Geometry-Fitted Fourier-Mellin Transform Pairs

Darren Crowdy

Abstract The construction of novel Fourier/Mellin-type transform pairs that are tailor-made for given planar regions within the special class of circular domains is surveyed. Circular domains are those having boundary components that are either circular arcs or straight lines. The new transform pairs generalize the classical Fourier and Mellin transforms. These geometry-fitted transform pairs can be used to great advantage in solving boundary value problems defined in these domains.

Keywords Fourier transform · Mellin transform · Geometric function theory

1 Introduction

This article surveys some of the mathematical ideas laid out in the author's plenary lecture at 10th International ISAAC Meeting in Macau in 2015. The topic is the construction of novel Fourier-Mellin type transform pairs that are "tailor-made" for given planar domains within a special class.

The class of domains amenable to the construction—at the time of writing at least—is the class of *circular domains*, either simply or multiply connected, having boundary components made up of straight lines, arcs of circles, or a mixture of both. Figure 1 shows examples: a simply connected convex quadrilateral (a polygon), a simply connected lens-shaped domain (a circular polygon), and the "disc-in-channel" geometry (a doubly connected circular domain) that arises in many applications.

The author's recent results in this area have been inspired by the extensive body of mathematical work over the last few decades pioneered by A.S. Fokas and collaborators, and now commonly referred to as the *Fokas method* [11, 12]. That work describes a unified transform approach to initial and boundary value problems for both linear and nonlinear integrable partial differential equations. In one strand of

D. Crowdy (✉)
Department of Mathematics, Imperial College London, 180 Queen's Gate,
London SW7 2AZ, UK
e-mail: d.crowdy@imperial.ac.uk

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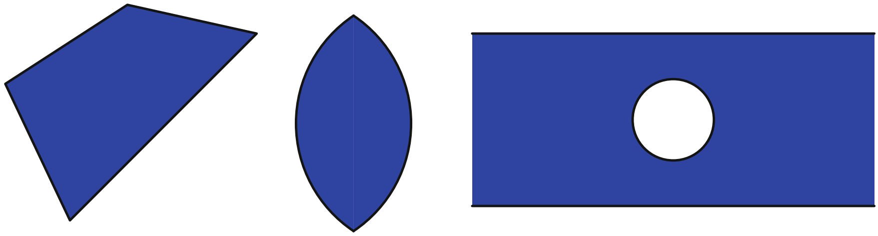

Fig. 1 Example *circular* domains: a convex polygon, a lens-shaped circular-arc domain, and the "disc-in-channel" geometry

this work a new constructive approach to the solution of boundary value problems for Laplace's equation in convex polygons has been described by Fokas and Kapaev [10] with the extension to biharmonic fields made by Crowdy and Fokas [9]. That work was based on an analytical formulation involving the spectral analysis of a Lax pair and use of Riemann-Hilbert methods. The new approach outlined here—and described in more detail in [2, 3]—has a more geometrical flavour and has led the way to generalization of results previously pertaining only to simply connected convex polygons to the much broader class of circular domains, including multiply connected cases.

We first review the results described in [2, 3]. Given a bounded N-sided convex polygon with straight line edges $\{S_n | n = 1, ..., N\}$ inclined at angles $\{\chi_n | n = 1, ..., N\}$ to the positive real axis (e.g., the quadrilateral shown in Fig. 1) we can derive the following transform pair to represent a function $f(z)$ analytic in the polygon:

$$f(z) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_L \rho_{jj}(k) e^{-i\chi_j} e^{ie^{-i\chi_j} kz} dk, \quad \rho_{mn}(k) = \int_{S_n} f(z') e^{-ie^{-i\chi_m} kz'} dz', \quad (1)$$

where L is the ray along the positive real axis in the k-plane (see Fig. 2) and where the *spectral functions* (or "transforms") satisfy the so-called *global relations* [2, 11, 12]

$$\sum_{n=1}^{N} \rho_{mn}(k) = 0, \qquad k \in \mathbb{C}, \qquad m = 1, ..., N. \quad (2)$$

Only the diagonal elements of what we call the *spectral matrix* $\rho_{mn}(k)$ appear in the inverse transform formula for $f(z)$.

In precise analogy, given a bounded convex N-sided circular polygon with edges that are arcs of circles with centres $\{\delta_n | n = 1, ..., N\}$ and radii $\{q_n | n = 1, ..., N\}$ (e.g., the lens-shaped domain of Fig. 1) we can derive the following transform pair to represent a function $f(z)$ analytic in it:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \left\{ \int_{L_1} \frac{\rho_{jj}(k)}{1 - e^{2\pi \mathrm{i} k}} \left[\frac{z - \delta_j}{q_j} \right]^k dk + \int_{L_2} \rho_{jj}(k) \left[\frac{z - \delta_j}{q_j} \right]^k dk \right.$$
$$\left. + \int_{L_3} \frac{\rho_{jj}(k) e^{2\pi \mathrm{i} k}}{1 - e^{2\pi \mathrm{i} k}} \left[\frac{z - \delta_j}{q_j} \right]^k dk \right\},$$
$$\rho_{mn}(k) = \frac{1}{q_m} \int_{C_n} \left[\frac{z' - \delta_m}{q_m} \right]^{-k-1} f(z') dz', \tag{3}$$

where L_1, L_2 and L_3 are the contours in the k-plane (see Fig. 2) and where the spectral functions satisfy the global relations

$$\sum_{n=1}^{N} \rho_{mn}(k) = 0, \qquad k \in -\mathbb{N}, \qquad m = 1, \ldots, N. \tag{4}$$

Again, only the diagonal elements of spectral matrix $\rho_{mn}(k)$ appear in the inverse transform formula for $f(z)$.

