follows:

$$\lambda_k(t_0) = \phi_{12_k}(T,t_0)^{-1}(P(T) - \phi_{11_k}(T,t_0)P(t_0)). \tag{2.87}$$

Given $P(t_0)$ and $\lambda_k(t_0)$, $P_k(t)$ and $\lambda_k(t)$ (the k^{th} estimations of $P(t)$ and $\lambda(t)$) can be updated by Eq. (2.85). Then, $\gamma_{1_k}(t)$ and $\gamma_{2_k}(t)$ can be updated by Eqs. (2.73) and (2.74). This process is continued until

$$\forall t \in [t_0, T],\ |\gamma_{i_k}(t) - \gamma_{i_{k-1}}(t)| \to 0,\ i = 1,\ 2. \tag{2.88}$$

Example 2.3. Suppose that leaders are initially placed at $(-5,0)$, $(5,0)$, and $(-1,10)$. They start their motion from rest at $t_0 = 0s$ and ultimately stop at $(7,-5)$, $(15,15)$, and $(7,7.5)$ at $T = 60s$. Leaders are restricted to satisfy the holonomic constraints C_1' and C_2 (Eqs. (2.63) and (2.64)) at any time $t \in [0,60]s$. The optimal paths of the leaders minimizing the objective function (2.65) are shown in Fig. 2.8. Also, the optimal control inputs q_i are shown versus time in Fig. 2.9.

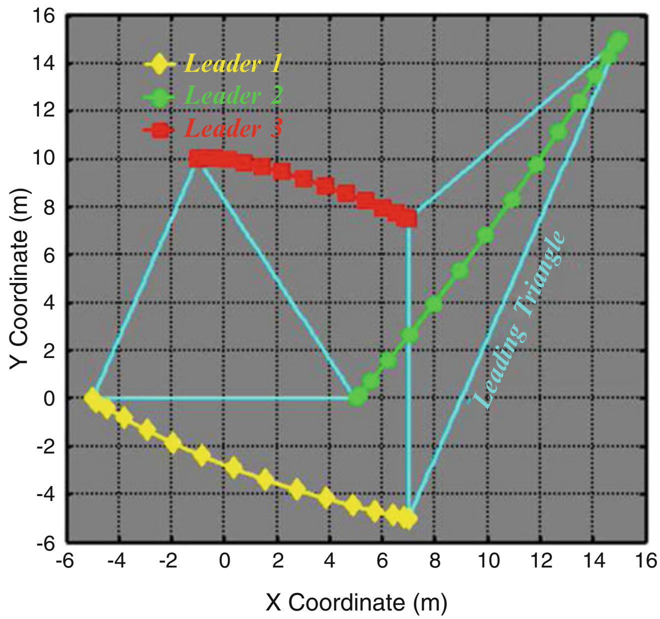

Fig. 2.8 Optimal paths of the leader agents

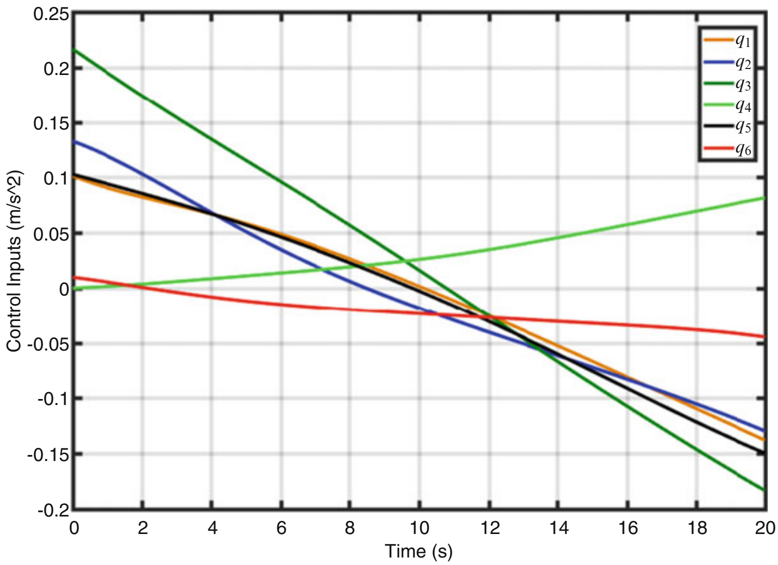

Fig. 2.9 Optimal control inputs

2.2 Evolution of a Multi-Agent System with Constrained Dynamics

Suppose that position of the follower $i \in V_F$ is updated by

$$\begin{cases} \dot{Z}_i = f(Z_i, u_i) \\ r_i = h(Z_i) \end{cases} \tag{2.89}$$

where the nonlinear dynamics (2.89) is both state controllable and state observable, $Z_i \in \mathbb{R}^{n_z}$ is the state vector, $u_i \in \mathbb{R}^{n_u}$ is the control input that belongs to a connected set $U_i \subset \mathbb{R}^{n_u}$, and actual position of the follower i, $r_i \in \mathbb{R}^n$, is considered as the control output.

Let ${\Phi_i}^k \subset \mathbb{R}^n$ be considered as the reachable displacement set for the follower i at the $t = t_{k+1}$. Then, ${\Phi_i}^k$ can be reached by the follower i at the time $t = t_{k+1}$, if the follower i chooses an admissible control $u_i \in U_i$ during $t \in [t_k, t_{k+1}]$. It is noted that the set ${\Phi_i}^k \subset \mathbb{R}^n$ is connected because the system is controllable. Also, ${\Phi_i}^k$ includes the origin since displacement of the follower i is $0 \in \mathbb{R}^n$ at the time t_k.

It is desired that

- MAS configuration at the time $t = t_{k+1}$ is a homogeneous deformation of the MAS formation at t_k, and
- $\forall t \in [t_k, t_{k+1}], \forall i \in V_F, ||r_i(t) - r_i(t_k)|| \in {\Phi_i}^k$.

In the theorem below, it is shown under what conditions the above objectives can be achieved.

Theorem 2.2. *Let*

(i) ${\Phi_i}^k$ *be the reachable displacement set of the follower i ($\forall i \in V_F$), where each follower is permitted to choose an admissible control input $u_i \in U_i$ during the time interval $[t_k, t_{k+1}]$,*
(ii) $\Phi_k = \bigcap_{i \in V_F} {\Phi_i}^k$, *and*
(iii) $\forall t \in [t_k, t_{k+1}], \forall i \in V_L, ||r_i(t) - r_i(t_k)|| \in {\Omega_r}^{k+1}$,

where

$${\Omega_r}^{k+1} = \{d \in \Phi_k | \; ||d|| \le r^{k+1}\}$$

is the biggest ball inside Φ_k with radius r^{k+1}. Then, followers can reach the desired positions defined by a homogeneous deformation at t_{k+1} ($k = 1, 2, \ldots$).

