Chapter 2
Computational Tools

In this chapter we introduce the computational methods needed for the analysis of the causal action principle in the continuum limit. These methods are the backbone of the analysis given in Chaps. 3–5. Nevertheless, in order to facilitate the reading of the book, we made the subsequent chapters accessible even without a detailed knowledge of the computational tools. To this end, all the technical computations are given in the appendices, whereas in the main Chaps. 3–5 these results are merely stated and explained. Therefore, a reader who is willing to accept the results of the detailed computations may skip the present chapter in a first reading.

Our main objective is to construct the fermionic projector in the presence of an external potential and to analyze it in position space. The first task is to define the unregularized fermionic projector $P(x, y)$ in the presence of the external potential. In this setting, the fermionic projector was constructed in a perturbation expansion in $\mathcal{B}$ in [F3, FG1, FT2]. More recently, a non-perturbative construction was given in [FR2, FR3, FMR] (see also the brief review in Sect. 1.5.1). For the explicit analysis of the causal action principle to be carried out in this book, we need the detailed formulas of the perturbation expansion. In order to focus on what is really needed in this book, we here restrict attention to the perturbative treatment (Sect. 2.1). The reader interested in non-perturbative methods is referred to the introduction in [FKT] or to the research papers [FR2, FR3, FMR].

Our next task is to derive detailed formulas for the fermionic projector in position space. Such formulas are most conveniently obtained using the so-called light-cone expansion as first developed in [F5, F6]. In Sect. 2.2 we give a self-contained introduction to the light-cone expansion.

In Sect. 2.3 the causal perturbation expansion and the light-cone expansion are adapted to the description of linearized gravity.

In Sect. 2.4 we turn attention to the ultraviolet regularization of the fermionic projector. This leads us to the so-called formalism of the continuum limit, which makes it possible to analyze how the different contributions to the causal action depend on the regularization. In order to make the presentation easily accessible, we begin with the example of an $i\varepsilon$-regularization (Sect. 2.4.1). Then we consider linear combinations of such regularizations (Sect. 2.4.2) and explain further regularization

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effects (Sect. 2.4.3). Then the formalism of the continuum is introduced (Sect. 2.4.4), and its derivation is outlined (Sect. 2.4.5). Our presentation is not as general as the original derivation as given in [F7, Chap. 4], but instead it aims at clarifying the main points of the construction.

In Sect. 2.5 we explain how to compute the local trace. This is important in view of the rescaling procedure explained in Sect. 1.4.1 (see (1.4.11)).

Finally, in Sect. 2.6 it is explained how the EL equations as derived in Sect. 1.4.1 can can be analyzed in the formalism of the continuum limit.

2.1 The Fermionic Projector in an External Potential

2.1.1 The Fermionic Projector of the Vacuum

Our starting point is the unregularized kernel of the fermionic projector of the vacuum which we already encountered in Sect. 1.2.5 (see Lemma 1.2.8, (1.2.25) and Lemma 1.2.9). For the later constructions, it is convenient to clarify that we are in the Minkowski vacuum by adding an index "vac." Moreover, we denote the mass by an additional index m. Thus we define the *kernel of the fermionic projector of the vacuum* as the bi-distribution

$$P_m^{\text{vac}}(x, y) = \int \frac{d^4k}{(2\pi)^4}\, P_m^{\text{vac}}(k)\, e^{-ik(x-y)}\,, \tag{2.1.1}$$

where $P_m^{\text{vac}}(k)$ is the distribution in momentum space

$$P_m^{\text{vac}}(k) = (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0) \tag{2.1.2}$$

(and Θ denotes the Heaviside function). We also consider the distribution $P_m^{\text{vac}}(x, y)$ as the integral kernel of an operator acting on wave functions in space-time, i.e.

$$P_m^{\text{vac}} \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to C^\infty(\mathscr{M}, S\mathscr{M})\,, \qquad (P_m^{\text{vac}}\psi)(x) = \int_M P_m^{\text{vac}}(x, y)\, \psi(y)\, d^4y\,. \tag{2.1.3}$$

This operator is the so-called fermionic projector of the vacuum.

Before going on, we briefly recall the physical picture. In (2.1.1) we integrate over all the plane-wave solutions of the Dirac equation of negative frequency (the decomposition into plane-wave solutions was explained in detail in Chap. 1; see (1.2.20) and Lemma 1.2.8). Thus P_m^{vac} describes the ensemble of all negative-frequency solutions of the Dirac equation. As already mentioned in Sect. 1.2.5, we use this Dirac sea configuration to describe the vacuum in Minkowski space. In order to describe a system with an additional particle, we simply add the corresponding bra/ket-combination by setting

$$P(x,y) = P_m^{\text{vac}}(x,y) - \frac{1}{2\pi}\psi(x)\,\overline{\psi(y)}\,,$$

where ψ is a positive-frequency solution of the Dirac equation (for the prefactor $-1/(2\pi)$ and the normalization of the wave function see Sect. 2.1.7). Similarly, we occupy several states by adding the bra/ket-combinations of several particle states,

$$P(x,y) = P_m^{\text{vac}}(x,y) - \frac{1}{2\pi}\sum_{k=1}^{n_{\text{p}}} \psi_k(x)\overline{\psi_k(y)}$$

(which need to be suitably ortho-normalized; see again Sect. 2.1.7). In order to introduce anti-particles, we similarly subtract bra/ket-combinations

$$P(x,y) = P_m^{\text{vac}}(x,y) - \frac{1}{2\pi}\sum_{k=1}^{n_{\text{p}}} \psi_k(x)\overline{\psi_k(y)} + \frac{1}{2\pi}\sum_{l=1}^{n_{\text{a}}} \phi_l(x)\overline{\phi_l(y)}\,, \tag{2.1.4}$$

where $\phi_1, \ldots, \phi_{n_{\text{a}}}$ are the wave functions of negative-frequency solutions. Thus in simple terms, we take Dirac's concept of the Dirac sea literally and describe particles by additional occupied states and anti-particles by "holes" in the sea.

With the methods introduced so far, this description of particles and anti-particles by occupying states and creating "holes" can only be performed in the non-interacting situation in which we can work with plane-wave solutions of the Dirac equation. But it is not obvious how the construction should be carried out if an external potential is present. In order to tackle this problem, we first analyze how to describe the completely filled Dirac sea in the presence of an external potential (see Sects. 2.1.2–2.1.6). Afterwards, we will come back to the description of systems involving particles and anti-particles (see Sect. 2.1.7).

2.1.2 The External Field Problem

We now return to the Dirac equation in the presence of an external potential (1.5.1),

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\psi(x) = 0\,, \tag{2.1.5}$$

where $\mathcal{B}$ is a smooth potential with suitable decay properties at spatial infinity and for large times (to be specified in Lemma 2.1.2). We now explain the basic problem in defining the fermionic projector in the presence of an external potential.

The definition of the fermionic projector of the vacuum (2.1.1) and (2.1.2) makes essential use of the fact that the solution space of the Dirac equation splits into two subspaces of negative and positive frequency, respectively. Indeed, this made it possible in (2.1.2) to integrate only over the solutions of negative frequency. In order to extend the definition of the fermionic projector to the case when an external

potential is present (2.1.5), one needs to again decompose the solution space into two subspaces. In the special case that $\mathcal{B}$ is static, one can still separate the time dependence by the plane wave ansatz $\psi(t, \vec{x}) = e^{-i\omega t}\, \psi_\omega(\vec{x})$, so that the sign of ω gives a canonical splitting of the solution space. This procedure is often referred to as the *frequency splitting*. In the general time-dependent setting, however, no plane wave ansatz can be used, so that the frequency splitting breaks down. Therefore, it is no longer obvious if there still is a canonical decomposition of the solution space into two subspaces.

This problem is sometimes referred to as the *external field problem* (for more details see Exercise 2.1 or the exposition in [F7, Sect. 2.1]). It is a common belief that in the presence of a general time-dependent external potential, there no longer exists a canonical decomposition of the solution space into two subspaces. Nevertheless, it is still possible to decompose the solution space into two subspaces, for example by using the sign of the spectrum of the Dirac Hamiltonian on a distinguished Cauchy surface. But the decomposition is no longer canonical in the sense that it involves an arbitrariness. This arbitrariness is often associated to an observer, so that the choice of the subspaces depends on the observer. As a consequence, the interpretation of the fermionic many-particle state in terms of particles and anti-particles also depends on the observer. This *observer dependence of the particle interpretation* becomes most apparent in the Unruh effect in which the vacuum of the observer at rest is described by a uniformly accelerated observer in terms of a thermal state involving particles and anti-particles.

Nevertheless, this reduction to particles and anti-particles as being objects associated to observers only tells part of the truth. Namely, as shall be developed in what follows, even in the presence of a time-dependent external potential there is a *canonical decomposition of the solution space into two subspaces*. In the static situation, this decomposition reduces to the frequency splitting. In the time-dependent situation, however, this decomposition depends on the global behavior of $\mathcal{B}$ in space-time. In particular, this decomposition cannot be associated to a local observer. Starting from the canonical decomposition of the solution space, one can again generate particles and holes, giving rise to an interpretation of the many-particle state in terms of particles and anti-particles. This particle interpretation is again independent of the choice of an observer. All constructions are explicitly covariant.

2.1.3 Main Ingredients to the Construction

Before entering the constructions, we explain a few ingredients and ideas. Generally speaking, we shall make use of additional properties of the fermionic projector of the vacuum, which are not immediately apparent in the Fourier decomposition (2.1.1) and (2.1.2). One ingredient is to use that *causality* is built into $P^{\text{vac}}(x, y)$. To see how this comes about, we decompose P_m^{vac} as

$$P_m^{\text{vac}}(x, y) = \frac{1}{2}\Big(p_m(x, y) - k_m(x, y)\Big)\,, \tag{2.1.6}$$

where $p_m(x,y)$ and $k_m(x,y)$ are the Fourier transforms of the distributions in momentum space

$$p_m(q) = (\not{q} + m)\, \delta(q^2 - m^2) \tag{2.1.7}$$
$$k_m(q) = (\not{q} + m)\, \delta(q^2 - m^2)\, \epsilon(q^0) \tag{2.1.8}$$

(and ϵ in (2.1.8) is again the sign function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise). All these Fourier integrals are well-defined tempered distributions, which are also distributional solutions of the vacuum Dirac equation. The point is that the distribution $k_m(x,y)$ is *causal* in the sense that it vanishes if x and y have spacelike separation. In order to see this, it is useful to introduce the *advanced* and the *retarded Green's functions* by

$$s_m^\vee(q) = \lim_{\nu \searrow 0} \frac{\not{q} + m}{q^2 - m^2 - i\nu q^0} \qquad \text{and} \qquad s_m^\wedge(q) = \lim_{\nu \searrow 0} \frac{\not{q} + m}{q^2 - m^2 + i\nu q^0}\,, \tag{2.1.9}$$

respectively (with the limit $\nu \searrow 0$ taken in the distributional sense). Taking their Fourier transform

$$s_m(x,y) = \int \frac{d^4q}{(2\pi)^4}\, s_m(q)\, e^{-iq(x-y)}\,, \tag{2.1.10}$$

we obtain corresponding bi-distributions $s_m^\vee(x,y)$ and $s_m^\wedge(x,y)$. By direct computation one verifies that these Green's functions satisfy the distributional equation

$$(i\not{\partial}_x - m)\, s_m(x,y) = \delta^4(x-y)\,. \tag{2.1.11}$$

Moreover, computing the Fourier integral (2.1.10) with residues, one sees that the support of these Green's functions lies in the upper respectively lower light cone, i.e.

$$\text{supp}\ s_m^\vee(x,.) \subset J_x^\vee\,, \qquad \text{supp}\ s_m^\wedge(x,.) \subset J_x^\wedge\,, \tag{2.1.12}$$

where $J_x^\vee$ and $J_x^\wedge$ denote the points in the causal future respectively past of x,

$$J_x^\vee = \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \geq 0\}$$
$$J_x^\wedge = \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \leq 0\}$$

(for details see Exercise 2.2 or [FKT, Chap. 4]). In view of (2.1.11), the difference of the advanced and retarded Greens' functions is a solution of the homogeneous Dirac equation. In order to compute it in detail, we again make use of (1.2.33) to obtain

$$\begin{aligned} s_m^\vee(q) - s_m^\wedge(q) &= (\not{q} + m) \lim_{\nu \searrow 0} \left[\frac{1}{q^2 - m^2 - i\nu q^0} - \frac{1}{q^2 - m^2 + i\nu q^0} \right] \\ &= (\not{q} + m) \lim_{\nu \searrow 0} \left[\frac{1}{q^2 - m^2 - i\nu} - \frac{1}{q^2 - m^2 + i\nu} \right] \epsilon(q^0) \end{aligned}$$

$$= 2\pi i\,(\not{q} + m)\,\delta(q^2 - m^2)\,\epsilon(q^0) \tag{2.1.13}$$

(for details see Exercise 2.4). Comparing with (2.1.8), we conclude that the difference of the advanced and retarded Green's functions is a multiple of k_m

$$k_m(x, y) = \frac{1}{2\pi i}\left(s^\vee(x, y) - s^\wedge(x, y)\right). \tag{2.1.14}$$

In particular, this shows that k_m is indeed causal, i.e.

$$\operatorname{supp} k_m(x, .) \subset J_x\,, \tag{2.1.15}$$

where $J_x := J_x^\vee \cup J_x^\wedge$. We refer to k_m as the *causal fundamental solution*.

Now (2.1.6) can be understood as the decomposition of the vacuum fermionic projector into a causal part (the distribution k_m) and a part which is not causal (the distribution p_m; note that the explicit formulas in (1.2.25) and Lemma 1.2.9 show that $p_m(x, y)$ is indeed non-zero for spacelike distances). One idea behind our constructions is to perform the perturbation expansion in such a way that the decomposition of $P(x, y)$ in to a causal and a non-causal part is preserved.

Another ingredient to our constructions is that the distributions p_m and k_m are related to each other by a *functional calculus*, as we now explain. We first point out that for the space-time integral in (2.1.3) to exist, we had to assume that the wave function ψ has suitable decay properties at infinity. More specifically, the time integral in (2.1.3) in general diverges if ψ is a physical wave function, being a solution of the Dirac equation. In particular, the operator in (2.1.3) cannot be defined as an operator from a vector space to itself, but it necessarily maps one function space to another function space. As a consequence, it is impossible to multiply the operator P_m by itself. This is obvious because the formal integral

$$\int P_m^{\text{vac}}(x, z)\, P_m^{\text{vac}}(z, y)\, d^4z \tag{2.1.16}$$

is ill-defined. This problem can be understood similarly in momentum space. Namely, using that convolution in position space corresponds to multiplication in momentum space, the integral in (2.1.16) corresponds to the formal product

$$P_m^{\text{vac}}(q)\, P_m^{\text{vac}}(q)\,,$$

which is again ill-defined because the square of the δ-distribution in (2.1.2) makes no mathematical sense. As we shall see, these obvious problems in the naive treatment of the fermionic projector are not only a mathematical subtlety. On the contrary, the methods for overcoming these problems will involve a careful analysis of the causal structure of the fermionic projector and of its proper normalization.

It is important to observe that the above operator product does make sense if we consider two different mass parameters. Namely,

$$\begin{aligned} P_m^{\text{vac}}(q)\, P_{m'}^{\text{vac}}(q) &= (\not{q} + m)\, \delta(q^2 - m^2)\, \Theta(-q^0)\, (\not{q} + m')\, \delta(q^2 - (m')^2)\, \Theta(-q^0) \\ &= \left(q^2 + (m+m')\, \not{q} + mm'\right) \delta(m^2 - (m')^2)\, \delta(q^2 - m^2)\, \Theta(-q^0) \\ &= \left(q^2 + (m+m')\, \not{q} + mm'\right) \frac{1}{2m}\, \delta(m - m')\, \delta(q^2 - m^2)\, \Theta(-q^0) \\ &= \delta(m - m')\, (\not{q} + m)\, \delta(q^2 - m^2)\, \Theta(-q^0)\,. \end{aligned}$$

giving rise to the distributional identity

$$P_m^{\text{vac}}\, P_{m'}^{\text{vac}} = \delta(m - m')\, P_m^{\text{vac}}\,. \tag{2.1.17}$$

This resembles idempotence, but it involves a δ-distribution in the mass parameter. We remark that this δ-normalization in the mass parameter can be treated in a mathematically convincing way using the notion of the mass oscillation property as introduced in [FR3]. For brevity, we shall not enter these constructions here. Instead, we are content with the fact that (2.1.17) is well-defined if we test in both m and q.

This calculus can be used similarly for the operators p_m and k_m obtained by considering the distributions (2.1.7) and (2.1.8) as multiplication operators in momentum space. In particular, this gives rise to the relation

$$k_m\, k_{m'} = \delta(m - m')\, p_m \tag{2.1.18}$$

(for details see Lemma 2.1.3). This identity is very useful because it allows us to deduce p_m from k_m. Therefore, our strategy is to first construct k_m in the presence of an external potential using the underlying causal structure (2.1.14). Then we take (2.1.18) to define p_m in the presence of the external potential. Finally, we use (2.1.6) to define the fermionic projector.

There is one subtle point in the construction which we want to mention here: the proper normalization of the states of the fermionic projector. The most obvious method is to interpret and use the identity (2.1.17) as a normalization condition. This so-called *mass normalization* was used in [F3, FG1]; see also [F7, Chap. 2]. More recently, the non-perturbative construction in [FR3] revealed that on a general globally hyperbolic manifold, the mass normalization cannot be used and should be replaced by the so-called *spatial normalization*. In [FT2] the causal perturbation expansion is worked out for both the mass and the spatial normalizations, and the methods and results are compared. In [FT2, Sect. 2.2] the advantages of the spatial normalization are discussed, but no decisive argument in favor of one of the normalization methods is given. Finally, the Noether-like theorems in [FK2] showed that the spatial normalization is the proper normalization method, because it reflects the intrinsic conservation laws of the causal fermion system (see [FK2, Remark 5.13] or the brief outline in Sect. 1.4.2).

With these results in mind, we here restrict attention to the spatial normalization, which we now introduce. Recall that for a Dirac wave function ψ, the quantity $(\overline{\psi}\gamma^0\psi)(t_0, \vec{x})$ has the interpretation as the probability density for the particle at time t_0 to be at position $\vec{x}$. Integrating over space and polarizing, we obtain the scalar product (1.2.2), which we also denote by

$$(\psi|\phi)_{t_0} = 2\pi \int_{\mathbb{R}^3} \overline{\psi(t_0, \vec{y})}\gamma^0\phi(t_0, \vec{y})\, d^3y\,. \tag{2.1.19}$$

It follows from current conservation that for any solutions ψ, ϕ of the Dirac equation, this scalar product is independent of the choice of t_0. This is the case even in the presence of an external potential (2.1.5), provided that the potential is symmetric with respect to the inner product on the spinors (1.2.18), i.e.

$$\prec\psi|\mathcal{B}\phi\succ \;=\prec\mathcal{B}\psi|\phi\succ \tag{2.1.20}$$

(see Exercise 2.5). Since the kernel of the fermionic projector is a solution of the Dirac equation, one is led to evaluating the integral in (2.1.19) for $\phi(y) = P(y, z)$ and $\overline{\psi(y)} = P(x, y)$. In the vacuum, the resulting integral can be computed, giving a simple result.

Lemma 2.1.1 *For any $t \in \mathbb{R}$, there is the distributional relation*

$$2\pi \int_{\mathbb{R}^3} P_m^{vac}\left(x, (t, \vec{y})\right)\gamma^0\, P_m^{vac}\left((t, \vec{y}), z\right) d^3y = -P_m^{vac}(x, z)\,. \tag{2.1.21}$$

Proof The identity follows by a straightforward computation, which was already given in the proof of Lemma 1.2.8 (see (1.2.24) and the computation thereafter). □

We refer to (2.1.21) as the **spatial normalization** of the fermionic projector. It has the advantage that it is well-defined even for fixed m. Moreover, the normalization method is closely related to the probabilistic interpretation of the Dirac equation.

In the following Sects. 2.1.4–2.1.6, we shall carry out the construction of the fermionic projector describing the completely filled Dirac sea in the presence of the external potential $\mathcal{B}$. Our method will make essential use of generalizations of the underlying causal structure (as is apparent in (2.1.6) and (2.1.12)), of the relation between k_m and p_m as expressed by (2.1.18), and of the spatial normalization (2.1.21). Finally, in Sect. 2.1.7 we shall extend the construction to allow for particles and anti-particles.

2.1.4 The Perturbation Expansion of the Causal Green's Functions

Using the causal support property, the advanced and retarded Green's functions $\tilde{s}_m^\vee$ and $\tilde{s}_m^\wedge$ are uniquely defined even in the presence of an external potential (2.1.5). They can be constructed non-perturbatively using the theory of symmetric hyperbolic systems (see [J] or [FKT, Chap. 5]). For our purposes, it is sufficient to work out their perturbation expansions: The retarded Green's function is characterized by the conditions

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{s}_m^\wedge(x, y) = \delta^4(x - y) \qquad \text{and} \qquad \operatorname{supp} \tilde{s}_m^\wedge(x, .) \subset J_x^\wedge \,.$$

Employing the perturbation ansatz

$$\tilde{s}_m^\wedge = \sum_{n=0}^{\infty} s_{(n)}^\wedge \qquad \text{with} \qquad s_{(0)}^\wedge = s_m^\wedge$$

(where the subscript (n) denotes the order of perturbation theory), we obtain for $n = 1, 2, \ldots$ the inductive conditions

$$(i\partial\!\!\!/ - m)\, s_{(n)}^\wedge = -\mathcal{B}\, s_{(n-1)}^\wedge \qquad \text{and} \qquad \operatorname{supp} \tilde{s}_{(n)}^\wedge(x, .) \subset J_x^\wedge \,. \tag{2.1.22}$$

Using the defining property of the Green's function (2.1.11), one sees that the left equation in (2.1.22) can be solved in the case $n = 1$ by

$$s_{(1)}^\wedge = -s_m\, \mathcal{B}\, s_m^\wedge \,, \tag{2.1.23}$$

where the operator product is defined as follows,

$$(s_m\, \mathcal{B}\, s_m^\wedge)(x, y) := \int d^4z\, s_m(x, z)\, \mathcal{B}(z)\, s_m^\wedge(z, y) \tag{2.1.24}$$

(the analytic justification of this and all other operator products in this section will be given in Lemma 2.1.2). The operator s_m in (2.1.23) is any Green's function (like the advanced, retarded or the symmetric Green's function). In order to determine which Green's function to choose, we evaluate the condition on the right side of (2.1.22). Namely, if we choose s_m in (2.1.23) again as the retarded Green's function, then the integral in (2.1.24) vanishes if x lies in the past of y because in this case the supports of the distributions $s_m^\wedge(x, .)$ and $s_m^\wedge(., y)$ do not intersect. This leads us to setting

$$s_{(1)}^\wedge = -s_m^\wedge\, \mathcal{B}\, s_m^\wedge \,.$$

Now we can evaluate (2.1.22) inductively to obtain

$$s^{\wedge}_{(n)} = -s^{\wedge}_m \mathcal{B}\, s^{\wedge}_{(n-1)} = \big(-s^{\wedge}_m \mathcal{B}\big)^n s^{\wedge}_m .$$

Proceeding similarly for the advanced Green's function, we obtain the unique perturbation series

$$\tilde{s}^{\vee}_m = \sum_{n=0}^{\infty} \big(-s^{\vee}_m \mathcal{B}\big)^n s^{\vee}_m , \qquad \tilde{s}^{\wedge}_m = \sum_{n=0}^{\infty} \big(-s^{\wedge}_m \mathcal{B}\big)^n s^{\wedge}_m . \tag{2.1.25}$$

Having derived a perturbation series for the causal Green's functions, we can also define the causal fundamental solution in generalization of (2.1.14) by

$$\tilde{k}_m := \frac{1}{2\pi i}(\tilde{s}^{\vee}_m - \tilde{s}^{\wedge}_m) , \tag{2.1.26}$$

We now specify a class of potentials for which all the operator products appearing here and later in this book are all well-defined in the distributional sense:

Lemma 2.1.2 *Let (C_j), $0 \le j \le n$, be a choice of operators $C_j \in \{k_m, p_m, s^{\vee}_m, s^{\wedge}_m\}$ (and p_m, k_m according to (2.1.7) and (2.1.8)). If the external potential $\mathcal{B}$ is smooth and decays so fast at infinity that the functions $\mathcal{B}(x)$, $x^i \mathcal{B}(x)$, and $x^i x^j \mathcal{B}(x)$ are integrable, then the operator product*

$$(C_n\, \mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_0)(x, y) \tag{2.1.27}$$

is a well-defined tempered distribution on $\mathbb{R}^4 \times \mathbb{R}^4$.

Proof Calculating the Fourier transform of (2.1.27) gives the formal expression

$$\begin{aligned} M(q_2, q_1) := \int \frac{d^4 p_1}{(2\pi)^4} \cdots \int \frac{d^4 p_{n-1}}{(2\pi)^4} C_n(q_2)\, \hat{\mathcal{B}}(q_2 - p_{n-1}) \\ \times\, C_{n-1}(p_{n-1})\, \hat{\mathcal{B}}(p_{n-1} - p_{n-2}) \cdots C_1(p_1)\, \hat{\mathcal{B}}(p_1 - q_1)\, C_0(q_1) , \end{aligned} \tag{2.1.28}$$

where we consider the C_j as multiplication operators in momentum space and where $\hat{\mathcal{B}}$ denotes the Fourier transform of the function $\mathcal{B}$ (it is more convenient to work in momentum space because the operators C_j are then diagonal). We will show that $M(q_2, q_1)$ is a well-defined tempered distribution; the Lemma then immediately follows by transforming back to position space.

The assumptions on $\mathcal{B}$ yield that $\hat{\mathcal{B}}$ is C^2 and has rapid decay at infinity, i.e.

$$\sup_{q \in \mathbb{R}^4,\, |\kappa| \le 2} |q^{i_1} \cdots q^{i_n}\, \partial_\kappa \hat{\mathcal{B}}(q)| \;<\; \infty$$

for all n, all tensor indices $i_1, \ldots, i_n$ and all multi-indices κ (with $\kappa = (\kappa^1, \ldots, \kappa^q)$, $|\kappa| := q$). As is verified explicitly in momentum space, the distributions k_m, p_m and s_m are bounded in the Schwartz norms of the test functions involving derivatives of only first order. More precisely,

$$|C(f)| \leq \text{const}\, \|f\|_{4,1} \qquad \text{with} \qquad C = k_m,\, p_m \text{ or } s_m \text{ and } f \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)\,,$$

where $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$ is the Schwartz space, and the Schwartz norms are defined as usual by

$$\|f\|_{p,q} = \max_{|I|\leq p,\, |J|\leq q} \sup_{x\in\mathbb{R}^4} |x^I\, \partial_J f(x)|$$

(for basics on the Schwartz space and distributions see for example [Fr]). As a consequence, we can apply the corresponding operators even to functions with rapid decay which are only C^1. Furthermore, we can form the convolution of such functions with C; this gives continuous functions (which will no longer have rapid decay, however). Since C involves first derivatives, a convolution decreases the order of differentiability of the function by one.

We consider the combination of multiplication and convolution

$$F(p_2) := \int \frac{d^4 p_1}{(2\pi)^4}\; f(p_2 - p_1)\, C(p_1)\, g(p_1)\,, \tag{2.1.29}$$

where we assume that $f \in C^2$ has rapid decay and $g \in C^1$ is bounded together with its first derivatives, $\|g\|_{0,1} < \infty$. For any fixed p_2, the integral in (2.1.29) is well-defined and finite because $f(p_2 - .)\, g(.)$ is C^1 and has rapid decay. The resulting function F is C^1 and bounded together with its first derivatives, more precisely

$$\|F\|_{0,1} \leq \text{const}\, \|f\|_{4,2}\, \|g\|_{0,1}\,. \tag{2.1.30}$$

After these preparations, we can estimate the integrals in (2.1.28) from the right to the left: We choose two test functions $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$ and introduce the functions

$$F_1(p_1) = \int \frac{d^4 q_2}{(2\pi)^4}\; \hat{\mathcal{B}}(p_1 - q_1)\, C_0(q_1)\, g(q_1) \tag{2.1.31}$$

$$F_j(p_j) = \int \frac{d^4 p_{j-1}}{(2\pi)^4}\; \hat{\mathcal{B}}(p_j - p_{j-1})\, C_{j-1}(p_{j-1})\, F_{j-1}(p_{j-1})\,, \quad 1 < j \leq n\,. \tag{2.1.32}$$

The integral (2.1.31) is of the form (2.1.29) and satisfies the above assumptions on the integrand. Using the bound (2.1.30), we can proceed inductively in (2.1.32). Finally, we perform the q_2-integration,

$$M(f, g) = \int \frac{d^4 q_2}{(2\pi)^4}\; f(q_2)\, C_n(q_2)\, F_n(q_2)\,. \tag{2.1.33}$$

We conclude that M is a linear functional on $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$, which is bounded in the Schwartz norm $\|.\|_{4,1}$ of the test functions. □

We remark that the assumptions in this lemma are stronger than what is needed for the operator products in (2.1.25) and (2.1.26) to be well-defined: First of all, the smoothness assumption for $\mathcal{B}$ is unnecessarily strong; for example, it would be sufficient to assume that $\mathcal{B}$ is twice differentiable. Moreover, using the causal structure, the contributions to the above perturbation expansions are well-defined even without the decay assumptions in Lemma 2.1.2. Namely, these perturbation expansions are all causal in the sense that for any given $x, y \in \mathscr{M}$, the distributions $\tilde{s}^\vee(x, y)$ and $\tilde{s}^\wedge(x, y)$ depend on the potential $\mathcal{B}$ only on in the so-called

$$\text{causal diamond} \qquad \big(J_x^\vee \cap J_y^\wedge\big) \cup \big(J_x^\wedge \cap J_y^\vee\big) .$$

Since the causal diamond is a bounded region of space-time, we may modify $\mathcal{B}$ outside this bounded set to arrange the decay assumptions without changing the contributions to the above perturbation expansions.

The reason why we prefer to impose with the stronger assumptions in Lemma 2.1.2 is that they will be needed later on. Indeed, for the operator products appearing in the causal perturbation expansion of the Dirac sea, the decay assumptions in Lemma 2.1.2 will be required. Moreover, the smoothness of $\mathcal{B}$ will be needed for the light-cone expansion.

The summands of the above perturbation expansions (2.1.25) and (2.1.26) arise similarly in quantum field theory and are then depicted by Feynman diagrams (see Fig. 2.1). Using the language of quantum field theory, we also refer to the summands of our perturbation expansions as *Feynman diagrams*. Then the result of the last lemma can be understood from the fact that in the presence of an external field one only encounters tree diagrams, which are all finite.

