

## Chapter 2

# Martin Davis and Hilbert's Tenth Problem

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**Abstract** The paper presents the history of the negative solution of Hilbert's tenth problem, the role played in it by Martin Davis, consequent modifications of the original proof of DPRM-theorem, its improvements and applications, and a new (2010) conjecture of Martin Davis.

**Keywords** Computability · Hilbert's Tenth Problem · DPRM-theorem

### 2.1 The Problem

Martin Davis will stay forever in the history of mathematics and computer science as a major contributor to the solution of Hilbert's tenth problem.

This was one among 23 problems which David Hilbert stated in his famous paper "Mathematical Problems" [18] delivered at the Second International Congress of Mathematicians. This meeting took place in Paris in 1900, on the turn of the century. These problems were, in Hilbert's opinion, among the most important problems that the passing nineteenth century was leaving open to the pending twentieth century.

The section of [18] devoted to the Tenth Problem is so short that it can be reproduced here in full:

#### 10. DETERMINATION OF THE SOLVABILITY OF A DIOPHANTINE EQUATION

Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

Equations from the statement of the problem have the form

$$P(x_1, x_2, \dots, x_n) = 0 \quad (2.1)$$

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where  $P$  is a polynomial with integer coefficients. The equations are named after the Greek mathematician Diophantus who lived, most likely, in the 3rd century A.D.

The Tenth problem is the only one of the 23 Hilbert's problems that is (in today's terminology) a *decision problem*, i.e., a problem consisting of infinitely many *individual problems* each of which requires a definite answer: YES or NO. The heart of a decision problem is the requirement to find a single method that will give an answer to any individual subproblem.

Since Diophantus's time, number-theorists have found solutions for a large amount of Diophantine equations, and also they have established the unsolvability of a lot of other equations. Unfortunately, for different classes of equations, and often even for different individual equations, it was necessary to invent specific methods. In his tenth problem, Hilbert asks for a *universal method* for deciding the solvability of all Diophantine equations.

A decision problem can be solved in a positive or in a negative sense, that is, either by discovering a required algorithm or by showing that none exists. Hilbert foresaw the possibility of negative solutions to some mathematical problems, in [18] he wrote:

Occasionally it happens that we seek the solution under insufficient hypotheses or in an incorrect sense, and for this reason do not succeed. The problem then arises: to show the impossibility of the solution under the given hypotheses, or in the sense contemplated. Such proofs of impossibility were effected by the ancients, for instance when they showed that the ratio of the hypotenuse to the side of an isosceles triangle is irrational. In later mathematics, the question as to the impossibility of certain solutions plays a preëminent part, and we perceive in this way that old and difficult problems, such as the proof of the axiom of parallels, the squaring of circle, or the solution of equations of the fifth degree by radicals have finally found fully satisfactory and rigorous solutions, although in another sense than that originally intended. It is probably this important fact along with other philosophical reasons that gives rise to conviction (which every mathematician shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts.

But in 1900 it was impossible even to state rigourously what would constitute a negative solution of Hilbert's tenth problem. The general mathematical notion of algorithm was developed by Alonzo Church, Kurt Gödel, Alan Turing, Emil Post, and other logicians only three decades after Hilbert's lecture [18].

The appearance of the general notion of algorithms gave the possibility to establish non-existence of algorithms for particular decision problems, and soon such *undecidable problems* were actually found. But these results didn't much impress "pure mathematicians" because the first discovered undecidable problems were from the realm of mathematical logic and the just emerging computer science.

The situation changed in 1947 when two mathematician, Andrei Andreevich Markov [28] in the USSR, and Emil Post [45] in the USA, independently proved that there is no algorithm for so called *Thue problem*. This problem was posed by Alex Thue [58] in 1914, much before the development of the general notion of an algorithm. Thue asked for a method for deciding, given a finitely presented semi-

group and two elements from it, whether the defining relations imply the equality of these two elements or not. Thus Thue problem, known also as *word problem for semigroups*, became the very first decision problem, born in mathematics proper and proved to be undecidable.

After the success with Thue problem, researchers were inspired to establish the undecidability of other long standing open mathematical problems. In particular, both Markov and Post were interested in Hilbert's tenth problem. Already in 1944 Post wrote in [44] that Hilbert's tenth problem "begs for an unsolvability proof". Post had a student to whom this statement produced great impression and he decided to tackle the problem.

The name of this student was Martin Davis.

## 2.2 Martin Davis Conjecture

### 2.2.1 Statement and Corollaries

Very soon Martin Davis came to a conjecture, first announced in [3], that would imply the undecidability of Hilbert's tenth problem. To be able to state this conjecture, we need to introduce a bit more terminology.

Hilbert asked for solving Diophantine equations with *numerical* coefficients. One can also consider equation with *symbolic* coefficient, that is, equations with parameters. Such an equation has the form

$$P(a_1, \dots, a_m, x_1, x_2, \dots, x_n) = 0 \quad (2.2)$$

similar to (2.1) but now the variables are split into two groups: *parameters*  $a_1, \dots, a_m$ , and *unknowns*  $x_1, \dots, x_n$ .

As another (minor technical) deviation from Hilbert's statement of the problem, we will assume that both the parameters and the unknowns range over the natural numbers; following the tradition of mathematical logic, we will consider 0 as a natural number.

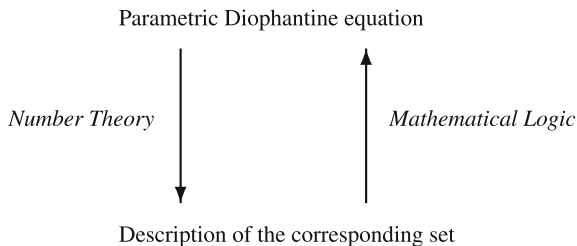
For some choice of the values of the parameters the Eq. (2.2) may have a solution in the unknowns, and for another choice may have no solution. We can consider the set  $\mathcal{M}$  of all  $m$ -tuples of the values of the parameters for which the Eq. (2.2) has a solution in the unknowns:

$$\langle a_1, \dots, a_m \rangle \in \mathcal{M} \iff \exists x_1 \dots x_n [P(a_1, \dots, a_m, x_1, x_2, \dots, x_n) = 0]. \quad (2.3)$$

Sets having such a *Diophantine representation* are also named *Diophantine*.

Traditionally, in Number Theory an equation is the primary object, and one is interested in a description of the set of the values of the parameters for which the equation has a solution. Martin Davis, in a sense, reversed the order of things taking

sets as primary objects—he decided to give a general characterization of the whole class of all Diophantine sets.