Proof. Let

$$\forall j \in V_L, \; {d_j}^{k+1} = {r_j}^{k+1} - {r_j}^k \in {\Omega_r}^{k+1},$$

denote displacement of the leader j, then desired displacement of the follower i, issued by a homogeneous deformation, is obtained as follows:

$$d_i^{k+1} = \sum_{j=1}^{n+1} \alpha_{i,j}(r_j{}^{k+1} - r_j{}^k) = \sum_{j=1}^{n+1} \alpha_{i,j} d_j{}^{k+1}. \tag{2.90}$$

Let

$$r^{k+1} = max\{d_1{}^{k+1},\ d_2{}^{k+1},\ \ldots,\ d_{n+1}{}^{k+1}\}$$

assign a maximum for leaders' displacements at t_{k+1}, then

$$||d_i^{k+1}|| = ||\sum_{j=1}^{n+1} \alpha_{i,j}(r_j{}^{k+1} - r_j{}^k)|| \leq \sum_{j=1}^{n+1} \alpha_{i,j}||d_j{}^{k+1}|| \leq r^{k+1},\ \forall i \in V_F. \tag{2.91}$$

Therefore,

$$\forall i \in V_F,\ ||d_i^{k+1}|| \in \Omega_r{}^{k+1} \subset \Phi_i{}^k \subset \Phi_k.$$

This implies that the desired position issued by a homogeneous deformation at t_{k+1} can be reached by the follower $i \in V_F$.

***Desired Positions of the Followers*:** It is assumed that each leader j ($j = 1,\ 2, \ldots,\ n+1$) moves on a line segment given by

$$r_j(t) = \frac{r_j{}^{k+1} - r_j{}^k}{t_{k+1} - t_k}(t - t_k) + r_j{}^k \tag{2.92}$$

where

$$||r_j{}^{k+1} - r_j{}^k|| \leq r^{k+1}. \tag{2.93}$$

Consequently, followers acquire homogeneous deformation under no interagent communication by knowing only positions of the leaders at distinct times t_0, t_1, $\ldots$, t_f. Desired position of the follower i, defined by a homogeneous deformation, becomes

$$\forall t \in [t_k, t_{k+1}],\ r_{iHT}{}^k(t) = \sum_{j=1}^{n+1} \alpha_{i,j} \frac{r_j{}^{k+1} - r_j{}^k}{t_{k+1} - t_k}(t - t_k) + r_j{}^k. \tag{2.94}$$

***Homogeneous Deformation of Wheeled Robots*:** Let each follower i be a wheeled robot with the dynamics

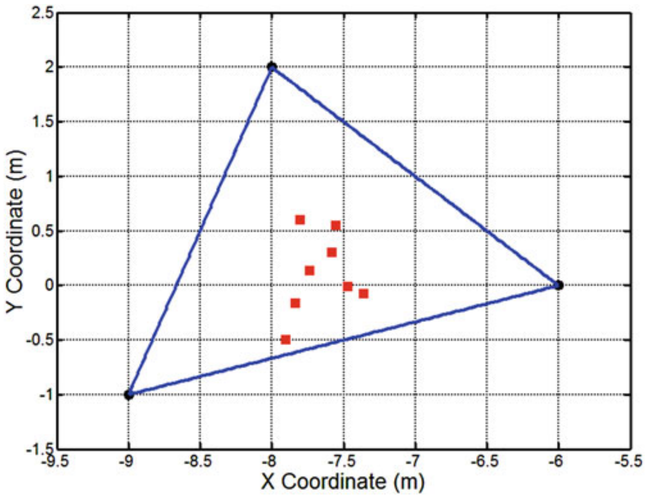

Fig. 2.10 Initial distribution of the agents in Example 2.4

Table 2.1 Initial heading angles of follower wheeled robots

Heading Angle	4	5	6	7	8	9	10	11
θ_i	45	30	87	123	135	130	65	100

$$\begin{cases} \dot{x}_i = u_i \cos\ \theta_i \\ \dot{y}_i = u_i \sin\ \theta_i \\ \dot{\theta}_i = \omega_i \end{cases}, \tag{2.95}$$

where the magnitude of u_i and ω_i are limited by

$$\begin{cases} |u_i| \leq 1 \\ |\omega_i| \leq 1 \end{cases}. \tag{2.96}$$

For simulation, evolution of an MAS containing 11 agents (3 leaders and 8 follower unicycle robots) is considered. The MAS has the initial formation that is shown in Fig. 2.10. Also, initial followers' orientations are listed in Table 2.1.

In Fig. 2.11, reachable displacement set ${\Phi_5}^0$ of the follower agent 5 at $t_1 = \pi s$ is illustrated. All reachable displacement sets (${\Phi_i}^0, i = 4,\ 5, \ldots,\ 11$) are also illustrated in Fig. 2.12. As it is observed, the disk Ω_0 has the radius $2m$ and it is centered at the origin. Notice that Ω_0 is a subset of

$$\Phi_0 = \bigcap_{i=4}^{11} {\Phi_i}^0.$$

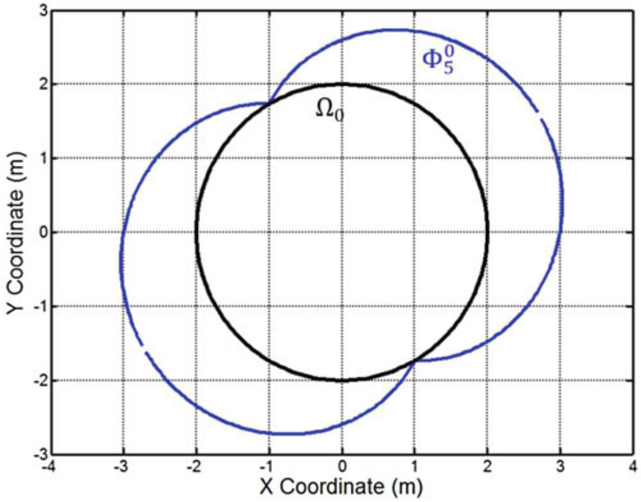

Fig. 2.11 Reachable displacement set $\Phi_5{}^0$

Table 2.2 Parameters $\alpha_{i,j}$ of the follower i with initial position shown in Fig. 2.10

	$X_i(m)$	$Y_i(m)$	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$
i=1	−9.0000	−1.0000	–	–	–
i=2	−6.0000	0	–	–	–
i=3	−8.0000	2.0000	–	–	–
i=4	−7.9000	−0.5000	0.6000	0.3500	0.0500
i=5	−7.8000	0.6000	0.3000	0.2500	0.4500
i=6	−7.7333	0.1333	0.4000	0.3333	0.2667
i=7	−7.5500	0.5500	0.2500	0.3500	0.4000
i=8	−7.5800	0.3000	0.3200	0.3700	0.3100
i=9	−7.3600	−0.0800	0.3600	0.5000	0.1400
i=10	−7.8333	−0.1667	0.5000	0.3333	0.1667
i=11	−7.4700	−0.0100	0.3700	0.4500	0.1800

Therefore, leaders are not permitted to transverse more than $2m$ during the time interval $t \in [0, \pi]$ seconds. By using Eq. (2.94), X and Y components of desired position of the follower $i \in V_F$ are obtained as follows:

$$\begin{cases} x_{i,HT}{}^k = \dfrac{t - t_k}{t_{k+1} - t_k} \sum_{j=1}^{n+1} \alpha_{i,j}(x_j{}^{k+1} - x_j{}^k) + x_j{}^k \\ y_{i,HT}{}^k = \dfrac{t - t_k}{t_{k+1} - t_k} \sum_{j=1}^{n+1} \alpha_{i,j}(y_j{}^{k+1} - y_j{}^k) + y_j{}^k \end{cases}. \tag{2.97}$$

In Eq. (2.97) t denotes time, $r_j{}^k = x_j{}^k \hat{\mathbf{e}}_x + y_j{}^k \hat{\mathbf{e}}_y$ and $r_j{}^{k+1} = x_j{}^{k+1} \hat{\mathbf{e}}_x + y_j{}^{k+1} \hat{\mathbf{e}}_y$. Initial centroid positions of the follower unicycles and the corresponding parameters $\alpha_{i,j}$, specified by Eq. (2.18), are listed in Table 2.2.