The plan of the article is as follows. First, in Sects. 2 and 3, we discuss the geometrical construction of the transform pairs (1) and (3). In Sect. 4 we combine those general ideas to construct a useful Fourier-Mellin type transform pair for the doubly connected disc-in-channel geometry of Fig. 1. Finally, Sect. 5 illustrates how the transform pairs can be used in practice by solving an accessory parameter problem in conformal mapping theory and finding a useful conformal mapping function associated with the disc-in-channel geometry of Fig. 1.

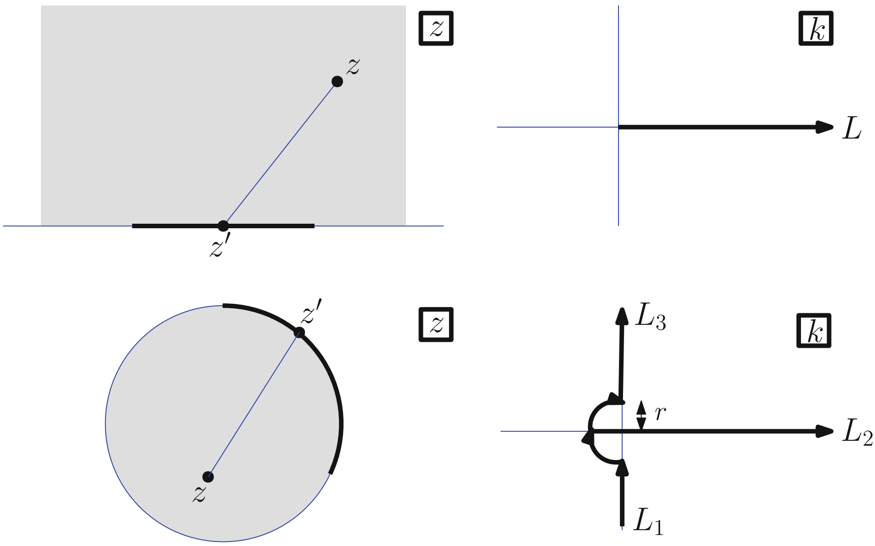

Fig. 2 The basic geometrical units in the z-plane, shown *left*, with the corresponding integration contours in the k-plane shown to the *right* $(0 < r < 1)$

2 Geometrical Approach to Transform Pairs

Suppose a point z' lies on some finite length slit on the real axis and z is in the upper-half plane (the left schematic of Fig. 3) then

$$0 < \arg[z - z'] < \pi. \tag{5}$$

It follows that

$$\int_L e^{\mathrm{i}k(z-z')} dk = \left[\frac{e^{\mathrm{i}k(z-z')}}{\mathrm{i}(z - z')} \right]_0^\infty = \frac{1}{\mathrm{i}(z' - z)} \tag{6}$$

or,

$$\frac{1}{z' - z} = \mathrm{i} \int_L e^{\mathrm{i}k(z-z')} dk, \qquad 0 < \arg[z - z'] < \pi. \tag{7}$$

It is easy to check that the contribution from the upper limit of integration vanishes for the particular choices of z' and z to which we have restricted consideration.

On the other hand, suppose z' lies on some other finite length slit making angle χ with the positive real axis and suppose that z is in the slanted half plane shown shaded in Fig. 3 (the half plane "to the left" of the slit as one follows its tangent with uniform inclination angle χ). Now the transformation

$$z' \mapsto e^{-\mathrm{i}\chi}(z' - \alpha), \qquad z \mapsto e^{-\mathrm{i}\chi}(z - \alpha), \tag{8}$$

for example, where the (unimportant) constant α is shown in Fig. 3, takes the slit to the real axis, and z to the upper-half plane, and

$$0 < \arg[e^{-\mathrm{i}\chi}(z - \alpha) - e^{-\mathrm{i}\chi}(z' - \alpha)] < \pi. \tag{9}$$

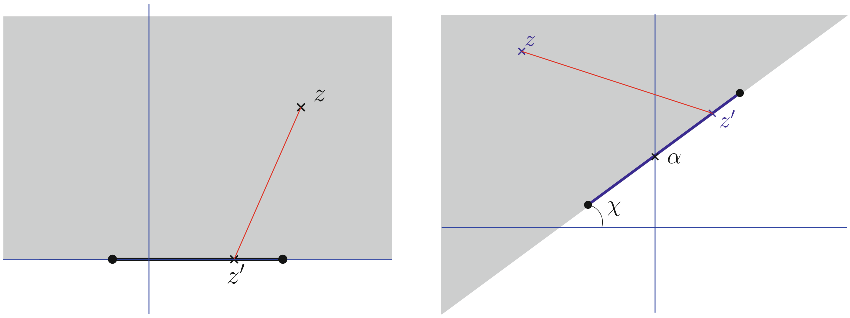

Fig. 3 Geometrical positioning of z and z' for the validity of (6) and (11)