2.1.5 Computation of Operator Products

We saw in (2.1.17) and (2.1.18) that operator products can be formed if the mass is considered as a variable parameter. We now develop this method more systematically. It is usually most convenient to work with the *symmetric Green's function* defined by

$$s_m = \frac{1}{2}(s_m^\vee + s_m^\wedge) . \tag{2.1.34}$$

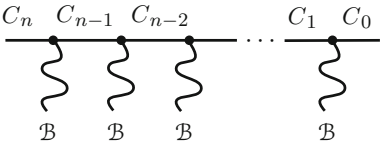

Fig. 2.1 A Feynman tree diagram

Lemma 2.1.3 *The following identities hold:*

$$p_m\, p_{m'} = k_m\, k_{m'} = \delta(m-m')\; p_m \tag{2.1.35}$$

$$p_m\, k_{m'} = k_m\, p_{m'} = \delta(m-m')\; k_m \tag{2.1.36}$$

$$p_m\, s_{m'} = s_{m'}\, p_m = \frac{PP}{m-m'}\; p_m \tag{2.1.37}$$

$$k_m\, s_{m'} = s_{m'}\, k_m = \frac{PP}{m-m'}\; k_m \tag{2.1.38}$$

$$s_m\, s_{m'} = \frac{PP}{m-m'}\; (s_m - s_{m'}) + \pi^2\, \delta(m-m')\; p_m\;, \tag{2.1.39}$$

where PP denotes the principal value defined in analogy to (1.2.27) alternatively by

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{PP}{m}\, \eta(m)\, dm &= \lim_{\nu \searrow 0} \left(\int_{-\infty}^{-\nu} + \int_{\nu}^{\infty} \right) \frac{\eta(m)}{m}\, dm \\ &= \lim_{\nu \searrow 0} \frac{1}{2} \sum_{\pm} \int_{-\infty}^{\infty} \frac{\eta(m)}{m \pm i\nu}\, dm\;. \end{aligned} \tag{2.1.40}$$

Proof Calculating pointwise in momentum space, we obtain

$$\begin{aligned} p_m(q)\, p_{m'}(q) &= (\not{q} + m)\, \delta(q^2 - m^2)\; (\not{q} + m')\, \delta(q^2 - m'^2) \\ &= \delta(m^2 - m'^2)\, \delta(q^2 - m^2)\, \left(q^2 + (m+m')\not{q} + mm' \right) \\ &= \frac{1}{2m}\, \delta(m-m')\, \delta(q^2 - m^2)\, \left(m^2 + (m+m')\not{q} + mm' \right) \\ &= \frac{1}{2m}\, \delta(m-m')\, \delta(q^2 - m^2)\; 2m\, (m + \not{q}) = \delta(m-m')\; p_m(q)\;. \end{aligned}$$

This gives the first part of (2.1.35). The second part of this formula as well as formula (2.1.36) are obtained analogously. The formulas (2.1.37) and (2.1.38) are obtained as follows:

$$\begin{aligned} 2\, p_m(q)\, s_{m'}(q) &= \lim_{\nu \searrow 0}\; \delta(q^2 - m^2)(\not{q} + m) \\ &\quad \times \left(\frac{\not{q} + m'}{q^2 - m'^2 - i\nu q^0} + \frac{\not{q} + m'}{q^2 - m'^2 + i\nu q^0} \right) \\ &= \lim_{\nu \searrow 0}\; \delta(q^2 - m^2)\, \left(q^2 + (m+m')\not{q} + mm' \right) \\ &\quad \times \left(\frac{1}{q^2 - m'^2 - i\nu q^0} + \frac{1}{q^2 - m'^2 + i\nu q^0} \right) \\ &= \lim_{\nu \searrow 0}\; \delta(q^2 - m^2)\, \left(m^2 + (m+m')\not{q} + mm' \right) \\ &\quad \times \left(\frac{1}{m^2 - m'^2 - i\nu q^0} + \frac{1}{m^2 - m'^2 + i\nu q^0} \right) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\nu \searrow 0} \delta(q^2 - m^2)(\not{q} + m) \\
&\quad \times \left(\frac{(m + m')}{(m + m')(m - m') - i\nu q^0} + \frac{(m + m')}{(m + m')(m - m') + i\nu q^0} \right) \\
&= 2 \, \frac{\mathrm{PP}}{m - m'} \, p_m(q) \, .
\end{aligned}
$$

The derivation of (2.1.39) is a bit more involved. Combining (2.1.14) and (2.1.34), we obtain

$$
s_m = s_m^\vee - i\pi k_m = s_m^\wedge + i\pi k_m \, . \tag{2.1.41}
$$

Thus we can express the product $s_m(q)\, s_{m'}(q)$ in two ways, namely as

$$
\begin{aligned}
s_m(q)\, s_{m'}(q) &= (s_m^\vee(q) - i\pi k_m(q))(s_{m'}^\vee(q) - i\pi k_{m'}(q)) \\
&= s_m^\vee(q)\, s_{m'}^\vee(q) - \pi^2 \delta(m - m')\, p_m(q) \\
&\quad - i\pi \lim_{\nu \searrow 0} \left(k_{m'}(q) \frac{1}{m' - m - i\nu q^0} + k_m(q) \frac{1}{m - m' - i\nu q^0} \right) ,
\end{aligned}
$$

or alternatively as

$$
\begin{aligned}
&s_m(q)\, s_{m'}(q) = (s_m^\wedge(q) + i\pi k_m(q))(s_{m'}^\wedge(q) + i\pi k_{m'}(q)) \\
&= s_m^\wedge(q)\, s_{m'}^\wedge(q) - \pi^2 \delta(m - m')\, p_m(q) \\
&\quad + i\pi \lim_{\nu \searrow 0} \left(k_{m'}(q) \frac{1}{m' - m + i\nu q^0} + k_m(q) \frac{1}{m - m' + i\nu q^0} \right) .
\end{aligned}
$$

Adding these two formulas yields

$$
\begin{aligned}
&2\, s_m(q)\, s_{m'}(q) - (s_m^\vee(q)\, s_{m'}^\vee(q) + s_m^\wedge(q)\, s_{m'}^\wedge(q)) + 2\pi^2 \delta(m - m')\, p_m(q) \\
&= i\pi \lim_{\nu \searrow 0} k_{m'}(q) \left(\frac{1}{m' - m + i\nu q^0} - \frac{1}{m' - m - i\nu q^0} \right) \\
&\quad + i\pi \lim_{\nu \searrow 0} k_m(q) \left(\frac{1}{m - m' + i\nu q^0} - \frac{1}{m - m' - i\nu q^0} \right) \\
&\overset{(*)}{=} i\pi k_{m'}(q) \epsilon(-q^0)\, 2\pi i \, \delta(m' - m) + i\pi k_m(q) \epsilon(-q^0)\, 2\pi i \, \delta(m - m') \\
&= -2\pi^2 \delta(m' - m)(-p_{m'}(q)) - 2\pi^2 \delta(m - m')(-p_m(q)) \, ,
\end{aligned}
$$

where in $(*)$ we applied (1.2.33), and in the last line we used the definitions of p_m and k_m. We thus obtain

$$
s_m\, s_{m'} = \frac{1}{2} \left(s_m^\vee\, s_{m'}^\vee + s_m^\wedge\, s_{m'}^\wedge \right) + \pi^2 \, \delta(m - m')\, p_m \, . \tag{2.1.42}
$$

It remains to derive the relations

$$
s_m^\vee\, s_{m'}^\vee = \frac{\mathrm{PP}}{m - m'} (s_m^\vee - s_{m'}^\vee) \qquad \text{and} \qquad s_m^\wedge\, s_{m'}^\wedge = \frac{\mathrm{PP}}{m - m'} (s_m^\wedge - s_{m'}^\wedge) \, , \tag{2.1.43}
$$

which can be regarded as "resolvent identities" for the causal Green's functions. It suffices to consider the case of the advanced Green's function. Clearly, the operators on the right side of (2.1.43) satisfy the support condition $\text{supp}((s^\vee_m - s^\vee_{m'})(x,.)) \subset J^\vee_x$, and from

$$s^\vee_m\, s^\vee_{m'}(x,y) = \int d^4z\, s^\vee_m(x,z)\, s^\vee_{m'}(z,y)$$

we see that the operators on the left side of (2.1.43) satisfy this support condition as well. Moreover, the calculations

$$(i\partial\!\!\!/_x - m)\, s^\vee_m\, s^\vee_{m'}(x,y) = s^\vee_{m'}(x,y)$$

and

$$\begin{aligned}(i\partial\!\!\!/_x - m)&\frac{\text{PP}}{m-m'}(s^\vee_m - s^\vee_{m'})(x,y) \\ &= \frac{\text{PP}}{m-m'}\Big(\delta(x-y) - (m'-m)s^\vee_{m'}(x,y) - \delta(x-y)\Big) = s^\vee_{m'}(x,y)\end{aligned}$$

show that both sides of (2.1.43) satisfy the same inhomogeneous Dirac equation. Hence their difference is a distributional solution of the homogeneous Dirac equation which vanishes outside $J^\vee_x$. The uniqueness of the solution of the Cauchy problem for hyperbolic PDEs yields that this difference vanishes identically. This proves (2.1.43) and thus concludes the proof of (2.1.39). □

In the above operator products we get contributions of two different forms: those involving a factor $\delta(m-m')$ and those involving the principal value of $1/(m-m')$. In order to simplify the structure of the multiplication rules, it is useful to get rid of the principal values by restricting attention to combinations in which all principal values drop out in telescopic sums. To this end, we introduce the series of operator products

$$b^<_m = \sum_{n=0}^\infty \big(-s_m \mathcal{B}\big)^n, \qquad b_m = \sum_{n=0}^\infty \big(-\mathcal{B} s_m\big)^n \mathcal{B}, \qquad b^>_m = \sum_{n=0}^\infty \big(-\mathcal{B} s_m\big)^n. \tag{2.1.44}$$

Corollary 2.1.4 *Let $C \in \{p_m, k_m\}$ and $C' \in \{p_{m'}, k_{m'}\}$ as well as $b^<_m$, $b^>_m$ as in (2.1.44). Then the following calculation rule holds:*

$$C\, b^>_m b^<_{m'}\, C' = CC' + \delta(m-m')\,\pi^2\, C\, b_m\, p_m\, b_m\, C'. \tag{2.1.45}$$

Proof Using the calculation rules of the previous lemma, we obtain

$$\begin{aligned}C\left(\sum_{l=0}^1 (\mathcal{B}s_m)^l (s_{m'}\mathcal{B})^{n-l}\right) C' &= C s_{m'}\,\mathcal{B}\,C' + C\,\mathcal{B}\,s_m\,C' \\ &= \frac{\text{PP}}{m-m'}\big(C\,\mathcal{B}\,C' - C\,\mathcal{B}\,C'\big) = 0\,.\end{aligned}$$

The same method also applies to higher order. We again get a telescopic sum, but the last summand in (2.1.39) gives additional contributions. More precisely, for any $n \geq 2$,

$$
\begin{aligned}
& C\left(\sum_{l=0}^{n}(\mathcal{B}s_m)^l(s_{m'}\mathcal{B})^{n-l}\right)C' \\
&= C(\mathcal{B}s_m)^nC' + C(s_{m'}\mathcal{B})^nC' + C\left[\sum_{l=1}^{n-1}(\mathcal{B}s_m)^l(s_{m'}\mathcal{B})^{n-l}\right]C' \\
&= C\,\frac{\mathrm{PP}}{m-m'}\left[-(\mathcal{B}s_m)^{n-1}\mathcal{B} + \mathcal{B}(s_{m'}\mathcal{B})^{n-1}\right]C' \\
&\quad + C\sum_{l=1}^{n-1}(\mathcal{B}s_m)^{l-1}\mathcal{B}\left(\frac{\mathrm{PP}}{m-m'}(s_m - s_{m'}) + \delta(m-m')\pi^2 p_m\right)\mathcal{B}(s_{m'}\mathcal{B})^{n-l-1}C' \\
&= \frac{\mathrm{PP}}{m-m'}\,C\left[-(\mathcal{B}s_m)^{n-1}\mathcal{B} + \mathcal{B}(s_{m'}\mathcal{B})^{n-1}\right]C' \\
&\quad + \frac{\mathrm{PP}}{m-m'}\,C\Big(\sum_{l=1}^{n-1}(\mathcal{B}s_m)^l(\mathcal{B}s_{m'})^{n-l-1}\mathcal{B} - \sum_{l=0}^{n-2}(\mathcal{B}s_m)^l(\mathcal{B}s_{m'})^{n-l-1}\mathcal{B}\Big)C' \\
&\quad + \delta(m-m')\,\pi^2\,C\sum_{l=1}^{n-1}(\mathcal{B}s_m)^{l-1}\mathcal{B}\,p_m\,\mathcal{B}(s_{m'}\mathcal{B})^{n-l-1}C' \\
&= \delta(m-m')\,\pi^2\,C\sum_{l=1}^{n-1}(\mathcal{B}s_m)^{l-1}\mathcal{B}\,p_m\,\mathcal{B}(s_{m'}\mathcal{B})^{n-l-1}C' \\
&= \delta(m-m')\,\pi^2\,C\sum_{l=0}^{n-2}(\mathcal{B}s_m)^{l}\mathcal{B}\,p_m\,\mathcal{B}(s_{m'}\mathcal{B})^{n-l-2}C'.
\end{aligned}
$$

Thus, performing an index shift, we obtain

$$
\begin{aligned}
C\,b_m^{>}b_{m'}^{<}\,C' &= C\sum_{n=0}^{\infty}(-\mathcal{B}s_m)^n\sum_{n'=0}^{\infty}(-s_{m'}\mathcal{B})^{n'}C' \\
&= C\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-\mathcal{B}s_m)^l(-s_{m'}\mathcal{B})^{n-l}C' \\
&= CC' + \delta(m-m')\pi^2\sum_{n=2}^{\infty}(-1)^nC\left(\sum_{l=0}^{n-2}(\mathcal{B}s_m)^l\mathcal{B}\,p_m\,\mathcal{B}(s_{m'}\mathcal{B})^{n-l-2}\right)C' \\
&= CC' + \delta(m-m')\pi^2\sum_{n=0}^{\infty}(-1)^nC\left(\sum_{l=0}^{n}(\mathcal{B}s_m)^l\mathcal{B}\,p_m\,\mathcal{B}(s_{m'}\mathcal{B})^{n-l}\right)C' \\
&= CC' + \delta(m-m')\pi^2C\,b_m\,p_m\,b_m\,C'.
\end{aligned}
$$

This concludes the proof. □

In what follows, we rewrite all operator products in terms of $\mathcal{B}$ and p_m, k_m as well as the above combinations $b_m^<$, b_m and $b_m^>$. In order to explain how this can be done, we rewrite the perturbation expansion for $\tilde{k}_m$ in this form.

Proposition 2.1.5 *The perturbation expansion for $\tilde{k}_m$ as given by (2.1.26) can be written as*

$$\tilde{k}_m = \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta} b_m^< k_m (b_m k_m)^{2\beta} b_m^> , \tag{2.1.46}$$

where the factors $b_m^<$, b_m and $b_m^>$ are again the operator products in (2.1.44),

Proof An explicit calculation shows that

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, b_m^< = 0 .$$

As all operator products in (2.1.46) have a factor $b_m^<$ at the left, the series in (2.1.46) is a solution of the Dirac equation.

From (2.1.14) and (2.1.34), we have

$$s_m^\vee = s_m + i\pi k_m , \qquad s_m^\wedge = s_m - i\pi k_m . \tag{2.1.47}$$

We substitute the series (2.1.25) into (2.1.26), insert (2.1.47) and expand. A reordering of the resulting sum gives the claim. The details of the reordering process can be found in [F3]. □

2.1.6 The Causal Perturbation Expansion

We follow the constructions in [FT2]. Recall that, in the presence of an external potential $\mathcal{B}$, the perturbation expansion of the advanced and retarded Green's functions is unique by causality (2.1.25). Moreover, Proposition 2.1.5 gave us a unique perturbation expansion of the causal fundamental solution (2.1.46).

In the following constructions, we need to multiply the operator products in (2.1.46). These products have a mathematical meaning as distributions in the involved mass parameters. Namely, according to Lemma 2.1.3 and Corollary 2.1.4,

$$p_m\, p_{m'} = k_m\, k_{m'} = \delta(m - m')\, p_m \tag{2.1.48}$$

$$p_m\, k_{m'} = k_m\, p_{m'} = \delta(m - m')\, k_m \tag{2.1.49}$$

$$k_m\, b_m^>\, b_{m'}^<\, k_{m'} = \delta(m - m') \Big(p_m + \pi^2\, k_m\, b_m\, p_m\, b_m\, k_m \Big) . \tag{2.1.50}$$

Since these formulas all involve a common prefactor $\delta(m - m')$, we can introduce a convenient notation by leaving out this factor and omitting the mass indices. For clarity, we denote this short notation with a dot, i.e. symbolically

$$A \cdot B = C \qquad \text{stands for} \qquad A_m \, B_{m'} = \delta(m - m') \, C_m \,. \tag{2.1.51}$$

With this short notation, the above multiplication rules can be written in the compact form

$$p \cdot p = k \cdot k = p \,, \qquad p \cdot k = k \cdot p = k \,, \qquad k\, b^> \cdot b^< k = p + \pi^2\, kbpbk \,. \tag{2.1.52}$$

Writing (2.1.46) as

$$\tilde{k} = \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\, b^< \, k \, (bk)^{2\beta}\, b^> \,, \tag{2.1.53}$$

powers of the operator $\tilde{k}$ with the product (2.1.51) are well-defined using the multiplication rules (2.1.52). This makes it possible to develop a spectral calculus for $\tilde{k}$. In particular, in [FG1] the operator P^{sea} is constructed as the projection operator on the negative spectral subspace of $\tilde{k}$. We now give an equivalent construction using contour integrals, which gives a more systematic procedure for computing all the contributions to the expansion (for basics on the resolvent and contour integrals see Exercise 2.6).

We introduce the resolvent by

$$\tilde{R}_\lambda = \big(\tilde{k} - \lambda\big)^{-1} \,. \tag{2.1.54}$$

Writing $\tilde{k}$ as

$$\tilde{k} = k + \Delta k \,, \tag{2.1.55}$$

(where k is the corresponding distribution in the vacuum), the resolvent $\tilde{R}_\lambda$ can be written as a Neumann series,

$$\tilde{R}_\lambda = (k - \lambda + \Delta k)^{-1} = (1 + R_\lambda \cdot \Delta k)^{-1} \cdot R_\lambda = \sum_{n=0}^{\infty} (-R_\lambda \cdot \Delta k)^n \cdot R_\lambda \,. \tag{2.1.56}$$

The multiplication rules (2.1.52) imply that p is idempotent and thus has the eigenvalues 1 and 0. Since the operator k commutes with p and its square equals p, it has the eigenvalues ± 1 and 0. A short computation shows that the corresponding spectral projection operators are $(p \pm k)/2$ and $\mathbb{1} - p$, respectively. Hence we can write the unperturbed resolvent $R_\lambda := (k - \lambda)^{-1}$ as

$$R_\lambda = \frac{p+k}{2} \left(\frac{1}{1-\lambda} \right) + \frac{p-k}{2} \left(\frac{1}{-1-\lambda} \right) - \frac{\mathbb{1} - p}{\lambda} \,. \tag{2.1.57}$$

Using this formula in (2.1.56), to every order in perturbation theory we obtain a meromorphic function in λ having poles only at $\lambda = 0$ and $\lambda = \pm 1$.

We now use contour integral methods to develop a spectral calculus. To this end, we choose a contour Γ_- which encloses the point -1 in counter-clockwise direction and does not enclose the points 1 and 0. Similarly, Γ_+ is a contour which encloses the point $+1$ in counter-clockwise direction and does not enclose the points -1 and 0. Moreover, we let f be a holomorphic function defined on an open neighborhood of the points ± 1. We define $f(\tilde{k})$ as the contour integral

$$f(\tilde{k}) := -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} f(\lambda)\, \tilde{R}_\lambda \, d\lambda \,. \tag{2.1.58}$$

Using (2.1.56) together with the fact that to every order in perturbation theory, the integrand is a meromorphic function in λ having poles only at $\lambda = 0$ and $\lambda = \pm 1$, one sees that the operator $f(\tilde{k})$ is well-defined to every order in perturbation theory and is independent of the choice of the contours Γ_+ and Γ_-.

Theorem 2.1.6 (functional calculus) *For any functions f, g which are holomorphic in discs around ± 1 which contain the contours $\Gamma_\pm$,*

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, f(\tilde{k}) = 0 \tag{2.1.59}$$

$$f(\tilde{k}) \cdot g(\tilde{k}) = (fg)(\tilde{k}) \,. \tag{2.1.60}$$

Proof Since the operator $\tilde{k}$ maps to solutions of the Dirac equation, we know that

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{R}_\lambda = (i\partial\!\!\!/ + \mathcal{B} - m)\, \big(-\lambda^{-1}\big).$$

Taking the contour integral (2.1.58) gives (2.1.59).

The starting point for proving (2.1.60) is the resolvent identity

$$\tilde{R}_\lambda \cdot \tilde{R}_{\lambda'} = \frac{1}{\lambda - \lambda'} \Big(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \Big) \,. \tag{2.1.61}$$

We set $\Gamma = \Gamma_+ \cup \Gamma_-$ and denote the corresponding contour for λ' by Γ'. Since the integral (2.1.58) is independent of the precise choice of the contour, we may choose

$$\Gamma = \partial B_\delta(1) \cup \partial B_\delta(-1) \qquad \text{and} \qquad \Gamma' = \partial B_{2\delta}(1) \cup \partial B_{2\delta}(-1)$$

for sufficiently small $\delta < 1/2$. Then Γ does not enclose any point of Γ', implying that

$$\oint_\Gamma \frac{f(\lambda)}{\lambda - \lambda'}\, d\lambda = 0 \qquad \text{for all } \lambda' \in \Gamma' \,. \tag{2.1.62}$$

On the other hand, Γ' encloses every point of Γ, so that

$$\oint_{\Gamma'} f(\lambda)\, g(\lambda') \, \frac{\tilde{R}_\lambda}{\lambda - \lambda'}\, d\lambda' = -2\pi i\, f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda \qquad \text{for all } \lambda \in \Gamma \,. \tag{2.1.63}$$

Combining (2.1.61) with (2.1.62) and (2.1.63), we obtain

$$f(\tilde{k}) \cdot g(\tilde{k}) = -\frac{1}{4\pi^2} \oint_\Gamma f(\lambda)\, d\lambda \oint_{\Gamma'} g(\lambda')\, d\lambda' \, \frac{1}{\lambda - \lambda'} \left(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \right)$$
$$= -\frac{1}{2\pi i} \oint_\Gamma f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda \, d\lambda = (fg)(\tilde{k}) \,.$$

This concludes the proof. □

The fermionic projector P^{sea} is obtained by choosing a specific function f, as we now explain. First, the desired splitting of the solution space of the Dirac equation into two subspaces (see Sect. 2.1.2) can now be obtained using the sign of the spectrum of $\tilde{k}$. More precisely, we choose P^{sea} such that its image coincides with the negative spectral subspace of $\tilde{k}$. To this end, we choose a function f which vanishes identically in a neighborhood of $+1$. In a neighborhood of -1, on the other hand, the form of f is determined by the spatial normalization condition (see (2.1.21)). Namely, the correct definition is

$$P^{\text{sea}} = -\frac{1}{2\pi i} \oint_{\Gamma_-} (-\lambda)\, \tilde{R}_\lambda \, d\lambda \,, \tag{2.1.64}$$

as becomes clear in the next proposition.

Proposition 2.1.7 *The expansion P^{sea} has the properties*

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, P^{\text{sea}} = 0 \tag{2.1.65}$$

$$2\pi \int_{\mathbb{R}^3} P^{\text{sea}}\big(x, (t, \vec{y})\big)\, \gamma^0\, P^{\text{sea}}\big((t, \vec{y}), z\big)\, d^3y = -P^{\text{sea}}(x, z) \,. \tag{2.1.66}$$

Moreover, P^{sea} is symmetric

$$(P^{sea})^* = P^{sea} \,, \tag{2.1.67}$$

where the star denotes the adjoint with respect to the space-time inner product (1.5.2).

We note for clarity that for the kernel of the fermionic projector, the symmetry property (2.1.67) means that

$$\big(P^{\text{sea}}(x, y)\big)^* = P^{\text{sea}}(y, x) \,, \tag{2.1.68}$$

where the star denotes the adjoint with respect to the spin scalar product (1.2.18).

In order to simplify the notation in the proof, we abbreviate the spatial integral in (2.1.66) by $|_t$, i.e.

$$(A\,|_t\, B)(x, z) := 2\pi \int_{\mathbb{R}^3} A\big(x, (t, \vec{y})\big)\, \gamma^0\, B\big((t, \vec{y}), z\big)\, d^3y \,.$$

We begin with a preparatory lemma.

Lemma 2.1.8 *For any $t_0 \in \mathbb{R}$, the distribution (2.1.26) has the property*

$$\tilde{k}_m \,|_{t_0}\, \tilde{k}_m = \tilde{k}_m \,.$$

Proof Clearly, it suffices to prove the relation when evaluated by a test function f. Then $\tilde{\phi} := \tilde{k}_m(f)$ is a smooth solution of the Dirac equation with spatially compact support. Therefore, it suffices to show that for any such solution,

$$\tilde{\phi}(t, \vec{x}) = 2\pi \int_{\mathbb{R}^3} \tilde{k}_m(t, \vec{x}; t_0, \vec{y})\, \gamma^0\, \tilde{\phi}_0(\vec{y})\, d^3y \,.$$

Since $\tilde{\phi}$ and $\tilde{k}_m$ satisfy the Dirac equation, it suffices to prove this equation in the case $t > t_0$. In this case, the equation simplifies in view of (2.1.26) to

$$\tilde{\phi}(x) = i \int_{\mathbb{R}^3} \tilde{s}_m^\wedge(x, y)\, \gamma^0\, \tilde{\phi}_0(y)\big|_{y=(t_0,\vec{y})}\, d^3y \,,$$

where we set $x = (t, \vec{x})$. This identity is derived as follows: We choose a non-negative function $\eta \in C^\infty(\mathbb{R})$ with $\eta|_{[t_0,t]} \equiv 1$ and $\eta_{(-\infty,t_0-1)} \equiv 0$. We also consider $\eta = \eta(x^0)$ as a function of the time variable in space-time. Then

$$\tilde{\phi}(x) = (\eta\tilde{\phi})(x) = \tilde{s}_m^\wedge\big((i\partial\!\!\!/ + \mathcal{B} - m)(\eta\tilde{\phi})\big) = \tilde{s}_m^\wedge\big(i\gamma^0\, \dot{\eta}\, \tilde{\phi})\big) \,,$$

where we used the defining equation of the Green's function $\tilde{s}_m^\wedge(i\partial\!\!\!/_x + \mathcal{B} - m) = \mathbb{1}$ together with the fact that $\tilde{\phi}$ is a solution of the Dirac equation. To conclude the proof, we choose a sequence η_l such that the sequence of derivatives $\dot{\eta}_l$ converges as $l \to \infty$ in the distributional sense to the δ-distribution δ_{t_0} supported at t_0. Then

$$\begin{aligned} \tilde{s}_m^\wedge\big(i\gamma^0\, \dot{\eta}\, \tilde{\phi})\big)(x) &= \int \Big(\tilde{s}_m^\wedge(x, y)\big(i\gamma^0\, \dot{\eta}(y^0)\, \tilde{\phi}(y)\big)\Big)\, d^4y \\ &\to \int_{\mathbb{R}^3} \Big(\tilde{s}_m^\wedge(x, y)\big(i\gamma^0\tilde{\phi}\big)\Big)\,\big|_{y=(t_0,\vec{y})}\, d^3y \,, \end{aligned}$$

giving the result. □

An alternative, more computational proof of this lemma is sketched in Exercise 2.7.

Proof of Proposition 2.1.7 The Dirac equation (2.1.65) follows immediately from the identity (2.1.59). In order to prove (2.1.66), we integrate the relations

$$\tilde{R}_\lambda \cdot (\tilde{k} - \lambda) = \mathbb{1} = (\tilde{k} - \lambda) \cdot \tilde{R}_\lambda \,,$$

to obtain

$$\oint_{\Gamma_-} \tilde{R}_\lambda \cdot \tilde{k}\, d\lambda = \oint_{\Gamma_-} \tilde{R}_\lambda\, \lambda\, d\lambda = \oint_{\Gamma_-} \tilde{k}\, \tilde{R}_\lambda\, d\lambda \,.$$

As a consequence,

$$P^{\text{sea}} \,|_t\, P^{\text{sea}} = -\frac{1}{4\pi^2} \oint_{\Gamma_-} d\lambda \oint_{\Gamma'_-} d\lambda' \, \tilde{R}_\lambda \cdot \tilde{k} \,|_t\, \tilde{k} \cdot \tilde{R}_{\lambda'} \,,$$

and applying Lemma 2.1.8 for $t_0 = t$ gives

$$P^{\text{sea}} \,|_t\, P^{\text{sea}} = -\frac{1}{4\pi^2} \oint_{\Gamma_-} d\lambda \oint_{\Gamma'_-} d\lambda' \, \tilde{R}_\lambda \cdot \tilde{k} \cdot \tilde{R}_{\lambda'} = -\frac{1}{4\pi^2} \oint_{\Gamma_-} \lambda \, d\lambda \oint_{\Gamma'_-} d\lambda' \, \tilde{R}_\lambda \cdot \tilde{R}_{\lambda'} \,.$$

Now we can again apply (2.1.61) and (2.1.62) (which remains valid if the integrand involves an additional factor λ) as well as (2.1.63). We thus obtain

$$P^{\text{sea}} \,|_t\, P^{\text{sea}} = -\frac{1}{2\pi i} \oint_{\Gamma_-} \lambda \, \tilde{R}_\lambda \, d\lambda = -P^{\text{sea}} \,.$$

It remains to prove the symmetry property (2.1.67). The operators p_m, k_m and s_m are obviously symmetric (with respect to the inner product (1.5.2)). According to (2.1.46), the operator $\tilde{k}_m$ is also symmetric. Hence the resolvent $\tilde{R}_\lambda$ defined by (2.1.54) has the property

$$\tilde{R}_\lambda^* = \tilde{R}_{\bar{\lambda}} \,.$$

This property implies that if we consider the Laurent expansion of $-\lambda \, \tilde{R}_\lambda$ around $\lambda = -1$,

$$-\lambda \, \tilde{R}_\lambda = \frac{A_{-1}}{\lambda + 1} + A_0 + A_1 \, (1 + \lambda) + \cdots \,,$$

then the operators $A_{-1}, A_0, \ldots$ are all symmetric with respect to (1.5.2). Since the contour integral (2.1.64) simply gives the residue $-A_{-1}$, we obtain (2.1.67). This concludes the proof. □

In order to illustrate the above constructions, we now compute the first orders of the perturbation expansion (2.1.64). We first recall that in the computation rules (2.1.48)–(2.1.50) no principal values occur. Using these rules in (2.1.56) and (2.1.64), one sees that also P^{sea} involves no principal values. With this in mind, we may omit all principal values in the computation, even if we consider other operator products. In particular, we may write the computation rules of Lemma 2.1.3 as

$$p \cdot s = s \cdot p = k \cdot s = s \cdot k = 0 \qquad \text{and} \qquad s \cdot s = \pi^2 \, p \,. \tag{2.1.69}$$

Combining (2.1.52) and (2.1.69) with (2.1.57), we obtain

$$\begin{aligned} R_\lambda \cdot s &= s \cdot R_\lambda = -\frac{1}{\lambda} \, s \\ R_\lambda \cdot k &= k \cdot R_\lambda = \frac{p+k}{2} \left(\frac{1}{1-\lambda} \right) - \frac{p-k}{2} \left(\frac{1}{-1-\lambda} \right) \end{aligned}$$

According to (2.1.53) and (2.1.55),

$$\Delta k = -s\mathcal{B}k - k\mathcal{B}s + k\mathcal{B}s\mathcal{B}s + s\mathcal{B}k\mathcal{B}s + s\mathcal{B}s\mathcal{B}k - \pi^2 k\mathcal{B}k\mathcal{B}k + \mathcal{O}(\mathcal{B}^3)\,.$$

Hence, using (2.1.56),

$$\begin{aligned}\tilde{R}_\lambda &= \sum_{n=0}^{\infty} (-R_\lambda \cdot \Delta k)^n \cdot R_\lambda = R_\lambda - R_\lambda \cdot \Delta k \cdot R_\lambda + R_\lambda \cdot \Delta k \cdot R_\lambda \cdot \Delta k \cdot R_\lambda + \mathcal{O}(\mathcal{B}^3) \\ &= R_\lambda - R_\lambda \cdot \Big(-s\mathcal{B}k - k\mathcal{B}s + k\mathcal{B}s\mathcal{B}s + s\mathcal{B}k\mathcal{B}s + s\mathcal{B}s\mathcal{B}k - \pi^2 k\mathcal{B}k\mathcal{B}k\Big) \cdot R_\lambda \\ &\quad + R_\lambda \cdot (-s\mathcal{B}k - k\mathcal{B}s) \cdot R_\lambda \cdot (-s\mathcal{B}k - k\mathcal{B}s) \cdot R_\lambda + \mathcal{O}(\mathcal{B}^3)\,.\end{aligned}$$

Using (2.1.57) and computing the contour integrals, one obtains to first order

$$\begin{aligned}P^{\text{sea}} &= -\lambda \frac{p-k}{2} - s\,\mathcal{B}\,\frac{p-k}{2} - \frac{p-k}{2}\,\mathcal{B}\,s\Big|_{\lambda=-1} + \mathcal{O}(\mathcal{B}^2) \\ &= \frac{p-k}{2} - s\,\mathcal{B}\,\frac{p-k}{2} - \frac{p-k}{2}\,\mathcal{B}\,s + \mathcal{O}(\mathcal{B}^2)\,. \qquad (2.1.70)\end{aligned}$$

To second and higher orders, the resolvent $\tilde{R}_\lambda$ involves higher poles at $\lambda = -1$. This gives rise to derivatives of the factor $(-\lambda)$ in (2.1.64), having an influence of the combinatorics of the perturbation expansion (see Exercise 2.8). The reader interested in more details is referred to [FT2, Appendix A]. A few structural results of the causal perturbation expansion are treated in Exercises 2.9–2.11.