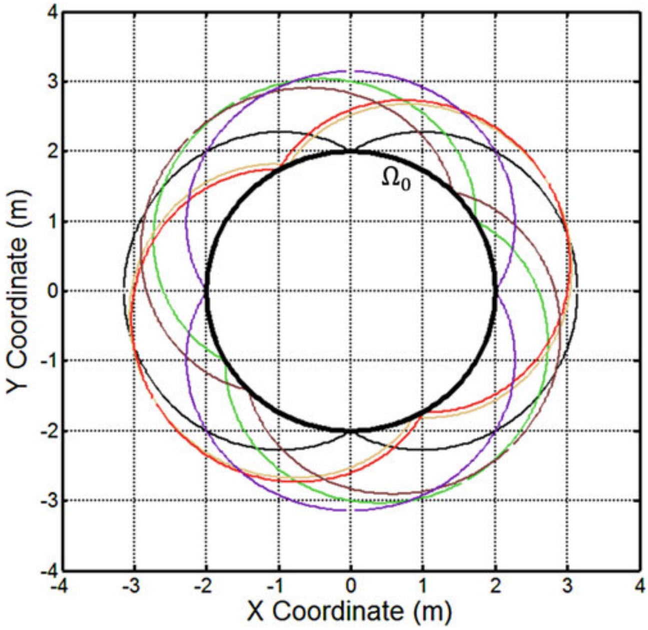

Fig. 2.12 The biggest disk Ω_0 inside the allowable displacement region

By taking second time derivative from Eq. (2.95), dynamics of evolution of the follower unicycle i becomes

$$\begin{bmatrix} \dddot{x}_i \\ \dddot{y}_i \end{bmatrix} = \begin{bmatrix} \cos\,\theta_i & -u_i \sin\,\theta_i \\ \sin\,\theta_i & u_i \cos\,\theta_i \end{bmatrix} \begin{bmatrix} \ddot{u}_i \\ \dot{\omega}_i \end{bmatrix} + \begin{bmatrix} -2\dot{u}_i\omega_i \sin\,\theta_i - u_i{\omega_i}^2 \cos\,\theta_i \\ 2\dot{u}_i\omega_i \cos\,\theta_i - u_i{\omega_i}^2 \sin\,\theta_i \end{bmatrix}. \tag{2.98}$$

If the linear velocity u_i does not vanish at any time t, then $\ddot{u}_i$ and $\dot{\omega}_i$ can be related to the centroid jerk (time derivative of the centroid acceleration) of the follower i by

$$\begin{bmatrix} \ddot{u}_i \\ \dot{\omega}_i \end{bmatrix} = \begin{bmatrix} \cos\,\theta_i & -u_i \sin\,\theta_i \\ \sin\,\theta_i & u_i \cos\,\theta_i \end{bmatrix}^{-1} \begin{bmatrix} \dddot{x}_i + 2\dot{u}_i\omega_i \sin\,\theta_i + u_i{\omega_i}^2 \cos\,\theta_i \\ \dddot{y}_i - 2\dot{u}_i\omega_i \cos\,\theta_i + u_i{\omega_i}^2 \sin\,\theta_i \end{bmatrix}. \tag{2.99}$$

Let

$$\begin{bmatrix} \dddot{x}_i \\ \dddot{y}_i \end{bmatrix} = \begin{bmatrix} \dddot{x}_{i,HT} \\ \dddot{y}_{i,HT} \end{bmatrix} + \zeta_i \begin{bmatrix} \ddot{x}_{i,HT} - \ddot{x}_i \\ \ddot{y}_{i,HT} - \ddot{y}_i \end{bmatrix} + \gamma_i \begin{bmatrix} \dot{x}_{i,HT} - \dot{x}_i \\ \dot{y}_{i,HT} - \dot{y}_i \end{bmatrix} + \eta_i \begin{bmatrix} x_{i,HT} - x_i \\ y_{i,HT} - y_i \end{bmatrix}, \tag{2.100}$$

then r_i asymptotically converges to $r_{i,HT}$ when $\zeta_i > 0$, $\gamma_i > 0$, $\eta_i > 0$ and $\gamma_i\zeta_i > \eta_i$. This is because the transient error (the difference between r_i and $r_{i,HT}$) converges to zero as $t \to \infty$. It is noted that $x_{i,HT}$ and $y_{i,HT}$ are the X and Y components of the desired centroid position of the robot $i \in V_F$ that is defined by a homogeneous transformation.

Discussion**:** As aforementioned, if leader agents do not leave the disk Ω_0 in π s, then followers can reach the desired positions prescribed by a homogeneous deformation at the time $t_1 = t_0 + \pi$. One possibility for a follower i is to move along the circular trajectory, connecting its ${r_{i,HT}}^0$ (the desired centroid position at the initial time t_0) and ${r_{i,HT}}^1$ (the desired centroid position at the time t_1), through choosing constant linear and angular velocities. For example, consider the follower 8 whose centroid is initially positioned at ${r_{8,HT}}^0 = -7.58\hat{\mathbf{e}}_x + 0.30\hat{\mathbf{e}}_y$. Then, it can reach ${r_{8,HT}}^1 = -7.58\hat{\mathbf{e}}_x + 2.30\hat{\mathbf{e}}_x$ by choosing

$$\begin{cases} \dot{x}_i = \dfrac{1}{\sqrt{2}}\cos(\dfrac{-t}{2} + \dfrac{3\pi}{4}) \\ \dot{y}_i = \dfrac{1}{\sqrt{2}}\sin(\dfrac{-t}{2} + \dfrac{3\pi}{4}) \\ \dot{\theta}_i = -\dfrac{1}{2} \end{cases} \tag{2.101}$$

In Fig. 2.13, the circular trajectory given by Eq. (2.101) is depicted by a continuous curve, where it connects the desired centroid positions of the follower 8 at the time $t_0 = 0s$ and $t_1 = \pi s$. As seen in Fig. 2.13, when circular trajectory is chosen, the follower 8 deviates from the desired trajectory that is shown by the dotted curve. Therefore, follower agents do not deform as homogeneous transformation during the time $t \in (0, \pi)$ and consequently interagent collision is not necessarily avoided.