Hence, on use of (7) with the substitutions (8), we can write

$$\frac{1}{e^{-\mathrm{i}\chi}(z'-\alpha)-e^{-\mathrm{i}\chi}(z-\alpha)} = \mathrm{i}\int_L e^{\mathrm{i}k(e^{-\mathrm{i}\chi}(z-\alpha)-e^{-\mathrm{i}\chi}(z'-\alpha))}dk, \tag{10}$$

or, on cancellation of α and rearrangement,

$$\frac{1}{z'-z} = \mathrm{i}\int_L e^{\mathrm{i}e^{-\mathrm{i}\chi}k(z-z')}e^{-\mathrm{i}\chi}dk. \tag{11}$$

Now consider a bounded convex polygon P with N sides $\{S_j \mid j = 1, \dots, N\}$. Figure 4 shows an example with $N = 3$. For a function $f(z)$ analytic in P, Cauchy's integral formula provides that for $z \in P$,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial P}\frac{f(z')dz'}{z'-z} \tag{12}$$

or, on separating the boundary integral into a sum over the N sides,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\int_{S_j} f(z')\,\frac{1}{(z'-z)}\,dz'. \tag{13}$$

But if side S_j has inclination χ_j then (11) can be used, with $\chi \mapsto \chi_j$, to reexpress the Cauchy kernel, that is $1/(z'-z)$, uniformly for all $z \in P$ and z' on the respective sides

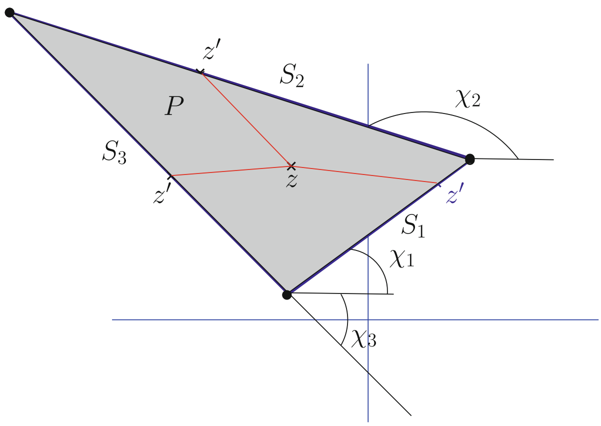

Fig. 4 A convex polygon P as an intersection of $N = 3$ half planes with N angles $\{\chi_j \mid j = 1, 2, 3\}$. Formula (11) can be used in the Cauchy integral formula with $\chi = \chi_j$ when z' is on side S_j (for $j = 1, 2, 3$)

$$f(z)=\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\int_{S_j}f(z')\left\{\mathrm{i}\int_L e^{\mathrm{i}e^{-\mathrm{i}\chi_j}k(z-z')}e^{-\mathrm{i}\chi_j}dk\right\}dz'. \tag{14}$$

On reversing the order of integration we can write

$$f(z)=\frac{1}{2\pi}\sum_{j=1}^{N}\int_L \rho_{jj}(k)e^{-\mathrm{i}\chi_j}e^{\mathrm{i}e^{-\mathrm{i}\chi_j}kz}dk, \tag{15}$$

where, for integers m, n between 1 and N, we define the *spectral matrix* [2] to be

$$\rho_{mn}(k)\equiv\int_{S_n}f(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_m}kz'}dz', \tag{16}$$

and where $L=[0,\infty)$ is the *fundamental contour* [2] for straight line edges shown in Fig. 2. We have then arrived at the transform pair (1).

The spectral matrix elements have their own analytical structure. Observe that, for any $k\in\mathbb{C}$, and for any $m=1,...,N$,

$$\sum_{n=1}^{N}\rho_{mn}(k)=\sum_{n=1}^{N}\int_{S_n}f(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_m}kz'}dz'=\int_{\partial P}f(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_m}kz'}dz'=0, \tag{17}$$

where we have used Cauchy's theorem and the fact that $f(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_m}kz'}$ (for $m=1,...,N$) is analytic inside P.

Special case: How does the traditional Fourier transform pair fit into this geometrical view? Transform pairs for unbounded polygons, such as strips and semi-strips, can be derived with minor modifications: the only difference is that the global relations are now valid in restricted parts of the spectral k-plane where the spectral functions are well defined. If P is the infinite strip $-l<\mathrm{Im}[z]<l$ then, geometrically, it is the intersection of two half planes, so $N=2$, with $\chi_1=0$ and $\chi_2=\pi$. For any $f(z)$ analytic in this strip the transform representation derived above is

$$f(z)=\frac{1}{2\pi}\int_L\rho_{11}(k)e^{\mathrm{i}kz}dk-\frac{1}{2\pi}\int_L\rho_{22}(k)e^{-\mathrm{i}kz}dk, \tag{18}$$

where the spectral functions are

$$\begin{aligned}\rho_{11}(k)&=\int_{-\infty}^{\infty}f(z)e^{-\mathrm{i}kz}dz, & \rho_{12}(k)&=\int_{\infty+\mathrm{i}l}^{-\infty+\mathrm{i}l}f(z)e^{-\mathrm{i}kz}dz,\\ \rho_{21}(k)&=\int_{-\infty}^{\infty}f(z)e^{\mathrm{i}kz}dz & \rho_{22}(k)&=\int_{+\infty+\mathrm{i}l}^{-\infty+\mathrm{i}l}f(z)e^{\mathrm{i}kz}dz.\end{aligned} \tag{19}$$