2.1.7 *Introducing Particles and Anti-Particles*

We shall now make the method of occupying particle and anti-particle states (2.1.4) precise in the presence of an external potential. To this end, it is useful to construct out of the kernel of the fermionic projector a projection operator on a Hilbert space, as we now explain. On the smooth solutions of the Dirac equation (2.1.5) with spatially compact support one can introduce the scalar product (1.2.2). Due to current conservation, this scalar product is again independent of the choice of t. Taking the completion, the solution space of the Dirac equation becomes a Hilbert space, which we denote by $(\mathcal{H}_m, (.|.)_m)$. We now introduce on the Dirac wave functions at time t the operator

$$\begin{aligned}&\Pi^{\text{sea}} \;:\; C_0^\infty(\mathcal{N}_t, S\mathcal{M}) \to C^\infty(\mathcal{M}, S\mathcal{M})\,, \\ &(\Pi^{\text{sea}}\psi)(x) = -2\pi \int_{\mathbb{R}^3} P^{\text{sea}}\big(x, (t, \vec{y})\big)\, \gamma^0\, \psi(\vec{y})\, d^3y\,, \qquad (2.1.71)\end{aligned}$$

where $\mathscr{N}_t := \{t\} \times \mathbb{R}^3 \subset \mathscr{M}$ denotes the spatial hyperplane at time t. According to (2.1.65), this operator maps to the solutions of the Dirac equation. Moreover, the spatial normalization property (2.1.66) implies that Π^{sea} can be extended by continuity to a projection operator on $\mathcal{H}_m$, i.e.

$$\Pi^{\text{sea}} : \mathcal{H}_m \to \mathcal{H}_m \qquad \text{with} \qquad (\Pi^{\text{sea}})^* = \Pi_{\text{sea}} = \Pi_{\text{sea}}^2$$

(where the star now denotes the adjoint with respect to the scalar product (1.2.2); note that the last equation follows from the symmetry of the kernel (2.1.68)).

Now we can form another operator by adding and subtracting projection operators. More precisely, the operator

$$\Pi := \Pi^{\text{sea}} + \Pi_{\text{span}(\psi_1,\ldots,\psi_{n_{\text{p}}})} - \Pi_{\text{span}(\phi_1,\ldots,\phi_{n_{\text{a}}})}$$

(where $\Pi_U : \mathcal{H}_m \to \mathcal{H}_m$ denotes the orthogonal projection to a subspace $U \subset \mathcal{H}_m$) is again a projection operator, provided that the functions ϕ_l are vectors in $\mathcal{H}_m$ which lie in the image of Π^{sea}, whereas the vectors $\psi_k \in \mathcal{H}_m$ are in the orthogonal complement of the image of Π^{sea}. In order to comply with the usual normalization of wave functions in quantum mechanics, we orthonormalize these vectors as follows,

$$(\psi_k | \psi_{k'})_m = 2\pi\, \delta_{k,k'} \qquad \text{and} \qquad (\phi_l | \phi_{l'})_m = 2\pi\, \delta_{l,l'} \tag{2.1.72}$$

(we included the factor 2π in order to account for the factor 2π in (1.2.2)). Then we can write Π more explicitly as

$$\Pi\psi := \Pi^{\text{sea}}\psi + \frac{1}{2\pi} \sum_{k=1}^{n_{\text{p}}} \psi_k\, (\psi_k | \psi)_m - \frac{1}{2\pi} \sum_{l=1}^{n_{\text{a}}} \phi_l\, (\phi_l | \psi)_m \,.$$

This new projection operator can again be written in the form (2.1.71) with the distribution

$$P(x,y) = P_m^{\text{vac}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_{\text{p}}} \psi_k(x) \overline{\psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_{\text{a}}} \phi_l(x) \overline{\phi_l(y)} \,.$$

This relation gives a mathematical justification for (2.1.4) in the presence of an external potential. Note that the wave functions ψ_k and ϕ_l must be solutions of the Dirac equation (2.1.5). Moreover, the ϕ_l must be in the image of Π^{sea}, whereas the ψ_k must be in the orthogonal complement of the image of Π^{sea}. Finally, the normalization conditions (2.1.72) can be written as

$$\int_{\mathbb{R}^3} (\overline{\psi_k} \gamma^0 \psi_{k'})(t, \vec{x})\, d^3x = \delta_{k,k'} \,, \qquad \int_{\mathbb{R}^3} (\overline{\phi_l} \gamma^0 \phi_{l'})(t, \vec{x})\, d^3x = \delta_{l,l'} \,.$$

2.2 The Light-Cone Expansion

The light-cone expansion is a powerful tool for analyzing the fermionic projector in position space. We now outline the constructions and results as first given in [F5] and [F6]. Before beginning, we point out that the light-cone expansion is closely tied to the causal perturbation expansion. Namely, we will shall see that the "causality" of the perturbation expansion (as built in via (2.1.26) into the resolvent (2.1.54)) will become apparent in the light-cone expansion of $P(x, y)$ in the fact that all appearing line integrals will be bounded integrals along the line segment $\overline{xy}$. This specific feature of the light-cone expansion is of central importance for the analysis of the continuum limit.

2.2.1 Basic Definition

We first give the basic definition of the light-cone expansion and explain it afterwards.

Definition 2.2.1 A distribution $A(x, y)$ on $M \times M$ is of the order $\mathcal{O}((y-x)^{2p})$ for $p \in \mathbb{Z}$ if the product

$$(y-x)^{-2p}\, A(x, y)$$

is a regular distribution (i.e. a locally integrable function). An expansion of the form

$$A(x, y) = \sum_{j=g}^{\infty} A^{[j]}(x, y) \tag{2.2.1}$$

with $g \in \mathbb{Z}$ is called **light-cone expansion** if the $A^{[j]}(x, y)$ are distributions of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p \geq g$,

$$A(x, y) - \sum_{j=g}^{p} A^{[j]}(x, y) \qquad \text{is of the order } \mathcal{O}\big((y-x)^{2p+2}\big)\,. \tag{2.2.2}$$

The parameter g gives the leading order of the singularity of $A(x, y)$ on the light cone. We point out that we do not demand that the infinite series in (2.2.1) converges. Thus, similar to a formal Taylor series, the series in (2.2.1) is defined only via the approximation by the partial sums (2.2.2). The notion of the light-cone expansion is illustrated in Exercise 2.12.

As a simple example for a light-cone expansion, we consider the distribution $T_{m^2}(x, y)$ as introduced in (1.2.26) and analyzed in Lemma 1.2.9. Expanding the Bessel functions in (1.2.29) in a power series, one obtains (see [OLBC, (10.2.2), (10.8.1) and (10.25.2), (10.31.1)])

$$T_{m^2}(x,y) = -\frac{1}{8\pi^3}\left(\frac{\mathrm{PP}}{(y-x)^2} + i\pi\delta\big((y-x)^2\big)\,\epsilon\big((y-x)^0\big)\right) + \frac{m^2}{32\pi^3}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{\big(m^2(y-x)^2\big)^j}{4^j} \times \Big(\log\big|m^2(y-x)^2\big| + c_j + i\pi\,\Theta\big((y-x)^2\big)\,\epsilon\big((y-x)^0\big)\Big) \tag{2.2.3}$$

with real coefficients c_j (here Θ and ϵ are again the Heaviside and the sign function, respectively). Due to the factors $(y-x)^{2j}$, this series representation is a light-cone expansion. The term with the leading singularity becomes integrable after multiplying by $(y-x)^2$, showing that $g = -1$.

The light-cone expansion of the kernel of the fermionic projector of the vacuum $P^{\mathrm{vac}}(x,y)$ (see (2.1.1) and (2.1.2)) is readily obtained using the relation (1.2.25). To this end, one simply applies the differential operator $i\partial\!\!\!/ + m$ to the above series expansion of T_{m^2} and computes the derivatives term by term. Since differentiation increases the order of the singularity on the light cone by one, we thus obtain a light-cone expansion of the form (2.2.1) with $g = -2$.

2.2.2 *Inductive Light-Cone Expansion of the Green's Functions*

We now return to the perturbation series for the causal Green's functions (2.1.25) derived in Sect. 2.1.4. Our goal is to develop a method for performing the light-cone expansion of each summand of this perturbation series. In order to get a first idea for how to proceed, we begin by considering the free advanced Green's function $s^\vee_m$ of a the Dirac equation of mass m in position space: Similar to (1.2.25), it is again convenient to pull the Dirac matrices out of $s^\vee_m$ by setting

$$s^\vee_m(x,y) = (i\partial\!\!\!/_x + m)\; S^\vee_{m^2}(x,y)\,, \tag{2.2.4}$$

where $S^\vee_{m^2}$ is the advanced Green's function of the Klein-Gordon operator,

$$S^\vee_{m^2}(x,y) = \lim_{\nu \searrow 0} \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2 - m^2 - i\nu p^0}\,e^{-ip(x-y)}\,. \tag{2.2.5}$$

Computing this Fourier integral and expanding the resulting Bessel function in a power series gives (for details see Exercise 2.13)

$$S^\vee_{m^2}(x,y) = -\frac{1}{2\pi}\,\delta\big((y-x)^2\big)\,\Theta\big(y^0-x^0\big) + \frac{m^2}{4\pi}\,\frac{J_1\Big(\sqrt{m^2\,(y-x)^2}\Big)}{\sqrt{m^2\,(y-x)^2}}\,\Theta\big((y-x)^2\big)\,\Theta\big(y^0-x^0\big) \tag{2.2.6}$$

$$= -\frac{1}{2\pi}\,\delta\big((y-x)^2\big)\,\Theta\big(y^0-x^0\big) + \frac{m^2}{8\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{\big(m^2(y-x)^2\big)^j}{4^j}\,\Theta\big((y-x)^2\big)\,\Theta\big(y^0-x^0\big)\,. \tag{2.2.7}$$

This computation shows that $S^\vee_{m^2}(x,y)$ has a $\delta((y-x)^2)$-like singularity on the light cone. Furthermore, one sees that $S^\vee_{m^2}$ is a power series in m^2. The important point for what follows is that the higher order contributions in m^2 contain more factors $(y-x)^2$ and are thus of higher order on the light cone. More precisely,

$$\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2}(x,y)\Big|_{m=0} \qquad \text{is of the order } \mathcal{O}\big((y-x)^{2n-2}\big)\,. \tag{2.2.8}$$

According to (2.2.4), the Dirac Green's function is obtained by computing the first partial derivatives of (2.2.7). Therefore, $s^\vee_m(x,y)$ has a singularity on the light cone which is even $\sim\delta'((y-x)^2)$. The higher order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (2.2.4) in m,

$$s^\vee_m(x,y) = \sum_{n=0}^{\infty}(i\partial\!\!\!/ + m)\,\frac{1}{n!}\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2}(x,y)\Big|_{m=0}\,,$$

as a light-cone expansion of the free Green's function. Our idea is to generalize this formula to the case with interaction. More precisely, we want to express the perturbed Green's function in the form

$$\tilde{s}^\vee(x,y) = \sum_{n=0}^{\infty} F_n(x,y)\,\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2}(x,y)\Big|_{m=0} \tag{2.2.9}$$

with factors F_n which depend on the external potential. We will see that this method is very convenient; especially, we can in this way avoid working with the rather complicated explicit formula (2.2.7). Apart from giving a motivation for the desired form (2.2.9) of the formulas of the light-cone expansion, the mass expansion (2.2.7) leads to the conjecture that even the higher order contributions in the mass to the *perturbed* Green's functions might be of higher order on the light cone. If this conjecture was true, it would be a good idea to expand the perturbation expansion for $\tilde{s}$ with respect to the parameter m. Therefore, our strategy is to first expand (2.1.25) with respect to

the mass and to try to express the contributions to the resulting expansion in a form similar to (2.2.9).

The expansion of (2.1.25) with respect to m gives a double sum over the orders in the mass parameter and in the external potential. It is convenient to combine these two expansions in a single perturbation series. To this end, we rewrite the Dirac operator as

$$i\partial\!\!\!/ + \mathcal{B} - m = i\partial\!\!\!/ + B \qquad \text{with} \qquad B := \mathcal{B} - m\,. \tag{2.2.10}$$

For the light-cone expansion of the Green's functions, we will always view B as the perturbation of the Dirac operator. This has the advantage that the unperturbed objects are massless. Expanding in powers of B gives the mass expansion and the perturbation expansion in one step. In order to simplify the notation, for the massless objects we usually omit the index m. Thus we write the Green's function of the massless Dirac equation in the Minkowski vacuum as

$$s^\vee(x,y) = i\partial\!\!\!/_x\, S^\vee_{m^2}(x,y)\big|_{m=0}\,, \qquad s^\wedge(x,y) = i\partial\!\!\!/_x\, S^\wedge_{m^2}(x,y)\big|_{m=0}\,. \tag{2.2.11}$$

Then the interacting Green's functions are given by the perturbation series

$$\tilde{s}^\vee = \sum_{k=0}^{\infty} (-s^\vee B)^k s^\vee\,, \qquad \tilde{s}^\wedge = \sum_{k=0}^{\infty} (-s^\wedge B)^k s^\wedge\,. \tag{2.2.12}$$

The constructions of the following subsections are exactly the same for the advanced and retarded Green's functions. In order to treat both cases at once, in the remainder of this section we will omit all superscripts '$\vee$', '$\wedge$'. The formulas for the advanced and retarded Green's functions are obtained by either adding '$\vee$' or '$\wedge$' to all factors s, S.

We now explain how the individual contributions to the perturbation expansion (2.2.12) can be written similar to the right side of (2.2.9) as a sum of terms of increasing order on the light cone. For the mass expansion of S_{m^2}, we set $a = m^2$ and use the notation

$$S^{(l)} = \left(\frac{d}{da}\right)^l S_a\big|_{a=0}\,. \tag{2.2.13}$$

In preparation, we derive some computation rules for the $S^{(l)}$: S_a satisfies the defining equation of a Klein-Gordon Green's function

$$(-\Box_x - a)\, S_a(x,y) = \delta^4(x-y)\,.$$

Differentiating with respect to a and setting $a = 0$ gives

$$-\Box_x S^{(l)}(x,y) = \delta_{l,0}\, \delta^4(x-y) + l\, S^{(l-1)}(x,y)\,, \qquad l \geq 0. \tag{2.2.14}$$

(For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. The expression is meaningful, however, if one keeps in mind that in this case the factor l is zero, and thus the whole second summand vanishes. We will also use this convention in the following calculations.) Next, we differentiate the formulas for S_a in momentum space,

$$S_a^\vee(p) = \frac{1}{p^2 - a - i\nu p^0}\,, \qquad S_a^\wedge(p) = \frac{1}{p^2 - a + i\nu p^0} \tag{2.2.15}$$

with respect to both p and a. Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) = -2p_k \,\frac{d}{da} S_a(p)\,,$$

or, after expanding in the parameter a,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) = -2p_k \, S^{(l+1)}(p)\,, \qquad l \geq 0. \tag{2.2.16}$$

This formula also determines the derivatives of $S^{(l)}$ in position space; namely

$$\begin{aligned}
\frac{\partial}{\partial x^k} S^{(l)}(x,y) &= \int \frac{d^4p}{(2\pi)^4}\, S^{(l)}(p)\, (-ip_k)\, e^{-ip(x-y)} \\
&\overset{(2.2.16)}{=} \frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, \frac{\partial}{\partial p^k} S^{(l-1)}(p)\, e^{-ip(x-y)} \\
&= -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, S^{(l-1)}(p)\, \frac{\partial}{\partial p^k} e^{-ip(x-y)} \\
&= \frac{1}{2}\,(y-x)_k \, S^{(l-1)}(x,y)\,, \qquad l \geq 1.
\end{aligned} \tag{2.2.17}$$

We iterate this relation to calculate the Laplacian,

$$\begin{aligned}
-\Box_x S^{(l)}(x,y) &= -\frac{1}{2}\, \frac{\partial}{\partial x^k} \left((y-x)^k \, S^{(l-1)}(x,y) \right) \\
&= 2\, S^{(l-1)}(x,y) + \frac{1}{4}\,(y-x)^2 \, S^{(l-2)}(x,y)\,, \qquad l \geq 2.
\end{aligned}$$

After comparing with (2.2.14), we conclude that

$$(y-x)^2 \, S^{(l)}(x,y) = -4l \, S^{(l+1)}(x,y)\,, \qquad l \geq 0\,. \tag{2.2.18}$$

Finally, $S^{(l)}(x,y)$ is only a function of $(y-x)$, which implies that

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) = -\frac{\partial}{\partial y^k} S^{(l)}(x,y)\,, \qquad l \geq 0\,. \tag{2.2.19}$$

The following lemma gives the light-cone expansion of an operator product which is linear in the external potential. We will later use it for the iterative light-cone expansion of more complicated operator products; in this case, the potential will be a composite expression in B and its partial derivatives. In order to avoid confusion then, we denote the external potential by V.

Lemma 2.2.2 (light-cone expansion to first order) *For any $l, r \geq 0$, the operator product $S^{(l)}\, V\, S^{(r)}$ has the light-cone expansion*

$$(S^{(l)}\, V\, S^{(r)})(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y)\,. \tag{2.2.20}$$

Proof The method of proof is to first compute the Laplacian of both sides of (2.2.20). The resulting formulas will have a similar structure, making it possible to proceed inductively.

On the left side of (2.2.20), we calculate the Laplacian with the help of (2.2.14) to

$$-\Box_x (S^{(l)}\, V\, S^{(r)})(x,y) = \delta_{l,0}\, V(x)\, S^{(r)}(x,y) + l\, (S^{(l-1)}\, V\, S^{(r)})(x,y)\,. \tag{2.2.21}$$

The Laplacian of the integral on the right side of (2.2.20) can be computed with (2.2.17) and (2.2.14),

$$\begin{aligned} -\Box_x \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y) \qquad (2.2.22)\\ = -\int_0^1 \alpha^l\, (1-\alpha)^{r+2}\, (\alpha-\alpha^2)^n\, (\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y)\\ - \int_0^1 \alpha^l\, (1-\alpha)^{r+1}\, (\alpha-\alpha^2)^n\, (\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\\ \times (y-x)^k\, S^{(n+l+r)}(x,y)\\ + (n+l+r+1) \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\\ \times\; S^{(n+l+r)}(x,y)\,. \end{aligned}$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α,

$$(y-x)^k (\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x} = \frac{d}{d\alpha} (\Box^n V)_{|\alpha y+(1-\alpha)x}$$

(as is verified immediately by computing the right side with the chain rule). This makes it possible to integrate in α by parts. We thus obtain

$$
\begin{aligned}
&\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,(\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k \\
&\quad= \int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,\frac{d}{d\alpha}\Big((\Box^n V)\big|_{\alpha y+(1-\alpha)x}\Big) d\alpha \\
&\quad= -\delta_{n,0}\,\delta_{l,0}\,V(x) \\
&\qquad - (n+l)\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\qquad + (n+r+1)\int_0^1 \alpha^l\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\quad= -\delta_{n,0}\,\delta_{l,0}\,V(x) \\
&\qquad - n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\qquad + (n+l+r+1)\int_0^1 \alpha^l\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\qquad - l\int_0^1 \alpha^{l-1}\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,.
\end{aligned}
$$

We substitute back into the original equation to obtain

$$
\begin{aligned}
(2.2.22) &= \delta_{n,0}\,\delta_{l,0}\,V(x)\,S^{(r)}(x,y) \\
&\quad + l\int_0^1 \alpha^{l-1}\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y) \\
&\quad - \int_0^1 \alpha^{l}\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n}\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r+1)}(x,y) \\
&\quad + n\int_0^1 \alpha^{l}\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^{n} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y)\,.
\end{aligned}
$$

After dividing by $n!$ and summation over n, the last two summands are telescopic and cancel each other. Thus one gets

$$
\begin{aligned}
&-\Box\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 \alpha^l\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r+1)}(x,y) \\
&= \delta_{l,0}\,V(x)\,S^{(r)}(x,y) \\
&\quad + l\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 \alpha^{l-1}\,(1-\alpha)^{r}\,(\alpha-\alpha^2)^{n}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y)\,.
\end{aligned}
\tag{2.2.23}
$$

We now compare the formulas (2.2.21) and (2.2.23) for the Laplacian of both sides of (2.2.20). In the special case $l=0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (2.2.20): We

assume that we consider the advanced Green's function (for the retarded Green's function, the argument is analogous). For given y, we denote the difference of both sides of (2.2.20) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \,|\, z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically zero according to (2.2.21) and (2.2.23). We conclude that

$$\Box F = 0 \qquad \text{and} \qquad F_{|\mathcal{H}} = \partial_k F_{|\mathcal{H}} = 0\,.$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l: Suppose that (2.2.20) holds for given $\hat{l}$ (and arbitrary r). Then, according to (2.2.21), (2.2.23), and the induction hypothesis, the Laplacian of both sides of (2.2.20) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (2.2.20). □

We recall for clarity that, according to (2.2.8), the higher a-derivatives of $S_a(x, y)$ are of higher order on the light cone. Thus the summands in (2.2.20) are of increasing order on the light cone, and the infinite sum is mathematically well-defined in the sense of Definition 2.2.1 via the approximation by the partial sums (2.2.2).

Lemma 2.2.2 can be used for the light-cone expansion of more complicated operator products. To explain the method, we look at the simplest example of three factors $S^{(0)}$ and two potentials V and W,

$$(S^{(0)}\, V\, S^{(0)}\, W\, S^{(0)})(x, y) = \int d^4z\, S^{(0)}(x, z)\, V(z)\, (S^{(0)}\, W\, S^{(0)})(z, y)\,.$$

Having split up the operator product in this form, we can apply Lemma 2.2.2 to the factor $S^{(0)} W S^{(0)}$,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4z\, S^{(0)}(x, z) \left\{ V(z) \int_0^1 (\alpha - \alpha^2)^n\, (\Box^n W)_{|\alpha y + (1-\alpha) z}\, d\alpha \right\} S^{(n+1)}(z, y)\,.$$

Now we rewrite the z-integral as the operator product $(S^{(0)} g_y S^{(0)})(x, y)$, where $g_y(z)$ is the function in the curly brackets. The y-dependence of g_y causes no problems because we can view y as a fixed parameter throughout the expansion. Thus we can simply apply Lemma 2.2.2 once again to obtain

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!\, n!} \int_0^1 d\beta\, (1-\beta)^{n+1}\, (\beta - \beta^2)^m \int_0^1 d\alpha\, (\alpha - \alpha^2)^n$$
$$\times\, \Box_z^m \left(V(z)\, (\Box^n W)_{|\alpha y + (1-\alpha) z} \right)_{|z = \beta y + (1-\beta) x}\, S^{(m+n+2)}(x, y)\,.$$

The Laplacian $\Box_z^m$ could be computed further with the Leibniz rule. Notice that the manipulations of the infinite sums are unproblematic because to every order on the light cone, the number of terms is actually finite (the situation would be more difficult if we studied the convergence of the sum (2.2.1), but, as pointed out earlier, the light-cone expansion is defined merely via the partial sums).

We want to iteratively perform the light-cone expansion of the operator products in (2.2.12). This is not possible directly with the method just described, because (2.2.12) contains the Dirac Green's function s (instead of S). We must think about how to deal with this complication. Relation (2.2.11) allows us to replace the factors s by S, but this gives additional partial derivatives in the operator product. These derivatives can be carried out after each iteration step by applying the Leibniz rule and using the differentiation rule (2.2.17). In the simplest example, we have

$$\begin{aligned}
&(s^{(0)}\; V\; S^{(0)})(x,y) = (i\partial\!\!\!/_x)(S^{(0)}\; V\; S^{(0)})(x,y) \\
&= i\partial\!\!\!/_x \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+1)}(x,y) \\
&= i \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (1-\alpha)\, (\alpha-\alpha^2)^n\, (\partial\!\!\!/\, \Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+1)}(x,y) \\
&\quad + \frac{i}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; (y-x)_j \gamma^j\; S^{(n)}(x,y)\,.
\end{aligned}$$

The only problem with this method is that the partial derivatives might hit a factor $S^{(0)}$, in which case the rule (2.2.17) cannot be applied. In order to resolve this problem, we extend our constructions in a way which allows us to use all previous formulas also in this special case. To this end, we take (2.2.17) as the defining equation for $(y-x)_k\, S^{(-1)}(x,y)$,

$$(y-x)_k\, S^{(-1)}(x,y) := 2\, \frac{\partial}{\partial x^k} S^{(0)}(x,y) \tag{2.2.24}$$

(notice that $S^{(-1)}$ itself remains undefined, only the combination $(y-x)_k\, S^{(-1)}(x,y)$ makes mathematical sense as the partial derivative of the distribution $S^{(0)}$). It turns out that with this definition, all our computation rules as well as the light-cone expansion of Lemma 2.2.2 remain valid for $S^{(-1)}$:

Lemma 2.2.3 (light-cone expansion to first order for $r=-1$) *The operator product* $(S^{(l)}\,.\,S^{(-1)})$, $l \geq 0$, *has the light-cone expansion*

$$\begin{aligned}
&\int d^4z\; S^{(l)}(x,z)\; V(z)\; (y-z)_k\; S^{(-1)}(z,y) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^{-1}\, (\alpha-\alpha^2)^n \\
&\qquad \times \Box_z^n \big(V(z)\, (y-z)_k\big)\big|_{z=\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l)}(x,y)\,.
\end{aligned}$$

Since the proof is straightforward, we omit it here but refer to Exercise 2.14 or [F6, proof of Lemma 2.2]. We note for clarity that the pole of the factor $(1-\alpha)^{-1}$ at $\alpha=1$ in the formula of the above lemma does not cause any problems. Namely, in the case $n=0$ it disappears because $(1-\alpha)^{-1}(y-z) = y-x$, whereas in the case $n>0$ it is compensated by the zero of the factor $(\alpha-\alpha^2)^n$.

2.2.3 Structural Results for Chiral Potentials

In the previous section, we gave a constructive procedure for performing the light-cone expansion of each summand of the perturbation expansion for the causal Green's functions (2.2.12). In this and the next section, we shall explain how to use this method to uncover the structure of the Green's functions in position space. To this end, we need to specify the form of the external potential $\mathcal{B}$ in the Dirac equation (2.1.5). We are mostly interested in the situation that $\mathcal{B}$ is composed of left- or right-handed potentials, i.e.

$$\mathcal{B} = \chi_L \,\mathcal{A}_R + \chi_R \,\mathcal{A}_L \,. \tag{2.2.25}$$

(here $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \Gamma)$ are the chiral projectors, and $\Gamma = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the usual pseudoscalar matrix). Such so-called *chiral potential* are of central interest because they allow for the description of *gauge fields*. For example, an electromagnetic field is described by choosing $A_L = A_R = A$, where A is the electromagnetic potential. A left-handed potential is needed for example for describing the weak interaction in the standard model. In this context, it is important to describe *non-abelian* gauge fields. In this case, the potentials A_L and A_R take values in a Lie algebra. For simplicity, we here always represent the potentials by matrices acting on $\mathbb{C}^g$ with $g \in \mathbb{N}$. In order to describe the coupling of the gauge gauge fields to the fermions, the Dirac wave functions must also carry an index $a = 1, \ldots, g$. Moreover, we want to allow for the situation that the system involves Dirac matrices of different rest masses, which we label again by an index a. This leads to the following setup. We define the fermionic projector of the vacuum and the Green's functions as direct sums of the corresponding operators with rest masses $m_1, \ldots, m_g$, i.e.

$$P^{\text{vac}} = \bigoplus_{a=1}^{g} P^{\text{vac}}_{m_a} \qquad \text{and} \qquad s = \bigoplus_{a=1}^{g} s_{m_a} \tag{2.2.26}$$

with $P^{\text{vac}}_{m_a}$ and s_m according to (2.1.2) and (2.1.9). We write the Dirac equation as

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\,\psi(x) = 0 \tag{2.2.27}$$

with $\mathcal{B}$ as in (2.2.25). Here Y is the *mass matrix* defined by

$$Y = \frac{1}{m}\,\text{diag}\big(m_1, \ldots, m_a\big)$$

(here m is introduced merely as an expansion parameter; the picture is that Y is dimensionless, whereas m carries the dimension of inverse length). For later use, it is also convenient to allow for *scalar* and *pseudoscalar potentials*. In order to built

these potentials into the Dirac equation (2.2.27), it is most convenient to replace the mass matrix by a space-time dependent matrix,[1]

$$Y = Y(x) := \chi_L Y_L(x) + \chi_R Y_R(x) \,, \tag{2.2.28}$$

referred to as the *dynamical mass matrix*.

In analogy to (2.2.10), we combine the mass term with the potential by setting

$$B = \chi_L \,\mathcal{A}_R + \chi_R \,\mathcal{A}_L - mY \,. \tag{2.2.29}$$

Then the perturbation expansion for the causal Green's functions can again be written in the form (2.2.12). The light-cone expansion can be carried out exactly as explained in the previous section. The only point to keep in mind is that the chiral potentials at different space-time points do not necessarily commute. Moreover, the chiral potentials in general do not commute with the mass matrix. Therefore, in what follows we need to be careful in keeping track of the order of multiplication.

Before going on, we explain our convention for the chiral indices of potentials in (2.2.28) and (2.2.29). We follow the usual rule that a left-handed potential couples to the left-handed component of the Dirac wave function, whereas the right-handed potential couples to the right-handed component of the wave function. Indeed, decomposing the Dirac wave function as

$$\psi = \chi_L \,\psi_L + \chi_R \,\psi_R \,, \tag{2.2.30}$$

the Dirac equation (2.2.27) becomes

$$\begin{aligned} 0 &= \Big(i\partial\!\!\!/ + \chi_L \,\mathcal{A}_R + \chi_R \,\mathcal{A}_L - m\chi_L Y_L(x) - m\chi_R Y_R(x)\Big)\Big(\chi_L \,\psi_L + \chi_R \,\psi_R\Big) \\ &= \chi_L \Big((i\partial\!\!\!/ + \mathcal{A}_R)\psi_R - mY_L\psi_L\Big) + \chi_R \Big((i\partial\!\!\!/ + \mathcal{A}_L)\psi_L - mY_R\psi_R\Big) \,. \end{aligned}$$

(here we use that the chirality is reversed at each Dirac matrix). This shows that our conventions (2.2.28) and (2.2.29) indeed imply that left-handed potentials couple to ψ_L and right-handed potentials to ψ_R.