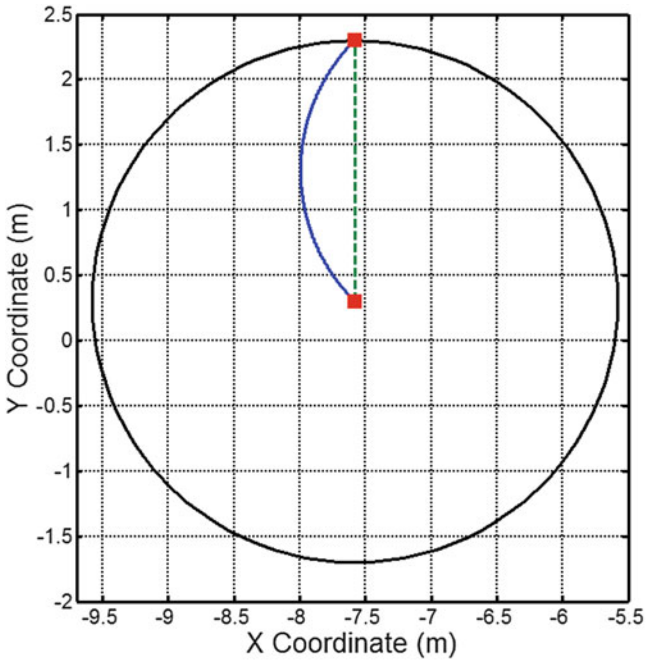

Fig. 2.13 Possible circular trajectory for the follower 8

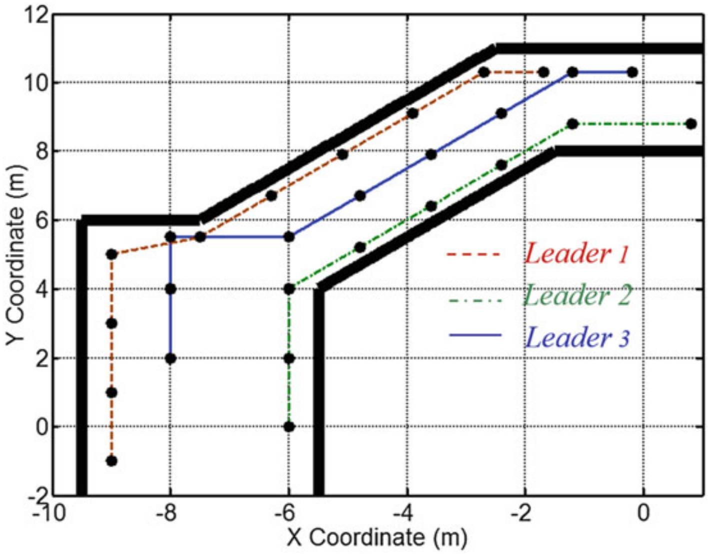

Fig. 2.14 Leaders' paths inside the narrow channel in Example 2.4

Notice that followers can fully track the desired line segment, connecting the desired positions at the time t_k and t_{k+1}, by first spinning with highest possible angular velocity, and then moving along the desired trajectory (which is the line segment connecting ${r_{i,HT}}^k$ and ${r_{i,HT}}^{k+1}$) by choosing the highest possible linear velocity. However, this trivial solution only works for the ground unicycles.

Updating centroid position of the unicycle i according to Eq. (2.100) is advantageous, because transient error (difference between actual and desired centroid positions of a follower) can be reduced during evolution. However, this requires choosing $t_{k+1} - t_k$ large enough in order to ensure that the constraints on the control inputs are not violated.

Example 2.4. In this example, motion of an MAS consisting 8 follower unicycle robots and three leaders are simulated. The initial positions and orientations of the followers are listed in Tables 2.1 and 2.2. In Fig. 2.14, the paths chosen by the leaders are shown. As it is seen, leaders guide collective motion of the follower unicycles inside the narrow channel. It is noticed that desired velocities of the leaders are piecewise constant for any time period $t \in [t_k, t_{k+1}]$, where $k \in \{0,\ 1, \ldots,\ 9\}$. In Figs. 2.15 and 2.16, elements of the Jacobian Q and rigid body displacement vector D, that are obtained based on the X and Y components of positions of the leaders by applying Eq. (2.21), are depicted.

Evolution of the follower unicycles: Desired centroid positions and velocities of the followers are determined by Eq. (2.97), and each follower i updates its current centroid position by using Eq. (2.100) with $\gamma_i = \zeta_i = \eta_i = 30$. In Fig. 2.17 desired and actual positions of the follower 8 inside the narrow channel are

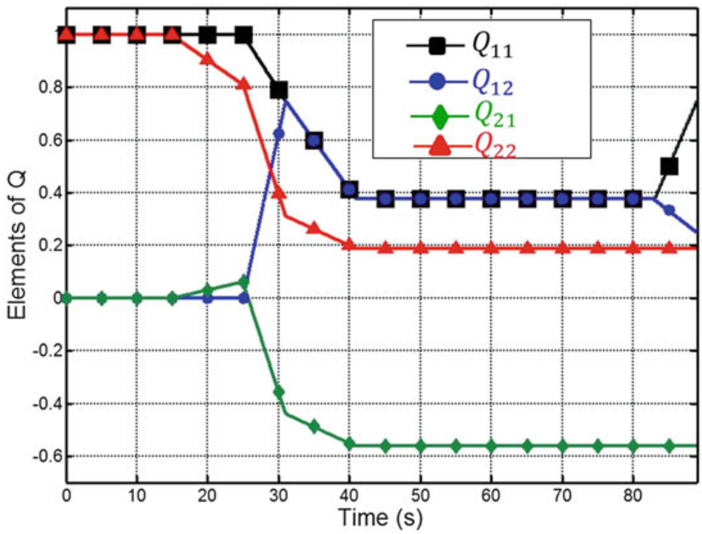

Fig. 2.15 Entries of Q versus time in Example 2.4

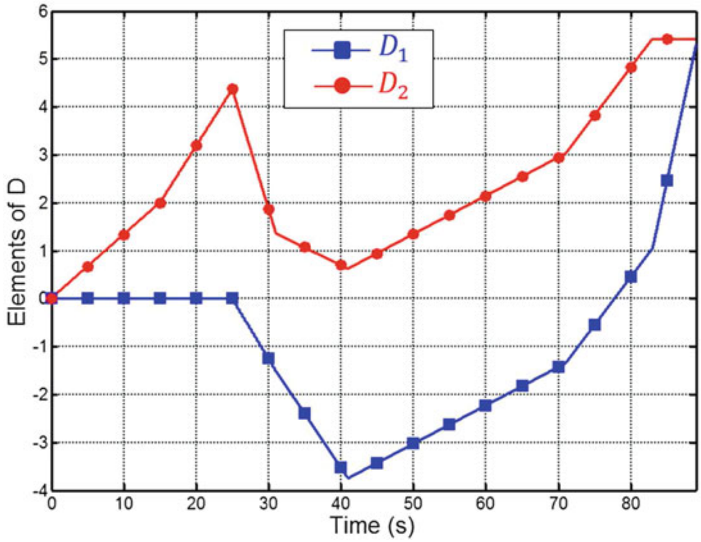

Fig. 2.16 Entries of D versus time in Example 2.4

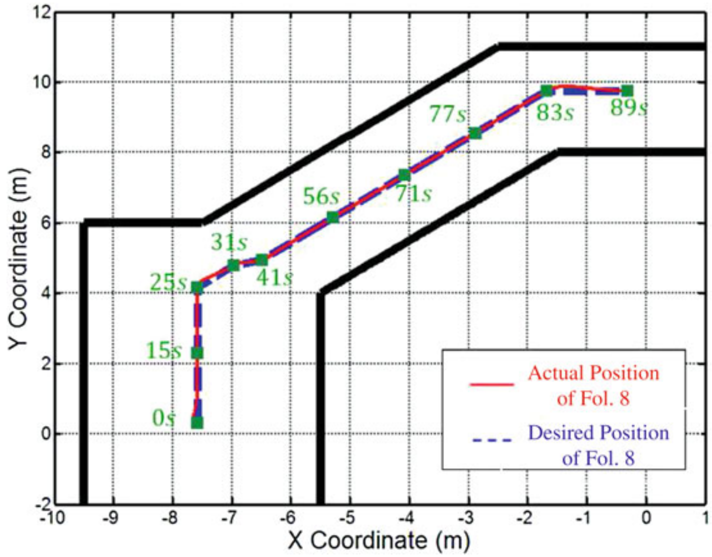

Fig. 2.17 Desired and actual paths of the centroid of follower unicycle 8

illustrated. As seen, followers reach desired way points, issued by the homogeneous transformation, at the times $t_k \in \{0,\ 15,\ 25,\ 31,\ 41,\ 56,\ 71,\ 77,\ 83,\ 89\}$.