The global relations in this case are

$$\rho_{11}(k) + \rho_{12}(k) = 0, \qquad \rho_{21}(k) + \rho_{22}(k) = 0, \qquad k \in \mathbb{R} \tag{20}$$

which, in contrast to the case of a bounded convex polygon (where the global relations are valid for all $k \in \mathbb{C}$), are only valid for $k \in \mathbb{R}$. It is clear from their definitions that

$$\rho_{22}(-k) = \rho_{12}(k) \tag{21}$$

implying, after a change of variable $k \mapsto -k$ in the second integral of (18), that we can write

$$f(z) = \frac{1}{2\pi}\int_L \rho_{11}(k)e^{\mathrm{i}kz}dk + \frac{1}{2\pi}\int_0^{-\infty} \rho_{12}(k)e^{\mathrm{i}kz}dk. \tag{22}$$

On use of the first global relation in (20) we can eliminate $\rho_{12}(k)$:

$$f(z) = \frac{1}{2\pi}\int_L \rho_{11}(k)e^{\mathrm{i}kz}dk - \frac{1}{2\pi}\int_0^{-\infty} \rho_{11}(k)e^{\mathrm{i}kz}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{11}(k)e^{\mathrm{i}kz}dk. \tag{23}$$

Dropping the (now unnecessary) subscripts on $\rho_{11}(k)$, we arrive at the well-known Fourier transform pair

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(k)e^{\mathrm{i}kz}dk, \qquad \rho(k) = \int_{-\infty}^{\infty} f(z)e^{-\mathrm{i}kz}dz. \tag{24}$$

In retrieving the classical Fourier transform in this way we see how our derivation generalizes it to produce “geometry-fitted” transform pairs for any simply connected convex polygon.

3 Transform Pairs for Circular Polygons

It is natural to ask if the construction extends to other domains beyond convex polygons. The answer is in the affirmative, and the author [2] has recently shown how to extend the construction to the much broader class of so-called *circular domains*, or circular polygons, including multiply connected ones [3]. The simple convex polygons just considered are a subset of this more general class.

A key step is to establish [2] the following formula valid for $|z| < 1$:

$$\frac{1}{1-z} = \int_{L_1} \frac{1}{1-e^{2\pi\mathrm{i}k}} z^k dk + \int_{L_2} z^k dk + \int_{L_3} \frac{e^{2\pi\mathrm{i}k}}{1-e^{2\pi\mathrm{i}k}} z^k dk. \tag{25}$$

This is the basic identity that replaces the result (7) in the construction of transform pairs for circular polygons. The *fundamental contour* (for circular-arc edges [2]) is now made up of three components labelled L_1, L_2 and L_3 and shown in Fig. 2. The parameter r is arbitrary but must be chosen so that $0 < r < 1$. In an appendix to [2] the author shows how to derive (25) in a natural way from the results of the previous section. We omit details here noting only that the appearance of the expressions z^k in (25) remind us of the classical Mellin transform.

Suppose, more generally, that z is a point inside some circle C_j with centre $\delta_j \in \mathbb{C}$ and radius $q_j \in \mathbb{R}$. Suppose too that z' is a point on the circle C_j. Then $|z-\delta_j| < |z'-\delta_j|$ and, on use of (25), the Cauchy kernel for z' on C_j and z inside C_j has the spectral representation

$$\begin{aligned}\frac{1}{z'-z} &= \frac{1}{(z'-\delta_j)-(z-\delta_j)} \\ &= \frac{1}{(z'-\delta_j)}\frac{1}{[1-(z-\delta_j)/(z'-\delta_j)]} \\ &= \int_{L_1}\frac{1}{1-e^{2\pi \mathrm{i}k}}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}dk + \int_{L_2}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}dk \\ &\quad + \int_{L_3}\frac{e^{2\pi \mathrm{i}k}}{1-e^{2\pi \mathrm{i}k}}\frac{(z-\delta_j)^k}{(z'-\delta_j)^{k+1}}dk. \end{aligned} \tag{26}$$

It is important, especially for numerical implementations, to write this as

$$\begin{aligned}\frac{1}{z'-z} &= \int_{L_1}\frac{1}{1-e^{2\pi \mathrm{i}k}}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1} dk \\ &\quad + \int_{L_2}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1} dk \\ &\quad + \int_{L_3}\frac{e^{2\pi \mathrm{i}k}}{1-e^{2\pi \mathrm{i}k}}\frac{1}{q_j}\left(\frac{z-\delta_j}{q_j}\right)^k\left[\frac{z'-\delta_j}{q_j}\right]^{-k-1} dk. \end{aligned} \tag{27}$$

Just as a convex N-sided polygon was interpreted geometrically as the intersection of N half plane regions, a convex N-sided *circular polygon* can be viewed as the intersection of N circular discs. Figure 5 shows an example circular polygon D with $N = 3$ bounded by circular arcs denoted by C_1, C_2 and C_3. The Cauchy integral formula for a function $f(z)$ analytic in this region is

$$f(z) = \frac{1}{2\pi \mathrm{i}}\oint_{\partial D}\frac{f(z')}{z'-z}dz' = \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{N}\int_{C_j}\frac{f(z')}{z'-z}dz', \tag{28}$$

where ∂D denotes the boundary of D and where, in the second equality, the integral around ∂D has been separated into the N separate integrals around the individual circular arcs $\{C_j | j = 1, ..., N\}$.