He approached this problem from a computational point of view. All Diophantine sets have one common property: they are *listable*, or, in another terminology, *effectively enumerable*. This means the following. Given a parametric Diophantine equation (2.2), we can start looking over, in some order, all possible  $(m + n)$ -tuples  $\langle a_1, \dots, a_m, x_1, \dots, x_n \rangle$ , for each such a tuple check if the value of the polynomial  $P$  is equal to zero, and if this happens to be the case, write down the tuple  $\langle a_1, \dots, a_m \rangle$  on some list. Sooner or later each tuple from the set (2.3) will appear on this list, maybe many times, and all tuples from the list will belong to this set.

Described above was a rather specific way of listing elements of Diophantine sets. For an arbitrary set to be listable no restriction is imposed on the method for generating its elements, the only requirement is that this should be done by an algorithm. For example, it is evident that the set of prime numbers is listable: it is easy to write a program that would print 2, 3, 5, ...

Thus, computability theory imposed an *obstacle* for a set to be Diophantine: *if a set is not listable, it cannot be Diophantine*. Martin Davis conjectured that this is the *only* obstacle.

**Martin Davis conjecture.** *Every listable set is Diophantine.*

I find that this was a rather daring conjecture because it has many corollaries, some of them quite striking.

For example, Martin Davis conjecture implied the existence of a one-parameter Diophantine equation

$$P(a, x_1, x_2, \dots, x_n) = 0 \quad (2.4)$$

having a solution if and only if  $a$  is a prime number. Hilary Putnam noticed in [47] that the same would be true for the equation

$$(x_0 + 1)(1 - P^2(x_0, x_1, x_2, \dots, x_n)) - 1 = a. \quad (2.5)$$

In other words, *the set of all prime numbers should be exactly the set of all non-negative values of the polynomial from the left-hand side of (2.5) assumed for all natural values of  $x_0, \dots, x_n$* . Number-theorists did not believe in such a possibility.

Some other consequences of Davis conjecture will be presented below. Of course, the undecidability of Hilbert's tenth problem is among of them. This is due to the

classical fundamental result, the existence of listable sets of natural numbers for which there is no algorithm for recognizing, given a natural number  $a$ , whether it belongs to the set or not.

Davis conjecture is much stronger than what would be sufficient for proving the undecidability of Hilbert's tenth problem. Namely, it would suffice to find for any particular undecidable set  $\mathcal{M}$  some representation similar to (2.2) with  $P$  being replaced by any function which becomes a polynomial in  $x_1, \dots, x_n$  after substituting numerical values for  $a_1, \dots, a_m$ . For example, we could allow parameters to appear in the exponents as it was done by Anatolyi Ivanovich Mal'tsev in [27].

### 2.2.2 The First Step to the Proof

Martin Davis had not much informal evidence in support of his conjecture. Slight support came from the following result announced in [3] and proved in [4–6].

**Theorem** (Martin Davis). *For every listable set  $\mathcal{M}$  there exists a polynomial  $Q$  with integer coefficients such that*

$$\langle a_1, \dots, a_m \rangle \in \mathcal{M} \iff \exists z \forall y_{\leq z} \exists x_1 \dots x_n [Q(a_1, \dots, a_m, x_1, x_2, \dots, x_n, y, z) = 0]. \quad (2.6)$$

Representation of type (2.6) became known as *Davis normal form*. They can be considered as an improvement of Kurt Gödel's technique [15] of arithmetization. This technique allowed him to represent any listable set by an arithmetical formula containing, possibly, many universal quantifiers. If all of them are bounded then such an arithmetical formula defines a listable set, and this can be used as another definition of them (this is the content of Theorem 2.7 from Martin Davis dissertation [4]).

### 2.2.3 A Milestone

In 1959 Martin Davis and Hilary Putnam [13] managed to eliminate the single universal quantifier from Davis normal form but this was not yet a proof of Davis conjecture for two reasons.

First, they were forced to consider a broader class of *exponential Diophantine equations*. They are equations of the form

$$E_L(a_1, \dots, a_m, x_1, \dots, x_n) = E_R(a_1, \dots, a_m, x_1, \dots, x_n) \quad (2.7)$$

where  $E_L$  and  $E_R$  are *exponential polynomials*, that is expression constructed by traditional rules from the variables and particular positive integers by addition, multiplication and exponentiation.

Second, the proof given by Martin Davis and Hilary Putnam was conditional: they assumed that *for every  $k$  there exist an arithmetical progression of length  $k$  consisting of different prime numbers*. In 1959 this hypothesis was considered plausible but it was proved by Ben Green and Terence Green only in 2004 [16]. Thus all what Davis and Putnam needed was to wait for 45 years!

Luckily, they had not to wait for so long. Julia Robinson [49] was able to modify the construction of Davis–Putnam and get an unconditional proof. In 1961 Martin Davis, Hilary Putnam, and Julia Robinson published a joint paper [14] with the following seminal result.

**DPR-theorem.** *Every listable set  $\mathcal{M}$  has an exponential Diophantine representation, i.e., a representation of the form*

$$\begin{aligned} \langle a_1, \dots, a_m \rangle \in \mathcal{M} &\iff \\ \exists x_1 \dots x_n [E_L(a_1, \dots, a_m, x_1, \dots, x_n) = E_R(a_1, \dots, a_m, x_1, \dots, x_n)] \end{aligned} \quad (2.8)$$

where  $E_L$  and  $E_R$  are *exponential polynomials*.

The elimination of the universal quantifier from Davis normal form immediately gave the undecidability of the counterpart of Hilbert’s tenth problem for the broader class of exponential Diophantine equations.

### 2.2.4 The Last Step

The DPR-theorem was a milestone on the way to proving Davis conjecture. All what remained to do was to prove a particular case of Davis conjecture, namely, to show that exponentiation is Diophantine. Indeed, suppose that we found a particular Diophantine equation

$$A(a, b, c, x_1, \dots, x_n) = 0 \quad (2.9)$$

which for given values of the parameters  $a, b$ , and  $c$  has a solution in  $x_1, \dots, x_n$  if and only if  $a = b^c$ . Using several copies of such an equation, one can easily transform an arbitrary exponential Diophantine equation into a genuine Diophantine equation (with additional unknowns) such that either both equations have solutions or none of them has.

In fact, Julia Robinson was tackling this problem from the beginning of the 1950s. It is instructive to note that her interest was originally stimulated by her teacher, Alfred Tarski, who asked one to prove that the set of all powers of 2 is *not* Diophantine. That is, the intuition of young Martin Davis was opposite to the intuition of venerable Alfred Tarski.