As mentioned in Section 2.1.1, if $p_{i,k}(t)$ remains unchanged at any time t during MAS evolution, then actual position of each follower i is coincided on the desired position prescribed by a homogeneous mapping. In other words, $p_{i,k}(t) - \alpha_{i,k}$ quantifies deviation of a follower i from the desired state prescribed by a homogeneous mapping (the parameter $\alpha_{i,k}$, obtained from Eq. (2.18), only depends on the initial positions of the leaders and the initial centroid position of a follower i). In Fig. 2.18, parameters $p_{8,1}(t)$, $p_{8,2}(t)$, and $p_{8,3}(t)$ are depicted versus time. As illustrated, the magnitude of the parameters $p_{8,1}(t)$, $p_{8,2}(t)$, and $p_{8,3}(t)$ at the times t_k ($k = 1,\ 2, \ldots,\ 9$) are the same as the parameters, $\alpha_{8,1}$, $\alpha_{8,2}$, and $\alpha_{8,3}$, respectively. This implies that MAS configuration at t_k ($k = 1,\ 2, \ldots,\ 9$) is a homogeneous deformation of the initial configuration.

In Figs. 2.19 and 2.20, linear and angular velocities of follower unicycle robots are shown. As it is seen, u_i and ω_i do not violate the input constraint in Eq. (2.96) during evolution of the follower i.

Moreover, orientations of the follower unicycles are depicted versus time in Fig. 2.21.

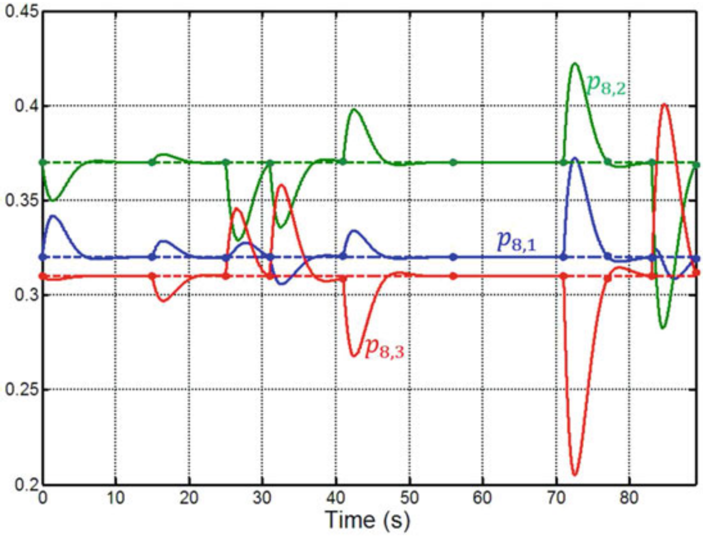

Fig. 2.18 Parameters $p_{8,1}(t)$, $p_{8,2}(t)$, and $p_{8,3}(t)$; Parameters $\alpha_{8,1}$, $\alpha_{8,2}$, and $\alpha_{8,3}$