The next theorem gives a structural result on the contributions to the light-cone expansion of the Green's functions. For the line integrals, we introduce the short notation

$$\int_x^y [l, r \,|\, n]\, dz \; f(z) := \int_0^1 d\alpha \; \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \; f(\alpha y + (1-\alpha)x) \,. \tag{2.2.31}$$

[1] To avoid confusion, we point out that our convention differs from that used in [F6, F7], where the dynamical mass matrix is defined instead by $Y = \chi_L Y_R + \chi_R Y_L$. Our convention fits to our general rule that left- and right-handed potentials should couple to the left- and right-handed component of the Dirac spinors, respectively (see also (2.2.30) and the explanation thereafter).

Furthermore, we abbreviate the following products with multi-indices,

$$\partial_z^J := \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_l}}, \qquad (y-x)^J := (y-x)^{j_1} \cdots (y-x)^{j_l}, \quad \gamma^J := \gamma^{j_1} \cdots \gamma^{j_l},$$

where $J = (j_1, \ldots, j_l)$.

Theorem 2.2.4 *In the presence of chiral potentials (2.2.29), the light-cone expansion of the kth order contribution $((-sB)^k s)(x, y)$ to the perturbation series (2.2.12) can be written as an infinite sum of expressions, each of which has the form*

$$\chi_{c_0} C (y-x)^I \int_x^y [l_1, r_1 \mid n_1]\, dz_1\, \partial_{z_1}^{I_1} \Box_{z_1}^{p_1} V^{(1)}_{J_1,c_1}(z_1) \int_{z_1}^y [l_2, r_2 \mid n_2]\, dz_2\, \partial_{z_2}^{I_2} \Box_{z_2}^{p_2} V^{(2)}_{J_2,c_2}(z_2) \cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box_{z_k}^{p_k} V^{(k)}_{J_k,c_k}(z_k)\, \gamma^J\, S^{(h)}(x, y)\,. \tag{2.2.32}$$

In this formula, C is a complex number and the parameters l_a, r_a, n_a, and p_a are non-negative integers; the indices c and c_a can take the two values L or R. The functions $V^{(a)}_{J_a,c_a}$ (where J_a is a multi-index and $c_a \in \{L, R\}$ is a chiral index) coincide with any of the individual potentials in (2.2.29) and (2.2.28) with chirality c_a, i.e.

$$\begin{aligned} V^{(a)}_{c_a} &= A_{c_a} \qquad \text{(in which case } |J_a| = 1\text{)} \quad \text{or} \\ V^{(a)}_{c_a} &= m Y_{c_a} \qquad \text{(in which case } |J_a| = 0\text{)}\,. \end{aligned} \tag{2.2.33}$$

The chirality c_a of the potentials is determined by the following rule:

(i) The chirality is reversed precisely at every mass matrix, i.e.

$$c_{a-1} \text{ and } c_a \quad \begin{cases} \text{coincide} & \text{if } V^{(a)}_{c_a} = A_{c_a} \\ \text{are opposite} & \text{if } V^{(a)}_{c_a} = m Y_{c_a} \end{cases}$$

for all $a = 1, \ldots, k$.

The tensor indices of the multi-indices in (2.2.32) are all contracted with each other, according to the following rules:

(a) No two tensor indices of the same multi-index are contracted with each other.
(b) The tensor indices of the factor γ^J are all contracted with different multi-indices, in the order of their appearance in the product (2.2.32) (i.e., for $J = (j_1, \ldots, j_l)$ and $1 \le a < b \le l$, the multi-index with which j_a is contracted must stand to the left of the multi-index corresponding to j_b).

The parameter h is given by

$$2h = k - 1 - |I| + \sum_{a=1}^{k} \Big(|I_a| + 2p_a \Big)\,. \tag{2.2.34}$$

The number of factors $(y-x)$ *is bounded by*

$$|I| \leq k+1-\sum_{a=1}^{k} |I_a| \,. \tag{2.2.35}$$

Basically, this theorem states that the light-cone expansion of the kth order Feynman diagrams can be written with k nested line integrals. Notice that the potentials $V^{(a)}(z_a)$ do in general not commute with each other, so that the order of multiplication is important in (2.2.32). In order to avoid misunderstandings, we point out that the derivatives $\partial_{z_a}^{I_a}$ and $\Box_{z_a}^{p_a}$ do not only act on $V^{(a)}(z_a)$, but also on all the following factors $V^{(a+1)}(z_{a+1})$, $V^{(a+2)}(z_{a+2}), \ldots$ (note that the variables $z_{a+1}, z_{a+2}, \ldots$ implicitly depend on z_a via the inductive definition of the line integrals). Clearly, these derivatives could be carried out further with the Leibniz rule, but it is easier not to do this at the moment. The restrictions *(a)* and *(b)* on the possible contractions of the tensor indices were imposed in order to avoid an abuse of our multi-index notation. More precisely, *(a)* prevents factors $(y-x)^2$ in $(y-x)^I$, an unnecessary large number of γ-matrices in γ^J, and "hidden" Laplacians in the partial derivatives $\partial_{z_a}^{I_a}$. The rule *(b)*, on the other hand, prevents factors $(y-x)^2$ and hidden Laplacians in combinations of the form $(y-x)_i\,(y-x)_j\,\gamma^i\,\gamma^j$ and $\partial_{ij} V_{J_a}^{(a)}\,\gamma^i\,\gamma^j$, respectively. Our ordering condition for the γ-matrices is just a matter of convenience. Relation (2.2.34) is very useful because it immediately tells for any configuration of the line integrals and potentials in (2.2.32) what the corresponding order on the light cone is. Notice that (2.2.34) and (2.2.35) imply the inequality

$$h \geq -1+\sum_{a=1}^{k} (|I_a|+p_a) \,. \tag{2.2.36}$$

In particular, one sees that $h \geq -1$. In the case $h=-1$, (2.2.34) yields that $|I|>0$, so that (2.2.32) must contain at least one factor $(y-x)$. Therefore, the factor $S^{(h)}$ in (2.2.32) is always well-defined by either (2.2.13) or (2.2.24).

We point out that, although the total number of summands (2.2.32) is infinite, the number of summands for any given value of the parameter h is finite. This is clear because, for fixed h, the relations (2.2.34) and (2.2.35) only allow for a finite number of possibilities to choose the parameters $|I|$, $|I_a|$, and p_a, giving rise to only a finite number of expressions of the form (2.2.32). Since, according to (2.2.8), the contributions for higher values of h are of higher order on the light cone, we conclude that the number of summands (2.2.32) is finite to every order on the light cone. Therefore, the light-cone expansion of Theorem 2.2.4 makes mathematical sense in terms of Definition 2.2.1.

Proof of Theorem 2.2.4 We proceed inductively in k. For $k=0$, the assumption is true because in view of (2.2.11) and (2.2.24) we can write the free Dirac Green's function as

$$s(x,y) = (\chi_L+\chi_R)\,\frac{i}{2}\,(y-x)^j \gamma_j\, S^{(-1)}(x,y) \,, \tag{2.2.37}$$

which is of the desired form (2.2.32). The conditions *(i)*, *(a)*, *(b)*, and the relations (2.2.34), (2.2.35) are clearly satisfied.

Assume that the theorem holds for a given k. With the formula

$$\big((-sB)^{k+1}\, s\big)(x,y) = -i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z)\; B(z)\; \big((-sB)^k\, s\big)(z,y)\,, \tag{2.2.38}$$

we can express the $(k+1)$st order contribution to the perturbation series (2.2.12) in terms of the kth order contribution. We must show that (2.2.38) can again be written as a sum of expressions of the form (2.2.32) (with k replaced by $k+1$), and that *(i)*, *(a)*, *(b)*, and (2.2.34), (2.2.35) are satisfied. This is done in several construction steps:

(1) Chiral decomposition:
We substitute the induction hypothesis (2.2.32) into (2.2.38). This gives a sum of expressions of the form

$$\begin{aligned} C\, i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z) \Big\{ (y-z)^I\; B(z)\; \chi_c \int_z^y [l_1, r_1 \,|\, n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box^{p_1}\; V^{(1)}_{J_1,c_1}(z_1) \\ \cdots \int_{z_{k-1}}^y [l_k, r_k \,|\, n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box^{p_k}\; V^{(k)}_{J_k,c_k}(z_k)\; \gamma^J \Big\}\; S^{(h)}(z,y)\,. \end{aligned} \tag{2.2.39}$$

We insert the specific form of the potential B, (2.2.29), and expand. Using the commutation rule $\gamma^i\; \chi_{L/R} = \chi_{R/L}\; \gamma^i$, we bring all chiral projectors to the very left, where they can be combined with the formula $\chi_c \chi_d = \delta_{cd}\; \chi_c$ to a single chiral projector. Next, we bring the γ-matrices of B to the right and write them together with the factor γ^J in (2.2.39) (notice that the Dirac matrices commute with the potentials $V^{(a)}_{c_a}$, which act non-trivially only on the Dirac sea index). Denoting the individual potentials of the factor B in (2.2.39) by $V^{(0)}_{J_0,c_0}$, we thus get for (2.2.39) a sum of expressions of the form

$$\begin{aligned} \chi_c\, C\, i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z) \Big\{ (y-z)^I\; V^{(0)}_{J_0,c_0}(z) \int_z^y [l_1, r_1 \,|\, n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box^{p_1}\; V^{(1)}_{J_1,c_1}(z_1) \\ \cdots \int_{z_{k-1}}^y [l_k, r_k \,|\, n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box^{p_k}\; V^{(k)}_{J_k,c_k}(z_k)\; \gamma^J \Big\}\; S^{(h)}(z,y)\,. \end{aligned} \tag{2.2.40}$$

The chiral decomposition in (2.2.29) and (2.2.28) imply that the chiralities in (2.2.40) satisfy the rule *(i)* (after relabeling the indices in an obvious way). The chirality of the potentials will not be affected in all the following construction steps; to simplify the notation, we will omit the indices c_a from now on.

(2) Light-cone expansion:
Since y can be considered as a fixed parameter, we can in (2.2.40) apply Lemma 2.2.2 with V given by the expression in the curly brackets,

$$(2.2.40) = \chi_c\, C\, i\partial\!\!\!/_x \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y [0, h \mid n]\, dz$$
$$\times\ \Box_z^n \Big((y-z)^I\, V_{J_0}^{(0)}(z) \int_z^y [l_1, r_1 \mid n_1]\, dz_1\, \partial_{z_1}^{I_1} \Box^{p_1}\, V_{J_1}^{(1)}(z_1)$$
$$\cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box^{p_k}\, V_{J_k}^{(k)}(z_k) \Big)\, \gamma^J\, S^{(n+h+1)}(x, y)\,. \quad (2.2.41)$$

(3) Computation of the Laplacian $\Box_z^n$:
We carry out the z-derivatives in (2.2.41) inductively with the Leibniz rule. Each derivative can act either on the factors $(y-z)^I$ or on the functions $V^{(a)}$. In the first case, one of the factors $(y-z)$ disappears. Thus we get a sum of expressions of the form

$$\chi_c\, C\, i\partial\!\!\!/_x \int_x^y [0, h \mid n]\, dz\, (y-z)^{\hat{I}}\, \partial_z^{I_0} \Box_z^{p_0}\, V_{J_0}^{(0)}(z) \int_z^y [l_1, r_1 \mid n_1]\, dz_1\, \partial_{z_1}^{I_1} \Box^{p_1}\, V_{J_1}^{(1)}(z_1)$$
$$\cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box^{p_k}\, V_{J_k}^{(k)}(z_k)\, \gamma^J\, S^{(n+h+1)}(x, y) \quad (2.2.42)$$

with $|\hat{I}| \leq |I|$ and

$$2n = |I| - |\hat{I}| + |I_0| + 2p_0\,. \quad (2.2.43)$$

We can assume that no tensor indices of $\partial_z^{I_0}$ are contracted with each other (otherwise we rewrite the corresponding partial derivatives as additional Laplacians). Then all the partial derivatives ∂_z in (2.2.42) were generated in the case when one derivative of a Laplacian $\Box_z$ in (2.2.41) hit a factor $(y-z)$ whereas the other derivative acted on the $V^{(a)}$. Thus the number of factors $(y-z)$ which disappeared by carrying out the Laplacians in (2.2.41) is larger or equal than the number of partial derivatives ∂_z,

$$|I| - |\hat{I}| \geq |I_0|\,. \quad (2.2.44)$$

(4) Extraction of the factors $(y-x)$:
In (2.2.42), we iteratively apply the identity

$$\int_x^y [0, r \mid n]\, dz\, (y-z) \cdots = (y-x) \int_x^y [0, r+1 \mid n]\, dz\, \cdots\,.$$

This gives $(k+1)$ nested line integrals of the form

$$(2.2.42) = \chi_c\, C\, i\partial\!\!\!/_x (y-x)^{\hat{I}}\, S^{(\hat{h})}(x, y) \int_x^y [l_0, r_0 \mid n_0]\, dz_0\, \partial_{z_0}^{I_0} \Box^{p_0}\, V_{J_0}^{(0)}(z_0)$$

$$\cdots \int_{z_{k-1}}^{y} [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k} \Box^{p_k}\; V_{J_k}^{(k)}(z_a)\; \gamma^J \tag{2.2.45}$$

with

$$l_0 = 0\,, \qquad r_0 = h + |\hat{I}|\,, \qquad n_0 = n \tag{2.2.46}$$

$$0 \le 2\hat{h} = 2(n + h + 1) \overset{(2.2.43)}{=} 2h + 2 + |I| - |\hat{I}| + |I_0| + 2p_0\,. \tag{2.2.47}$$

We can arrange that the parameters l_0, r_0, and n_0 are all positive: The only parameter which might be negative is r_0; in this case, $h = -1$, $|\hat{I}| = 0$, and thus $r_0 = -1$. The induction hypothesis (2.2.34) yields that $|I| > 0$. Thus $|I| > |\hat{I}|$, and relation (2.2.43) gives that $(n_0 =)n > 0$. Therefore, we can apply the identity

$$[l_0, r_0 \mid n_0] = [l_0 + 1, r_0 + 1 \mid n_0 - 1]$$

to make all the parameters in this bracket positive.

(5) Computation of the partial derivative $\partial\!\!\!/_x$:
The x-derivative in (2.2.45) can act on the factors $S^{(\hat{h})}$, $(y - x)^{\hat{I}}$, or $V^{(a)}(z_a)$. The first case can be computed with the rules (2.2.17) or (2.2.24); it decreases $\hat{h}$ by one and gives one additional factor $(y - x)$. In the second case, one factor $(y - x)$ disappears, and thus $|\hat{I}|$ is decremented. The last case can be handled with the rule

$$\frac{\partial}{\partial x^k} \int_x^y [l, r \mid n]\, dz\; f(z, y) = \int_x^y [l, r + 1 \mid n]\; \frac{\partial}{\partial z^k} f(z, y)\,, \tag{2.2.48}$$

which increases $|I_0|$ by one. As is immediately verified in each of these cases, equation (2.2.47) transforms into

$$2\hat{h} = 2h + 1 + |I| - |\hat{I}| + |I_0| + 2p_0\,, \tag{2.2.49}$$

whereas inequality (2.2.44) must be weakened to

$$|\hat{I}| \le 1 + |I| - |I_0|\,. \tag{2.2.50}$$

Finally, we combine the γ-matrix of the factor $\partial\!\!\!/_x$ with γ^J.

After these transformations, the $(k+1)^{\text{st}}$ order Feynman diagram consists of a sum of terms of the form

$$\chi_c\; C\; (y - x)^{\hat{I}} \int_x^y [l_0, r_0 \mid n_0]\, dz_0\; \partial_{z_0}^{I_0} \Box_{z_0}^{p_0}\; V_{J_0}^{(0)}(z_0)$$
$$\cdots \int_{z_{k-1}}^{y} [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k} \Box_{z_k}^{p_k}\; V_{J_k}^{(k)}(z_k)\; \gamma^J\; S^{(\hat{h})}(x, y)\,. \tag{2.2.51}$$

Notice that the parameters I_a, p_a, $a = 1, \ldots, k$, were not changed by the above construction steps; they are still the same as in the induction hypothesis (2.2.32). After renaming the indices and the integration variables, (2.2.51) is of the required form (2.2.32). The conditions *(a)* and *(b)* for the contractions of the tensor indices, however, will in general be violated. Therefore we need two further computation steps:

(6) Simplification of the Dirac matrices:
If any two of the tensor indices of the factor γ^J are contracted with each other, we reorder the γ-matrices with the anti-commutation relations

$$\{\gamma^i,\ \gamma^j\} = 2\, g^{ij}\ \mathbb{1} \tag{2.2.52}$$

until the corresponding matrices are next to each other. Applying the identity $\gamma^i \gamma_i = 4\ \mathbb{1}$, both Dirac matrices disappear. We iterate this procedure until no tensor indices of γ^J are contracted with each other (notice that the iteration comes to an end because the number of γ-factors is decreased by two in each step). Again using the anti-commutation rule (2.2.52), we reorder the Dirac matrices until they are in the same order in which the factors to which their tensor indices are contracted appear in the product (2.2.51). If any two of the γ-matrices are contracted with the same multi-index, these γ-matrices are next to each other, and we can use the symmetry in the tensor indices to eliminate them both, more precisely

$$(y-x)_i\ (y-x)_j \cdots \gamma^i \gamma^j = (y-x)^2 \cdots \mathbb{1} \tag{2.2.53}$$

$$\partial_{ij} V^{(a)} \cdots \gamma^i \gamma^j = \Box V^{(a)} \cdots \mathbb{1}\ . \tag{2.2.54}$$

After all these transformations, condition *(b)* is satisfied.
Notice that the parameters $|I_a|$ and p_a are in general changed in this construction step. More precisely, each transformation (2.2.54) modifies the parameters according to

$$|I_a| \to |I_a| - 2 \qquad \text{and} \qquad p_a \to p_a + 1\ . \tag{2.2.55}$$

(7) Handling of the new contractions:
If any two tensor indices of a factor $\partial_{z_a}^{I_a}$ are contracted with each other, we rewrite the corresponding partial derivatives as a Laplacian; this changes the parameters $|I_a|$ and p_a according to (2.2.55). If two tensor indices of the factor $(y-x)^{\hat{I}}$ are contracted with each other, this gives a factor $(y-x)^2$. Using the identity (2.2.18), we inductively absorb the factors $(y-x)^2$ into $S^{(\hat{h})}(x, y)$, which transforms $\hat{h}$ and $|\hat{I}|$ as

$$\hat{h}\ \to\ \hat{h} + 1 \qquad \text{and} \qquad |\hat{I}|\ \to\ |\hat{I}| - 2\ . \tag{2.2.56}$$

After these transformations, condition *(a)* is also satisfied.

After all these construction steps, the $(k+1)^{\text{st}}$ order Feynman diagram is a sum of terms of the form (2.2.51) satisfying the conditions *(a)* and *(b)*. It remains to show that the relations (2.2.34) and (2.2.35) remain valid in our inductive construction: As mentioned earlier, the parameters I_a, p_a, $a = 1, \ldots, k$ are not changed in the construction steps *(1)* to *(5)*. In the steps *(6)* and *(7)*, the transformations (2.2.55) and (2.2.56) preserve both the induction hypothesis (2.2.34), (2.2.35) and the relations (2.2.49), (2.2.50), as is immediately verified. By substituting (2.2.49) and (2.2.50) into (2.2.55), (2.2.56), we obtain

$$2\hat{h} = (k+1) - 1 - |\hat{I}| + \sum_{a=0}^{k} |I_a| + 2p_a\,, \qquad |\hat{I}| \leq (k+1) + 1 - \sum_{a=0}^{k} |I_a|\,.$$

This concludes the proof. □

2.2.4 *Reduction to the Phase-Free Contribution*

The shortcoming of the constructions of the previous section is that the resulting formulas become more and more involved to higher order in perturbation theory. Moreover, to any order on the light cone, one gets an infinite number of contributions. In order to clarify the structure of the singularities on the light-cone, it is therefore essential to collect and rearrange the different contribution to the light-cone expansion. This procedure is called *resummation* of the light-cone expansion. After the resummation, the light-cone expansion of $\tilde{s}(x, y)$ will, to every order on the light cone, consist of only a finite number of terms. Before beginning, we remark that the resummation technique can also be understood from underlying gauge symmetries. In order no to mix mathematical constructions with physical considerations, we postpone the explanation of gauge phases and gauge transformations to Sect. 3.6.2 (however, the idea of working with local transformations will be used in our constructions; see (2.2.82) and the computations thereafter).

In order to give a first idea of how the resummation works, we consider the leading singularity on the light cone by neglecting all terms of the order $\mathcal{O}((y-x)^{-2})$. According to (2.2.8), we need to take into account only the contributions (2.2.32) with $h = -1$. The inequality (2.2.36) implies that no derivatives of the potentials appear. Moreover, we obtain from (2.2.34) that $|I| = k+1$. Using the rules *(a)* and *(b)*, we conclude that one tensor index of the multi-index I is contracted with a Dirac matrix, whereas all the remaining k indices of I are contracted with chiral potentials. Therefore, all k potentials are chiral, and no dynamical mass matrices appear. A detailed calculation yields for the kth order Feynman diagram a term of precisely this structure,

$$\chi_c \left((-sB)^k s\right)(x,y) = \chi_c\, (-i)^k \int_x^y dz_1\, (y-x)_{j_1}\, A_c^{j_1}(z_1)$$
$$\times \int_{z_1}^y dz_2\, (y-z_1)_{j_2}\, A_c^{j_2}(z_2) \cdots \int_{z_{k-1}}^y dz_k\, (y-z_k)_{j_k}\, A_c^{j_k}(z_k)\, s(x,y) + \mathcal{O}\big((y-x)^{-2}\big)\,.$$

The obtained nested line integrals can be identified with the summands of the familiar Dyson series. This allows us to carry out the sum over all Feynman diagrams,

$$\chi_c\, \tilde{s}(x,y) = \chi_c\, \mathrm{Pexp}\left(-i \int_x^y (y-x)_j\, A_c^j(z)\, dz\right) s(x,y) \;+\; \mathcal{O}((y-x)^{-2})\,, \tag{2.2.57}$$

where Pexp is defined as follows.

Definition 2.2.5 For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the **ordered exponential** $\mathrm{Pexp}(\int F(\alpha)\, d\alpha)$ is given by the Dyson series

$$\begin{aligned}\mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha\right) &= \mathbb{1} + \int_a^b F(t_0)\, dt_0 \;+\; \int_a^b dt_0\, F(t_0) \int_{t_0}^b dt_1\, F(t_1) \\ &\quad + \int_a^b dt_0\, F(t_0) \int_{t_0}^b dt_1\, F(t_1) \int_{t_1}^b dt_2\, F(t_2) + \cdots\,.\end{aligned}$$

For ordered exponentials over the chiral potentials, we use the short notations

$$\begin{aligned}&\mathrm{Pexp}\left(-i \int_x^y (y-x)_j\, A_c^j(z)\, dz\right) = \mathrm{Pexp}\left(-i \int_x^y A_c^j\, (y-x)_j\right) = \mathrm{Pe}^{-i \int_x^y A_c^j\, (y-x)_j} \\ &:= \mathrm{Pexp}\left(-i \int_0^1 A_c^j|_{\alpha y + (1-\alpha) x}\, (y-x)_j\, d\alpha\right).\end{aligned}$$

Sometimes, we shall find it more convenient to write $\mathrm{Pexp}(\cdots)$ as $\mathrm{Pe}^{(\cdots)}$. For elementary properties of the ordered exponentials we refer to Exercise 2.15. For the general background on the ordered exponential we refer to [RS2, X.12] or to the closely related time-ordered or path-ordered exponential in the physics literature (see for example [PS, Sect. 4.2]). The connection to local gauge transformations is explained in Exercise 2.16.

To lower order on the light cone, the situation clearly is more complicated. The idea is to rearrange the contributions of the light-cone expansion in a such a way that certain subseries can be summed up to again obtain ordered exponentials of the chiral potentials. This idea is made precise in the following proposition and theorem, which we state and explain before giving their proofs.

Note that the partial derivatives in (2.2.32) may be contracted with the factors $y - x$. If this is the case, the corresponding combination

$$(y-x)^j\, \frac{\partial}{\partial z_k^j} \tag{2.2.58}$$

is a derivative in the direction of the vector $y - x$. Since the direction $y - x$ is tangential to the corresponding line integral, such so-called *tangential derivatives* can be rewritten as derivatives with respect to the corresponding integration variable (for details see Exercise 2.17 or the proof of Proposition 2.2.6). Integrating by parts,

the tangential derivatives disappear. Proceeding in this way, one can in fact eliminate all tangential derivatives, as is made precise in the following Proposition.

Proposition 2.2.6 (elimination of tangential derivatives)
Every contribution (2.2.32) to the light cone expansion of Theorem 2.2.4 can be written as a finite sum of expressions of the form

$$\chi_c\, C\, (y-x)^K\; W^{(0)}(x) \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\; W^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \,|\, n_2]\, dz_2\; W^{(2)}(z_2) \\ \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha]\, dz_\alpha\; W^{(\alpha)}(z_\alpha)\; \gamma^J\; S^{(h)}(x,y) \quad (2.2.59)$$

with $\alpha \leq k$, where the factors $W^{(\beta)}$ are composed of the potentials and their partial derivatives,

$$W^{(\beta)} = (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) \quad (2.2.60)$$

with $a_1 = 1$, $a_{\beta+1} = b_\beta + 1$, $b_\beta \geq a_\beta - 1$ (in the case $b_\beta = a_\beta - 1$, $W^{(\beta)}$ is identically one), and $b_\alpha = k$. The parameters l_a, r_a, and n_a are non-negative integers, C is a complex number, and $c = L/R$, $c_a = L/R$ are chiral indices. The potentials $V^{(a)}$ are again given by (2.2.33); their chirality is determined by the rule (i) in Theorem 2.2.4. The tensor indices of the multi-indices J, K, J_a, and K_a are all contracted with each other, according to the rules (a),(b) of Theorem 2.2.4 and

(c) The tensor indices of $(y-x)^K$ are all contracted with the tensor indices of the factors $V^{(a)}_{J_a}$ or γ^J (but not with the factors ∂^{K_a}).

We have the relation

$$2h = k - 1 - |K| + \sum_{a=1}^{k} \big(|K_a| + 2p_a\big)\,. \quad (2.2.61)$$

Before coming to the proof, we make precise how this proposition can be used to simplify the light-cone expansion.

Definition 2.2.7 A contribution of the form (2.2.32) to the light-cone expansion of Theorem 2.2.6 is called **phase-free** if all the tangential potentials $V^{(a)}_{J_a}$ are differentiated, i.e.

$$|K_a| + 2p_a > 0 \qquad \text{whenever} \qquad J_a \text{ is contracted with } (y-x)^K\,.$$

From every phase-free contribution the corresponding **phase-inserted** contribution is obtained as follows: We insert ordered exponentials according to the replacement rule

$$W^{(\beta)}(z_\beta) \;\longrightarrow\; W^{(\beta)}(z_\beta)\; \mathrm{Pexp}\Big(-i \int_{z_\beta}^{z_{\beta+1}} A^{j_\beta}_{c_\beta}\,(z_{\beta+1}-z_\beta)_{j_\beta}\Big), \quad \beta = 0, \ldots, \alpha\,, \tag{2.2.62}$$

where we set $z_0 = x$ and $z_{\alpha+1} = y$. The chiralities c_β are determined by the relations $c_0 = c$ and

$$c_{\beta-1} \text{ and } c_\beta \left\{ \begin{array}{l} \text{coincide} \\ \text{are opposite} \end{array} \right\} \\ \text{if } W^{(\beta-1)} \text{ contains an } \left\{ \begin{array}{l} \text{even} \\ \text{odd} \end{array} \right\} \text{number of factors } Y_\cdot\,. \tag{2.2.63}$$

Theorem 2.2.8 *To every order on the light cone, the number of phase-free contributions is finite. The light-cone expansion of the Green's function $\tilde{s}(x, y)$ is given by the sum of the corresponding phase-inserted contributions.*

This theorem gives a convenient procedure for performing the light-cone expansion of the Green's function. The only task is to compute to any order on the light cone the finite number of phase-free contributions. Then one inserts ordered exponentials according to Definition 2.2.7. Note that this method is constructive in the sense that it gives a procedure with which the light-cone expansion of every Feynman diagram can be carried out explicitly. Indeed, this procedure is implemented in the C++-program class_commute.[2] These computations are illustrated in Exercise 2.18.

The remainder of this section is devoted to the proof of Proposition 2.2.6 and Theorem 2.2.8. We begin with a preparatory lemma which controls the number of tangential derivatives in the contributions (2.2.32) in Theorem 2.2.4.

Lemma 2.2.9 *For any $a \in \{1, \ldots, k\}$, we let t_a be the number of tensor indices of the multi-index I_a in (2.2.32) which are contracted with the factor $(y-x)^l$. Then the following inequalities hold for all $a = 1, \ldots, k$:*

$$l_a + n_a \geq t_a - 1 \qquad \text{and} \qquad r_a + n_a \geq \sum_{b=a}^{k} t_b\,. \tag{2.2.64}$$

Proof As in the proof of Theorem 2.2.4, we proceed inductively in the order k of the perturbation theory. For $k = 0$, the inequalities (2.2.64) are trivially satisfied according to (2.2.37). Assume that (2.2.64) is true for a given k. We go through the construction steps *(1)* to *(7)* of Theorem 2.2.4 and check that the inequalities (2.2.64) then also hold in (2.2.51) for $a = 0, \ldots, k$.

We first consider the case $a > 0$. The parameters l_a, r_a, and n_a remain unchanged in all the construction steps of Theorem 2.2.4. Furthermore, it is obvious that the parameters t_a are not affected in the steps *(1)*, *(2)*, *(4)* and *(7)*. In the steps *(3)* and *(5)*,

[2] The C++ program class_commute and its computational output as well as the resulting Mathematica worksheets were included as ancillary files to the arXiv submission arXiv:1211.3351 [math-ph].

the computation of the derivatives $\Box_z^n$ and $\partial\!\!\!/_x$ might annihilate some of the factors $(y-x)$ which were contracted with the factors $\partial_{z_a}^{I_a}$; this may decrease the parameters t_a. For the analysis of step *(6)*, note that all γ-matrices which are contracted with factors $(y-x)$ stand to the left of those γ-matrices which are contracted with the $\partial_{z_a}^{I_a}$, $a=1,\ldots,k$ (this follows from the ordering condition *(b)* in the induction hypothesis and the fact that additional factors $(y-x)^j \cdots \gamma_j$ are only generated during the construction if the partial derivative $\partial\!\!\!/_x$ hits $S^{(\hat{h})}$ in step *(5)*; in this case, the corresponding γ-matrix stands at the very left in γ^J). Therefore the commutations of the Dirac matrices do not lead to additional contractions between factors $(y-x)$ and $\partial_{z_a}^{I_a}$, which implies that the parameters t_a remain unchanged in step *(6)*. We conclude that the l_a, r_a, and n_a remain unchanged whereas the t_a may only decrease, and thus (2.2.64) holds for $a=1,\ldots,k$ throughout all the construction steps.