Julia Robinson was not able to construct the required equation (2.9) but she found [48, 50] a number of conditions sufficient for its existence. In particular she proved that in order to construct such an  $A$ , it is sufficient to have an equation

$$B(a, b, x_1, \dots, x_m) = 0 \quad (2.10)$$

which defines a relation  $J(a, b)$  with the following two properties:

- for any  $a$  and  $b$ ,  $J(a, b)$  implies that  $a < b^b$ ;
- for any  $k$ , there exist  $a$  and  $b$  such that  $J(a, b)$  and  $a > b^k$ .

Julia Robinson called a relation  $J$  with these two properties a *relation of exponential growth*; Martin Davis named them *Julia Robinson predicates*.

In 1970 I [29] was able to construct the first example of a relation of exponential growth, and it was the last link in the proof of Davis conjecture. Nowadays it is often referred to as

**DPRM-theorem.** *Every listable set of  $m$ -tuples of natural numbers has a Diophantine representation.*

This theorem implies, in particular, the undecidability of Hilbert's tenth problem: *There is no algorithm for deciding whether a given Diophantine equation has a solution.*

## 2.3 Further Modifications of Original Proofs

### 2.3.1 Pell Equation

My original construction of a Diophantine relation of exponential growth was based on the study of Fibonacci numbers defined by recurrent relations

$$\varphi_0 = 0, \quad \varphi_1 = 1, \quad \varphi_{n+1} = \varphi_n + \varphi_{n-1}, \quad (2.11)$$

while Julia Robinson worked with solutions of the following special kind of *Pell equation*:

$$x^2 - (a^2 - 1)y^2 = 1. \quad (2.12)$$

Solutions of this equation  $\langle \chi_0, \psi_0 \rangle, \langle \chi_1, \psi_1 \rangle, \dots, \langle \chi_n, \psi_n \rangle, \dots$  listed in the order of growth, satisfy the recurrence relations

$$\chi_0 = 1, \quad \chi_1 = a, \quad \chi_{n+1} = 2a\chi_n - \chi_{n-1}, \quad (2.13)$$

$$\psi_0 = 0, \quad \psi_1 = 1, \quad \psi_{n+1} = 2a\psi_n - \psi_{n-1}. \quad (2.14)$$

Sequences  $\varphi_0, \varphi_1, \dots$  and  $\psi_0, \psi_1, \dots$  have many similar properties, for example, they grow up exponentially fast. Immediately after the acquaintance with my construction for Fibonacci numbers, Martin Davis gave in [8] a Diophantine definition of the sequence of solutions of the Pell equation (2.12). The freedom in selection of the value of the parameter  $a$  allowed Martin Davis to construct a Diophantine definition (2.9) of the exponentiation directly, that is, without using the general method proposed by Julia Robinson starting with an arbitrary Diophantine relation of exponential growth. Today the use of the Pell equation for defining the exponentiation by a Diophantine equation has become a standard.

### 2.3.2 *Eliminating Bounded Universal Quantifier*

In [13] the necessity to work with long arithmetical progressions consisting of primes only was due to the usage of a version of Gödel's technique for coding arbitrary long sequences of natural numbers via the Chinese Remainder Theorem. Julia Robinson has managed to replace such progressions by arithmetical progressions composed of pairwise relatively prime numbers having arbitrary large prime factors. Much later, in 1972, using a multiplicative version of Dirichlet principle, I [30] made further modification allowing one to work just with arithmetical progressions of arbitrary big relatively prime numbers.

In [37, Sect. 6.3] I introduced a quite different technique for eliminating bounded universal quantifier based on replacing  $\forall y \leq z$  by  $\sum_{y=0}^z$  with a suitable summand allowing one to find a closed form for the corresponding sum.

### 2.3.3 *Existential Arithmetization*

The method for constructing the Davis normal form (2.6) presented in [5] starts with a representation of the set  $\mathcal{M}$  by an arithmetical formula in prenex form with any number of bounded universal quantifiers constructed, for example, by Gödel's technique. Two tools are repeatedly applied to such a formula, one tool allowing us to glue two consecutive existential or universal quantifiers, and the other tool giving the possibility to change the order of consecutive universal and existential quantifiers. The footnote on page 36 in [5] tells us that the idea of this construction belongs to an unknown referee of the paper.

Nevertheless, the main theorem from [5] does belong to Martin Davis, but his original proof presented in [4] was quite different. Namely, for the initial representation of listable sets he used *normal systems* introduced by his teacher Post. Thanks to the great simplicity of the normal systems Martin Davis was able to arithmetize them in a very economical way using only one bounded universal quantifier.

While Martin Davis remarks in the same footnote that the proof presented in the paper is shorter than his original proof, the latter was very appealing to me: *now that*



*we know that there is no need to use bounded universal quantifiers at all, could not we perform completely existential arithmetizing, thus avoiding universal quantifiers?*

Finally I was able to give such a quite different proof of the DPR-theorem based on arithmetization of the work of Turing machines [33]. In [37] I presented another way of simulating Turing machines by means of exponential Diophantine equations. Peter van Emde Boas proposed in [59] yet another, rather different way of doing it. However, James P. Jones and I found [22] that so called *register machines* are even more suitable for existential arithmetization; several versions of such “visual proof” are given in [23, 24, 35, 39].

The “advantage” of register machines over Turing machines for constructing Diophantine representations is due to the fact that the former operate directly with integers. However, register machines are not such a “classical” tool as Turing machines are. Esteemed *partial recursive function* have both properties: on the one hand they are defined on natural numbers, on the other hand, they are quite “classical”. In [38] I used exponential Diophantine equations for simulating partial recursive function thus giving yet another proof of the DPR-theorem.

My paper [41] presents a unifying technique allowing one to eliminate bounded universal quantifier and simulate by means of exponential Diophantine equations Turing machines, register machines, and partial recursive functions in the same “algebraic” style.

Thus today we have quite a few very different proofs of the celebrated DPR-theorem. In contrast, it is a bit strange that no radically new techniques were found for transforming exponential Diophantine equations into genuine Diophantine ones: all known proofs in fact are minor variations of the construction presented by Martin Davis in [8] (Maxim Vsemirnov [60] made a generalization from (2.11) and (2.14) to some recurrent sequences of orders 3 and 4 but this gives no advantage for constructing Diophantine representations).