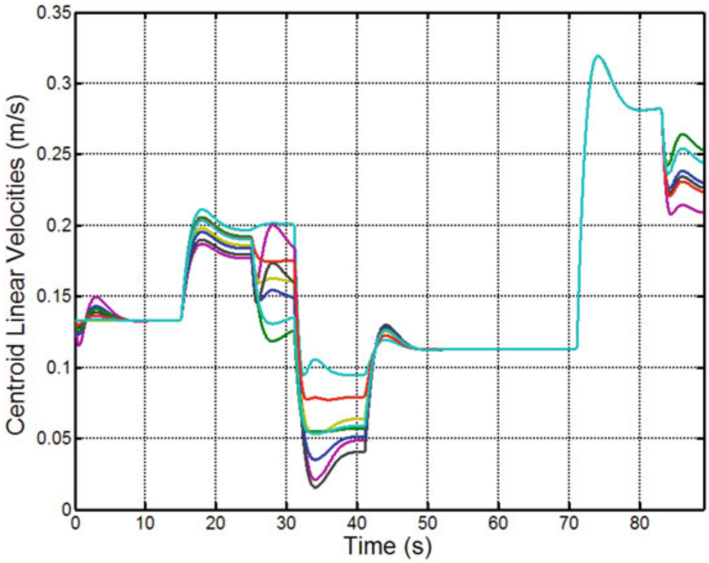

Fig. 2.19 Linear velocities of follower unicycles

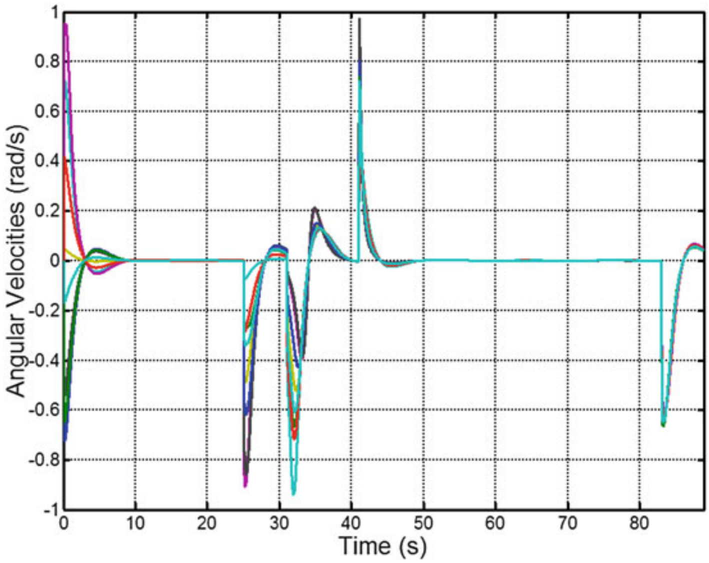

Fig. 2.20 Angular velocities of follower unicycles

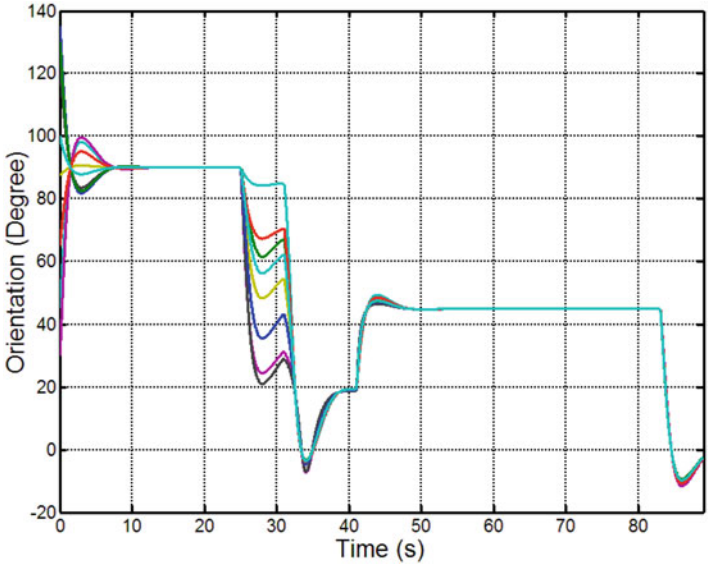

Fig. 2.21 Heading angles of the follower unicycles versus time

2.3 Homogeneous Deformation of Followers with Linear Dynamics

2.3.1 Continuous Time

Suppose the leader $j \in V_L$ moves with constant velocity on a straight line-connecting two consecutive way points at ${r_j}^k \in \mathbb{R}^n$ and ${r_j}^{k+1} \in \mathbb{R}^n$. Then, velocity of the leader j is determined by

$$\forall t \in [t_k, t_{k+1}],\ v_j(t) = \frac{{r_j}^{k+1} - {r_j}^k}{t_{k+1} - t_k} \in \mathbb{R}^n. \tag{2.102}$$

and desired position of the follower $i \in V_F$ becomes

$$\forall t \in [t_k, t_{k+1}],\ {r_{i,HT}}^k(t) = \sum_{j=1}^{n+1} \alpha_{i,j} \left[\frac{{r_j}^{k+1} - {r_j}^k}{t_{k+1} - t_k}(t - t_k) + {r_j}^k \right]. \tag{2.103}$$

It is also assumed that a follower i has a linear dynamics that is defined by

$$\begin{cases} \dot{Z}_i = A_i Z_i + B_i W_i \\ r_i = C_i Z_i \end{cases}, \tag{2.104}$$

where $Z_i \in \mathbb{R}^{n_z}$ is the control state, $W_i \in \mathbb{R}^{n_W}$ is the control input, and position of the follower i (which is denoted by $r_i \in \mathbb{R}^n$) is the control output of the linear system (2.104). Notice that the dynamics of the follower i is both state controllable and state observable.

***Robust Tracking and Disturbance Rejection*:** It is desired that r_i asymptotically tracks ${r_{i,HT}}^k(t)$ given by Eq. (2.103), while the disturbance $d_i(t)$ is rejected [17]. The plant transfer function of the dynamics of the follower $i \in V_F$ is obtained as follows:

$$G_i(s) = D_i^{-1}(s) N_i(s) = C_i (sI - A_i)^{-1} B_i. \tag{2.105}$$

It is noted that $G_i(s) = D_i^{-1}(s) N_i(s)$ is left coprime. This is because the dynamics of the follower i is both state controllable and state observable. The Laplace transform of ${r_{i,HT}}^k(t)$ becomes

$$\hat{r}^k_{i,HT}(s) = 1/s^2 ({\mu_i}^k s + {v_i}^k), \tag{2.106}$$

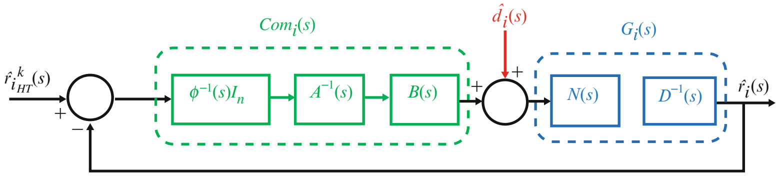

Fig. 2.22 Block diagram for robust tracking and disturbance rejection

where

$$\mu_i{}^k = \sum_{j=1}^{n+1} \alpha_{i,j} \left(r_j{}^k - t_k \frac{r_j{}^{k+1} - r_j{}^k}{t_{k+1} - t_k} \right) \tag{2.107}$$

$$\nu_i{}^k = \sum_{j=1}^{n+1} \alpha_{i,j} \frac{r_j{}^{k+1} - r_j{}^k}{t_{k+1} - t_k}. \tag{2.108}$$

Additionally, the Laplace transform of $d_i(t)$ is expressed by

$$\hat{d}_i(s) = N_{d_i} D_{d_i}^{-1}. \tag{2.109}$$

It is assumed that $\Phi_i(s) = s^2 p_i(s)$ is the least common denominator of the unstable poles $\hat{r}_{i,HT}^k(s)$ and $\hat{d}_i(s)$. Therefore, roots of $p_i(s)$ encompass unstable poles of $\hat{d}_i(s)$.

Block diagram for asymptotic tracking of desired position and disturbance rejection is shown in Fig. 2.22.

***Design of the Compensator*:** For a given plant transfer function $G_i(s) = D_i{}^{-1}(s)N_i(s)$, the compensator $Com_i(s) = B(s)(\Phi(s)A(s))^{-1}$ is designed such that

1. the transfer function from $\hat{r}_{i,HT}^k(s)$ to $\hat{r}_i^k(s)$,

$$\begin{aligned} G_{0_i}(s) &= G_i(s)Com_i(s)[I_n + G_i(s)Com_i(s)]^{-1} \\ &= [I_n + G_i(s)Com_i(s)]^{-1} G_i(s)Com_i(s) \\ &= [I_n + D^{-1}NB(\Phi A)^{-1}]^{-1} D^{-1}NB(\Phi A)^{-1}, \end{aligned} \tag{2.110}$$

   is asymptotically stable, and
2. the desired position $r_{i,HT}{}^k(t)$ is asymptotically tracked by $r_i(t)$ and the disturbance $d_i(t)$ is rejected.

   Suppose

$$D\Phi A + NB = F \tag{2.111}$$

then Eq. (2.110) simplifies to

$$G_{0_i}(s) = I_n - \Phi A F^{-1} D. \tag{2.112}$$

Remark 2.1. If the roots of the polynomial matrix F are all placed in the open left half s-plane, then the transfer function $G_{0_i}(s)$ is stable. Notice that the desired position ${r_{i,HT}}^k(t)$ is asymptotically tracked and the disturbance $d_i(t)$ is rejected, if $\Phi_i(s) = s^2 p_i(s)$.

Example 2.5. Let each follower be a double integrator that moves in a plane, then the dynamics of the follower $i \in V_F$ becomes

$$\begin{cases} \dot{Z}_i = A_i Z_i + B_i W_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} Z_i + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} W_i \\ r_i = C_i Z_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} Z_i \end{cases}. \tag{2.113}$$

It is desired that each follower asymptotically tracks the desired position specified by Eq. (2.103).