Now for $z \in D$ we can substitute (27) into the Cauchy integral formula (28) when z' sits on each of the separate boundary arcs $\{C_j | j = 1, ..., N\}$ to find

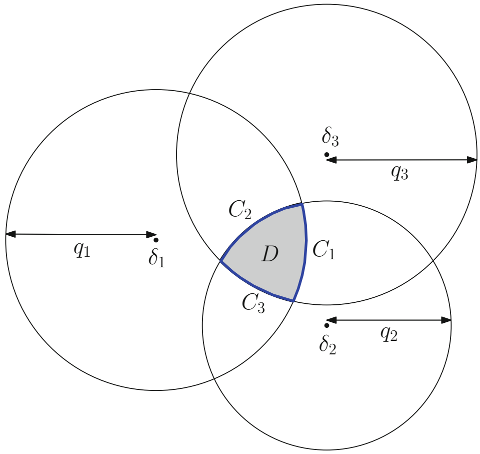

Fig. 5 A circular polygon D with $N = 3$ sides denoted by C_1, C_2 and C_3

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \left\{ \int_{L_1} \frac{\rho_{jj}(k)}{1 - e^{2\pi \mathrm{i} k}} \left[\frac{z - \delta_j}{q_j} \right]^k dk + \int_{L_2} \rho_{jj}(k) \left[\frac{z - \delta_j}{q_j} \right]^k dk \right.$$
$$\left. + \int_{L_3} \frac{\rho_{jj}(k) e^{2\pi \mathrm{i} k}}{1 - e^{2\pi \mathrm{i} k}} \left[\frac{z - \delta_j}{q_j} \right]^k dk \right\}, \tag{29}$$

where we have swapped the order of integration and introduced the N-by-N *spectral matrix*

$$\rho_{mn}(k) \equiv \frac{1}{q_m} \int_{C_n} \left[\frac{z' - \delta_m}{q_m} \right]^{-k-1} f(z') dz'. \tag{30}$$

Global relations for this system are

$$\sum_{n=1}^{N} \rho_{mn}(k) = 0, \qquad k \in -\mathbb{N} \tag{31}$$

for any $m = 1, 2, \ldots, N$. There are N such global relations but each is an equivalent statement of the analyticity of $f(z)$ in the domain D. In this way, we have constructed the "tailor-made" transform pair (3) for a circular polygon.

4 Disc-in-Channel Geometry

The geometrical construction can be extended to *multiply connected* circular domains [3], and to domains whose boundaries are a combination of straight line and circular-arc edges. An example geometry, important in applications [4, 5, 13, 15, 16], is the disc-in-channel geometry of Fig. 2. The transform representation of a function $\hat{\chi}(z)$ that is analytic *and single-valued* in such a domain can be shown [3] to be

$$\hat{\chi}(z) = \underbrace{\frac{1}{2\pi}\int_0^{\infty} \rho_{11}(k)e^{\mathrm{i}kz}dk + \frac{1}{2\pi}\int_0^{-\infty} \rho_{33}(k)e^{\mathrm{i}kz}dk}_{\text{Fourier-type transform}}$$
$$\underbrace{-\frac{1}{2\pi\mathrm{i}}\left\{\int_{L_1} \frac{\rho_{22}(k)}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk + \int_{L_2} \rho_{22}(k)\frac{1}{z^{k+1}}dk + \int_{L_3} \frac{\rho_{22}(k)e^{2\pi\mathrm{i}k}}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk\right\}}_{\text{Mellin-type transform}}, \tag{32}$$

where the simultaneous appearance of both "Fourier-type" and "Mellin-type" contributions naturally reflects the hybrid geometry of the domain (and motivates the designation "Fourier-Mellin transforms" [2]). The elements of the spectral matrix are defined as follows:

$$\rho_{11}(k) = \int_{-\infty-\mathrm{i}l}^{+\infty-\mathrm{i}l} \hat{\chi}(z)e^{-\mathrm{i}kz}dz = \rho_{31}(k) \qquad \rho_{22}(k) = -\oint_{|z|=1} \hat{\chi}(z)z^k dz, \tag{33}$$

and

$$\rho_{33}(k) = \int_{\infty+\mathrm{i}l}^{-\infty+\mathrm{i}l} \hat{\chi}(z)e^{-\mathrm{i}kz}dz = \rho_{13}(k), \tag{34}$$

with

$$\rho_{21}(k) = \int_{-\infty-\mathrm{i}l}^{+\infty-\mathrm{i}l} \hat{\chi}(z)z^k dz, \qquad \rho_{23}(k) = \int_{\infty+\mathrm{i}l}^{-\infty+\mathrm{i}l} \hat{\chi}(z)z^k dz, \tag{35}$$

and

$$\rho_{12}(k) = \rho_{32}(k) = -\oint_{|z|=1} \hat{\chi}(z)e^{-\mathrm{i}kz}dz. \tag{36}$$

The functions appearing in the spectral matrix satisfy the *global relations*

$$\begin{aligned} \rho_{11}(k) + \rho_{12}(k) + \rho_{13}(k) &= 0, \qquad k \in \mathbb{R}, \\ \rho_{31}(k) + \rho_{32}(k) + \rho_{33}(k) &= 0, \qquad k \in \mathbb{R}, \end{aligned} \tag{37}$$

which are equivalent, and

$$\rho_{21}(k) + \rho_{22}(k) + \rho_{23}(k) = 0, \qquad k \in -\mathbb{N}. \tag{38}$$

As discussed in [3], the doubly connected nature of the domain means that both (37) and (38) must be analysed to find the unknown spectral functions.