## 2.4 Improvements

### 2.4.1 Single-Fold Representations

I [32] was also able to improve the DPR-theorem in another direction, namely, to show the existence of a *single-fold* exponential Diophantine representation for every listable set, that is, a representation of the form (2.8) in which the values of  $x_1, \dots, x_n$ , if they exist, are uniquely determined by the values of  $a_1, \dots, a_m$ .

However, the two improvements to the DPR-theorem—to the DPRM-theorem and to single-fold representations—have not been so far combined, that is, the question about the existence of single-fold Diophantine representations for all listable sets still remains open. This is so because all today known methods of constructing Diophantine representation (2.9) are based on the study of behavior of sequences like (2.11) and (2.14) taken some modulo; clearly, this behavior is periodic and

as a consequence each known Diophantine representation of exponentiation is infinite-fold—as soon as the corresponding equation (2.9) has a solution, it has infinitely many of them.

Single-fold representations have important applications (one of them is given in Sect. 2.6.2), and for this reason Martin Davis paper [7] titled “*One equation to rule them all*” remains of interest. The equation from the title is

$$9(u^2 + 7v^2)^2 - 7(r^2 + 7s^2)^2 = 2, \quad (2.15)$$

and it has a trivial solution  $u = r = 1, v = s = 0$ . Martin Davis proved that if this is the only solution, then some Diophantine relation has exponential growth. His expectations were broken by Oskar Herrman [17] who established the existence of another solution. The equation attracted interest of other researches, Daniel Shanks [53] was first in writing down two solutions explicitly and later he and Samuel S. Wagstaff, Jr. [54] found 48 more solutions.

The discovery of non-trivial solutions did not spoil Martin Davis approach completely. In fact, it can be shown that if (2.15) has only *finitely* many solutions then every listable set has a single-fold Diophantine representation.

### 2.4.2 Representations with a Small Number of Quantifiers

The existence of *universal listable sets* together with the DPRM-theorem implies that we can bound the number of unknowns in a Diophantine representation (2.3) of an arbitrary listable set  $\mathcal{M}$ ; today’s record  $n = 9$  was obtained by me [34] (a detailed proof is presented in [19]). Accordingly, Hilbert’s tenth problem remains undecidable even if we restrict ourselves to equations in 9 unknowns.

With present techniques, in order to get results for even smaller number of variables, one has to broaden the class of admissible formulas.

For example, for the DPR-theorem 3 unknowns are sufficient; originally this was proved in [36], and even for single-fold representations. Later this result was improved in [20, 21] to representations of the form

$$\begin{aligned} \langle a_1, \dots, a_m \rangle \in \mathcal{M} &\iff \\ \exists x_1 x_2 [E_L(a_1, \dots, a_m, x_1, x_2) \leq E_R(a_1, \dots, a_m, x_1, x_2)] \end{aligned} \quad (2.16)$$

where exponential polynomials  $E_L$  and  $E_R$  are constructed by using *unary* exponentiation  $2^c$  only (rather than general *binary* exponentiation  $b^c$ ). Harry R. Levitz proved in [26] that this result cannot be further improved to single unknown.

Soon after Martin Davis introduction of the normal form (2.6), Raphael Robinson [51] gave a rather different proof and showed that one can always take  $n = 4$ . In the same paper he gave another representation with 6 quantified variables, namely,

$$\langle a_1, \dots, a_m \rangle \in \mathcal{M} \iff \exists z_1 z_2 \forall y_{\leq B(a_1, \dots, a_m, z_1, z_2)} \exists x_1 x_2 x_3 [Q(a_1, \dots, a_m, x_1, x_2, x_3, y, z_1, z_2) = 0]. \quad (2.17)$$

Much later, exploiting the power of the DPRM-theorem, he [52] improved the bound for Davis normal form (2.6) to  $n = 3$  and showed that  $x_3$  can be dropped from (2.17). Both of these results were further improved: in [31] to  $n = 2$ , in (2.6) and in [42] both  $x_2$  and  $x_3$  were dropped from (2.17).

More interesting is the possibility to replace the bounded universal quantifier in (2.6) and (2.17) by finite conjunction. For example, it was shown in [31] that every listable set has a representation of the form

$$\langle a_1, \dots, a_m \rangle \in \mathcal{M} \iff \exists z_1 z_2 \&_{y=1}^l \exists x_1 x_2 [Q_y(a_1, \dots, a_m, x_1, x_2, z_1, z_2) = 0] \quad (2.18)$$

where  $l$  is a fixed number and  $Q_1, \dots, Q_l$  are polynomials with integer coefficients. Clearly, the right-hand side of (2.18) can be rewritten as a system of Diophantine equations in  $2l + 2$  unknowns. While this quantity is high, each single equation has only 4 unknowns. This implies, for example, the following. Consider the class  $\mathcal{D}_{2,2}$  of Diophantine sets that can be defined by formulas of the form

$$\langle x_1, x_2 \rangle \in \mathcal{M} \iff \exists z_1 z_2 [Q(x_1, x_2, z_1, z_2) = 0]. \quad (2.19)$$

Clearly, *we cannot decide whether a given intersection of finitely many sets from class  $\mathcal{D}_{2,2}$  is empty or not*. Informally, this means that among sets of pairs of natural numbers defined by Diophantine equations with just 2 unknowns there are sets with complicated structure having no “transparent” description.

In the above cited results the variables range over natural numbers; for the case of integer-valued variables corresponding results are at present somewhat weaker (in terms of the number of unknowns).

## 2.5 “Positive Aspects of a Negative Solution”

The title of this section reproduces part of the title of [12], the joint paper of Martin Davis, Julia Robinson, and myself written for the Proceedings of Symposium on Hilbert's problems [2]. The undecidability of Hilbert's tenth problem is just one of the corollaries of the DPRM-theorem. Actually it can serve as bridge for transferring ideas and results from Computability Theory to Number Theory; a few of such applications are given below.

### 2.5.1 Speeding Up Diophantine Equations

A simplest form of such transfer is as follows: *take any theorem about listable sets and replace them by Diophantine sets*. For example, one can explicitly write down a polynomial (2.5) with the set of its positive values being exactly the set of all prime numbers; the supposed impossibility of such a definition of primes was considered by many number-theorists as an informal argument against Martin Davis Conjecture.

It is quite typical that the map “listable”  $\rightarrow$  “Diophantine” produces theorems not conventional for Number Theory. For example, Martin Davis published in [10] the following Diophantine counterpart of Manuel Blum’s [1] *speed-up theorem*.