It remains to show that the inequalities (2.2.64) hold in (2.2.51) for $a=0$. We first look at the situation after step *(4)* in (2.2.45): The values (2.2.46) for l_0, r_0, and n_0 give in combination with (2.2.43) the equations

$$l_0 + n_0 = \frac{1}{2}\Big(|I| - |\hat{I}| + |I_0| + 2p_0\Big) \tag{2.2.65}$$

$$r_0 + n_0 = h + \frac{1}{2}\Big(|I| + |\hat{I}| + |I_0| + 2p_0\Big)\,. \tag{2.2.66}$$

Moreover, the number of tangential derivatives t_0 at the first potential is clearly bounded by the total number of derivatives there,

$$|I_0| \geq t_0\,. \tag{2.2.67}$$

Furthermore, the total number of tangential derivatives is smaller than the number of factors $(y-x)$,

$$|\hat{I}| \geq \sum_{a=0}^{k} t_a\,. \tag{2.2.68}$$

Substituting (2.2.44) and (2.2.67) into (2.2.65) yields the inequalities

$$l_0 + n_0 \geq |I_0| + p_0 \geq t_0\,. \tag{2.2.69}$$

In order to get a bound for $r_0 + n_0$, we must distinguish two cases. If $h \geq 0$, we substitute (2.2.44) into (2.2.66) and get with (2.2.68) the inequality

$$r_0 + n_0 \geq |\hat{I}| + |I_0| + p_0 \geq |\hat{I}| \geq \sum_{a=0}^{k} t_a\,. \tag{2.2.70}$$

In the case $h=-1$, (2.2.36) shows that $|I_a|$, and consequently also t_a, vanish for $1 \leq a \leq k$. Furthermore, (2.2.34) yields that $|I| \neq 0$. Thus (2.2.66) and (2.2.67), (2.2.68) give the bound

$$r_0 + n_0 \geq h + \frac{|I|}{2} + \frac{1}{2} \sum_{a=0}^{k} t_a + \frac{1}{2}\, t_0 \geq \frac{1}{2} \sum_{a=0}^{k} t_a + \frac{1}{2}\, t_0 \,,$$

where we used in the last inequality that $h + |I|/2 \geq -1/2$ and that all the other terms are integers. Since $t_0 = \sum_{a=0}^{k} t_a$, we conclude that inequality (2.2.70) also holds in the case $h = -1$.

We finally consider how the bounds (2.2.69) and (2.2.70) for $l_0 + n_0$ and $r_0 + n_0$ must be modified in the subsequent construction steps. In step *(5)*, the partial derivative $\partial\!\!\!/_x$ may annihilate a factor $(y-x)$, in which case the parameters t_a might decrease. On the other hand, the partial derivatives $\partial\!\!\!/_x$ may produce an additional factor ∂_{z_0}; in this case, r_0 is incremented according to (2.2.48). In step *(6)*, only this additional factor ∂_{z_0} may be contracted with $(y-x)^{\hat{I}}$. Step *(7)* does not change l_0, r_0, n_0, and t_0. Putting these transformations together, we conclude that the inequality (2.2.69) for $l_0 + n_0$ must be weakened by one, whereas the bound (2.2.70) for $r_0 + n_0$ remains valid as it is. This gives precisely the inequalities (2.2.64) for $a = 0$. □

Proof of Proposition 2.2.6 The basic method for the proof is to iteratively eliminate those partial derivatives $\partial_{z_a}^{I_a}$ in (2.2.32) which are contracted with a factor $(y-x)$. This is accomplished with the integration-by-parts formula

$$\begin{aligned}
&(y-x)^j \int_x^y [l, r \mid n]\, dz\, \partial_j f(z) \overset{(2.2.31)}{=} \int_0^1 d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, \frac{d}{d\alpha} f(\alpha y + (1-\alpha)x) \\
&= \delta_{r+n,0}\, f(y) \;-\; \delta_{l+n,0}\, f(x) \\
&\quad -\; (l+n) \int_x^y [l-1, r \mid n]\, dz\; f(z) \;+\; (r+n) \int_x^y [l, r-1 \mid n]\, dz\; f(z)\,.
\end{aligned}$$

In order to see the main difficulty, we consider the example of two nested line integrals with two tangential derivatives

$$(y-x)^j\, (y-x)^k \int_x^y [0, 1 \mid 0]\, dz_1\; V^{(1)}(z_1) \int_{z_1}^y [0, 1 \mid 0]\, dz_2\; \partial_{jk} V^{(2)}(z_2) \tag{2.2.71}$$

$$= (y-x)^j \int_x^y [0, 0 \mid 0]\, dz_1\; V^{(1)}(z_1)\, (y-z_1)^k \int_{z_1}^y [0, 1 \mid 0]\, dz_2\; \partial_{jk} V^{(2)}(z_2)$$

$$= -(y-x)^j \int_x^y dz_1\; V^{(1)}(z_1)\, \partial_j V^{(2)}(z_1) \tag{2.2.72}$$

$$+\; (y-x)^j \int_x^y dz_1\; V^{(1)}(z_1) \int_{z_1}^y dz_2\; \partial_j V^{(2)}(z_2)\,. \tag{2.2.73}$$

Although the line integrals in (2.2.71) satisfy the conditions of Theorem 2.2.4, the expression cannot be transformed into the required form (2.2.59). Namely, in (2.2.72) we cannot eliminate the remaining tangential derivative (because partial integration would yield a term $(y-x)^j\, \partial_j V^{(1)}(z_1)$). In (2.2.73), on the other hand, we can successfully perform a second partial integration

$$(2.2.73) = \int_x^y [0, -1 \mid 0]\, dz_1\; V^{(1)}(z_1)\; (V^{(2)}(y) - V^{(2)}(z_1))\;,$$

but then the second parameter in the bracket $[.,.\mid.]$ becomes negative. More generally, we must ensure that the boundary terms contain no tangential derivatives, and that the parameters l_a, r_a, and n_a stay positive in the construction.

Since the chirality of the potentials is not affected by the partial integrations, it is obvious that the rule *(i)* in Theorem 2.2.4 will remain valid. For ease in notation, in the remainder of the proof we usually omit the indices c_a.

First of all, we split up the factor $(y-x)^I$ in (2.2.32) in the form $(y-x)^I = (y-x)^K\,(y-x)^L$, where L are those tensor indices which are contracted with the partial derivatives $\partial_{z_a}^{I_a}$, $a = 1, \ldots, k$. Setting $b = 1$ and $z_0 = x$, the first line integral in (2.2.32) can be written as

$$\cdots\; (y - z_{b-1})^L \int_{z_{b-1}}^y [l_b, r_b \mid n_b]\, dz_b\; \partial_{z_b}^{I_b}\, \Box_{z_b}^{p_b}\, V_{J_b}^{(b)}(z_b)\; \cdots\;. \tag{2.2.74}$$

We rewrite the tangential derivatives in this line integral as derivatives in the integration variable,

$$= \cdots\; (y - z_{b-1})^N \int_0^1 d\alpha\; \alpha^l\, (1-\alpha)^r \left(\frac{d}{d\alpha}\right)^q \partial_{z_b}^{K_b}\, \Box_{z_b}^{p_b}\, V_{J_b}^{(b)}(z_b)\; \cdots \tag{2.2.75}$$

with $|L| = |N| + q$ and $l = l_b + n_b$, $r = r_b + n_b$. Lemma 2.2.9 gives the bounds

$$l \geq q - 1 \qquad \text{and} \qquad r \geq q + |N|\;. \tag{2.2.76}$$

More generally, we use (2.2.75) and (2.2.76) as our induction hypothesis, where the left factor '$\cdots$' stands for all previous line integrals (which contain *no* tangential derivatives), and the right factor '$\cdots$' stands for subsequent line integrals. The tensor indices of the factor $(y - z_{a-1})^N$ must all be contracted with the partial derivatives $\partial_{z_a}^{I_a}$ for $a > b$ and thus give tangential derivatives in the subsequent line integrals. The induction step is to show that all the α-derivatives in (2.2.75) can be eliminated, and that we can write the resulting expressions again in the form (2.2.75) and (2.2.76) with b replaced by $b+1$. Under the assumption that this induction step holds, we can eliminate all tangential derivatives in k steps. The resulting expressions are very similar to (2.2.59) and (2.2.60). The only difference is that the derivatives ∂^{K_a} and $\Box^{p_a}$ in the resulting expressions are differential operators acting on all the following factors $V^{(a)}$, $V^{(a+1)}$, ...; in (2.2.60), on the other hand, the partial derivatives act only on the adjacent potential $V^{(a)}$. In order to bring the resulting expressions into the required form, we finally carry out all the derivatives with the Leibniz rule and the chain rule (2.2.48).

For the proof of the induction step, we integrate in (2.2.75) q times by parts (if q is zero, we can skip the partial integrations; our expression is then of the form (2.2.78)). Since the powers of the factors α and $(1-\alpha)$ are decreased at most by one in each

partial integration step, (2.2.76) implies that the boundary values vanish unless in the last step for $\alpha = 0$. We thus obtain a sum of terms of the form

$$\cdots (y - z_{b-1})^N \, \partial_{z_b}^{K_b} \, \Box_{z_b}^{p_b} \, V_{J_b}^{(b)}(z_b) \, \cdots_{|z_b \equiv z_{b-1}} \tag{2.2.77}$$

and

$$\cdots (y - z_{b-1})^N \int_{z_{b-1}}^{y} [l, r \,|\, n = 0] \, dz_b \, \partial_{z_b}^{K_b} \, \Box_{z_b}^{p_b} \, V_{J_b}^{(b)}(z_b) \qquad \text{with } l \geq 0, r \geq |N| \,. \tag{2.2.78}$$

In (2.2.78), we iteratively use the relation

$$(y - x)^j \int_x^y [l, r \,|\, n] \, dz \, \cdots = \int_x^y [l, r - 1 \,|\, n] \, dz \, (y - z)^j \, \cdots$$

to bring all factors $(y - z_{b-1})$ to the right. We thus obtain expressions of the form

$$(2.2.78) = \cdots \int_{z_{b-1}}^{y} [l, r \,|\, n = 0] \, dz_b \, (y - z_b)^N \, \partial_{z_b}^{K_b} \, \Box_{z_b}^{p_b} \, V_{J_b}^{(b)}(z_b) \cdots \quad \text{with } l, r \geq 0 \,. \tag{2.2.79}$$

In both cases (2.2.77) and (2.2.79), we have an expression of the form

$$\cdots (y - z_b)^N \, \partial_{z_b}^{K_b} \, \Box_{z_b}^{p_b} \, V_{J_b}^{(b)}(z_b) \cdots \,, \tag{2.2.80}$$

where the first factor '$\cdots$' stands for line integrals without tangential derivatives, and where none of the factors $(y - z_b)$ are contracted with $\partial_{z_b}^{K_b}$. Applying the "inverse Leibniz rules"

$$(y - x)^j \, \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^k} \, (y - x)^j + \delta_k^j$$
$$(y - x)_j \, \Box_x = \Box_x \, (y - x)_j + 2 \frac{\partial}{\partial x^j} \,,$$

we iteratively commute all factors $(y - z_b)$ in (2.2.80) to the right. This gives a sum of expressions of the form

$$\cdots \partial_{z_b}^{K_b} \, \Box_{z_b}^{p_b} \, V_{J_b}^{(b)}(z_b) \, (y - z_b)^L \cdots \,, \tag{2.2.81}$$

where the factors $(y - z_b)$ are all contracted with the partial derivatives $\partial_{z_a}^{I_a}$, $a = b + 1, \ldots, k$. The Leibniz rules may have annihilated some factors $(y - z_b)$ (i.e., $|L|$ might be smaller than $|N|$); in this case, the parameters t_a, $a = b + 1, \ldots, k$ have decreased. As a consequence, the inequalities of Lemma 2.2.9 are still valid for all expressions (2.2.81). If we write (2.2.81) in the form (2.2.74) with b replaced by

$b+1$, we can thus split up the tangential derivatives in the form (2.2.75) and (2.2.76). This concludes the proof of the induction step.

It remains to derive equation (2.2.61): Note that each integration by parts decreases both the number of factors $(y - z_{a-1})$ and the total number of partial derivatives by one. If we carry out the remaining derivatives with the Leibniz rule (in the last step of the proof), this does not change the total order $\sum_{a=1}^{k} |K_a| + 2p_a$ of the derivatives. Therefore, relation (2.2.34) in Theorem 2.2.4 transforms into (2.2.61). □

We come to the proof of Theorem 2.2.8. A possible method would be to rearrange all the contributions to the light-cone expansion of Theorem 2.2.4 until recovering the Dyson series of the ordered exponentials in (2.2.62). However, this method has the disadvantage of being rather involved. It is more elegant to use a particular form of *local gauge invariance* of the Green's function for the proof (for basics see Exercise 2.16). To this end, for given x and y we will transform the spinors locally. The transformation will be such that the light-cone expansion for the transformed Green's function $\hat{s}(x, y)$ consists precisely of all phase-free contributions. Using the transformation law of the Green's function, we then show that the light-cone expansion of $\tilde{s}(x, y)$ is obtained from that of $\hat{s}(x, y)$ by inserting unitary matrices into the line integrals. Finally, we prove that these unitary matrices coincide with the ordered exponentials in Definition 2.2.7.

In preparation, we consider the transformation law of the Dirac operator and the Green's function under generalized local phase transformations of the spinors. We let $U_L(x)$ and $U_R(x)$ be two unitary matrices acting on the Lie algebra index of the gauge potential. We transform the wave functions according to

$$\psi(x) \to \hat{\psi}(x) = U(x)\, \psi(x) \qquad \text{with} \qquad U(x) = \chi_L\, U_L(x) + \chi_R\, U_R(x)\,. \tag{2.2.82}$$

Thus U_L and U_R transform the left and right handed component of the wave functions, respectively. We point out that transformation U is *not* unitary with respect to the spin scalar product because $\chi_L^* = \chi_R$ and therefore

$$\begin{aligned} V := U^{-1} &= \chi_L\, U_L^{-1} + \chi_R\, U_R^{-1} \qquad \text{but} \\ U^* &= \gamma^0\, U^\dagger\, \gamma^0 = \chi_R\, U_L^{-1} + \chi_L\, U_R^{-1}\,. \end{aligned}$$

Therefore, in what follows we carefully distinguish between U, U^* and their inverses V and V^*. As an immediate consequence of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\psi = 0$, the transformed wave functions $\hat{\psi}$ satisfies the equation

$$V^*(i\partial\!\!\!/ + B)V\, \hat{\psi} = 0\,.$$

A short computation yields for the transformed Dirac operator

$$V^*(i\partial\!\!\!/ + B)V = i\partial\!\!\!/ + \hat{B}$$

with

$$\hat{B} = \chi_L \, (\hat{\not{A}}_R - m \, \hat{Y}_L) \; + \; \chi_R \, (\hat{\not{A}}_L - m \, \hat{Y}_R) \,,$$

where $\hat{A}_{L/R}$ and $\hat{Y}_{L/R}$ are the potentials

$$\hat{A}^j_{L/R} = U_{L/R} \, A^j_{L/R} \, U^{-1}_{L/R} \; + \; i U_{L/R} (\partial^j U^{-1}_{L/R}) \tag{2.2.83}$$

$$\hat{Y}_{L/R} = U_{R/L} \, Y \, U^{-1}_{L/R} \,. \tag{2.2.84}$$

We denote the advanced and retarded Green's functions of the transformed Dirac operator $i\hat{\not{\partial}} + \hat{B}$ by $\hat{s}$. They satisfy the equation

$$\big(i\hat{\not{\partial}}_x \; + \; \hat{B}(x)\big) \, \hat{s}(x, y) = \delta^4(x - y) \,. \tag{2.2.85}$$

Since we can view $\hat{B}$ as the perturbation of the Dirac operator, the Green's function $\hat{s}$ has, in analogy to (2.2.12), the perturbation expansion

$$\hat{s} = \sum_{n=0}^{\infty} (-s \hat{B})^n \, s \,. \tag{2.2.86}$$

The important point for what follows is that the Green's functions $\tilde{s}$ and $\hat{s}$ are related to each other by the local transformation

$$\hat{s}(x, y) = U(x) \, \tilde{s}(x, y) \, U(y)^* \,. \tag{2.2.87}$$

This is verified as follows: The right side of (2.2.87) also satisfies the defining equation (2.2.85) of the Green's functions; namely

$$\begin{aligned}
&(i\hat{\not{\partial}}_x + \hat{B}(x)) \, U(x) \, \tilde{s}(x, y) \, U(y)^* = V(x)^* \, (i\hat{\not{\partial}}_x + B(x)) \, V(x) \, U(x) \, \tilde{s}(x, y) \, U(y)^* \\
&\quad = V(x)^* \, (i\hat{\not{\partial}}_x + B(x)) \, \tilde{s}(x, y) \, U(y)^* = V(x)^* \, \delta^4(x - y) \, U(y)^* \\
&\quad = V(x)^* \, U(x)^* \, \delta^4(x - y) = \delta^4(x - y) \,.
\end{aligned}$$

Furthermore, the supports of both sides of (2.2.87) lie (depending on whether we consider the advanced or retarded Green's functions) either in the upper or in the lower light cone. A uniqueness argument for the solutions of hyperbolic differential equations yields that both sides of (2.2.87) coincide.

We next specify the unitary transformations U_L and U_R: We fix the points x and y. For any point z on the line segment $\overline{xy}$, we chose $U_{L/R}(z)$ as

$$U_{L/R}(z) = \mathrm{Pexp}\left(-i \int_x^z A^j_{L/R} \, (z - x)_j\right) \,. \tag{2.2.88}$$

Using the differential equation for the ordered exponential (see Exercise 2.15)

$$(y-x)^k \frac{\partial}{\partial x^k} \operatorname{Pe}^{-i\int_x^y A_c^j\,(y-x)_j} = i(y-x)_k\; A_c^k(x)\;\operatorname{Pe}^{-i\int_x^y A_c^j\,(y-x)_j}\,, \tag{2.2.89}$$

we obtain

$$\begin{aligned}(y-x)^j\; U_c(z)\,(\partial_j U_c(z)^{-1}) &= \operatorname{Pe}^{-i\int_x^z A_c^k\,(z-x)_k}\;(y-x)^j \frac{\partial}{\partial z^j} \operatorname{Pe}^{-i\int_z^x A_c^k\,(x-z)_k} \\ &= \operatorname{Pe}^{-i\int_x^z A_c^k\,(z-x)_k}\; i(y-x)_j\; A_c^j(z)\;\operatorname{Pe}^{-i\int_z^x A_c^k\,(x-z)_k} \\ &= i(y-x)_j\; U_c(z)\; A_c^j(z)\; U_c(z)^{-1}\,.\end{aligned}$$

Using this formula in (2.2.83) gives

$$\hat{A}^j_{L/R}(z)\;(y-x)_j = 0 \qquad \text{for } z \in \overline{xy}\,. \tag{2.2.90}$$

Thus our choice of U_L and U_R makes the potentials $\hat{A}_L(z)$ and $\hat{A}_R(z)$ for $z \in \overline{xy}$ orthogonal to the vector $(y-x)$. Before going on, we point out that we did not specify $U_{L/R}(z)$ away from the line segment $z \in \overline{xy}$; the unitary transformation $U_{L/R}$ may be arbitrary there. This also implies that also $\hat{A}_{L/R}$ is undetermined outside the line segment $\overline{xy}$. In particular, all the non-tangential derivatives of $\hat{A}_{L/R}(z)$ for $z \in \overline{xy}$ are undetermined. However, (2.2.88) does give constraints for the tangential derivatives. For example, differentiating (2.2.90) in the direction $(y-x)$ yields

$$(y-x)^j\;(y-x)_k\;\partial_j \hat{A}^k_{L/R}(z) = 0 \qquad \text{for} \qquad z \in \overline{xy}\,.$$

We now consider the perturbation expansion (2.2.86). The light-cone expansion of all Feynman diagrams according to Theorem 2.2.4 gives a sum of terms of the form

$$\begin{aligned}&\chi_c\; C\;(y-x)^K\;\hat{W}^{(0)}(x) \int_x^y [l_1, r_1 \mid n_1]\, dz_1\;\hat{W}^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \mid n_2]\, dz_2\;\hat{W}^{(2)}(z_2) \\ &\quad \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \mid n_\alpha]\, dz_\alpha\;\hat{W}^{(\alpha)}(z_\alpha)\;\gamma^J\;S^{(h)}(x,y)\,,\end{aligned} \tag{2.2.91}$$

where the factors $\hat{W}^{(\beta)}$ are of the form

$$\hat{W}^{(\beta)} = (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} \hat{V}^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} \hat{V}^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}})\,. \tag{2.2.92}$$

Because of (2.2.90), all the contributions which are not phase-free vanish. Furthermore, according to Theorem 2.2.4, the contributions (2.2.91) and (2.2.92) contain no tangential derivatives. Clearly, the derivatives in these formulas may have a component in direction of $(y-x)$. But the contribution of the derivatives transversal

to $(y-x)$ uniquely determines the form of each derivative term. Therefore, all the phase-free contributions of the form (2.2.91) and (2.2.92) are independent in the sense that we have no algebraic relations between them. We conclude that, as long as the potentials $\hat{A}_{L/R}$ and $\hat{Y}_{L/R}$ are only specified by (2.2.83), (2.2.84) and (2.2.88), the light-cone expansion (2.2.91) and (2.2.92) consists precisely of all phase-free contributions.

Next, we exploit the local transformation law (2.2.87) of the Green's functions: We solve this equation for $\tilde{s}$,

$$\tilde{s}(x,y) = V(x)\,\hat{s}(x,y)\,V(y)^* . \tag{2.2.93}$$

The transformation $U_{L/R}$ does not enter on the left side of this equation. Thus the right side of (2.2.93) is also independent of $U_{L/R}$. In particular, we conclude that the light-cone expansion of $\hat{s}(x,y)$ must be independent of the derivatives of $U_{L/R}$ along the line segment $\overline{xy}$. At first sight, this might seem inconsistent because the individual contributions (2.2.91) and (2.2.92) do depend on the derivatives of $U_{L/R}$ (this is obvious if one substitutes (2.2.83) and (2.2.84) into (2.2.92) and carries out the derivatives with the Leibniz rule). The right way to understand the independence of $\hat{s}(x,y)$ on the derivatives of $U_{L/R}$ is that all derivative terms of $U_{L/R}$ cancel each other to every order on the light cone if the (finite) sum over all contributions (2.2.91) to the light-cone expansion of $\hat{s}(x,y)$ is carried out. Since we will form the sum over all contributions to the light-cone expansion in the end, it suffices to consider only those contributions to the light-cone expansion which contain no derivatives of $U_{L/R}$. This means that we can substitute (2.2.83) and (2.2.84) into (2.2.92), forget about the derivative term $iU_{L/R}(\partial^j U_{L/R}^{-1})$ in (2.2.83), and pull the unitary transformations $U_{L/R}$, $U_{L/R}^{-1}$ out of the derivatives. In other words, we can replace $\hat{W}^{(\beta)}$, (2.2.92), by

$$\hat{W}^{(\beta)} = U_{d_{a_\beta}} (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) U^{-1}_{c_{a_\beta}} \cdots U_{d_{b_\beta}} (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) U^{-1}_{c_{b_\beta}} \tag{2.2.94}$$

with chiral indices $c_a, d_a = L/R$. The light-cone expansion for $\hat{s}(x,y)$ consists precisely of the sum of all phase-free contributions of the form (2.2.91) and (2.2.94).

The chiralities c_a, d_a of the unitary transformations $U_{L/R}$, $U_{L/R}^{-1}$ in (2.2.94) are determined by the rule *(i)* in Theorem 2.2.4 and by (2.2.83) and (2.2.84). According to this rule, the indices c_{a-1} and c_a coincide iff $V^{(a)}$ is a chiral potential. According to (2.2.83) and (2.2.84), on the other hand, the indices d_a and c_a coincide iff $V^{(a)} = A_{L/R}$. We conclude that the indices c_{a-1} and d_a always coincide. Thus all the intermediate factors $U^{-1}_{c_{a-1}} U_{d_a}$ give the identity, and (2.2.94) simplifies to

$$\hat{W}^{(\beta)} = U_{d_\beta}\, W^{(\beta)}\, U^{-1}_{c_\beta} . \tag{2.2.95}$$

Furthermore, the chiralities d_β and c_β coincide if and only if $W^{(\beta)}$ contains an even number of dynamic mass matrices.

Finally, we substitute the light-cone expansion (2.2.91) for $\hat{s}(x, y)$ as well as (2.2.95) into (2.2.93). This gives for the light-cone expansion of $\tilde{s}(x, y)$ a sum of expressions of the form

$$\chi_c\, C\, (y-x)^K\, U_c^{-1}(x)\, (U_{d_0} W^{(0)} U_{c_0}^{-1})(x) \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\, (U_{d_1} W^{(1)} U_{c_1}^{-1})(z_1) \\ \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha]\, dz_\alpha\, (U_{d_\alpha} W^{(0)} U_{c_\alpha}^{-1})(z_\alpha)\, U_{c_{\alpha+1}}(y)\, \gamma^J\, S^{(h)}(x, y)\,, \tag{2.2.96}$$

where the sum runs over all phase-free contributions of this type. Similar to the considerations before (2.2.95), one sees that adjacent unitary transformations always have the same chirality. Therefore, renaming the chiral indices, the expressions (2.2.96) can be written in the simpler form

$$\chi_c\, C\, (y-x)^K\, W^{(0)}(x) \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\, U_{c_1}(x)^{-1}\, U_{c_1}(z_1)\, W^{(1)} \\ \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha]\, dz_\alpha\, U_{c_\alpha}(z_{\alpha-1})^{-1}\, U_{c_\alpha}(z_\alpha)\, W^{(0)}(z_\alpha)\, U_{c_{\alpha+1}}(z_\alpha)^{-1} \\ \times\, U_{c_{\alpha+1}}(y)\, \gamma^J\, S^{(h)}(x, y)\,,$$

where the chiral indices c_a satisfy the rule (2.2.63). According to (2.2.88), the factors $U_c^{-1}(.)\, U_c(.)$ coincide with the ordered exponentials in (2.2.62). This concludes the proof of Theorem 2.2.8.

2.2.5 *The Residual Argument*

In the previous sections, the light-cone expansion was performed for the causal Green's functions. We now want to extend our methods and results to the fermionic projector. We begin by describing how the light-cone expansion of the Green's functions can be understood in momentum space. Apart from giving a different point of view, this will make it possible to get a connection to the light-cone expansion of the fermionic projector. For notational simplicity, we restrict attention to the case $g = 1$ where in (2.2.26) there is only one direct summand (the generalization to several direct summands is obtained in a straightforward way by replacing all vacuum operators as in (2.2.26) by corresponding direct sums). As in (2.2.10), we again combine the rest mass and the external potential in a potential B. Furthermore, we only consider the advanced Green's function; for the retarded Green's function, the calculation is analogous.

Suppose that we want to perform the light-cone expansion of the kth order contribution to the perturbation series (2.2.12). Using that the Green's function is diagonal in momentum space and that multiplying by B in position space corresponds to a

convolution in momentum space, we can write the contribution as a multiple Fourier integral,

$$\begin{aligned}&\big((-s^\vee B)^k s^\vee\big)(x,y)\\&\quad=\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q_1}{(2\pi)^4}\cdots\int\frac{d^4q_k}{(2\pi)^4}\,\Delta s^\vee(p;q_1,\ldots,q_k)\,e^{-i(p+q_1+\cdots+q_k)x+ipy}\,,\end{aligned}\tag{2.2.97}$$

where the distribution $\Delta s^\vee(p;q_1,\ldots,q_k)$ is the Feynman diagram in momentum space,

$$\begin{aligned}\Delta s^\vee(p;q_1,\ldots,q_k)=(-1)^k\,&s^\vee(p+q_1+\cdots+q_k)\,\hat B(q_k)\,s^\vee(p+q_1+\cdots+q_{k-1})\,\hat B(q_{k-1})\\&\cdots\,\hat B(q_2)\,s^\vee(p+q_1)\,\hat B(q_1)\,s^\vee(p)\end{aligned}\tag{2.2.98}$$

(here $\hat B$ denotes the Fourier transform of the potential B, and $s^\vee(p)$ is the multiplication operator in momentum space). For the arguments of the Green's functions, we introduce the abbreviation

$$p_0:=p\qquad\text{and}\qquad p_l:=p+q_1+\cdots+q_l\,,\quad 1\le l\le k\,.\tag{2.2.99}$$

Substituting the explicit formulas (2.2.4) and (2.2.15) into (2.2.98), we obtain

$$\begin{aligned}\Delta s^\vee(p;q_1,\ldots,q_k)=(-1)^k\,&\not p_k\,\hat B(q_k)\,\not p_{k-1}\,\cdots\,\not p_1\,\hat B(q_1)\,\not p_0\\&\times\lim_{\nu_0,\ldots,\nu_k\searrow 0}\frac{1}{(p_k)^2-i\nu_k p_k^0}\,\frac{1}{(p_{k-1})^2-i\nu_{k-1}p_{k-1}^0}\cdots\frac{1}{(p_0)^2-i\nu_0 p_0^0}\,.\end{aligned}$$

We already know that the limits $\nu_0,\ldots,\nu_k\searrow 0$ exist in the distributional sense. This can be understood directly from the fact that, fixing the momenta $q_1,\ldots q_k$ as well as $\vec p$, the above expression for $\Delta s^\vee$ is a meromorphic function in p^0 having poles only in the lower half plane. Computing the Fourier transform with residues, we obtain a well-defined expression which remains finite as $\nu_0,\ldots,\nu_k\searrow 0$. This consideration also shows that we may choose the $\nu_0,\ldots,\nu_k$ to be equal, i.e.

$$\begin{aligned}\Delta s^\vee(p;q_1,\ldots,q_k)=(-1)^k\,&\not p_k\,\hat B(q_k)\,\not p_{k-1}\,\cdots\,\not p_1\,\hat B(q_1)\,\not p_0\\&\times\lim_{\nu\searrow 0}\frac{1}{(p_k)^2-i\nu p_k^0}\,\frac{1}{(p_{k-1})^2-i\nu p_{k-1}^0}\cdots\frac{1}{(p_0)^2-i\nu p_0^0}\,.\end{aligned}\tag{2.2.100}$$

We now expand the Klein-Gordon Green's functions in (2.2.100) with respect to the momenta p_l-p. If we expand the terms $i\nu p_l^0$ with a geometric series,

$$\frac{1}{(p_l)^2-i\nu p_l^0}=\sum_{n=0}^\infty\frac{(i\nu\,(p_l^0-p^0))^n}{((p_l)^2-i\nu p^0)^{1+n}}\,,$$

all contributions with $n \geq 1$ contain factors ν and vanish in the limit $\nu \searrow 0$. Therefore, we must only expand with respect to the parameters $((p_l)^2 - p^2)$. This gives, again with geometric series,

$$\Delta s^{\vee}(p; q_1, \ldots, q_k) = (-1)^k \not{p}_k \hat{B}(q_k) \not{p}_{k-1} \cdots \not{p}_1 \hat{B}(q_1) \not{p}_0 \times \sum_{n_1,\ldots,n_k=0}^{\infty} (p^2 - p_k^2)^{n_k} \cdots (p^2 - p_1^2)^{n_1} \lim_{\nu \searrow 0} \frac{1}{(p^2 - i\nu p^0)^{1+k+n_1+\cdots+n_k}} .$$

Rewriting the negative power of $(p^2 - i\nu p^0)$ as a mass-derivative,

$$\frac{1}{(p^2 - i\nu p^0)^{1+k+n_1+\cdots+n_k}} = \frac{1}{(k+n_1+\cdots+n_k)!} \left(\frac{d}{da}\right)^{k+n_1+\cdots+n_k} \frac{1}{p^2 - a - i\nu p^0}\bigg|_{a=0} , \tag{2.2.101}$$

we obtain a formula containing only one Green's function. Namely, using the notation (2.2.13), we get

$$\Delta s^{\vee}(p; q_1, \ldots, q_k) = (-1)^k \not{p}_k \hat{B}(q_k) \not{p}_{k-1} \cdots \not{p}_1 \hat{B}(q_1) \not{p}_0 \times \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{(k+n_1+\cdots+n_k)!} (p^2 - p_k^2)^{n_k} \cdots (p^2 - p_1^2)^{n_1} S^{\vee(k+n_1+\cdots+n_k)}(p) . \tag{2.2.102}$$

This is the basic equation for the light-cone expansion of the Green's functions in momentum space. Similar to the light-cone expansion of the previous section, (2.2.102) involves the mass derivatives of the Green's functions $S^{\vee(\cdot)}$. In order to get a connection to the nested line integrals of, say, Theorem 2.2.4, it remains to transform the polynomials in the momenta $p_0, \ldots, p_k$ as follows: Using (2.2.99), we rewrite (2.2.102) in terms of the momenta $p, q_1, \ldots, q_k$ and multiply out. Furthermore, we simplify the Dirac matrices with the anti-commutation rules (2.2.52). This gives for (2.2.102) a sum of terms of the form

$$\chi_c \, C \, \gamma^I \, q_k^{I_k} \cdots q_1^{I_1} \, \tilde{V}^{(k)}_{J_k,c_k}(q_k) \cdots \tilde{V}^{(1)}_{J_1,c_1}(q_1) \, p^L \, S^{\vee(h)}(p) \qquad \big(h \geq \big[|L|/2\big]\big) , \tag{2.2.103}$$

where the tensor indices of the multi-indices I, I_l, J_l, and L are contracted with each other (similar to the notation of Theorem 2.2.4, the factors $\tilde{V}^{(l)}_{J_l,c_l}$ stand for the individual potentials of $\hat{B}$). If tensor indices of the power p^L are contracted with each other, we can eliminate the corresponding factors p^2 iteratively with the rule (2.2.14), more precisely

$$p^2 \, S^{\vee(h)}(p) = h \, S^{\vee(h-1)}(p) \qquad (h \geq 1) . \tag{2.2.104}$$

In this way, we can arrange that the tensor indices of p^L in (2.2.103) are all contracted with tensor indices of the factors γ^I, $q_l^{I_l}$, or $\tilde{V}^{(l)}_{J_l,c_l}$. By iteratively applying the differentiation rule (2.2.16), we can now rewrite the power p^L in (2.2.103) with p-derivatives, e.g.

$$p_j\, p_k\, S^{\vee(2)}(p) = -\frac{1}{2}\, p_j\, \frac{\partial}{\partial p^k} S^{\vee(1)}(p) = -\frac{1}{2}\, \frac{\partial}{\partial p^k}(p_j\, S^{\vee(1)}(p)) + \frac{1}{2}\, g_{jk}\, S^{\vee(1)}(p)$$
$$= \frac{1}{4}\, \frac{\partial^2}{\partial p^j\, \partial p^k} S^{(0)}(p) + \frac{1}{2}\, g_{jk}\, S^{(1)}(p)\,.$$

In this way, we obtain for $\Delta s^\vee(p; q_1, \dots, q_k)$ a sum of terms of the form

$$\chi_c\, C\, \gamma^I\, q_k^{I_k} \cdots q_1^{I_1}\, \tilde{V}^{(k)}_{J_k,c_k}(q_k) \cdots \tilde{V}^{(1)}_{J_1,c_1}(q_1)\, \partial_p^K\, S^{\vee(h)}(p)\,, \tag{2.2.105}$$

where no tensor indices of the derivatives ∂_p^K are contracted with each other. We substitute these terms into (2.2.97) and transform them to position space. Integrating the derivatives ∂_p^K by parts gives factors $(y-x)^K$. The factors $q_l^{I_l}$, on the other hand, can be written as partial derivatives ∂^{I_l} acting on the potentials $V^{(l)}$. More precisely, substituting into (2.2.97), the term (2.2.105) gives the contribution

$$\chi_c\, C\, i^{|I_1|+\cdots+|I_k|}\, (-i)^{|K|}\, \gamma^I\, (\partial^{I_k} V^{(k)}_{J_k,c_k}(x)) \cdots (\partial^{I_1} V^{(1)}_{J_1,c_1}(x))\, (y-x)^K\, S^{\vee(h)}(x,y)\,, \tag{2.2.106}$$

where the tensor indices of the factor $(y-x)^K$ are all contracted with tensor indices of the multi-indices I, I_l, or J_l. The Feynman diagram $((-sB)^k s)(x,y)$ coincides with the sum of all these contributions.