The plant transfer function

$$G_i(s) = C_i(sI - A_i)^{-1} B_i = \begin{bmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix} = D^{-1} N \tag{2.114}$$

is strictly proper. The fraction $G_i(s) = D^{-1}N$ is left coprime, where $D = s^2 I_2$ and $N = I_2$. Let

$$F(s) = D\Phi A + NB = \begin{bmatrix} (s+3)^4 & 0 \\ 0 & (s+3)^4 \end{bmatrix} \tag{2.115}$$

then $\Phi = I_2$,

$$A(s) = \begin{bmatrix} s^2 + 12s + 54 & 0 \\ 0 & s^2 + 12s + 54 \end{bmatrix},$$

and

$$B(s) = \begin{bmatrix} 108s + 1 & 0 \\ 0 & 108s + 1 \end{bmatrix}.$$

Therefore,

$$G_0(s) = \begin{bmatrix} \frac{108s+81}{(s+3)^4} & 0 \\ 0 & \frac{108s+81}{(s+3)^4} \end{bmatrix} \tag{2.116}$$

and

$$\hat{r}_{i,HT}^k(s) - \hat{r}_i^k(s) = \begin{bmatrix} \frac{s^4+12s^3+54s^2}{(s+3)^4} & 0 \\ 0 & \frac{s^4+12s^3+54s^2}{(s+3)^4} \end{bmatrix} \hat{r}_{i,HT}^k(s). \tag{2.117}$$

Consequently, the transient error $({r_{i,HT}}^k(t) - r_i(t))$ converges to zero, and the desired position ${r_{i,HT}}^k(t)$ is asymptotically tracked by the follower $i \in V_F$ during the time interval $t \in [t_k, t_{k+1}]$.

2.3.2 Discrete Time Finite Time Reachability Model

In this subsection, it is demonstrated how followers can reach desired way points, specified by a homogeneous deformation, in a finite horizon of time.

Suppose that dynamics of the follower $i \in V_F$ is updated by

$$\begin{cases} Z_i[K+1] = A_i Z_i[K] + B_i W_i[K] + d_i[K] \\ r_i[K] = C_i Z_i[K] + n_i[K] \end{cases} \tag{2.118}$$

where $K = 1,\ 2, \ldots,$ denotes time steps, $Z_i \in \mathbb{R}^{n_z}$, $W_i \in \mathbb{R}^{n_w}$, and $r_i \in \mathbb{R}^n$ $(n = 1,2,3)$ are control state, input and output, respectively. Furthermore, $d_i \in \mathbb{R}^{n_z}$ and $n_i \in \mathbb{R}^n$ are zero-mean disturbance and measurement noise, respectively. The dynamics (2.118) is controllable, therefore, the matrix

$$S_i = \left[{A_i}^{p-1} \ldots B_i\right] \tag{2.119}$$

has the rank n_z, if $p \geq n_z$. Let $K = (J-1)p + l$ $(J = 1,\ 2, \ldots$ and $l = 1,\ 2, \ldots,\ p)$. Then the dynamics (2.119) can be written in the following p-step ahead form:

$$\begin{cases} Z_i[Jp] = {A_i}^p Z_i[(J-1)p] + S_i CO_i[(J-1)p] + D_i[(J-1)p] \\ r_i[Jp] = C_i Z_i[Jp] + n_i[Jp] \end{cases} . \tag{2.120}$$

Note that

$$D_i = \sum_{l=0}^{p-1} A_i^{p-1-l} d_i[(J-1)p+l] \in \mathbb{R}^{n_z} \tag{2.121}$$

$$CO_i = \begin{bmatrix} W_i[(J-1)p] \\ \vdots \\ W_i[Jp-1] \end{bmatrix} \in \mathbb{R}^{n_w p}. \tag{2.122}$$

***Deterministic Dynamics*:** If disturbance and measurement noise are both zero, then it can be assured that the desired output

$$r_{i,HT}[Jp] = \sum_{k=1}^{n+1} \alpha_{i,k} r_k[Jp] \tag{2.123}$$

can be reached at $K = p,\ 2p, 3p, \ldots$. This implies that there exists a control vector CO_i^* assuring the reachability condition,

$$r_i[Jp] - r_{i,HT}[Jp] = 0. \tag{2.124}$$

at $J = 1,\ 2, \ldots$. The control $CO_i{}^*$ also minimizes the cost function

$$\begin{aligned} cost = \frac{1}{2} CO_i[(J-1)p]^T \Omega_{J-1} CO_i[(J-1)p] \\ + \sum_{l=0}^{p-1} \Big[r_i[(J-1)p+l] - r_{i,HT}[(J-1)p+l] \Big]^T \Gamma_{J-1} \Big[r_i[(J-1)p+l] - r_{i,HT}[(J-1)p+l] \Big]. \end{aligned} \tag{2.125}$$

where $\Omega_{J-1} \in \mathbb{R}^{n_w \times n_w}$ and $\Gamma_{J-1} \in \mathbb{R}^{n \times n}$ are positive definite weight matrices.

Equation (2.125) can be rewritten as

$$\begin{aligned} cost = \frac{1}{2} CO_i[(J-1)p]^T (\Omega_{J-1} + P_{J-1}) CO_i[(J-1)p] + E_{J-1}^T CO_i[(J-1)p] \\ + \sum_{l=1}^{p} (C_i A_i^l Z[(J-1)p] \\ - r_{i,HT}[(J-1)p+l]) \Gamma_{J-1} (C_i A_i^l Z[(J-1)p] - r_{i,HT}[(J-1)p+l]). \end{aligned} \tag{2.126}$$

where the j^{th} $(j = 1, 2, \ldots, p)$ block of $E_{J-1} \in \mathbb{R}^{p.n_w}$ and the $i_1 i_2$ $(i_1, i_2 = 1, 2, \ldots, p)$ block of $P_{J-1} \in \mathbb{R}^{p.n_w \times n.n_w}$ are obtained as follows:

$$E_{J-1} = 2 \sum_{\substack{l=1 \\ j \leq l}}^{p} \left(C_i A_i^{l-j} B_i\right)^T \Gamma_{J-1} \left(C_i A_i^l B_i Z_i[(J-1)p] - r_{i,HT}[(J-1)p+l]\right) \in \mathbb{R}^{n_w \times n_w} \tag{2.127}$$

$$\{P_{J-1}\}_{i_1 i_2} = \sum_{l=max\{i_1,i_2\}}^{p} \left(C_i {A_i}^{l-i_1} B_i\right)^T \Gamma_{J-1} \left(C_i {A_i}^{l-i_2} B_i\right) \in \mathbb{R}^{n_u \times n_u}. \tag{2.128}$$

To specify ${CO_i}^*$ the augmented cost function

$$\begin{aligned} COST = \frac{1}{2} CO_i[(J-1)p]^T (\Omega_{J-1} + P_{J-1}) CO_i[(J-1)p] + \\ {E_J}^T CO_i[(J-1)p] + \sum_{l=1}^{p} (C_i {A_i}^l Z[(J-1)p] - \\ r_{i,HT}[(J-1)p+l])^T \Gamma_{J-1} (C_i {A_i}^l Z_i[(J-1)p] - r_{i,HT}[(J-1)p+l]) + \\ \lambda_i[J-1]^T \left(r_{i,HT}[Jp] - C_i({A_i}^p Z_i[(J-1)p] + S_i CO_i[(J-1)p])\right) \end{aligned} \tag{2.129}$$

should be minimized, where $\lambda_i[J-1] \in \mathbb{R}^n$ is the Lagrange multiplier vector. Therefore,

$$CO_i = {CO_i}^* = (\Omega_{J-1} + P_{J-1})^{-1} ({S_i}^T C_i \lambda_i [J-1]^* - E_{J-1}) \tag{2.130}$$

with

$$\begin{aligned} \lambda_i[J-1] = \lambda_i[J-1]^* = (C_i S_i (\Omega_{J-1} + P_{J-1})^{-1} {S_i}^T {C_i}^T)^{-1} \\ \times (r_{i,HT}[Jp] + C_i S_i (\Omega_{J-1} + P_{J-1})^{-1} E[J-1] - C_i {A_i}^p Z[(J-1)p]) \end{aligned} \tag{2.131}$$

minimizes the cost (2.126), and finite time reachability of the desired position $r_{i,HT}$ is guaranteed.