**Theorem** *For every general recursive function  $\alpha(a, w)$  there are Diophantine equations*

$$B(a, x_1, \dots, x_n) = 0, \quad (2.20)$$

$$C(a, y_1, \dots, y_m) = 0 \quad (2.21)$$

*such that:*

- *for every value of  $a$  one and only one of these two equations has a solution;*
- *if equations*

$$B'(a, x'_1, \dots, x'_{n'}) = 0, \quad (2.22)$$

$$C'(a, y'_1, \dots, y'_{m'}) = 0 \quad (2.23)$$

*are solvable exactly for the same values of the parameter  $a$  as Eqs. (2.20) and (2.21) respectively, then there is third pair of equations*

$$B''(a, x''_1, \dots, x''_{n''}) = 0, \quad (2.24)$$

$$C''(a, y''_1, \dots, y''_{m''}) = 0 \quad (2.25)$$

*such that:*

- *these equations are also solvable exactly for the same values of the parameter  $a$  as Eqs. (2.20) and (2.21) respectively;*
- *for almost all  $a$  for every solution of equation (2.22) (Eq. (2.23)) there is solution of equation (2.24) (respectively, Eq. (2.25)) such that*

$$x'_1 + \dots + x'_{n'} > \alpha(a, x''_1 + \dots + x''_{n''}) \quad (2.26)$$

*(or*

$$y'_1 + \dots + y'_{m'} > \alpha(a, y''_1 + \dots + y''_{m''}) \quad (2.27)$$

*respectively).*

This theorem in its full generality is about an arbitrary general recursive function; replacing it by any particular (growing fast) function we obtain theorems which are purely number-theoretical but quite non-standard for Number Theory.

### 2.5.2 Universal Equations

One of the fundamental notion in Computability Theory is that of *universal Turing machine* or, equivalently, its counterpart *universal listable set*. Now the DPRM-theorem brings the idea of such kind of universality into the realm of Diophantine equations. Namely, for every fixed  $n$ , we can construct a particular Diophantine equation

$$U_n(k, a_1, \dots, a_n, x_1, x_2, \dots, x_m) = 0 \quad (2.28)$$

which is *universal* in the following sense: *solving an arbitrary Diophantine equation with  $n$  parameters*

$$D(a_1, \dots, a_n, x_1, x_2, \dots) = 0 \quad (2.29)$$

*is equivalent to solving the equation*

$$U_n(k_D, a_1, \dots, a_n, x_1, x_2, \dots, x_m) = 0 \quad (2.30)$$

*resulting from the Eq. (2.28) by choosing a particular value  $k_D$  for the first parameter, that is, for this fixed value of  $k$  and for any choice of the values of the parameters  $a_1, \dots, a_m$  either both of the Eqs. (2.29) and (2.30) have a solution or neither of them has any.*

What is remarkable in this reduction of one equation to another is the following: the degree and the number of unknowns of the Eq. (2.30) is fixed while the Eq. (2.29) can have any number of unknowns and be of arbitrarily large degree. This implies that hierarchies of Diophantine equations traditional for Number Theory (with 1, 2, 3, ... unknowns; of degree 1, 2, 3, ...) collapse at some level.

Not only number-theorists never anticipated universal Diophantine equations, their possibility was incredible even for some logicians as it can be seen from the review in *Mathematical Reviews* on the celebrated paper by Martin Davis, Hilary Putnam, and Julia Robinson [14].

We can look at universal Diophantine equations as a purely number-theoretical result *inspired* by Computability Theory. But do we really need the general notion of listable sets for proving the existence of universal Diophantine equations or could we construct such equations by purely number-theoretical tools? In my book [37] I managed to prove the existence of universal Diophantine equations before proving the DPRM-theorem; in [41] I introduced another purely number-theoretical construction of universal Diophantine equations.

### 2.5.3 Hilbert's Eighth and Tenth Problems

The notion of listable set is very broad and can be found in a surprising variety of contexts. Here is one such example.

Hilbert included into his 8th problem an outstanding conjecture, the famous *Riemann's hypothesis*. In its original formulation it is a statement about complex zeros of Riemann's *zeta function* which is the analytical continuation of the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \quad (2.31)$$

Much as almost every great problem, Riemann's hypothesis has many equivalent restatements. Georg Kreisel [25] managed to reformulate it as an assertion about the emptiness of a particular listable set (each element of this set would produce a counterexample to the hypothesis). Respectively, we can construct a Diophantine representation of this set and obtain a particular Diophantine equation

$$R(x_0, \dots, x_m) = 0 \quad (2.32)$$

which has no solution if and only if the Riemann hypothesis is true.

It was my share to write Sect. 2 of [12] devoted to reductions of Riemann's hypothesis and some other famous problems to Diophantine equations, but I failed to present the equation whose unsolvability is equivalent to Riemann's Hypothesis. Kreisel's main construction was very general, applicable to any analytical function, and some details of how to transfer it to a Diophantine equation were cumbersome. Luckily, Harold N. Shapiro, a colleague of Martin Davis, came to help and suggested a simpler construction, specific to the zeta function, based on the relationship of Riemann's hypothesis and distribution of prime numbers, and the corresponding part of Sect. 2 from [12] was written by Martin Davis.

In [37] I present a reductions of Riemann's hypothesis to Diophantine equations that is a bit simpler than the construction in [12], the simplification was due to certain new explicit constants related to distribution of primes that were obtained at that time in Number Theory.

Thus, Riemann's hypothesis can be viewed as a very particular case of Hilbert's tenth problem; such a relationship between it and Hilbert's eighth problem was not known before the DPRM-theorem was proved.

Hardly one can hope to prove or to disprove Riemann's hypothesis by examining a corresponding Diophantine equation. On the other hand, such a reduction gives an informal "explanation" of why Hilbert's tenth problem is undecidable: it would be rather surprising if such a long-standing open problem could be solved by a mechanical procedure required by Hilbert.

## 2.6 Other Impossibilities

DPR-theorem and DPRM-theorem turned out to be very powerful tools for establishing that many other things cannot be done algorithmically. Only a few examples will be mentioned here, surveys of many others can be found, for example, in [37, 40].

### 2.6.1 The Number of Solutions

In his tenth problem Hilbert demanded to find a method for deciding whether a given Diophantine equation has a solution or not. But one can ask many other similar questions, for example:

- is the number of solutions of a given Diophantine equation finite or infinite?
- is the number of solutions of a given Diophantine equation odd or even?
- is the number of solutions of a given Diophantine equation a prime number?

Martin Davis showed in [9] that Hilbert's tenth problem can be reduced to the above and analogous decision problems, and hence all of them are undecidable. Namely, the following theorem holds.