This expansion has much similarity with the light-cone expansion of Theorem 2.2.4. Namely, if one expands the nested line integrals in (2.2.32) in a Taylor series around x, one gets precisely the expansion into terms of the form (2.2.106). Clearly, the light-cone expansion of Theorem 2.2.4 goes far beyond the expansion (2.2.106), because the dependence on the external potential is described by non-local line integrals. Nevertheless, the expansion in momentum space (2.2.102) and subsequent Fourier transformation give an easy way of understanding in principle how the formulas of the light-cone expansion come about. We remark that, after going through the details of the combinatorics and rearranging the contributions (2.2.106), one can indeed recover the Taylor series of the line integrals in (2.2.32). This gives an alternative method for proving Theorem 2.2.4. However, it is obvious that this becomes complicated and does not yield the most elegant approach (the reader interested in the details of this method is referred to [F5], where a very similar technique is used for the light-cone expansion to first order in the external potential).

Next, we want to generalize the previous construction to other types of Green's functions. Since, similar to (2.2.101), we must rewrite a product of Green's functions as the mass derivative of a single Green's function, we can only expect the construction to work if all Green's functions in the product (2.2.98) are of the same type (e.g. the construction breaks down for a "mixed" operator product containing both

advanced and retarded Green's functions). But we need not necessarily work with the advanced or retarded Green's functions. Instead, we can use Green's functions with a different location of the poles in the complex p^0-plane: We consider the Green's functions

$$s^{\pm}(p) = \not{p}\, S^{\pm}_{a\,|\,a=0}(p) \qquad \text{with} \qquad S^{\pm}_a(p) = \lim_{\nu \searrow 0} \frac{1}{p^2 - a \mp i\nu} \tag{2.2.107}$$

and again use the notation (2.2.13),

$$S^{\pm\,(l)} = \left(\frac{d}{da}\right)^l S^{\pm}_{a\,|\,a=0}\,.$$

The distribution s^- is referred to as the *Feynman propagator* (see Exercise 2.3). The perturbation expansion for these Dirac Green's functions is, similar to (2.1.25) or (2.2.12), given by the formal series

$$\tilde{s}^+ := \sum_{n=0}^{\infty} (-s^+\, B)^n s^+ \qquad \text{and} \qquad \tilde{s}^- := \sum_{n=0}^{\infty} (-s^-\, B)^n s^-\,. \tag{2.2.108}$$

The light-cone expansion in momentum space is performed exactly as for the advanced and retarded Green's functions. In analogy to (2.2.97) and (2.2.102), we thus obtain the formula

$$\begin{aligned}&((-s^{\pm}\, B)^k\, s^{\pm})(x,y)\\ &\quad = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q_1}{(2\pi)^4} \cdots \int \frac{d^4q_k}{(2\pi)^4}\, \Delta s^{\pm}(p; q_1, \ldots, q_k)\, e^{-i(p+q_1+\cdots+q_k)x + ipy}\end{aligned}$$

with

$$\begin{aligned}\Delta s^{\pm}(p; q_1, \ldots, q_k) &= (-1)^k\, \not{p}_k\, \hat{B}(q_k)\, \not{p}_{k-1} \cdots \not{p}_1\, \hat{B}(q_1)\, \not{p}_0\\ &\quad \times \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{(k+n_1+\cdots+n_k)!}\, (p^2 - p_k^2)^{n_k} \cdots (p^2 - p_1^2)^{n_1}\, S^{\pm\,(k+n_1+\cdots+n_k)}\,.\end{aligned}$$

Since $S^{\pm}$ are Green's functions of the Klein-Gordon equation, they clearly also satisfy the identity (2.2.104). Furthermore, the differentiation rule (2.2.16) is also valid for $S^{\pm}$; namely

$$\begin{aligned}\frac{\partial}{\partial p^j} S^{\pm\,(l)}(p) &= \left(\frac{d}{da}\right)^l \lim_{\nu \searrow 0} \frac{\partial}{\partial p^j} \left(\frac{1}{p^2 - a \mp i\nu}\right)\bigg|_{a=0}\\ &= \left(\frac{d}{da}\right)^l \lim_{\nu \searrow 0} \frac{-2p_j}{(p^2 - a \mp i\nu)^2}\bigg|_{a=0} = -2p_j\, S^{\pm\,(l+1)}(p)\,.\end{aligned}$$

Therefore we can, exactly as in (2.2.105), rewrite the power p^L with p-derivatives. Thus the expansion (2.2.106) is valid in the same way for the Green's functions $s^\pm$ if one only replaces the index "$\vee$" in (2.2.106) by "$\pm$". As explained before, the expansion (2.2.106) can be obtained from the light-cone expansion of Theorem 2.2.4 by expanding the potentials around the space-time point x. Since the formulas of the light-cone expansion are uniquely determined by this Taylor expansion, we immediately conclude that the statement of Theorem 2.2.4 is also valid for the kth order contribution to the perturbation expansion (2.2.108) if the factor $S^{(h)}$ in (2.2.32) stands more generally for $S^{+\,(h)}$ or $S^{-\,(h)}$, respectively. This simple analogy between the formulas of the light-cone expansions of the Feynman diagrams $((-s^{\vee/\wedge}\, B)^k\, s^{\vee/\wedge})$ and $((-s^\pm\, B)^k\, s^\pm)$, which are obtained by changing the location of the poles of the vacuum Green's functions in momentum space, is called the *residual argument* (the name is motivated by the fact that the effect of changing the location of the poles becomes apparent when taking the Fourier integral with residues).

Having other Green's functions to our disposal, one can also form more general solutions of the homogeneous equation. Namely, taking the difference of s^+ and s^-, we obtain similar to (2.1.13),

$$s^+(q) - s^-(q) = \not{q} \lim_{\nu \searrow 0} \left[\frac{1}{q^2 - i\nu} - \frac{1}{q^2 + i\nu} \right] = 2\pi i\, \not{q}\, \delta(q^2) = 2\pi i\, p(q) \tag{2.2.109}$$

with p according to (2.1.7). Replacing the Green's functions by those in the external potential, one gets a canonical perturbation series for p. As we shall see below (see Sect. 2.2.7), this perturbation series does *not* agree with the causal perturbation expansion (2.1.64). Therefore, we denote the obtained operator with an additional index res. Similar to (2.1.26), we thus introduce the *residual fundamental solution* $\tilde{p}^{\text{res}}$ by

$$\tilde{p}^{\text{res}} := \frac{1}{2\pi i} \left(\tilde{s}^+ - \tilde{s}^- \right). \tag{2.2.110}$$

We now introduce the residual fermionic projector by replacing the operators p_m and k_m in (2.1.6) by the corresponding perturbation series.

Definition 2.2.10 The **residual fermionic projector** $\tilde{P}^{\text{res}}(x, y)$ is defined by

$$\tilde{P}^{\text{res}}(x, y) = \frac{1}{2} \left(\tilde{p}^{\text{res}} - \tilde{k} \right)(x, y), \tag{2.2.111}$$

where the operator $\tilde{p}^{\text{res}}$ is defined in (2.2.110), and $\tilde{k}$ is again given by (2.1.53).

Similar to (2.1.64), the residual fermionic projector also has a contour integral representation (see Exercise 2.19).

Applying the residual argument, the light-cone expansion of the Green's functions immediately carries over to $\tilde{P}^{\text{res}}$: As in (1.2.26) we denote the lower mass shell by T_a, i.e. in momentum space

$$T_a(q) = \Theta(-q^0)\, \delta(q^2 - a)\,. \tag{2.2.112}$$

In analogy to the mass expansion of the Green's functions (2.2.13), we set

$$T^{(l)}_{\text{formal}} = \left(\frac{d}{da}\right)^l T_a\big|_{a=0}\,. \tag{2.2.113}$$

In order not to distract from the main idea, we postpone the analysis of whether these derivatives exist to Sect. 2.2.6. This is why we added the index "formal."

Proposition 2.2.11 *The light-cone expansion for the causal Green's functions also holds for the residual fermionic projector* $\tilde{P}^{res}(x, y)$ *if one simply replaces* $S^{(l)} \to T^{(l)}_{formal}$.

Proof The starting point is the light-cone expansion for the causal Green's functions (see Theorems 2.2.4, 2.2.6 and 2.2.8). By linearity, this light-cone expansion also hold for $\tilde{k}$ defined by (2.1.26), after the replacements

$$S^{(l)} \to \frac{1}{2\pi i}\left(S^{\vee\,(l)} - S^{\wedge\,(l)}\right).$$

Using the residual argument, the light-cone expansion of the Green's functions $\tilde{s}^{\pm}$ is obtained by the replacements $S^{(l)} \to S^{\pm\,(l)}$. It follows by linearity that $\tilde{p}^{\text{res}}$ as defined by (2.2.110) also has a light-cone expansion obtained by the replacements

$$S^{(l)} \to \frac{1}{2\pi i}\left(S^{+\,(l)} - S^{-\,(l)}\right).$$

Finally, again by linearity, we obtain the light-cone expansion of residual fermionic projector (2.2.111) by the replacements

$$S^{(l)} \to \frac{1}{4\pi i}\left(S^{+\,(l)} - S^{-\,(l)} - S^{\vee\,(l)} + S^{\wedge\,(l)}\right).$$

A direct computation in analogy to (2.1.13) and (2.2.109) shows that

$$\frac{1}{4\pi i}\left(S^{+} - S^{-} - S^{\vee} + S^{\wedge}\right) = T_a\,.$$

This concludes the proof. □

We point out that the result of Proposition 2.2.11 is only formal because we have not yet analyzed whether the factors $T^{(l)}_{\text{formal}}$ are mathematically well-defined. This will be done in the next section.

2.2.6 The Non-causal Low Energy Contribution

We now want to put the residual argument and the formal light-cone expansion of Proposition 2.2.11 on a satisfying mathematical basis. In order to explain what precisely we need to do, we first recall how the light-cone expansion of the Green's functions makes mathematical sense: Theorem 2.2.4 gives a representation of every Feynman diagram of the perturbation series (2.2.12) as an infinite sum of contributions of the form (2.2.32). According to the bound (2.2.36), there are, for any given h, only a finite number of possibilities to choose I_a and p_a; as a consequence, we get, for fixed h, only a finite number of contributions (2.2.32). Thus we can write the light-cone expansion in the symbolic form

$$\big((-sB)^k\, s\big)(x,y) = \sum_{h=-1}^{\infty} \sum_{\text{finite}} \cdots\; S^{(h)}(x,y)\,, \tag{2.2.114}$$

where '$\cdots$' stands for a configuration of the γ-matrices and nested line integrals in (2.2.59). According to the explicit formula (2.2.7), the higher a-derivatives of $S_a(x,y)$ contain more factors $(y-x)^2$ and are thus of higher order on the light cone. This makes it possible to make mathematical sense of the infinite series in (2.2.114) as a light-cone expansion.

According to Proposition 2.2.11, all the results for the Green's function are, on a formal level, also valid for the residual fermionic projector. We begin by considering the light-cone expansion of the individual Feynman diagrams in more detail. Similar to (2.2.114), the kth order contribution ΔP^{res} to the residual fermionic projector has an expansion of the form

$$\Delta P^{\text{res}}(x,y) = \sum_{h=-1}^{\infty} \sum_{\text{finite}} \cdots\; T^{(h)}_{\text{formal}}(x,y)\,, \tag{2.2.115}$$

where $T^{(h)}_{\text{formal}}$ is the a-derivative (2.2.113) of the lower mass shell T_a, (2.2.112). In position space, T_a is given explicitly in (2.2.3). The basic difference between the light-cone expansions (2.2.114) and (2.2.115) is related to the logarithmic pole $\log|a|$ in (2.2.3). Namely, as a consequence of this logarithm, the higher a-derivatives of T_a are *not* of higher order on the light cone. To the order $\mathcal{O}((y-x)^2)$, for example, one has

$$\left(\frac{d}{da}\right)^n T_a(x,y) = \frac{1}{32\pi^3}\left(\frac{d}{da}\right)^n (a\,\log|a|) + \mathcal{O}((y-x)^2) \qquad (n \geq 2)\,. \tag{2.2.116}$$

In our context of an expansion around $a=0$, the situation is even worse, because the a-derivatives of T_a are singular for $a \to 0$ (as one sees e.g. in (2.2.116)). Thus not even the individual contributions to the light-cone expansion make mathematical sense. These difficulties arising from the logarithm in (2.2.3) are called the *logarithmic*

mass problem (see [F5] for a more detailed discussion in a slightly different setting). Since we know from Lemma 2.1.2 that the Feynman diagrams are all well-defined, the logarithmic mass problem is not a problem of the perturbation expansion, but shows that something is wrong with the light-cone expansion of Proposition 2.2.11.

In order to resolve the logarithmic mass problem, we first "regularize" the formal light-cone expansion by taking out the problematic $\log|a|$ term. By resumming the formal light-cone expansion, we then show that the difference between the residual Dirac sea and the "regularized" Dirac sea is a smooth function in position space. We introduce the notation

$$T_a^{\text{reg}}(x,y) = T_a(x,y) - \frac{a}{32\pi^3}\,\log|a| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(a\xi^2)^j}{4^j} \tag{2.2.117}$$

$$T^{(l)} = \left(\frac{d}{da}\right)^l T^{\text{reg}}_{a\,|\,a=0} \tag{2.2.118}$$

(where $\xi^2 \equiv \xi^j \xi_j$ denotes again the Minkowski inner product).

Definition 2.2.12 The **causal contribution** $\tilde{P}^{\text{causal}}$ to the fermionic projector is obtained from the residual Dirac sea $\tilde{P}^{\text{res}}$ by replacing all factors $T^{(h)}_{\text{formal}}$ in the formal light-cone expansion by $T^{(h)}$. The **non-causal low energy contribution** $\tilde{P}^{\text{le}}$ to the fermionic projector is given by

$$\tilde{P}^{\text{le}}(x,y) = \tilde{P}^{\text{res}}(x,y) - \tilde{P}^{\text{causal}}(x,y)\,.$$

By the replacement $T^{(h)}_{\text{formal}} \to T^{(h)}$, the formal light-cone expansion of Proposition 2.2.11 becomes mathematically meaningful in the sense of Definition 2.2.1. Thus we can restate this result as a theorem, leaving out the word "formal."

Theorem 2.2.13 *The light-cone expansion for the causal Green's functions also holds for the causal contribution $\tilde{P}^{causal}$ to the fermionic projector if one simply replaces $S^{(l)} \to T^{(l)}$ with $T^{(l)}$ according to (2.2.118).*

Since $T_a - T_a^{\text{reg}}$ is a smooth function in x and y, it is natural to expect that the non-causal low energy contribution should also be smooth. This is indeed the case, in the following sense.

Theorem 2.2.14 *To every order in the external potential $\mathcal{B}$, the non-causal low energy contribution $\tilde{P}^{le}(x,y)$ is a smooth function in x and y.*

The subtle point in the proof is that, to every order in perturbation theory, the non-causal low energy contribution involves an infinite number of summands. Although each summand is smooth, it is not clear whether the infinite sum converges and gives rise to a smooth function. This makes it necessary to use a *resummation technique for the smooth contributions to the light-cone expansion*. For brevity, we do not enter these constructions here but instead refer the interested reader to [F6, Proof of Theorem 3.8]. The resummation technique will also be introduced and applied in Appendix D.

2.2.7 The Non-causal High Energy Contribution

In the previous sections (Sects. 2.2.5 and 2.2.6) we performed the light-cone expansion for the residual fermionic projector $\tilde{P}^{\text{res}}$ (see Definition 2.2.10). The remaining task is to deduce the light-cone expansion of the fermionic projector P^{sea} with spatial normalization (as defined by (2.1.64)). We now prove that P^{sea} and $\tilde{P}^{\text{res}}$ have the same light-cone expansion.

We begin by giving the difference between the fermionic projector and the residual fermionic projector a name.

Definition 2.2.15 The **non-causal high energy contribution** $\tilde{P}^{\text{he}}(x, y)$ to the fermionic projector is given by

$$\tilde{P}^{\text{he}}(x, y) = P^{\text{sea}}(x, y) - \tilde{P}^{\text{res}}(x, y) .$$

Theorem 2.2.16 *To every order in the external potential $\mathcal{B}$, the non-causal high energy contribution $\tilde{P}^{he}(x, y)$ is a smooth function in x and y.*

Proof Our first task is to rewrite the perturbation expansion for $\tilde{P}^{\text{res}}$ in terms of the potential $\mathcal{B}$. To this end, one combines the rest masses of the Dirac particles with the unperturbed Green's functions. Thus for the advanced and retarded Green's functions, we return to the perturbation expansions (2.1.25). Similarly, for the Green's functions $\tilde{s}^{\pm}$, we rewrite (2.2.108) as

$$\tilde{s}^+_m = \sum_{n=0}^{\infty} (-s^+_m \mathcal{B})^n s^+_m \qquad \text{and} \qquad \tilde{s}^-_m = \sum_{n=0}^{\infty} (-s^-_m \mathcal{B})^n s^-_m .$$

Then $\tilde{k}$ and $\tilde{p}^{\text{res}}$ are defined again by (2.1.26) and (2.2.110), respectively. As a result, the operators $\tilde{k}$ and $\tilde{p}^{\text{res}}$ are defined as sums of operator products of the form

$$C_n \mathcal{B} C_{n-1} \mathcal{B} \cdots \mathcal{B} C_0 , \tag{2.2.119}$$

where the factors C_l coincide with either k, p or s.

Next, we need a few structural properties of the causal perturbation expansion. These results are derived in Exercises 2.9–2.11. Alternatively, these results are obvious from the detailed formulas in the research papers [FG1, FT2]. First, the operator $\tilde{k}$ has the contour integral representation (see Exercise 2.9 (a))

$$\tilde{k} = -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} \lambda \, \tilde{R}_\lambda \, d\lambda .$$

As a consequence, the fermionic projector P^{sea}, (2.1.64), can be represented as

$$P^{\text{sea}} = \frac{1}{2} \left(\tilde{p} - \tilde{k} \right) ,$$

where $\tilde{p}$ is defined by

$$\tilde{p} := -\frac{1}{2\pi i}\left(\oint_{\Gamma_+} - \oint_{\Gamma_-}\right) \lambda\, \tilde{R}_\lambda\, d\lambda$$

(see Exercise 2.9 (b)). Comparing with (2.2.111) and Definition 2.2.15, we conclude that

$$\tilde{P}^{\mathrm{he}} = \frac{1}{2}\left(\tilde{p} - \tilde{p}^{\mathrm{res}}\right).$$

Next, the operator $\tilde{p}$ has the following properties:

(i) Every contribution to the perturbation expansion of $\tilde{p}$ contains an even number of factors k.
(ii) If in the perturbation series for $\tilde{p}$ one replaces all factors k by factors p, one gets precisely the perturbation series for $\tilde{p}^{\mathrm{res}}$.

These properties can be read off from the explicit formulas for $\tilde{p}$ and $\tilde{p}^{\mathrm{res}}$ given in [FG1, FT2]. For abstract proofs, one can proceed as follows. Property (i) is shown in Exercise 2.10. In order to prove (ii), we first bring the perturbation expansion for the residual fundamental solution into a more explicit form. Comparing (2.2.110) with (2.1.26) and noting that in view of (2.2.109), the Green's functions $s_m^{\pm}$ satisfy in analogy to (2.1.41) the relations

$$s = s^+ - i\pi p = s^- + i\pi p\,,$$

we find that the perturbation expansion for $\tilde{p}^{\mathrm{res}}$ is obtained from that for $\tilde{k}$, (2.1.46), simply by replacing all factors k by factors p,

$$\tilde{p}^{\mathrm{res}} = \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\, b^<\, p\, (b\, p)^{2\beta}\, b^>\,.$$

In Exercise 2.11 it is shown that exactly the same perturbation series is obtained if in the perturbation series for $\tilde{p}$ one replaces all factors k by factors p. This proves (ii).

Using the above properties (i) and (ii), we can convert the perturbation series for $\tilde{p}$ into that for $\tilde{p}^{\mathrm{res}}$ by iteratively replacing pairs of factors k in the operator products by pairs of factors p. Thus the difference $\tilde{p} - \tilde{p}^{\mathrm{res}}$ can, to every order in perturbation theory, be written as a finite sum of expressions of the form

$$C_n\, \mathcal{B} \cdots C_{b+1}\, \mathcal{B} \Big(p\, \mathcal{B}\, C_{b-1} \cdots C_{a+1}\, \mathcal{B}\, p - k\, \mathcal{B}\, C_{b-1} \cdots C_{a+1}\, \mathcal{B}\, k \Big)\, \mathcal{B}\, C_{a-1} \cdots \mathcal{B}\, C_0\,, \tag{2.2.120}$$

where the factors C_l again stand for k, p or s. Therefore, it remains to show that (2.2.120) is a smooth function in position space.

We first simplify our problem: Once we have shown that the bracket in (2.2.120) is smooth and bounded in position space, the additional multiplications to the very left and right can be carried out by iteratively multiplying with $\mathcal{B}$ and forming the convolution with C_l, which again gives a smooth and bounded function in each step (notice that, according to the assumptions of Lemma 2.1.2, $\mathcal{B}$ decays sufficiently fast at infinity). Thus we must only consider the bracket in (2.2.120). We rewrite this bracket with the projectors $\frac{1}{2}(p-k)$ and $\frac{1}{2}(p+k)$ on the lower and upper mass shells,

$$\begin{aligned} &p\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,p \;-\; k\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,k \\ &\quad= \frac{1}{2}\,(p+k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p-k) \;+\; \frac{1}{2}\,(p-k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p+k)\,. \end{aligned}$$

For symmetry reasons, it suffices to consider the first summand of this decomposition,

$$((p+k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p-k))(x,y)\,, \tag{2.2.121}$$

where the factors C_l again stand for k, p, or s. Our task is to show that (2.2.121) is a smooth function in position space.

We proceed in momentum space. We say that a function $f(q)$ has *rapid decay for positive frequency* if it is C^1, bounded together with its first derivatives (i.e. $\sup|f|, \sup|\partial_l f| < \infty$), and satisfies for every $\alpha > 0$ the bounds

$$\sup_{\omega>0,\,\vec{k}\in\mathbb{R}^3} |\omega^\alpha\, f(\omega,\vec{k})|, \qquad \sup_{\omega>0,\,\vec{k}\in\mathbb{R}^3} |\omega^\alpha\, \partial_l f(\omega,\vec{k})| \;<\; \infty\,. \tag{2.2.122}$$

After setting $C_0 = p-k$ and $C_n = p+k$, the operator product (2.2.121) is of the form (2.1.27). We choose a function g with rapid decay for positive frequency and decompose the operator product in the form (2.1.31), (2.1.32). It follows by induction that the functions F_j all have rapid decay for positive frequency: The induction hypothesis is obvious by setting $F_0 = g$. The induction step is to show that for a function F_{j-1} with rapid decay for positive frequency, the convolution

$$F_j(\omega,\vec{k}) = \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^3}\; \hat{B}(\omega-\omega',\vec{k}-\vec{k}')\; C_{j-1}(\omega',\vec{k}')\; F_{j-1}(\omega',\vec{k}') \tag{2.2.123}$$

also has rapid decay for positive frequency. In Lemma 2.1.2, it was shown that F_j is C^1 and bounded together with its first derivatives. As a consequence, we must only establish the bounds (2.2.122) for $\omega > 1$. Moreover, because of the monotonicity $\omega^\alpha < \omega^\beta$ for $\alpha < \beta$ (and $\omega > 1$), it suffices to show that there are arbitrarily large numbers α satisfying the bounds (2.2.122); we only consider $\alpha = 2n$ with $n \in \mathbb{N}$. For $\omega > 1$ and $\omega' \in \mathbb{R}$, we have the inequality

$$\omega^{2n} \leq (2\omega')^{2n}\,\Theta(\omega') + (2(\omega-\omega'))^{2n}\,,$$

as is immediately verified by checking the three regions $\omega' \le 0$, $0 < \omega' \le \omega/2$, and $\omega' > \omega/2$. We combine this inequality with (2.2.123) and obtain for $\omega > 1$ the estimate

$$|\omega^{2n}\, F_j(\omega, \vec{k})| \le \left| \int \frac{d\omega'}{(2\pi)} \int \frac{d\vec{k}'}{(2\pi)^3} \left(E_1 + E_2\right) \right| , \tag{2.2.124}$$

where E_1 and E_2 are given by

$$E_1 = \hat{B}(\omega - \omega', \vec{k} - \vec{k}')\; C_{j-1}(\omega', \vec{k}') \left[(2\omega')^{2n}\; \Theta(\omega')\; F_{j-1}(\omega', \vec{k}) \right] \tag{2.2.125}$$

$$E_2 = \left[(2(\omega - \omega'))^{2n}\; \hat{B}(\omega - \omega', \vec{k} - \vec{k}') \right]\; C_{j-1}(\omega', \vec{k}')\; F_{j-1}(\omega', \vec{k})\; . \tag{2.2.126}$$

According to the induction hypothesis, the square bracket in (2.2.125) is bounded together with its first derivatives. Since $\hat{B}$ has rapid decay at infinity, the square bracket in (2.2.126) also has rapid decay at infinity. As a consequence, the integral in (2.2.124) satisfies the hypothesis considered in Lemma 2.1.2 for (2.1.29) and is therefore bounded. In order to estimate the expression $|\omega^{2n} \partial_l F_j|$, we differentiate (2.2.123) and obtain similar to (2.2.125) and (2.2.126) the inequality

$$\begin{aligned} &|\omega^{2n}\; \partial_l F_j(\omega, \vec{k})| \\ &\le \left| \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^3}\; \partial_l \hat{B}(\omega - \omega', \vec{k} - \vec{k}')\; C_{j-1}(\omega', \vec{k}') \left[(2\omega')^{2n}\; \Theta(\omega')\; F_{j-1}(\omega', \vec{k}) \right] \right| \\ &\quad + \left| \int \frac{d\omega'}{d\omega} \int \frac{d\vec{k}'}{(2\pi)^3} \left[(2(\omega - \omega'))^{2n}\; \partial_l \hat{B}(\omega - \omega', \vec{k} - \vec{k}') \right]\; C_{j-1}(\omega', \vec{k}')\; F_{j-1}(\omega', \vec{k}) \right| . \end{aligned}$$

This concludes the proof of the induction step.