5 Application to Conformal Mapping

Suppose an application demands the solution for a harmonic field satisfying some boundary value problem in the disc-in-channel geometry of Fig. 1. If that boundary value problem is conformally invariant one might think to make use of conformal mapping. Even so, this is more easily said than done. The domain is doubly connected so a suitable pre-image domain is some annulus $\rho < |\zeta| < 1$ in a parametric complex ζ-plane (see Fig. 6); the conformal modulus ρ must be found as part of the construction of the conformal mapping. The domain is also a circular-arc domain and the construction of conformal mappings from the unit disc, say, to simply connected circular-arc domains is treated in standard texts [1, 14]. The extension of that theory to doubly connected domains (relevant to this example) has been presented more recently by Crowdy and Fokas [6], with the extension to arbitrary multiply connected domains given by Crowdy, Fokas and Green [7]. A well-known difficulty in all these conformal mapping constructions is solving for the *accessory parameters* [1, 6, 7, 14]. In this example ρ is one such accessory parameter.

We now show how that accessory parameter problem can be conveniently solved—linearized, in fact—by the generalized Fourier-Mellin transform pairs derived earlier. The main idea is to use the transform method to construct not the conformal mapping $z = z(\zeta)$, say, from the annulus $\rho < |\zeta| < 1$ to the given disc-in-channel geometry, but its inverse, which we denote by $\zeta = \zeta(z)$. Actually, the latter function is often more useful in applications since if a conformally invariant boundary value problem can be solved in the more convenient annulus geometry $\rho < |\zeta| < 1$ then knowledge of the transformation $\zeta = \zeta(z)$ allows immediate solution of the boundary value problem in the original disc-in-channel geometry.

To proceed with the construction we define the subsidiary function

$$\chi(z) \equiv \log \zeta(z). \tag{39}$$

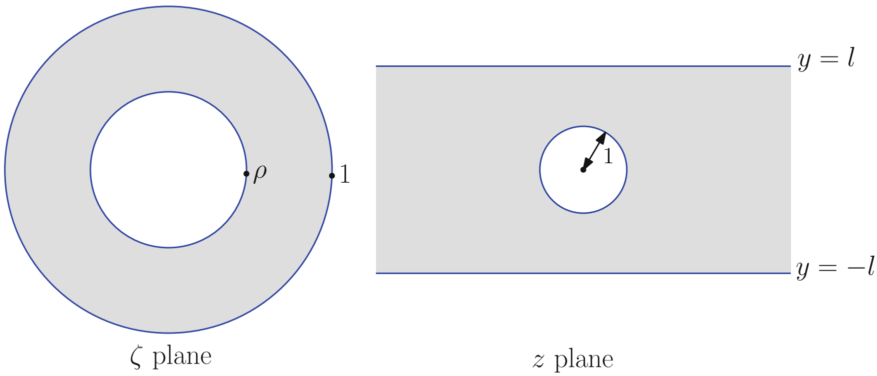

Fig. 6 Conformal mapping problem: to construct the mapping $\zeta = \zeta(z)$ from the disc-in-channel geometry to the concentric annulus $\rho < |\zeta| < 1$

It is reasonable to assume, on grounds of symmetry, that $\zeta = 1$ maps to the end of the channel as $x \to \infty$ and $\zeta = -1$ maps to $x \to -\infty$. We can write

$$\chi(z) = \log \tanh\left(\frac{\pi z}{4l}\right) + \hat{\chi}(z), \tag{40}$$

where $\hat{\chi}(z) \to 0$ as $x \to \pm\infty$. The first term is just the logarithm of the conformal mapping of the channel (without the circular hole) to a unit disc; the latter is easily derived using elementary considerations (e.g., the classical Schwarz-Christoffel formula [1, 8]). Since the boundary conditions on $\chi(z)$ are

$$\mathrm{Re}[\chi(z)] = \begin{cases} 0, & \text{on } y = \pm l, \\ \log\rho, & \text{on } |z| = 1, \end{cases} \tag{41}$$

then, on use of (40), the following boundary conditions on $\hat{\chi}(z)$ pertain:

$$\mathrm{Re}[\hat{\chi}(z)] = \begin{cases} \log\left|\coth\left(\dfrac{\pi z}{4l}\right)\right|, & \text{on } y = \pm l, \\ \\ \log\rho + \log\left|\coth\left(\dfrac{\pi z}{4l}\right)\right|, & \text{on } |z| = 1. \end{cases} \tag{42}$$

Recall that ρ is not known in advance and must be found.