**Theorem** (Martin Davis). *Let  $\mathcal{N} = \{0, 1, 2, \dots, \infty\}$  and let  $\mathcal{M}$  be a proper subset of  $\mathcal{N}$ ; there is no algorithm for deciding, for given Diophantine equation, whether the number of its solutions belongs to  $\mathcal{M}$  or not.*

Clearly, the case  $\mathcal{M} = \{0\}$  is the original Hilbert's tenth problem.

### 2.6.2 Non-effectivizable Estimates

Suppose that we have an equation

$$P(a, x_1, \dots, x_n) = 0, \tag{2.33}$$

which for every value of the parameter  $a$  has at most finitely many solutions in  $x_1, \dots, x_n$ . This fact can be expressed in two form:

- Equation (2.33) has at most  $\nu(a)$  solutions;
- in every solution of (2.33)  $x_1 < \sigma(a), \dots, x_n < \sigma(a)$

for suitable functions  $\nu$  and  $\sigma$ .

From a mathematical point of view these two statements are equivalent. However, they are rather different computationally. Having  $\sigma(a)$  we can calculate  $\nu(a)$  but not *vice versa*. Number-theorists have found many classes of Diophantine equations with

computable  $\nu(a)$  for which they fail to compute  $\sigma(a)$ . In such cases number-theorists say that “the estimate of the size of solutions is *non-effective*”.

Now let us take some undecidable set  $\mathcal{M}$  and construct an exponential Diophantine equation

$$E_L(a, x_1, x_2, \dots, x_n) = E_R(a, x_1, x_2, \dots, x_n) \quad (2.34)$$

giving a single-fold representation for  $\mathcal{M}$ . Clearly, Eq. (2.34) has the following two properties:

- for every value of the parameter  $a$ , Eq. (2.34) has at most one solution in  $x_1, \dots, x_n$ ;
- for every effectively computable function  $\sigma$  there is a value of  $a$  for which the Eq. (2.34) has a solution  $x_1, \dots, x_n$  such that  $\max\{x_1, \dots, x_n\} > \sigma(a)$  (otherwise we would be able to determine whether  $a$  belongs to  $\mathcal{M}$  or not).

In other words, the boundedness of solutions of equation (2.34) cannot be made effective in principle. This relationship between undecidability and non-effectivizability is one of the main stimuli to improve the DPRM-theorem to single-fold (or at least to finite-fold) representations and thus establish the existence of non-effectivizable estimates for genuine Diophantine equations.

### 2.6.3 Solutions in Other Rings

Most likely, Hilbert expected a positive solution of his tenth problem. This would allow us to recognize solvability of polynomial equations in many other rings, for example, in the ring of algebraic integers from any finite extension of the field of rational numbers, and in the ring of rational numbers. However, the obtained negative solution of Hilbert’s tenth problem does not imply *directly* undecidability results for other rings. Nevertheless, different researchers were able to reduce the Tenth problem to solvability of equations in many classes of rings and thus establish the undecidability of analogs of the Tenth problem for them (for survey see book [55] or [56] in this volume).

Such reductions can be made by constructing a polynomial equation solvable in a considered ring if and only if the parameter is a rational integer, or, more generally, by constructing a *Diophantine model* of integers in that ring. Such an approach exploits the mere undecidability of the original Hilbert’s tenth problem and does not require any new ideas from Computability Theory. However, number-theorists foresee some deep obstacles for the existence of such models for certain rings including, maybe the most interesting, the ring  $\mathbb{Q}$  of rational numbers.

Recently Martin Davis proposed in [11] a quite different approach based on the existence of a special kind of undecidable sets constructed by Emil Post who named them *simple*. A listable set  $S$  is called simple if its complement to the set  $\mathbb{N}$  of all natural numbers is infinite but contains no infinite listable set. In [43] Bjorn Poonen proved the undecidability of a counterpart of Hilbert tenth problem for a ring  $\mathcal{U}$  of rational numbers denominators of which are allowed to contain “almost all” prime



factors. His technique allows us to define a simple set  $S$  by a formula of the form

$$\{a \in \mathbb{N} \mid \exists x_1 \dots x_m [p(y_a, x_1, \dots, x_m) = 0]\} \quad (2.35)$$

where  $y_a$  is a computable function of  $a$ ,  $p$  is a polynomial, and  $x_1, \dots, x_m$  range over  $\sim \mathcal{U}$ . When these variables are allowed to range over all rational numbers, the same formula (2.35) defines some set  $\widehat{S}_p$ ; clearly,  $S \subseteq \widehat{S}_p$ .

**Martin Davis Conjecture [2010].** *There is a Diophantine definition of a simple set  $S$  for which  $\mathbb{N} - \widehat{S}_p$  is infinite, so that  $\widehat{S}_p$  is undecidable.*

Martin Davis wrote in [11]:

This conjecture implies the unsolvability of H10 [Hilbert's tenth problem] over  $\mathbb{Q}$ . The conjecture seems plausible because although it is easy to construct simple sets, and there are a number of ways to do so, and if the conjecture is false, then no matter how  $S$  is constructed, and no matter what Diophantine definition of  $S$  is provided,  $\widehat{S}_p$  differs from  $\mathbb{N}$  by only finitely many elements. Because the additional primes permitted in denominators in the transition from  $\mathcal{U}$  to  $\mathbb{Q}$  form a sparse set, this seems implausible.

Let us believe in the wisdom of the celebrated guru.

## References

1. Blum, M. (1967). A machine-independent theory of the complexity of recursive functions. *Journal of the ACM*, 14(2), 322–336.
2. Browder, (Ed.). (1976). Mathematical Developments arising from Hilbert Problems. *Proceedings of Symposia in Pure Mathematics* (vol. 28). American Mathematical Society.
3. Davis, M. (1950). Arithmetical problems and recursively enumerable predicates (abstract). *Journal of Symbolic Logic*, 15(1), 77–78.
4. Davis, M. (1950). *On the theory of recursive unsolvability*. PhD thesis, Princeton University.
5. Davis, M. (1953). Arithmetical problems and recursively enumerable predicates. *Journal of Symbolic Logic*, 18(1), 33–41.
6. Davis, M. (1958). *Computability and unsolvability*. New York: McGraw-Hill. Reprinted with an additional appendix, Dover 1983.
7. Davis, M. (1968). One equation to rule them all. *Transactions of the New York Academy of Sciences. Series II*, 30(6), 766–773.
8. Davis, M. (1971). An explicit Diophantine definition of the exponential function. *Communications on Pure and Applied Mathematics*, 24(2), 137–145.
9. Davis, M. (1972). On the number of solutions of Diophantine equations. *Proceedings of the American Mathematical Society*, 35(2), 552–554.
10. Davis, M. (1973). Speed-up theorems and Diophantine equations. In R. Rustin (Ed.), *Courant computer science symposium 7: Computational complexity* (pp. 87–95). New York: Algorithms Press.
11. Davis, M. (2010). Representation theorems for r.e. sets and a conjecture related to Poonen's larges subring of  $\mathbb{Q}$ . *Zapiski Nauchnykh Seminarov Peterburgskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova RAN (POMI)*, 377, 50–54. Reproduced in: *Journal of Mathematical Sciences*, 171(6), 728–730 (2010).
12. Davis, M., Matijasevich, Yu., & Robinson, J. Hilbert's tenth problem. Diophantine equations: Positive aspects of a negative solution, pp. 323–378 in [2]. Reprinted in [51, pp. 269–324].