We just showed that for a function g with rapid decay for positive frequency, the function

$$F_n(q) = \int \frac{d^4 q_1}{(2\pi)^4}\; (\mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, C_0)\; (q, q_1)\; g(q_1) \tag{2.2.127}$$

has rapid decay for positive frequency. We now consider what this means for our operator product (2.2.121) in position space. For a given four-vector $y = (y^0, \vec{y})$, we choose

$$g(\omega, \vec{k}) = \eta(\omega)\; e^{-i(\omega y^0 - \vec{k}\vec{y})} ,$$

where η is a smooth function with $\eta(\omega) = 1$ for $\omega \le 0$ and $\eta(\omega) = 0$ for $\omega > 1$ (this choice of g clearly has rapid decay for positive frequency). Since the support of the factor $C_0 = (p - k)$ is the lower mass cone $\{q^2 \ge 0,\; q^0 \le 0\}$, $g(\omega, \vec{k})$ enters into the integral (2.2.127) only for negative ω. But for $\omega \le 0$, the cutoff function η is identically one. Thus the integral (2.2.127) is simply a Fourier integral; i.e., with a mixed notation in momentum and position space,

$$F_n(q) = (\mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, (p-k))\,(q,y)\,.$$

Next, we multiply from the left with the operator $(p+k)$,

$$((p+k)\, \mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, (p-k))\,(q,y) = (p+k)(q)\; F_n(q)\,. \tag{2.2.128}$$

Since F_n has rapid decay for positive frequency and $(p+k)$ has its support in the upper mass cone $\{q^2 \geq 0,\ q^0 > 0\}$, their product decays fast at infinity. More precisely,

$$\left|q^I\; (p+k)(q)\; F_n(q)\right| \leq \text{const}(I)\; (p+k)(q)$$

for any multi-index I. As a consequence, the Fourier transform of (2.2.128) is even finite after multiplying with an arbitrary number of factors q, i.e.

$$\left| \int \frac{d^4q}{(2\pi)^4}\; q^I\; (p+k)(q)\; F_n(q)\; e^{-iqx} \right| \leq \text{const}(I) \;<\; \infty$$

for all x and I. This shows that our operator product in position space (2.2.121) is bounded and, for fixed y, a smooth function in x (with derivative bounds which are uniform in y). Similarly, one obtains that (2.2.121) is, for fixed x, a smooth function in y. We conclude that the distribution (2.2.121) is a smooth and bounded function. □

2.2.8 *The Unregularized Fermionic Projector in Position Space*

The previous constructions give a representation of the fermionic projector in the presence of chiral and scalar/pseudoscalar potentials (see (2.2.27), (2.2.25) and (2.2.28)) of the form

$$\begin{aligned} P^{\text{sea}}(x,y) = \sum_{n=-1}^{\infty} &(\text{phase-inserted line integrals}) \times T^{(n)}(x,y) \\ &+ \tilde{P}^{\text{le}}(x,y) + \tilde{P}^{\text{he}}(x,y)\,. \end{aligned} \tag{2.2.129}$$

Here the series is a light-cone expansion which describes the singular behavior of the fermionic projector on the light cone non-perturbatively. It is obtained from the light-cone expansion of the Green's functions by the simple replacement rule

$$S^{(n)} \longrightarrow T^{(n)}$$

(with $T^{(n)}$ as defined in (2.2.118)). In particular, the phase-inserted line integrals are exactly the same as those for the Green's functions (see Definition 2.2.7). The

contributions $\tilde{P}^{\text{le}}$ and $\tilde{P}^{\text{he}}$, on the other hand, are both given perturbatively by a series of terms which are all smooth on the light cone. The "causality" of the causal perturbation expansion can be understood from the fact that the phase-inserted line integrals in (2.2.129) are all bounded integrals along the line segment joining the points x and y (whereas the light-cone expansion of general operator products involves unbounded line integrals). In particular, when y lies in the causal future or past of x, the light-cone expansion in (2.2.129) depends on the external potential only inside the causal diamond $(J_x^\vee \cap J_y^\wedge) \cup (J_x^\wedge \cap J_y^\vee)$. Nevertheless, the light-cone expansion is not causal in this strict sense because there are contributions for $y \notin J_x$. Furthermore, the low and high energy contributions cannot be described with line integrals and violate locality as well as causality. This non-locality can be understood from the fact that the fermionic projector is a global object in space-time. We conclude that the singular behavior of the fermionic projector on the light-cone can be described explicitly by causal line integrals, whereas the smooth contributions to the fermionic projector are governed by non-local effects.

We finally remark that the decomposition (2.2.129) is also a suitable starting point for analyzing the smooth contributions to the fermionic projector. Indeed, the low energy contribution $\tilde{P}^{\text{le}}$ can be computed effectively by resumming the perturbation expansion, as is explained in Appendix D. The high energy contribution $\tilde{P}^{\text{he}}$, on the other hand, is given in terms of operator products, which can be analyzed with Fourier methods.

2.3 Description of Linearized Gravity

We now outline how our computational tools apply in the presence of a gravitational field. Note that so far, the external potential $\mathcal{B}$ in the Dirac equation (2.1.5) was assumed to be a multiplication operator. When describing a gravitational field, however, the derivative terms in the Dirac equation are modified. The gravitational field can still be described by the Dirac equation (2.1.5) if we allow for $\mathcal{B}$ to be a first order differential operator. This means that the causal perturbation expansion of Sect. 2.1.6 still applies. An analysis similar to that in Lemma 2.1.2 shows that the contributions to the perturbation series are again all well-defined and finite, provided that the gravitational field is smooth and decays sufficiently fast at infinity. In order to perform the light-cone expansion of the Green's functions, it is convenient to commute the differential operators contained in $\mathcal{B}$ to the very left to obtain operator products of the form

$$\frac{\partial}{\partial x^I} \big[s\, Z_1 \cdots Z_n\, s \big](x, y) \,,$$

where the $Z_1, \ldots, Z_n$ are again multiplication operators (which contain tensor indices contracted with the multi-index I). This makes it possible to perform the light-cone expansion of the square brackets with the inductive procedure described

in Sect. 2.2.2. Carrying out the derivatives ∂_x^l gives the desired light-cone expansion of the Green's function.

The basic difficulty with this construction is that, due to the additional derivatives, the contributions to higher order in perturbation theory become more and more singular on the light cone. In particular, the structural results of Sect. 2.2.3 no longer hold, and the resummation method of Sect. 2.2.4 no longer applies. These difficulties are closely related to the fact that in the presence of a gravitational field, the light cone is no longer the light cone of Minkowski space, but it is generated by the null geodesics of the Lorentzian metric. This "deformation of the light cone" by the gravitational field is an effect which cannot be properly described by a light-cone expansion in Minkowski space. A possible way out is to use to use the non-perturbative construction in [FR3, FMR]. The structure of the singularities on the light-cone can then be analyzed with the so-called Hadamard expansion (for explicit computations for the fermionic projector we refer to [FG2, Appendix A]). Since we do not want to enter these techniques here, we simply describe how *linearized gravity* can be described with our methods. We refer to more details to [F5, Appendix B].

For the metric, we consider a first order perturbation h_{jk} of the Minkowski metric $\eta_{jk} = \text{diag}(1, -1, -1, -1)$,

$$g_{jk}(x) = \eta_{jk} + h_{jk}(x) \,.$$

As in the usual formalism (see for example [LL, Sects. 105 and 107]), we raise and lower tensor indices with the Minkowski metric. Using the transformation of h_{jk} under infinitesimal coordinate transformations, we can assume [LL, Sect. 105] that

$$\partial^k h_{jk} = \frac{1}{2}\, \partial_j h \qquad \text{with} \qquad h := h^k_k \,.$$

A straightforward computation (using for example the formalism introduced in [F4]) shows that in the so-called symmetric gauge, the Dirac operator takes the form

$$i \partial\!\!\!/_x - \frac{i}{2}\, \gamma^j\, h_{jk}\, \eta^{kl}\, \frac{\partial}{\partial x^l} + \frac{i}{8}\, (\partial\!\!\!/ h) \,.$$

In contrast to (2.1.5), now the perturbation itself is a differential operator.

One complication arises from the fact that the integration measure in curved space is $\sqrt{|g|}\, d^4x = (1 + \frac{h}{2})\, d^4x$, whereas the formula (2.1.70) for the perturbation of the fermionic projector is valid only if one has the integration measure d^4x of Minkowski space. Therefore we first transform the system such that the integration measure becomes d^4x, then apply (2.1.70), and finally transform back to the original integration measure $\sqrt{|g|}\, d^4x$. Rewriting the space-time inner product (1.5.2) as

$$\int_{\mathscr{M}} \prec \psi | \phi \succ d\mu(x) = \int_{\mathbb{R}^4} \prec \psi | \phi \succ \sqrt{|g|}\, d^4x = \int_{\mathbb{R}^4} \prec (|g|^{\frac{1}{4}} \psi) \,|\, (|g|^{\frac{1}{4}}\, \phi \succ)\, d^4x \,,$$

the transformation to the measure d^4x is accomplished by

$$\begin{aligned} \psi(x) &\to \hat{\psi}(x) = |g|^{\frac{1}{4}}(x)\, \psi(x) \\ i\partial\!\!\!/_x - \frac{i}{2}\,\gamma^j\, h_j^k\, \partial_k + \frac{i}{8}\,(\partial\!\!\!/ h) &\to |g|^{\frac{1}{4}} \left(i\partial\!\!\!/_x - \frac{i}{2}\,\gamma^j\, h_j^k\, \partial_k + \frac{i}{8}\,(\partial\!\!\!/ h) \right) |g|^{-\frac{1}{4}} \\ &= i\partial\!\!\!/_x - \frac{i}{2}\,\gamma^j\, h_j^k\, \partial_k - \frac{i}{8}\,(\partial\!\!\!/ h)\,. \end{aligned}$$

The perturbation $\Delta P^{(d^4x)}$ of the transformed system is given by (2.1.70),

$$\begin{aligned} \Delta P^{(d^4x)}(x,y) = -\int d^4z \,\Big\{ & s(x,z)\Big(-\frac{i}{2}\,\gamma^j\, h_j^k \frac{\partial}{\partial z^k} - \frac{i}{8}(\partial\!\!\!/ h)(z)\Big) P(z,y) \\ & + P(x,z)\Big(-\frac{i}{2}\,\gamma^j\, h_j^k \frac{\partial}{\partial z^k} - \frac{i}{8}(\partial\!\!\!/ h)(z)\Big) s(z,y) \Big\}\,. \end{aligned} \tag{2.3.1}$$

The formula for the transformation of the Dirac sea to the original integration measure $\sqrt{|g|}\, d^4x$ is

$$P(x,y) \;+\; \Delta P(x,y) = |g|^{-\frac{1}{4}}(x)\, |g|^{-\frac{1}{4}}(y) \Big(P(x,y) + \Delta P^{(d^4x)}(x,y) \Big)\,.$$

Thus

$$\Delta P(x,y) = \Delta P^{(d^4x)}(x,y) - \frac{1}{4}\,(h(x)+h(y))\; P(x,y)\,.$$

Since the factors $P(z,y)$ and $s(z,y)$ in (2.3.1) only depend on the difference vector $z-y$, we can rewrite the z-derivatives as y-derivatives,

$$\frac{\partial}{\partial z^k} P(z,y) = -\frac{\partial}{\partial y^k} P(z,y)\,, \qquad \frac{\partial}{\partial z^k} s(z,y) = -\frac{\partial}{\partial y^k} s(z,y)\,,$$

which can be pulled out of the integral. Furthermore, the relations

$$\begin{aligned} \int d^4z\; P(x,z)\, \big(i\partial\!\!\!/_z h(z)\big)\, s(z,y) &= \int d^4z\; P(x,z)\, \big[(i\partial\!\!\!/_z - m),\; h(z)\big]\, s(z,y) \\ &= -P(x,y)\, h(y) \\ \int d^4z\; s(x,z)\, \big(i\partial\!\!\!/_z h(z)\big)\; P(z,y) &= h(x)\; P(x,y) \end{aligned}$$

make it possible to simplify the factors $(\partial\!\!\!/ h)$ in the integral. In the resulting formula for $\Delta P(x,y)$, one recovers the perturbation by an electromagnetic potential. More precisely,

$$\Delta P(x,y) = \Big(-\frac{1}{8}\, h(x) \;-\; \frac{3}{8}\, h(y) \Big)\, P(x,y) \;-\; \frac{i}{2}\, \frac{\partial}{\partial y^k}\, \Delta P[\gamma^j h_j^k](x,y)\,, \tag{2.3.2}$$

where $\Delta P[\gamma^j h_j^k](x,y)$ is the perturbation (2.1.70) of the Dirac sea corresponding to the electromagnetic potential $\mathcal{B} = \gamma^j h_j^k$. The light-cone expansion of $\Delta P(x,y)$ is obtained by substituting the light-cone expansion of $\Delta P[\gamma^j h_j^k](x,y)$ into (2.3.2) and computing the y-derivative.

2.4 The Formalism of the Continuum Limit

In Sect. 2.2 we developed a method for analyzing the unregularized kernel of the fermionic projector in position space (see the summary in Sect. 2.2.8). Our next goal is to extend these methods in order to include an *ultraviolet regularization*. Following the method of variable regularization (see Remark 1.2.1), the allowed class of regularizations should be as large as possible. Moreover, we need to analyze in detail how the causal action and the corresponding EL equations depend on the regularization. As we shall see, these issues can be treated conveniently in the so-called *formalism of the continuum limit*, which is also most suitable for explicit computations.

The formalism of the continuum limit was first introduced in [F7, Chap. 4], based on earlier considerations in the unpublished preprint [F1]. In particular, the analysis in [F7, Sects. 4.3–4.5] puts the formalism on a rigorous basis. For better readability, we here follow the original ideas in [F1] and develop the formalism from a more computational perspective. This makes it possible to explain the main points of the formalism in a non-technical way. Generalizing the concepts, we then obtain the formalism of the continuum limit. In order avoid repetitions, we only outline the general derivation and refer the reader interested in the details to [F7, Sects. 4.3–4.5] and Appendix F.

2.4.1 Example: The $i\varepsilon$-Regularization

In Sect. 1.2 we introduced the UV regularization in Minkowski space using general regularization operators (see Definition 1.2.3 and the resulting regularized kernel in Proposition 1.2.7). In order to get a better idea of what the effect of the regularization is, we now consider an explicit example. To this end, we assume that the regularized kernel of the fermionic projector, denoted again by $P^\varepsilon(x,y)$, is *homogeneous* in the sense that it depends only on the difference vector $\xi := y - x$. Then the kernel can be written as a Fourier integral

$$P^\varepsilon(x,y) = \int \frac{d^4k}{(2\pi)^4}\; \hat{P}^\varepsilon(k)\, e^{-ik(x-y)} \tag{2.4.1}$$

with a distribution $\hat{P}^\varepsilon(k)$. From the computational point of view, the simplest possible regularization method is to modify the unregularized kernel (1.2.23) by inserting a convergence-generating exponential factor. This leads us to choosing

$$\hat{P}^\varepsilon(k) = (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, \exp\left(\varepsilon k^0\right), \tag{2.4.2}$$

where $\varepsilon > 0$ is the regularization length. The convergence-generating factor ensures that the Fourier integral (2.4.1) converges pointwise for any vector $\xi \in \mathscr{M}$. Moreover, differentiating (2.4.1) with respect to x or y gives rise to powers of k. Since these polynomial factors are dominated by the convergence-generating exponential factor, the Fourier integral again converges pointwise. We thus conclude that $P^\varepsilon(x, y)$ is a smooth function,

$$P^\varepsilon(.,.) \in C^\infty(\mathscr{M} \times \mathscr{M}) .$$

Therefore, all composite expressions in the fermionic are well-defined (like the closed chain (1.1.14), its eigenvalues $\lambda_1^{xy}, \ldots \lambda_{2n}^{xy}$, the Lagrangian (1.1.9), the integrands in (1.1.4) and (1.1.5) as well as the kernel $Q(x, y)$ in (1.4.16)). But clearly, the singularities on the light-cone reappear in the limit $\varepsilon \searrow 0$, and the composite expressions will diverge. In other words, the limit $\varepsilon \searrow 0$ is a *singular limit*. Our goal is to analyze this singular limit in detail.

The effect of the convergence-generating factor in (2.4.2) can be described conveniently in position space. Namely, introducing the short notations

$$\omega = k^0 \qquad \text{and} \qquad \xi = (t, \vec{x}) ,$$

one can combine the exponential with the phase factor of the Fourier transform,

$$\exp(\varepsilon k^0)\, e^{ik\xi} = e^{i\omega(t - i\varepsilon) - i\vec{k}\vec{x}} .$$

This shows that the regularization amounts to the replacement

$$t \to t - i\varepsilon . \tag{2.4.3}$$

This simple replacement rule motivates the name $i\varepsilon$-**regularization**.

In order to illustrate how to work with this regularization, we next derive explicit formulas for the fermionic projector of the vacuum with this regularization. Our starting point is the light-cone expansion of the unregularized fermionic projector (2.2.129). More specifically, pulling the Dirac matrices out of the Fourier integral (1.2.25) and expanding in a Taylor series in the mass parameter using (2.2.118) and (2.2.117), we obtain

$$P^{\text{vac}}(x, y) = (i\partial\!\!\!/ + m)\, T_{m^2} = (i\partial\!\!\!/ + m) \left(\sum_{n=0}^{\infty} \frac{m^{2n}}{n!}\, T^{(n)} + (\text{smooth contributions}) \right).$$

Since we are again interested mainly in the behavior of the singularities, for simplicity we shall disregard the smooth contributions. Clearly, such smooth contributions are important, and they also affect the singularities of composite expressions on the light cone (for example if multiplied by a singular contribution when forming the closed chain). But of course, smooth contributions can be treated in composite expressions in a straightforward way. Therefore, we now focus on the singularities and do all computations modulo smooth contributions. Then the residual argument shows that the $T^{(n)}$ satisfy the same computation rules as the Green's functions in (2.2.17) and (2.2.19),

$$\frac{\partial}{\partial x^k} T^{(l)}(x,y) = -\frac{\partial}{\partial y^k} T^{(l)}(x,y) = \frac{1}{2}\, \xi_k\, T^{(l-1)}(x,y) \tag{2.4.4}$$

(again valid up to smooth contributions; for an explicit derivation see Exercise 2.21). We thus obtain the light-cone expansion

$$P^{\text{vac}}(x,y) = \frac{i\not{\xi}}{2} \sum_{n=0}^{\infty} \frac{m^{2n}}{n!}\, T^{(-1+n)} + \sum_{n=0}^{\infty} \frac{m^{2n+1}}{n!}\, T^{(n)} \tag{2.4.5}$$

(where in analogy to (2.2.24) we use (2.4.4) to *define* $T^{(-1)}$).

The next step is to apply the replacement rule (2.4.3). The factor $\not{\xi}$ becomes

$$\not{\xi} \to \not{\xi}^{\varepsilon} := (t - i\varepsilon)\gamma^0 - \vec{\xi}\vec{\gamma}\,. \tag{2.4.6}$$

In order to regularize the factors $T^{(l)}$, we first note that, applying the replacement rule (2.4.3) to the distribution T_a computed in Lemma 1.2.9, one really obtains a smooth function. Moreover, using the series expansion (2.2.3), one can compute the factors $T^{(n)}$ as defined by (2.2.118) and (2.2.117). When doing so, it is most convenient to combine the principal part with the δ-contribution as well as the logarithm with the Heaviside function by using the identities

$$\begin{aligned} \frac{\text{PP}}{\xi^2} + i\pi\delta\big(\xi^2\big)\,\epsilon\big(\xi^0\big) &= \lim_{\nu \searrow 0} \frac{1}{\xi^2 - i\nu\xi^0} \\ \log\big|(y-x)^2\big| + i\pi\,\Theta\big(\xi^2\big)\,\epsilon\big(\xi^0\big) &= \lim_{\nu \searrow 0} \Big(\log\big(\xi^2 - i\nu\xi^0\big) - i\pi \Big), \end{aligned}$$

where the logarithm is understood in the complex plane which is cut along the positive real axis such that $\lim_{\nu \searrow 0} \log(x + i\nu)$ is real for $x > 0$. This gives

$$T^{(0)} \to -\frac{1}{8\pi^3}\, \frac{1}{(t-i\varepsilon)^2 - \big|\vec{\xi}\big|^2} \tag{2.4.7}$$

$$T^{(1)} \to \frac{1}{32\pi^3}\, \log\Big((t-i\varepsilon)^2 - \big|\vec{\xi}\big|^2 \Big)\,, \tag{2.4.8}$$

and similar for the other distributions $T^{(n)}$. These replacement rules are compatible with our earlier computation rules like (2.4.4). These rules can also be used to compute $T^{(-1)}$ via (2.2.24) to obtain

$$T^{(-1)} \to -\frac{1}{2\pi^3} \frac{1}{\big((t-i\varepsilon)^2 - r^2\big)^2}\,, \tag{2.4.9}$$

where we set $r = |\vec{\xi}|$.

Next, in (2.4.5) we apply the replacement rule (2.4.6) and replace the factors $T^{(l)}$ according to rules like (2.4.7)–(2.4.9). We thus obtain the regularized fermionic projector of the vacuum $P^\varepsilon(x,y)$. The kernel $P^\varepsilon(y,x)$ is obtained by taking the conjugate with respect to the spin scalar product (see (1.1.15) or (2.1.68)). Then one can form the closed chain A_{xy} by (1.1.14) and compute all other quantities of interest. In order to give a concrete example, let us consider the massless case. Then

$$\begin{aligned}
P(x,y) &= \frac{i}{2}\, \not{\xi}\, T^{(-1)} \qquad \text{and thus} \\
P^\varepsilon(x,y) &= -\frac{i}{4\pi^3} \frac{(t-i\varepsilon)\gamma^0 - \vec{\xi}\vec{\gamma}}{\big((t-i\varepsilon)^2 - r^2\big)^2} \\
P^\varepsilon(y,x) &= P^\varepsilon(x,y)^* = \frac{i}{4\pi^3} \frac{(t+i\varepsilon)\gamma^0 - \vec{\xi}\vec{\gamma}}{\big((t+i\varepsilon)^2 - r^2\big)^2} \\
A^\varepsilon_{xy} &= P^\varepsilon(x,y)\, P^\varepsilon(y,x) \\
&= \frac{1}{16\pi^6} \frac{1}{\big|(t-i\varepsilon)^2 - r^2\big|^4} \Big((t-i\varepsilon)\gamma^0 - \vec{\xi}\vec{\gamma}\Big)\Big((t+i\varepsilon)\gamma^0 - \vec{\xi}\vec{\gamma}\Big)\,.
\end{aligned}$$

Simplifying the Dirac matrices according to

$$\big(\not{\xi} - i\varepsilon\gamma^0\big)\big(\not{\xi} + i\varepsilon\gamma^0\big) = \xi^2 - i\varepsilon[\gamma^0, \not{\xi}] + \varepsilon^2\,, \tag{2.4.10}$$

we obtain

$$A^\varepsilon_{xy} = \frac{1}{16\pi^6} \frac{(t^2 - r^2) - i\varepsilon[\gamma^0, \not{\xi}] + \varepsilon^2}{\big|(t-i\varepsilon)^2 - r^2\big|^4}\,. \tag{2.4.11}$$

In order to compute the eigenvalues of this matrix, the task is to diagonalize the bilinear contribution $i\varepsilon[\gamma^0, \not{\xi}]$. The calculation

$$\big(i\varepsilon[\gamma^0, \not{\xi}]\big)^2 = -4\varepsilon^2\, \gamma^0(\vec{\xi}\vec{\gamma})\gamma^0(\vec{\xi}\vec{\gamma}) = 4\varepsilon^2\, \gamma^0\gamma^0(\vec{\xi}\vec{\gamma})(\vec{\xi}\vec{\gamma}) = -4\varepsilon^2\, |\vec{\xi}|^2 < 0$$

shows that this bilinear contribution has complex eigenvalues. Thus the *regularization makes the spacelike region larger*. As we shall see below, this happens in a much more general setting. It is a desirable effect because it decreases the causal action.

Clearly, the singular behavior of the resulting expressions in the limit $\varepsilon \searrow 0$ is rather complicated. However, one limiting case, which will be important later on, is relatively easy to handle. This limiting case is to consider the region *close to the light cone* and *away from the origin*. For simplicity, we restrict attention to the *upper light cone* $t \approx r$ (but clearly, the lower light cone can be treated similarly). Then "close to the light cone" means that $t - r$ is much smaller than r, whereas "away from the origin" means that ε is much smaller that r. Under these assumptions, we have approximately

$$(t - i\varepsilon)^2 - r^2 = (t + r - i\varepsilon)(t - r - i\varepsilon) \approx 2r\,(t - r - i\varepsilon)\,.$$

In order to make the approximation precise, we write the error term as

$$(t - i\varepsilon)^2 - r^2 = 2r\,(t - r - i\varepsilon)\left(1 + \mathcal{O}\Big(\frac{t-r}{r}\Big) + \mathcal{O}\Big(\frac{\varepsilon}{r}\Big)\right). \tag{2.4.12}$$

Computing up to error terms of this type, the above formulas (2.4.7)–(2.4.9) can be simplified to

$$T^{(0)} \to -\frac{1}{8\pi^3}\,\frac{1}{2r\,(t - r - i\varepsilon)} \tag{2.4.13}$$

$$T^{(1)} \to \frac{1}{32\pi^3}\,\log\big(2r\,(t - r - i\varepsilon)\big) \tag{2.4.14}$$

$$T^{(-1)} \to -\frac{1}{8\pi^3\, r^2}\,\frac{1}{(t - r - i\varepsilon)^2}\,. \tag{2.4.15}$$

Using this approximation, the closed chain (2.4.11) simplifies to

$$A^\varepsilon_{xy} = \frac{1}{256\pi^6\, r^4}\,\frac{(t^2 - r^2) - i\varepsilon[\gamma^0, \not\xi] + \varepsilon^2}{|t - r - i\varepsilon|^4}\,.$$

Moreover, the numerator can be further simplified. We first note that, since ξ is close to the light cone, the factor ξ^2 can be arbitrarily small. Therefore, despite the factor ε, the summand $\varepsilon[\gamma^0, \not\xi]$ cannot be left out. But the summand ε^2 is of higher order in ε/r and can be omitted. We conclude that

$$A^\varepsilon_{xy} = \frac{1}{256\pi^6\, r^4}\,\frac{(t^2 - r^2) - i\varepsilon[\gamma^0, \not\xi]}{|t - r - i\varepsilon|^4}\left(1 + \mathcal{O}\Big(\frac{t-r}{r}\Big) + \mathcal{O}\Big(\frac{\varepsilon}{r}\Big)\right). \tag{2.4.16}$$

Clearly, composite expressions diverge in the limit $\varepsilon \searrow 0$. In order to analyze this singular behavior, the proper method is to evaluate weakly in t for fixed r. Thus one considers integrals of the form

$$\int_{-\infty}^{\infty} \eta(t)\,\big(\cdots\big)\,dt \tag{2.4.17}$$

for a smooth test function η, where "$\cdots$" stands for a composite expression in the $T^{(n)}$ and $\overline{T^{(n)}}$. Then "$\cdots$" is a meromorphic function in t with poles at $t = \pm r \pm i\varepsilon$. This makes it possible to compute the integral with the help of residues. The reader interested in an explicit example is referred to Exercise 2.20. Here we proceed by compiling and explaining a few general conclusions which will be important later on.

(a) The integrand in (2.4.17) has poles at $t = \pm r \pm i\varepsilon$. Again restricting attention to the upper light cone, we only need to consider the poles at $t = r \pm i\varepsilon$. When computing the residues at these points, the variable $t - r$ is of the order ε. Therefore, the two error terms in (2.4.12) become the same. For convenience, we usually write the error terms as

$$\cdots + (\text{higher orders in } \varepsilon/|\vec{\xi}|)\,. \tag{2.4.18}$$

Moreover, the theorem of residues gives rise to contributions where the test function η is differentiated. Every such derivative gives rise to an additional factor of ε. In order to keep the dimensions of length, we write the resulting error terms in the form

$$+ (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,, \tag{2.4.19}$$

where ℓ_{macro} denotes the "macroscopic" length scale on which η varies.

(b) The scaling of the integral (2.4.17) in ε and r can be described by

$$T^{(n)} \sim \big(\varepsilon\,|\vec{\xi}|\big)^{n-1} \qquad \text{and} \qquad dt \sim \varepsilon\,. \tag{2.4.20}$$

The resulting scaling of a composite expression in powers of $1/(\varepsilon\,|\vec{\xi}|)$ is referred to as the *degree* of the expression. One should carefully distinguish the powers of $1/(\varepsilon\,|\vec{\xi}|)$ defining the degree from the factors $\varepsilon/|\vec{\xi}|$ appearing in the error terms in (2.4.18). To make this distinction, it is important that we have two independent variables ε and $|\vec{\xi}|$, and that we consider the scaling behavior in both variables. In this way, when evaluating a sum of expressions of different degrees, our methods make it possible to evaluate each degree separately, each with error terms of the form (2.4.18) and (2.4.19).

(c) The scaling behavior of the factors ξ^ε is more subtle, as we now explain. If a factor ξ^ε is contracted to Dirac matrices or to a macroscopic function (like a gauge potential or the Dirac current), we may simply disregard the regularization (2.4.6), i.e.

$$\begin{aligned} \not{\xi}^\varepsilon &= \not{\xi} + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \\ \xi^\varepsilon_j\, f^j &= \xi_j\, f^j + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \end{aligned}$$

(where f_j is a macroscopic vector field). We refer to such factors ξ^ε as *outer factors*.

(d) Two factors ξ^ε which are contracted to each other are called *inner factors*. Since the resulting function ξ^2 is very small on the light cone, the factor ε in (2.4.6) must be taken into account, i.e.

$$\left(\xi^\varepsilon\right)^2 = (t - i\varepsilon)^2 - |\vec{\xi}|^2 = t^2 - |\vec{\xi}|^2 - 2i\varepsilon t - \varepsilon^2 \,. \tag{2.4.21}$$

But similar as in (2.4.12), the quadratic term in ε may be dropped, i.e.

$$\left(\xi^\varepsilon\right)^2 = t^2 - |\vec{\xi}|^2 - 2i\varepsilon t + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \,. \tag{2.4.22}$$

The general rule is that in every contraction, the factors $i\varepsilon$ must be taken into account linearly. This means in particular that the regularized factors ξ^ε are no longer real, but must be treated as complex-valued vectors. Taking their complex conjugate corresponds to flipping the sign of ε, i.e.

$$\overline{\xi^\varepsilon} = (t + i\varepsilon, \vec{\xi}) \,.$$

Taking the adjoint of $\not{\xi}^\varepsilon$ (with respect to the spin scalar product), we need to take the complex conjugate of ξ^ε, i.e.

$$\left(\overline{\xi}^\varepsilon\right)^* = \overline{\not{\xi}^\varepsilon} \,.$$

One must carefully distinguish ξ^ε and $\overline{\xi^\varepsilon}$ in all computations.

(e) Clearly, a factor ξ^ε may also be contracted to a factor $\overline{\xi}^\varepsilon$, or two factors $\overline{\xi}^\varepsilon$ may be contracted to each other. In these cases, we again refer to the factors ξ^ε and $\overline{\xi}^\varepsilon$ as *inner factors*. Since we only take into account ε linearly, we get

$$\begin{aligned} \left(\overline{\xi}^\varepsilon\right)^2 &= t^2 - |\vec{\xi}|^2 + 2i\varepsilon t + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \\ (\xi^\varepsilon)_j \, (\overline{\xi}^\varepsilon)^j &= t^2 - |\vec{\xi}|^2 + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \,. \end{aligned}$$

Comparing these formulas with (2.4.22), one sees that

$$(\xi^\varepsilon)_j \, (\overline{\xi}^\varepsilon)^j = \frac{1}{2}\Big(\left(\xi^\varepsilon\right)^2 + \left(\overline{\xi}^\varepsilon\right)^2\Big) + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \,. \tag{2.4.23}$$

This identity will appear later in a much more general context as the so-called *contraction rule*.

After applying this contraction rule, one gets products of the form $(\xi^\varepsilon)^2 \, T^{(l)}$. We remark that such products can be further simplified. Namely, according to the residual argument, the rule (2.2.18) also holds for $S^{(l)}$ replaced by $T^{(l)}$, up to smooth contributions. In fact, this rule even holds with regularization, i.e. for all $l \geq 0$

$$\big(\xi^{(p)}\big)^2\, T^{(l)} = -4p\, T^{(l-1)} + (\text{smooth contributions}) \tag{2.4.24}$$

(the smooth contributions are of course important, but they can be treated together with the other smooth contributions to the fermionic projector as outlined in Sect. 2.2.8). The reader interested in the details of the derivation of the identity (2.4.24) is referred to Exercise 2.21.

(f) We mention one more structure which in the present example is easy to understand, and which will come up in a more general context later on. Namely, suppose that the composite expression "$\cdots$" in (2.4.17) can be written as a time derivative. Then we can integrate by parts,

$$\int_{-\infty}^{\infty} \eta(t)\, \frac{\partial F}{\partial t}\, dt = -\int_{-\infty}^{\infty} \big(\partial_t \eta(t)\big)\, F(t)\, dt\,.$$

Since derivatives of the test function scale like factors $1/\ell_{\text{macro}}$, this contribution is much smaller than expected from the scalings (2.4.20). We write

$$\int_{-\infty}^{\infty} \eta(t)\, \frac{\partial F}{\partial t}\, dt = 0 \;+\; (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,. \tag{2.4.25}$$

This relation shows that certain composite expressions in the factors $T^{(n)}$ and $\overline{T^{(n)}}$ vanish when evaluated weakly on the light cone. In other words, there are relations between composite expressions.