By the symmetries of the proposed mapping between regions we expect that if a point $\zeta = e^{\mathrm{i}\theta}$ on the upper-half unit circle corresponds to $z = x + \mathrm{i}l$ then the point $\overline{\zeta}$ will correspond to $z = x - \mathrm{i}l$. This means that, for each x,

$$\chi(x + \mathrm{i}l) = \log\zeta = \mathrm{i}\theta = -\chi(x - \mathrm{i}l), \tag{43}$$

implying the relation

$$\chi(z + 2\mathrm{i}l) = -\chi(z). \tag{44}$$

Since the first term in (40) also satisfies this identity then we infer

$$\hat{\chi}(z + 2\mathrm{i}l) = -\hat{\chi}(z). \tag{45}$$

It follows that

$$\hat{\chi} = \begin{cases} -G(x), & \text{on } y = l, \\ G(x), & \text{on } y = -l, \end{cases} \tag{46}$$

for some (purely imaginary) function $G(x)$. On $|z| = 1$ we will write

$$\hat{\chi}(z) = r(z) + \mathrm{i}H(z) \tag{47}$$

where

$$r(z) = \log \rho + \log \left| \coth \left(\frac{\pi z}{4l} \right) \right|, \qquad H(z) = a_0 + \sum_{m \geq 1} a_m z^m + \frac{\overline{a_m}}{z^m}, \tag{48}$$

and where the coefficients $a_0 \in \mathbb{R}$, $\{a_m \in \mathbb{C} | m \geq 1\}$ are to be found. We also expect, on grounds of symmetry,

$$\overline{\chi(z)} = \chi(z), \qquad \text{on } \bar{z} = z \tag{49}$$

implying that

$$\bar{r}(z) - \mathrm{i}\overline{H}(z) = r(z) + \mathrm{i}H(z), \quad \text{or} \quad \overline{H}(z) = -H(z). \tag{50}$$

This condition implies $a_0 = 0$ and

$$a_n = \mathrm{i}b_n, \tag{51}$$

for some real set $\{b_n\}$. We will make use of these facts later.

The key observation is this: the function $\hat{\chi}(z)$ is single-valued and analytic in the fluid region D. It therefore has a transform representation of the form (32). To find it, we must determine the unknown spectral functions. This can be done by analysis of the global relations (37) and (38).

Now (37) and (38) give, respectively,

$$\int_{-\infty}^{\infty} G(x) e^{-\mathrm{i}kx} \left[e^{-kl} + e^{kl} \right] dx - \oint_{|z|=1} e^{-\mathrm{i}kz} \hat{\chi}(z) dz = 0, \quad k \in \mathbb{R}, \tag{52}$$

$$\int_{-\infty}^{\infty} G(x) \left[\frac{1}{(x - \mathrm{i}l)^n} + \frac{1}{(x + \mathrm{i}l)^n} \right] dx - \oint_{|z|=1} \hat{\chi}(z) \frac{dz}{z^n} = 0, \quad n \in \mathbb{N}. \tag{53}$$

Equation (52) implies that

$$2 \cosh(kl) \mathcal{G}(k) = B(k) + R_1(k), \tag{54}$$

where

$$\mathcal{G}(k) \equiv \int_{-\infty}^{\infty} G(x) e^{-\mathrm{i}kx} dx \tag{55}$$

and

$$B(k) \equiv \oint_{|z|=1} \mathrm{i}H(z) e^{-\mathrm{i}kz} dz, \qquad R_1(k) \equiv \oint_{|z|=1} r(z) e^{-\mathrm{i}kz} dz. \tag{56}$$

It follows that

$$\mathcal{G}(k) = \frac{B(k) + R_1(k)}{2\cosh(kl)} \tag{57}$$

and the inverse Fourier transform provides

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{B(k) + R_1(k)}{2\cosh(kl)}\right]e^{\mathrm{i}kx}dk. \tag{58}$$

The second global relation (53) implies that, for $n \in \mathbb{N}$,

$$\oint_{|z|=1}\frac{\mathrm{i}H(z)}{z^n}dz + R_2(n-1) = \int_{-\infty}^{\infty}G(x)\left[\frac{1}{(x-\mathrm{i}l)^n} + \frac{1}{(x+\mathrm{i}l)^n}\right]dx, \tag{59}$$

where we define

$$R_2(n) \equiv \oint_{|z|=1}\frac{r(z)}{z^{n+1}}dz. \tag{60}$$

It is easy to show that for $n \geq 1$,

$$\oint_{|z|=1}\frac{\mathrm{i}H(z)}{z^n}dz = -2\pi a_{n-1}. \tag{61}$$

Equation (59) then implies that

$$a_{n-1} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}G(x)\left[\frac{1}{(x-\mathrm{i}l)^n} + \frac{1}{(x+\mathrm{i}l)^n}\right]dx + \frac{R_2(n-1)}{2\pi}. \tag{62}$$

On substitution of (58) for $G(x)$, we find

$$a_{n-1} = \int_{-\infty}^{\infty}J(k,n-1)\left[\frac{B(k) + R_1(k)}{2\cosh(kl)}\right]dk + \frac{R_2(n-1)}{2\pi}, \qquad n \geq 1, \tag{63}$$

where we define

$$J(k,n) \equiv -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{\mathrm{i}kx}\left[\frac{1}{(x-\mathrm{i}l)^{n+1}} + \frac{1}{(x+\mathrm{i}l)^{n+1}}\right]dx. \tag{64}$$

Some residue calculus reveals that for $n \geq 1$,

$$J(k,n) = \begin{cases} -\dfrac{\mathrm{i}e^{-kl}(\mathrm{i}k)^n}{2\pi n!}, & k \geq 0, \\[2ex] \dfrac{\mathrm{i}e^{kl}(\mathrm{i}k)^n}{2\pi n!}, & k < 0, \end{cases} \tag{65}$$

and, for $n = 0$,

$$J(k,0) = \begin{cases} -\dfrac{\mathrm{i}e^{-kl}}{2\pi}, & k > 0, \\ 0, & k = 0, \\ \dfrac{\mathrm{i}e^{kl}}{2\pi}, & k < 0. \end{cases} \tag{66}$$