13. Davis, M., & Putnam, H. (1959). A computational proof procedure; Axioms for number theory; Research on Hilbert's Tenth Problem. O.S.R. Report AFOSR TR59-124, U.S. Air Force.
14. Davis, M., Putnam, H., & Robinson, J. (1961). The decision problem for exponential Diophantine equations. *Annals of Mathematics*, 74(2), 425–436. Reprinted in [51].
15. Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme. I. Monatshefte für Mathematik und Physik bf 38(1), 173–198. Reprinted with English translation in: S. Feferman et al., (Eds.) (1986). *Kurt Gödel. Collected Works* (vol. I, pp. 144–195). Oxford University Press. English translation can also be found in: M. Davis, (Ed.) (1965). *The Undecidable* (pp. 4–38). Raven Press, Hewlett, New York and in: J. van Heijenoort, (Ed.) (1967). *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* (pp. 596–616). Harvard University Press, Cambridge, Massachusetts.
16. Green, B., & Tao, T. (2008). The primes contain arbitrarily long arithmetic progressions, *Annals of Mathematics*, 167, 481–547. doi:[10.4007/annals.2008.167.481](https://doi.org/10.4007/annals.2008.167.481), [arXiv:math.NT/0404188](https://arxiv.org/abs/math.NT/0404188).
17. Herrman, O. (1971). A non-trivial solution of the Diophantine equation  $9(x^2 + 7y^2)^2 - 7(u^2 + 7v^2)^2 = 2$ . In A. O. L. Atkin & B. J. Birch (Eds.), *Computers in number theory* (pp. 207–212). London: Academic Press.
18. Hilbert, D. (1900) Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker Kongress zu Paris 1900. *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Kl.* 253–297. See also Hilbert, D. (1935). *Gesammelte Abhandlungen* (vol. 3), Berlin: Springer. (Reprinted: Chelsea, New York (1965)). English translation: *Bulletin of the American Mathematical Society*, 8, 437–479 (1901–1902); reprinted in: [2, pp. 1–34].
19. Jones, J. P. (1982). Universal Diophantine equation. *Journal Symbolic Logic*, 47, 549–571.
20. Jones, J. P., & Matiyasevič, Ju V. (1982). Exponential Diophantine representation of recursively enumerable sets. In J. Stern (Ed.), *Proceedings of the Herbrand Symposium: Logic Colloquium '81, Studies in Logic and the Foundations of Mathematics*, 107, 159–177. Amsterdam: North Holland.
21. Jones, J. P., & Matiyasevič, Ju V. (1982). A new representation for the symmetric binomial coefficient and its applications. *Annales Sci. Mathém. du Québec*, 6(1), 81–97.
22. Jones, J. P., & Matiyasevič, Ju V. (1983). Direct translation of register machines into exponential Diophantine equations. In L. Priese (Ed.), *Report on the 1st GTI-workshop* (pp. 117–130). Reihe Theoretische Informatik: Universität-Gesamthochschule Paderborn.
23. Jones, J. P., & Matiyasevič, Ju V. (1984). Register machine proof of the theorem on exponential Diophantine representation of enumerable sets. *Journal Symbolic Logic*, 49(3), 818–829.
24. Jones, J. P., & Matiyasevič, Ju V. (1991). Proof of recursive unsolvability of Hilbert's tenth problem. *American Mathematical Monthly*, 98(8), 689–709.
25. Kreisel, G. (1958). Mathematical significance of consistency proofs. *Journal of Symbolic Logic*, 23(2), 155–182.
26. Levitz, H. (1985). Decidability of some problem pertaining to base 2 exponential Diophantine equations. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 31(2), 109–115.
27. Mal'tsev, A. I. (1968). On some boundary questions between algebra and logic (in Russian). *Trudy Mezhdunarodnogo Kongressa Matematikov (Moskva, 1966)*, pp. 217–231. Translated in: American Mathematical Society Translations. Series 2, Vol. 70: 31 invited addresses (8 in abstract) at the International Congress of Mathematicians in Moscow, 1966. American Mathematical Society, Providence, R.I. 1968, p. 266.
28. Markov, A. A. (1947). Impossibility of certain algorithms in the theory of associative systems (in Russian), *Doklady Akademii Nauk SSSR*, 55(7), 587–590 (1947). Translated in: *Compte rendus de l'Académie des Sciences de l'U.R.S.S.*, 55, 583–586 (1947).
29. Matiyasevich, Yu. V. (1970). Enumerable sets are Diophantine (in Russian). *Dokl. AN SSSR*, 191(2), 278–282; Translated in: *Soviet Math. Doklady*, 11(2), 354–358. Correction *ibid* 11 (6) (1970), vi. Reprinted on pp. 269–273 in: *Mathematical logic in the 20th century*, G. E. Sacks, (Ed.), (2003). Singapore University Press and World Scientific Publishing Co., Singapore and River Edge, NJ.