Stochastic Dynamics**:** It is assumed D_i and n_i represent nonzero zero-mean disturbance and measurement noise, respectively, where they are uncorrelated and

$$\mathbb{E}({D_i}^T D_i) = Q_i \tag{2.132}$$

$$\mathbb{E}({n_i}^T n_i) = R_i. \tag{2.133}$$

denote the expected values of ${D_i}^T D_i$ and ${n_i}^T n_i$, respectively. In presence of disturbance and measurement noise, Kalman filter [130] is used to estimate the state Z_i. Let $\hat{Z}_i^-$ and $\hat{Z}_i^+$ denote prediction and measurement update of the state Z_i. Then,

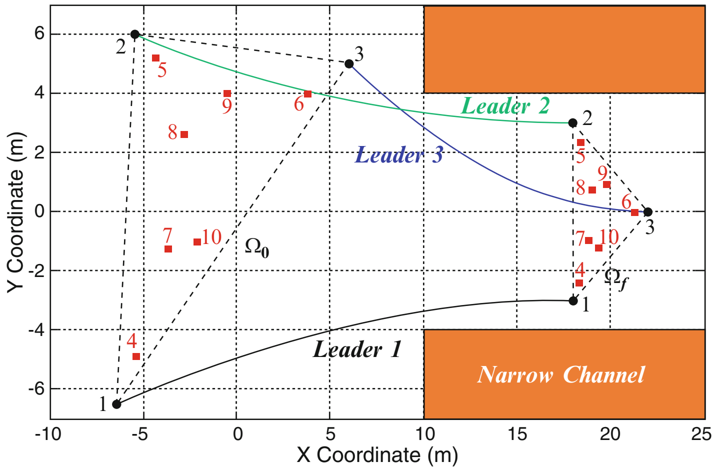

Fig. 2.23 Leaders' paths in Example 2.6

$$\hat{CO}_i = (\Omega_{J-1} + P_{J-1})^{-1}(S_i{}^T C_i \hat{\lambda}_i[J-1]^* - E_{J-1})$$

$$\hat{\lambda}_i[J-1] = \lambda_i[J-1]^* = (C_i S_i (\Omega_{J-1} + P_{J-1})^{-1} S_i{}^T C_i{}^T)^{-1} \times (r_{i,HT}[Jp] + C_i S_i (\Omega_{J-1} + P_{J-1})^{-1} E[J-1] - C_i A_i{}^p \hat{Z}_i^+[(J-1)p]) \tag{2.134}$$

and states are estimated by using Kalman filter as follows:

$$\begin{cases} P_i{}^-[Jp] = A_i{}^p P_i{}^+[(J-1)p] A_i^{pT} \\ K_i[Jp] = P_i{}^-[Jp] C_i{}^T (C_i P_i{}^-[Jp] C_i{}^T + R_i) \\ \hat{Z}_i^-[Jp] = A_i{}^p \hat{Z}_i^+[(J-1)p] + S_i \widehat{CO}_i \\ \hat{Z}_i^+[Jp] = \hat{Z}_i^-[Jp] + K_i[Jp](r_{i,HT}[Jp] - C_i \hat{Z}_i^-[Jp]) \\ P_i{}^+[Jp] = (I - K_i[Jp]C_i) P_i{}^- (I - K_i[Jp]C_i)^T + K_i[Jp] R_i K_i{}^T[Jp] \end{cases} . \tag{2.135}$$

Example 2.6 ([105]). Consider an MAS consisting of 10 agents (3 leaders and 7 followers) with the initial and final formations shown in Fig. 2.23. Leaders choose the paths shown in Fig. 2.23, where they start their motion from rest at $K = 1$ and they stop at $K = 3000$.

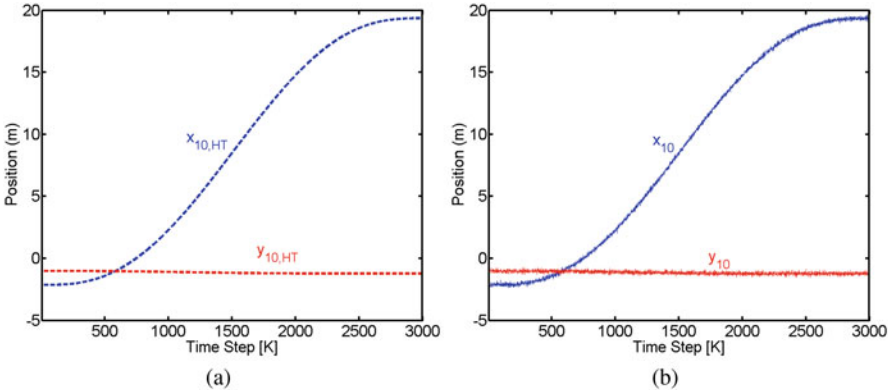

Fig. 2.24 (a) X and Y components of the desired position $r_{10,HT} = x_{10,HT}\hat{\mathbf{e}}_x + y_{10,HT}\hat{\mathbf{e}}_y$; (b) actual positions $r_{10} = x_{10}\hat{\mathbf{e}}_x + y_{10}\hat{\mathbf{e}}_y$

It is assumed that followers have the linear dynamics defined by Eq. (2.120), where

$$A_i = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and $K = (J-1)p + l = 1,\ 2,\ \ldots,\ 3000$ ($p = 4,\ J = 1,\ 2, \ldots,\ 750$). Followers update their states according to Eqs. (2.134) and (2.135), where disturbance and measurement noise are characterized by $Q_i = 0.1I \in \mathbb{R}^{4\times 4}$ and $R_i = 0.1I \in \mathbb{R}^{2\times 2}$.

Given leaders' positions at $K = 1,\ 2, \ldots,\ 3000$, desired position of the follower 10 is obtained from Eq. (2.17). Shown in Fig. 2.24(a) are the X and Y components of the desired position $r_{10,HT} = x_{10,HT}\hat{\mathbf{e}}_x + y_{10,HT}\hat{\mathbf{e}}_y$. Furthermore, components of actual position $r_i = x_{10}\hat{\mathbf{e}}_x + y_{10}\hat{\mathbf{e}}_y$ are illustrated versus time in Fig. 2.24(b).

Springer
http://www.springer.com/978-3-319-41593-2

Continuum Deformation of Multi-Agent Systems
Rastgoftar, H.
2016, XIII, 217 p. 136 illus., 99 illus. in color., Hardcover
ISBN: 978-3-319-41593-2
A product of Birkhäuser Basel