It follows that, for $n \geq 1$,

$$a_{n-1} = \int_{-\infty}^{\infty} \frac{J(k,n-1)B(k)}{2\cosh(kl)}dk + \int_{-\infty}^{\infty} \frac{J(k,n-1)R_1(k)}{2\cosh(kl)}dk + \frac{R_2(n-1)}{2\pi}. \tag{67}$$

But

$$\begin{aligned} B(k) &= \oint_{|z|=1} e^{-\mathrm{i}kz}\mathrm{i}H(z)dz = \oint_{|z|=1} \mathrm{i}e^{-\mathrm{i}kz}\left[a_0 + \sum_{m\geq 1} a_m z^m + \frac{\overline{a_m}}{z^m}\right]dz \\ &= -2\pi \sum_{m\geq 1} \frac{\overline{a_m}(-\mathrm{i}k)^{m-1}}{(m-1)!}. \end{aligned} \tag{68}$$

Hence

$$a_{n-1} + \sum_{m\geq 1} A_{n-1,m}\overline{a_m} = E_{n-1}, \qquad n \geq 1, \tag{69}$$

where

$$\begin{aligned} A_{n,m} &= \pi \int_{-\infty}^{\infty} \frac{J(k,n)(-\mathrm{i}k)^{m-1}}{(m-1)!\cosh(kl)}dk, \\ E_n &= \int_{-\infty}^{\infty} \frac{J(k,n)R_1(k)}{2\cosh(kl)}dk + \frac{R_2(n)}{2\pi}. \end{aligned} \tag{70}$$

Finally, on use of (51), system (69) becomes the system of real equations

$$\sum_{m\geq 1} A_{0m}b_m = \mathrm{i}E_0, \tag{71}$$

$$b_n - \sum_{m\geq 1} A_{nm}b_m = -\mathrm{i}E_n, \qquad n \geq 1. \tag{72}$$

It can be checked easily that E_0 is the only quantity that depends on the unknown $\log\rho$. We can therefore solve (72) for the set of coefficients $\{b_n | n \geq 1\}$ and then use (71) *a posteriori* to determine $\log\rho$. With the coefficients determined from this simple linear system all the spectral functions needed in the representation (32) of the required function $\hat{\chi}(z)$ can be found. The required inverse conformal mapping $\zeta(z)$ then follows.

The construction above was originally presented in an appendix to [5] where it was used as a check on a solution given there.

6 Summary

It is hoped that this article provides a useful overview of recent developments concerning these "geometry-fitted" Fourier-Mellin transform pairs, and how to make constructive use of them. The author has recently employed the new transform pairs described here in a number of different applications [2–5] with much earlier work on solving various PDEs in convex polygons carried out by other authors [11, 12]. We believe that the full scope and implications of the method for applications, and its various extensions, have yet to be explored.

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References

1. Ablowitz, M., Fokas, A.S.: Complex Variables and Applications. Cambridge University Press (1997)
2. Crowdy, D.G.: Fourier-Mellin transforms for circular domains. Comput. Meth. Funct. Theory **15**(4), 655–687 (2015)
3. Crowdy, D.G.: A transform method for Laplace's equation in multiply connected circular domains. IMA J. Appl. Math. **80**(6), 1902–1931 (2015)
4. Crowdy, D.G.: Effective slip lengths for longitudinal shear flow over partial-slip circular bubble mattresses. Fluid Dyn. Res. **47**, 065507 (2015)
5. Crowdy, D.G.: Uniform flow past a periodic array of cylinders. Eur. J. Mech. B/Fluids **56**, 120–129 (2015)
6. Crowdy, D.G., Fokas, A.S.: Conformal mappings to a doubly connected polycircular arc domain. Proc. R. Soc. A **463**, 1885–1907 (2007)
7. Crowdy, D.G., Fokas, A.S., Green, C.C.: Conformal mappings to multiply connected polycircular arc domains. Comput. Meth. Funct. Theory **11**(2), 685–706 (2011)
8. Crowdy, D.G.: The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains. Proc. R. Soc. A **461**, 2653–2678 (2005)
9. Crowdy, D.G., Fokas, A.S.: Explicit integral solutions for the plane elastostatic semi-strip. Proc. R. Soc. A **460**, 1285–1310 (2004)
10. Fokas, A.S., Kapaev, A.A.: On a transform method for the Laplace equation in a polygon. IMA J. Appl. Math. **68**, 355–408 (2003)
11. Fokas, A.S.: A unified approach to boundary value problems. In: CBMS-NSF Regional Conference Series in Applied Mathematics, No 78. SIAM, Philadelphia (2008)
12. Fokas, A.S., Pelloni, B. (eds.): Unified Transform for Boundary Value Problems: Applications and Advances. SIAM Monographs, Philadelphia (2015)

13. Martin, P.A., Dalrymple, R.A.: Scattering of long waves by cylindrical obstacles and gratings using matched asymptotic expansions. J. Fluid Mech. **188**, 465–490 (1988)
14. Nehari, Z.: Conformal Mapping. Dover Publications, New York (2011)
15. Poritsky, H.: Potential of a charged cylinder between two parallel grounded planes. J. Math. Phys. **39**, 35–48 (1960)
16. Richmond, H.W.: On the electrostatic field of a plane or circular grating formed of thick rounded bars. Proc. Lond. Math. Soc. **22**(2), 389–403 (1923)

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