30. Matiyasevich, Yu. V. (1972). Diophantine sets (in Russian). *Uspekhi Matematicheskikh Nauk*, 27(5), 185–222. English translation: *Russian Mathematical Surveys*, 27(5), 124–164 (1972).
31. Matiyasevich, Yu. V. (1972). Arithmetical representations of enumerable sets with a small number of quantifiers (in Russian). *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR (LOMI)*, 32, 77–84. English translation: *Journal of Soviet Mathematics*, 6(4), 410–416 (1976).
32. Matiyasevich, Yu. V. (1974). Existence of noneffectivizable estimates in the theory of exponential Diophantine equations (in Russian). *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR (LOMI)*, 40, 77–93; Translated in: *Journal of Soviet Mathematics*, 8(3), 299–311 (1977).
33. Matiyasevich, Yu. V. (1976). A new proof of the theorem on exponential Diophantine representation of enumerable sets (in Russian). *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR (LOMI)*, 60, 75–92; Translated in: *Journal of Soviet Mathematics*, 14(5), 1475–1486 (1980).
34. Matiyasevich, Yu. V. (1977). Some purely mathematical results inspired by mathematical logic. *Proceedings of Fifth International Congress on Logic, Methodology and Philosophy of science, London, Ontario, 1975* (pp. 121–127), Reidel, Dordrecht.
35. Matiyasevich, Yu. V. (1984). On investigations on some algorithmic problems in algebra and number theory. *Trudy Matematicheskogo instituta im. V. A. Steklova of Academy of Sciences of the USSR*, 168, 218–235. Translated in *Proceedings of the Steklov Institute of Mathematics*, 168, 227–252 (1986).
36. Matiyasevich, Yu. V. (1979). Algorithmic unsolvability of exponential Diophantine equations in three unknowns (in Russian). In A. A. Markov, & V. I. Homich (Eds.), *Studies in the Theory of Algorithms and Mathematical Logic*, Computing Center Russian Academy Sci., Moscow, pp. 69–78; Translated in *Selecta Mathematica Sovietica*, 3, 223–232 (1983/1984).
37. Matiyasevich, Yu. V. (1993). *Hilbert's Tenth Problem (in Russian)*. Fizmatlit, Moscow. English translation: MIT Press, Cambridge (Massachusetts) London (1993). French translation: *Le dixième Problème de Hilbert*, Masson, Paris Milan Barcelone (1995). <http://logic.pdmi.ras.ru/~yumat/H10Pbook>.
38. Matiyasevich, Yu. (1994). A direct method for simulating partial recursive functions by Diophantine equations. *Annals of Pure and Applied Logic*, 67, 325–348.
39. Matiyasevich, Yu. (2000). Hilbert's tenth problem: What was done and what is to be done. *Contemporary Mathematics*, 270, 1–47.
40. Matiyasevich, Yu. (2005). Hilbert's tenth problem and paradigms of computation. *Lecture Notes in Computer Science*, 3526, 310–321.
41. Matiyasevich, Yu. (2009). Existential arithmetization of Diophantine equations. *Annals of Pure and Applied Logic*, 157(2–3), 225–233.
42. Matiyasevich, Yu., & Robinson, J. (1974). Two universal 3-quantifier representations of recursively enumerable sets (in Russian). In B. A. Kushner & N. M. Nagornyi (Eds.), *Teoriya algoritmov i matematicheskaya logika*, Vychislitel'nyi Tsentr, Akademiya Nauk SSSR, Moscow, pp. 112–123. Reprinted in [51, pp. 223–234]. English translation in <http://arxiv.org/abs/0802.1052>.
43. Poonen, B. (2003). Hilbert's tenth problem and Mazur's conjecture for large subrings of  $\mathbb{Q}$ . *Journal of the American Mathematical Society*, 16(4), 981–990.
44. Post, E. L. (1944). Recursively enumerable sets of positive integers and their decision problems. *Bulletin of American Mathematical Society*, 50, 284–316. Reprinted in [46, pp. 461–494].
45. Post, E. L. (1947). Recursive unsolvability of a problem of Thue. *Journal of Symbolic Logic*, 12, 1–11. Reprinted in [46, pp. 503–513].
46. Post, E. L. (1994). *Solvability, provability, definability: The collected works of E. L. Post*. In M. Davis, (Ed.), Birkhäuser, Boston.
47. Putnam, H. (1960). An unsolvable problem in number theory. *Journal of Symbolic Logic*, 25(3), 220–232.
48. Robinson, J. (1952). Existential definability in arithmetic, *Transactions of the American Mathematical Society*, 72, 437–449. Reprinted in [51, pp. 47–59].

49. Robinson, J. B. (1960). The undecidability of exponential Diophantine equations. *Notices of the American Mathematical Society*, 7(1), 75.
50. Robinson, J. Unsolvability of Diophantine problems. *Proceedings of the American Mathematical Society*, 22(2), 534–538. Reprinted in [51, pp. 195–199].
51. Robinson, R. M. (1956). Arithmetical representation of recursively enumerable sets. *Journal of Symbolic Logic*, 21(2), 162–186.
52. Robinson, R. M. (1972). Some representations of Diophantine sets. *Journal of Symbolic Logic*, 37(3), 572–578.
53. Shanks, D. (1973). Five number-theoretic algorithms. In R. S. D. Thomas & H. C. Williams (Eds.), *Proceedings of the Second Manitoba Conference on Numerical Mathematics (University of Manitoba)* (pp. 51–70). Winnipeg, Manitoba, 5–7 October 1972, volume VII of Congressus Numerantium Winnipeg, Manitoba: Utilitas Mathematica Publishing.
54. Shanks, D., & Wagstaff, S. S., Jr. (1995). 48 more solutions of Martin Davis’s quaternary quartic equation. *Mathematics of Computation*, 64(212), 1717–1731.
55. Shlapentokh, A. (2007). *Hilbert’s tenth problem, New Mathematical Monographs* (vol. 7). Cambridge: Cambridge University Press.
56. Shlapentokh, A. Extensions of Hilbert’s Tenth Problem: Definability and Decidability in Number Theory. This book, pp. 57–91
57. The collected works of Julia Robinson. (1996). *S. Feferman, ed.* Providence, RI: American Mathematical Society. ISBN 0-8218-0575-4. (Collected Works, vol. 6).
58. Thue, A. (1914). Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln. *Skrifter utgit av Videnskapsselskapet i Kristiana, I in Matematisk naturvidenskabelig klasse, Norske Videnskaps-Akademi, Oslo, 10*, 493–524. Reprinted in: Nagell, T., Selberg, A., Selberg, S., & Thalberg, K. (Eds.). *Selected Mathematical Papers of Axel Thue*. Universitetsforlaget, Oslo, 1977.
59. van Emde Boas, P. (1997). The convenience of tiling. In A. Sorbi, (Ed.), *Complexity, logic and recursion theory, Lecture Notes in Pure and Applied Mathematics* (vol. 187, pp. 331–363).
60. Vsemirnov, M. A. (1995). Diophantine representations of linear recurrent relations (in Russian). *Zapiski Nauchnykh Seminarov Peterburgskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova RAN*, 227, 52–60. Translated in *Journal of Mathematical Sciences (New York)*, 89(2), 1113–1118 (1998).

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