These relations are expressed most conveniently in terms of so-called *integration-by-parts rules*. The starting point for deriving these rules is the identity (2.4.4) which holds up to smooth contributions, i.e. for all $l \geq 0$

$$\frac{\partial}{\partial x^k} T^{(l)}(x,y) = \frac{1}{2}\,(y-x)_k\, T^{(l-1)}(x,y) + (\text{smooth contributions}) \tag{2.4.26}$$

(recall that in the case $l = 0$, this relation serves as the *definition* of $T^{(-1)}$). For an explicit derivation of the identity (2.4.26) we again refer to Exercise 2.21. Considering a derivative in time direction (and noting that $\partial_t = -\partial_{x^0}$), we obtain

$$\frac{\partial}{\partial t} T^{(l)}(x,y) = -\frac{1}{2}\, t\, T^{(l-1)}(x,y) + (\text{smooth contributions})\,.$$

Near the upper light cone, we can write this identity as

$$\frac{1}{r}\,\frac{\partial}{\partial t} T^{(l)}(x,y) = -\frac{1}{2}\, T^{(l-1)}(x,y) + (\text{smooth contributions}) + (\text{higher orders in } \varepsilon/|\vec{\xi}|)\,.$$

Introducing the abbreviation

$$\nabla := \frac{1}{t} \frac{\partial}{\partial t} , \tag{2.4.27}$$

we thus obtain the relations

$$\nabla T^{(l)} = -\frac{1}{2}\, T^{(l-1)} . \tag{2.4.28}$$

Moreover, the identity (2.4.25) can be written in the short symbolic form

$$\nabla\big(\cdots\big) = 0 + (\text{smooth contributions}) + (\text{higher orders in } \varepsilon/|\vec{\xi}|) , \tag{2.4.29}$$

where "$\cdots$" again stands for a composite expression in the $T^{(n)}$ and $\overline{T^{(n)}}$.

We finally remark that, at this stage, neglecting all terms of the order (2.4.18) merely is a matter of convenience. In fact, one can also take into account the higher orders in $\varepsilon/|\vec{\xi}||$ by performing an expansion in powers of $\varepsilon/|\vec{\xi}|$. Such an expansion is called *regularization expansion*. We will come back to the regularization expansion in Sect. 2.4.5. But before, we analyze the situation for more general regularizations.

2.4.2 Example: Linear Combinations of $i\varepsilon$-Regularizations

Clearly, the $i\varepsilon$-regularization is very special and ad-hoc. In order to get a first idea on what happens for more general regularizations, it is instructive to consider linear combinations of $i\varepsilon$-regularizations. To this end, we choose an integer N and generalize (2.4.2) to

$$\hat{P}^\varepsilon(k) = (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0) \left(\sum_{a=1}^{N} c_a \, \exp\left(\varepsilon\, d_a\, k^0\right) \right) \tag{2.4.30}$$

with positive parameters $d_1, \ldots, d_N$ and real numbers $c_1, \ldots, c_N$ which add up to one,

$$c_1 + \cdots + c_N = 1 .$$

In fact, by choosing N sufficiently large, with this ansatz one can approximate any regularization of the form

$$\hat{P}^\varepsilon(k) = (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, \hat{h}\big(k^0\big) , \tag{2.4.31}$$

corresponding to a regularization by convolution with a function $h(t)$ (being a special case of the regularizations in Example 1.2.4).

For regularizations of the form (2.4.30), we can again evaluate weakly on the light cone (2.4.17). It turns out that the scalings in ε and $|\vec{\xi}|$ are exactly the same as for the $i\varepsilon$-regularization. In order to see this in a simple setting, one can consider a polynomial in $T^{(n)}$ and $\overline{T^{(n)}}$,

$$T^{(l_1)} \cdots T^{(l_\alpha)}\, \overline{T^{(n_1)} \cdots T^{(n_\beta)}}\,.$$

When evaluating weakly on the light cone, one can pull the sums of the linear combinations in (2.4.30) out of the integral, i.e.

$$\begin{aligned}&\int_{-\infty}^{\infty} \eta(t)\, T^{(l_1)} \cdots T^{(l_\alpha)}\, \overline{T^{(n_1)} \cdots T^{(n_\beta)}}\, dt \\ &= \sum_{a_1,\ldots,a_\alpha,b_1,\ldots,b_\beta=1}^{N} c_{a_1} \cdots c_{a_\alpha}\, c_{b_1} \cdots c_{b_\beta} \int_{-\infty}^{\infty} \eta(t)\, T_{d_{a_1}}^{(l_1)} \cdots T_{d_{a_\alpha}}^{(l_\alpha)}\, \overline{T_{d_{b_1}}^{(n_1)} \cdots T_{d_{b_\beta}}^{(n_\beta)}}\, dt\,,\end{aligned} \tag{2.4.32}$$

where $T_d^{(n)}$ denotes the $i\varepsilon$-regularization with ε replaced by εd. Again computing up to the error terms (2.4.18) and (2.4.19), one can again use the explicit formulas for $T^{(n)}$ like (2.4.13)–(2.4.15) and analyze the integral with residues. The only difference compared to the analysis of the $i\varepsilon$-regularization is that one has many poles at positions $t = r \pm i\varepsilon d_a$, and the residue theorem gives sums over these poles. But obviously, this has no effect on all scalings.

The contraction of the inner factors must be handled with care, as we now explain. Using (2.4.6) and forming linear combinations, one sees that the factor $\not{\xi} T^{(n)}$ is to be regularized according to

$$\not{\xi}\, T^{(n)} \to \sum_{a=1}^{N} c_a \Big(\big(t - i\varepsilon d_a\big)\gamma^0 - \vec{\xi}\vec{\gamma} \Big)\, T_{d_a}^{(n)} \tag{2.4.33}$$

(with $T_d^{(n)}$ again as in (2.4.32)). When forming composite expressions, one must take into account that the regularized factors ξ and $T^{(n)}$ both carry the same summation index. Therefore, one should regard the factors $T^{(n)}$ and $\not{\xi}$ as belonging together. It is useful to make this connection explicit in the notation. Therefore, we discard (2.4.6) and introduce instead the more general rule

$$\not{\xi}\, T^{(n)} \to \not{\xi}^{(n)}\, T^{(n)}\,,$$

where the right side is a short notation for the sum in (2.4.33).

Contracting two inner factors ξ in this formalism gives

$$\begin{aligned}\left(\xi^{(l)}\right)_j T^{(l)} \left(\xi^{(n)}\right)^j T^{(n)} &= \sum_{a,b=1}^{N} c_a c_b \left(\left(t - i\varepsilon d_a\right), \vec{\xi}\right)_j T^{(l)}_{d_a} \left(\left(t - i\varepsilon d_b\right), \vec{\xi}\right)^j T^{(n)}_{d_b} \\ &= \sum_{a,b=1}^{N} c_a c_b \, T^{(l)}_{d_a} T^{(n)}_{d_b} \left(t^2 - i\varepsilon t d_a - i\varepsilon t d_b - \varepsilon^2 d_a d_b - |\vec{\xi}|^2\right). \qquad (2.4.34)\end{aligned}$$

This is considerably more complicated than (2.4.21). However, if as in (2.4.22) we drop the term quadratic in ε, the formula can be simplified to

$$\begin{aligned}&\left(\xi^{(p)}\right)_j T^{(l)} \left(\xi^{(q)}\right)^j T^{(n)} \\ &= \sum_{a,b=1}^{N} c_a c_b \, T^{(l)}_{d_a} T^{(n)}_{d_b} \left(t^2 - i\varepsilon t d_a - i\varepsilon t d_b - |\vec{\xi}|^2\right) \qquad (2.4.35) \\ &\quad + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \\ &= \frac{1}{2} \sum_{a,b=1}^{N} c_a c_b \, T^{(l)}_{d_a} T^{(n)}_{d_b} \left(\left(\left(t - i\varepsilon d_a\right), \vec{\xi}\right)^2 + \left(\left(t - i\varepsilon d_b\right), \vec{\xi}\right)^2\right) \qquad (2.4.36) \\ &\quad + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \\ &= \frac{1}{2}\left(\left(\xi^{(l)}\right)^2 + \left(\xi^{(n)}\right)^2\right) T^{(l)} T^{(n)} + (\text{higher orders in } \varepsilon/|\vec{\xi}|)\,, \qquad (2.4.37)\end{aligned}$$

where the squares in (2.4.36) denote the Minkowski inner product, and where in the last step we introduced the notation

$$\left(\xi^{(l)}\right)^2 T^{(l)} = \sum_{a=1}^{N} c_a \left(\left(t - i\varepsilon d_a\right)^2 - |\vec{\xi}|^2\right) T^{(l)}_{d_a}\,. \qquad (2.4.38)$$

In this way, the contraction rules (2.4.22) can be generalized to

$$\left(\xi^{(l)}\right)^j \left(\xi^{(n)}\right)_j = \frac{1}{2}\left(\left(\xi^{(l)}\right)^2 + \left(\xi^{(n)}\right)^2\right) \qquad (2.4.39)$$

Similarly the contraction rule (2.4.23) becomes

$$\left(\xi^{(l)}\right)^j \overline{\left(\xi^{(n)}\right)_j} = \frac{1}{2}\left(\left(\xi^{(l)}\right)^2 + \overline{\left(\xi^{(n)}\right)^2}\right). \qquad (2.4.40)$$

We remark that this product can again be simplified using (2.4.24), giving rise to the computation rule

$$\left(\xi^{(p)}\right)^2 T^{(l)} = -4p\, T^{(l-1)} + (\text{smooth contributions})\,.$$

We also remark that the integration-by-parts rules (2.4.28) and (2.4.29) with ∇ according to (2.4.27) remain valid, as one sees immediately by applying (2.4.28) to each summand in (2.4.30) and by noting that (2.4.25) holds for any regularization.

Working with linear combinations of $i\varepsilon$-regularization gives a first hint why one should disregard error terms of the form (2.4.18) and (2.4.19), as we now explain. Using the method of variable regularization (see Remark 1.2.1), we must show that the structure of the effective equations in the continuum limit does not depend on the details of the regularization. Evaluating weakly on the light cone and neglecting error terms of the form (2.4.18) and (2.4.19), one gets relatively simple computation rules (like (2.4.39), (2.4.40) or (2.4.24)), giving rise to a formalism which captures the structure of the EL equation independent of regularization details. However, for example the quadratic term in ε in (2.4.34)

$$-\varepsilon^2 \sum_{a,b=1}^{N} c_a c_b \, d_a d_b \, T^{(l)}_{d_a} \, T^{(n)}_{d_b} \tag{2.4.41}$$

has a different structure. Namely, even after prescribing linear moments as they appear in (2.4.33), there is a lot of freedom to give the quadratic term in (2.4.41) an arbitrary value. More generally, if we computed the terms (2.4.18) or (2.4.19), these contributions would depend on the regularization in a complicated way, so much so that without knowing the regularization in detail, it would be impossible to evaluate these contributions. This is the reason why we shall disregard these contributions. Clearly, at this stage, the above argument is not quite satisfying because notions like "complicated" and "knowing the regularization in detail" are somewhat vague. The argument will be made more precise in Sect. 2.4.5 using Fourier methods.

2.4.3 Further Regularization Effects

Working with linear combinations of $i\varepsilon$-regularizations, one is still in the restrictive class of regularizations of the form (2.4.31) where the unregularized distribution is multiplied in momentum space by a convergence-generating function $\hat{h}(k^0)$. Considering more general regularizations gives rise to additional effects. We now list those regularization effects will be important later on:

- The support of the distribution in (2.4.31) can be slightly deformed from the hyperboloid to another hypersurface. It turns out that in this case, one can still perform a mass expansion of the form (2.4.5). But the regularization of the factors $T^{(n)}$ also depends on the power of the mass in the corresponding contribution to the fermionic projector. In order to implement this effect into our formalism, one adds a subscript [.] to the factors $T^{(n)}$ which counts the power in m. Thus we regularize the contributions to the light-cone expansion according to the rule

$$m^p \, T^{(n)} \to m^p \, T^{(n)}_{[p]} \,.$$

For example, the regularization of the light-cone expansion of the vacuum (2.4.5) now takes the form

$$P^\varepsilon(x,y) = \sum_{n=0}^{\infty} \frac{m^{2n}}{n!}\, \frac{i\not{g}^{(-1+n)}}{2}\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^{\infty} \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]}\,.$$

Regularizing the fermionic projector in the presence of an external potential, one gets contributions involving factors $T^{(n)}_{[p]}$ with the same n but different values of p. These factors must be treated as being different (although they clearly coincide without regularization).

- The direction of the vector k which appears in the factor $\not{k}$ in (2.4.31) can be slightly changed by the regularization. This leads to the notion of the *shear of surface states*.This effect is of importance when inner factors are contracted. More precisely, one needs to modify the calculation rule (2.4.24) to

$$\big(\xi^{(p)}\big)^2\, T^{(n)}_{[p]} = -4\left(n\, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}}\right) + (\text{smooth contributions})\,,$$

where the factors $T^{(l)}_{\{p\}}$ with curly brackets have the same scaling behavior as the corresponding factors with square brackets but are regularized differently.

- There may be additional contributions to $\hat{P}(k)$ which lie outside the hyperboloid in (2.4.31) or the deformation thereof. It turns out that the resulting contributions can be absorbed into the error terms (2.4.18) and (2.4.19) (for details see Sect. 2.4.5).

We also remark that the regularization of neutrinos is more involved because the regularization must break the chiral symmetry and because the corresponding Dirac sea can "mimic" a Dirac sea of a different mass. In order not to distract from the main points of our construction, these extensions of the formalism will be introduced later when we need them (see Sect. 4.2).

2.4.4 *The Formalism of the Continuum Limit*

After the above motivation and preparations, we now present the formalism of the continuum limit. In Sect. 2.4.5 we shall outline the derivation of this formalism as first given in [F7, Chap. 4].

Before beginning, we point out that we work *modulo smooth contributions* throughout. The reason for this procedure is that the smooth contributions can be computed in a straightforward manner by first evaluating composite expressions away from the light cone (where they are smooth) and taking the limit when $y - x$ approaches the light cone. Clearly, computing the smooth contributions is important and not always easy (for details see Appendix D). But these computations are not related to the problem of the singularities on the light cone to be considered here.

Our starting point is the light-cone expansion of the unregularized fermionic projector $P(x, y)$ (as given in Sect. 2.2.8). In order to regularize the light-cone expansion on the length scale ε, we proceed as follows. The smooth contributions are all left unchanged. For the regularization of the factors $T^{(n)}$, we employ the replacement rule

$$m^p\, T^{(n)} \to m^p\, T^{(n)}_{[p]}\,, \tag{2.4.42}$$

where the factors $T^{(n)}_{[p]}$ are smooth functions of ξ. Fortunately, the rather complicated detailed form of the factors $T^{(n)}_{[p]}$ will not be needed here, because these functions can be treated symbolically using the following simple calculation rules. In computations one may treat the $T^{(n)}_{[p]}$ like complex functions. However, one must be careful when tensor indices of factors $\not{\xi}$ are contracted with each other. Naively, this gives a factor ξ^2 which vanishes on the light cone and thus changes the singular behavior on the light cone. In order to describe this effect correctly, we first write every summand of the light cone expansion (2.2.129) such that it involves at most one factor $\not{\xi}$ (this can always be arranged using the anti-commutation relations of the Dirac matrices). We now associate every factor $\not{\xi}$ to the corresponding factor $T^{(n)}_{[p]}$. In short calculations, this can be indicated by putting brackets around the two factors, whereas in the general situation we add corresponding indices to the factor $\not{\xi}$, giving rise to the replacement rule

$$m^p\, \not{\xi} T^{(n)} \to m^p\, \not{\xi}^{(n)}_{[p]}\, T^{(n)}_{[p]}\,. \tag{2.4.43}$$

For example, we write the regularized fermionic projector of the vacuum as

$$P^\varepsilon = \frac{i}{2} \sum_{n=0}^\infty \frac{m^{2n}}{n!}\, \not{\xi}^{(-1+n)}_{[2n]}\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]}\,.$$

The kernel $P(y, x)$ is obtained by taking the conjugate (see (2.1.68)). The conjugates of the factors $T^{(n)}_{[p]}$ and $\xi^{(n)}_{[p]}$ are the complex conjugates,

$$\overline{T^{(n)}_{[p]}} := \big(T^{(n)}_{[p]}\big)^* \qquad \text{and} \qquad \overline{\xi^{(n)}_{[p]}} := \big(\xi^{(n)}_{[p]}\big)^*\,.$$

One must carefully distinguish between these factors with and without complex conjugation. In particular, the factors $\not{\xi}^{(n)}_{[p]}$ need not be symmetric,

$$\big(\not{\xi}^{(n)}_{[p]}\big)^* \neq \not{\xi}^{(n)}_{[p]} \qquad \text{in general}\,.$$

When forming composite expressions, the tensor indices of the factors ξ are contracted to other tensor indices. The factors ξ which are contracted to other factors ξ are called *inner factors*. The contractions of the inner factors are handled with the so-called *contraction rules*

$$(\xi^{(n)}_{[p]})^j \, (\xi^{(n')}_{[p']})_j = \frac{1}{2} \left(z^{(n)}_{[p]} + z^{(n')}_{[p']} \right) \tag{2.4.44}$$

$$(\xi^{(n)}_{[p]})^j \, \overline{(\xi^{(n')}_{[p']})_j} = \frac{1}{2} \left(z^{(n)}_{[p]} + \overline{z^{(n')}_{[p']}} \right) \tag{2.4.45}$$

$$z^{(n)}_{[p]} \, T^{(n)}_{[p]} = -4 \left(n \, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}} \right), \tag{2.4.46}$$

which are to be complemented by the complex conjugates of these equations. Here the factors $z^{(n)}_{[p]}$ can be regarded simply as a book-keeping device to ensure the correct application of the rule (2.4.46). The factors $T^{(n)}_{\{p\}}$ have the same scaling behavior as the $T^{(n)}_{[p]}$, but their detailed form is somewhat different; we simply treat them as a new class of symbols. In cases where the lower index does not need to be specified we write $T^{(n)}_\circ$. After applying the contraction rules, all inner factors ξ have disappeared. The remaining so-called *outer factors* ξ need no special attention and are treated like smooth functions.

Next, to any factor $T^{(n)}_\circ$ we associate the *degree* $\deg T^{(n)}_\circ$ by

$$\deg T^{(n)}_\circ = 1 - n \,.$$

The degree is additive in products, whereas the degree of a quotient is defined as the difference of the degrees of numerator and denominator. The degree of an expression can be thought of as describing the order of its singularity on the light cone, in the sense that a larger degree corresponds to a stronger singularity (for example, the contraction rule (2.4.46) increments n and thus decrements the degree, in agreement with the naive observation that the function $z = \xi^2$ vanishes on the light cone). Using formal Taylor series, we can expand in the degree. In all our applications, this will give rise to terms of the form

$$\eta(x, y) \, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad \text{with } \eta(x, y) \text{ smooth} \,. \tag{2.4.47}$$

The quotient of the two monomials in this equation is referred to as a *simple fraction*.

A simple fraction can be given a quantitative meaning by considering one-dimensional integrals along curves which cross the light cone transversely away from the origin $\xi = 0$. This procedure is called *weak evaluation on the light cone*. For our purpose, it suffices to integrate over the time coordinate $t = \xi^0$ for fixed $\vec{\xi} \neq 0$. Moreover, using the symmetry under reflections $\xi \to -\xi$, it suffices to consider the upper light cone $t \approx |\xi|$. The resulting integrals diverge if the regularization is removed. The leading contribution for small ε can be written as

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt \; \eta(t, \vec{\xi}) \, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \;\approx\; \eta(|\vec{\xi}|, \vec{\xi}) \, \frac{c_{\text{reg}}}{(i|\vec{\xi}|)^L} \, \frac{\log^r(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}} \,, \tag{2.4.48}$$

where L is the degree of the simple fraction and c_{reg}, the so-called *regularization parameter*, is a real-valued function of the spatial direction $\vec{\xi}/|\vec{\xi}|$ which also depends on the simple fraction and on the regularization details (the error of the approximation will be specified below). The integer r describes a possible logarithmic divergence. Apart from this logarithmic divergence, the scalings in both ξ and ε are described by the degree.

When analyzing a sum of expressions of the form (2.4.47), one must know if the corresponding regularization parameters are related to each other. In this respect, the *integration-by-parts rules* are important, which are described symbolically as follows. On the factors $T_\circ^{(n)}$ we introduce a derivation ∇ by

$$\nabla T_\circ^{(n)} = T_\circ^{(n-1)} .$$

Extending this derivation with the Leibniz and quotient rules to simple fractions, the integration-by-parts rules state that

$$\nabla \left(\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \right) = 0 . \tag{2.4.49}$$

These rules give relations between simple fractions. The name is motivated by the integration-by-parts method as explained for the $i\varepsilon$-regularization in (2.4.25). Simple fractions which are not related to each other by the integration-by-parts rules are called *basic fractions*. As shown in [F7, Appendix E], there are no further relations between the basic fractions. Thus the corresponding *basic regularization parameters* are linearly independent.

The above symbolic computation rules give a convenient procedure to evaluate composite expressions in the fermionic projector, referred to as the *analysis in the continuum limit*: After applying the contraction rules and expanding in the degree, the EL equations can be rewritten as equations involving a finite number of terms of the form (2.4.47). By applying the integration-by-parts rules, we can arrange that all simple fractions are basic fractions. We evaluate weakly on the light cone (2.4.48) and collect the terms according to their scaling in ξ. Taking for every given scaling in ξ only the leading pole in ε, we obtain equations which involve linear combinations of smooth functions and basic regularization parameters. We consider the basic regularization parameters as empirical parameters describing the unknown microscopic structure of space-time. We thus end up with equations involving smooth functions and a finite number of free parameters. We point out that these free parameters cannot be chosen arbitrarily because they might be constrained by inequalities (see the discussion after [F7, Theorem E.1]). Also, the values of the basic regularization parameters should ultimately be justified by an analysis of vacuum minimizers of the causal action principle.

We finally specify the error of the above expansions. By not regularizing the bosonic potentials and fermionic wave functions, we clearly disregard the

$$\text{higher orders in } \varepsilon/\ell_{\text{macro}} \,. \tag{2.4.50}$$

Furthermore, in (2.4.48) we must stay away from the origin, meaning that we neglect the

$$\text{higher orders in } \varepsilon/|\vec{\xi}| \,. \tag{2.4.51}$$

The higher order corrections in $\varepsilon/|\vec{\xi}|$ depend on the fine structure of the regularization and thus seem unknown for principal reasons. Neglecting the terms in (2.4.50) and (2.4.51) also justifies the formal Taylor expansion in the degree. Clearly, leaving out the terms (2.4.51) is justified only if $|\vec{\xi}| \gg \varepsilon$. Therefore, whenever using the above formalism, we must always ensure that $|\vec{\xi}|$ is much larger than ε (we will come back to this point in Sects. 2.6.5, 3.5.2 and Appendix A).

2.4.5 *Outline of the Derivation*

We now outline the derivation of the formalism of the continuum limit (for more details see [F7, Chap. 4]). The method relies on an asymptotic analysis of the Fourier integral (2.4.1),

$$P^\varepsilon(x, y) = \int \frac{d^4k}{(2\pi)^4}\, \hat{P}^\varepsilon(k)\, e^{ik\xi} \,. \tag{2.4.52}$$

For simplicity, we begin the analysis for the scalar component, i.e. we consider the case

$$\hat{P}^\varepsilon(p) = \phi(p)\, f(p) \tag{2.4.53}$$

(the vector component will be treated after (2.4.82) below). We may assume that the spatial component of the vector ξ points in the direction of the x-axis of our Cartesian coordinate system, i.e. $y - x = (t, r, 0, 0)$ with $r > 0$. Choosing cylindrical coordinates ω, k, ρ and φ in momentum space, defined by $p = (\omega, \vec{p})$ and $\vec{p} = (k,\ \rho\cos\varphi,\ \rho\sin\varphi)$, the Fourier integral becomes

$$P(x, y) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\ \hat{P}^\varepsilon(\omega, k, \rho, \varphi)\, e^{i\omega t - ikr} \,. \tag{2.4.54}$$

Since the exponential factor in this formula is independent of ρ and φ, we can write the fermionic projector as the two-dimensional Fourier transform

$$P(x, y) = 2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk\ h(\omega, k)\, e^{i\omega t - ikr} \tag{2.4.55}$$

of a function h defined by

$$h(\omega, k) = \frac{1}{2\,(2\pi)^4} \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\; (\phi\, f)(\omega, k, \rho, \varphi)\,. \tag{2.4.56}$$

We want to analyze $P(x, y)$ close to the light cone $(y-x)^2 = 0$ away from the origin $y = x$. Without loss of generality, we may restrict attention to the upper light cone $t = r$. Thus we are interested in the region $t \approx r > 0$. The "light-cone coordinates"

$$s = \frac{1}{2}\,(t-r)\,, \qquad l = \frac{1}{2}\,(t+r) \tag{2.4.57}$$

are well-suited to this region, because the "small" variable s vanishes for $t = r$, whereas the "large" variable l is positive and non-zero. Introducing also the associated momenta

$$u = -k - \omega\,, \qquad v = k - \omega\,, \tag{2.4.58}$$

we can write the fermionic projector as

$$P(s, l) = \int_{-\infty}^\infty du \int_{-\infty}^\infty dv\; h(u, v)\, e^{-i(us+vl)}\,. \tag{2.4.59}$$

Let us briefly discuss the qualitative form of the function h, (2.4.56). Without regularization, the scalar component is given by the δ-distribution on the lower mass shell $\hat{P} = m\,\delta(p^2 - m^2)\,\Theta(-p^0)$. In this case, the integral (2.4.56) can be evaluated to be

$$\begin{aligned} h &= \frac{m}{2\,(2\pi)^4} \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\; \delta(\omega^2 - k^2 - \rho^2 - m^2)\,\Theta(-\omega) \\ &= \frac{m}{4\,(2\pi)^3}\,\Theta(\omega^2 - k^2 - m^2)\,\Theta(-\omega) = \frac{m}{32\pi^3}\,\Theta(uv - m^2)\,\Theta(u)\,. \end{aligned} \tag{2.4.60}$$

Thus integrating over ρ and φ yields a constant function in the interior of the two-dimensional "lower mass shell" $\omega^2 - k^2 = m^2$, $\omega < 0$. From this we conclude that for small momenta, where the regularization should play no role, the function h should have a discontinuity along the hyperbola $\{uv = m^2,\ u > 0\}$, be zero below (i.e. for $uv < m^2$) and be nearly constant above. The precise form of h for large energy or momentum can be arbitrary. We only know that h decays at infinity.

It is instructive to discuss the energy scales. Clearly, one scale is given by the regularization length ε. In momentum space, this corresponds to the high energy scale ε^{-1}. We sometimes refer to the region $|\omega| + |k| \gtrsim \varepsilon^{-1}$ as the high energy region. The relevant low energy scale, on the other hand, is εm^2 (it is zero for massless fermions). This is because the hyperbola $uv = m^2$ comes as close to the v-axis as as $v \sim \varepsilon m^2$ before entering the high energy region. Finally, the Compton scale m lies between the low- and high energy scales,

$$\varepsilon m^2 \lesssim m \lesssim \varepsilon^{-1} .$$

Since we want to analyze the situation close to the light cone, we choose the "small" light-cone parameter s on the regularization scale, i.e.

$$s \lesssim \varepsilon . \tag{2.4.61}$$

The "large" light-cone parameter l, on the other hand, is non-zero. We shall always choose this scale between the regularization scale and the Compton scale,

$$\varepsilon \ll l \ll \frac{1}{m} . \tag{2.4.62}$$

Since $\varepsilon m \ll 1$, the inequalities in (2.4.62) still leave us the freedom to vary l on many orders of magnitude.

Our task is to evaluate the Fourier integral (2.4.59) using the scales (2.4.61) and (2.4.62). In preparation, we discuss and specify the function $h(u, v)$ for fixed u, also denoted by $h_u(v)$. Without regularization (2.4.60), the function h_u has a discontinuous "jump" from zero to a finite value on the hyperbola. Therefore, we cannot expect that h_u is continuous when a regularization is present. On the contrary, the decay for large v suggests that h_u might have another discontinuity for large v, where it might "jump" to zero. In order to keep the presentation reasonably simple, we assume that h_u is always of this general form, i.e.

$$h_u(v) = \begin{cases} 0 & \text{for } v < \alpha_u \text{ or } v > \beta_u \\ \text{smooth} & \text{for } \alpha_u \leq v \leq \beta_u \end{cases} \tag{2.4.63}$$

with parameters $\alpha_u < \beta_u$. The case of less than two discontinuities can be obtained from (2.4.63) by setting $h_u(\alpha_u)$ or $h_u(\beta_u)$ equal to zero, or alternatively by moving the position of the discontinuities α_u or β_u to infinity. We remark that the discontinuity at $v = \beta_u$ will become irrelevant later; it is taken into account only in order to explain why the behavior of the fermionic projector on the light cone is independent of many regularization details.

Without regularization (2.4.60), the function $h_u(v)$ is constant for $v \geq \alpha_u$. Thus the v-dependence of $h_u(v)$ for $\alpha_u \leq v \leq \beta_u$ merely is a consequence of the regularization, and it is therefore reasonable to assume that the v-derivatives of $h_u(v)$ scale in powers of the regularization length ε. More precisely, we assume that there is a constant $c_1 \ll l/\varepsilon$ such that

$$|h_u^{(n)}(v)| \leq \big(c_1\, \varepsilon\big)^n \ \max |h_u| \qquad \text{for } \alpha_u \leq v \leq \beta_u , \tag{2.4.64}$$

where the derivatives at $v = \alpha_u$ and β_u are understood as the right- and left-sided limits, respectively. This regularity condition is typically satisfied for polynomial, exponential and trigonometric functions, but it excludes the case that the function h_u has small-scale fluctuations. Clearly, we could also consider a more general ansatz

for h_u with more than two discontinuities or weaker regularity assumptions. But this does not seem to be the point because all interesting effects, namely the influence of discontinuities for small and large v as well as of smooth regions, can already be studied in the setting (2.4.63), (2.4.64).

Let us analyze the v-integral of the Fourier transform (2.4.59),

$$P_u(l) \;:=\; \int_{-\infty}^{\infty} h_u(v)\, e^{-ivl}\, dv\,. \tag{2.4.65}$$

According to the left inequality in (2.4.62), the exponential factor in (2.4.65) is highly oscillatory on the scale $v \sim 1/\varepsilon$. Thus we can expect that the smooth component of h_u only gives a small contribution to the integral (2.4.65), so that the discontinuities at α_u and β_u should play the dominant role. In order to make this picture mathematically precise, in (2.4.65) we iteratively integrate K times by parts,

$$\begin{aligned} P_u(l) &= \int_{\alpha_u}^{\beta_u} h_u(v)\, e^{-ivl}\, dv = -\frac{1}{il} \int_{\alpha_u}^{\beta_u} dv\; h_u(v)\, \frac{d}{dv} e^{-ivl} \\ &= -\frac{1}{il}\, h_u(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \frac{1}{il} \int_{\alpha_u}^{\beta_u} h_u'(v)\, e^{-ivl}\, dl = \cdots = \\ &= -\frac{1}{il} \sum_{n=0}^{K-1} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \left(\frac{1}{il}\right)^K \int_{\alpha_u}^{\beta_u} h_u^{(K)}(v)\, e^{-ivl}\, dl\,. \end{aligned} \tag{2.4.66}$$

If we bound all summands in (2.4.66) using the first inequality in (2.4.62) and the regularity condition (2.4.64), each v-derivative appears in combination with a power of l^{-1}, and giving a factor $c_1\varepsilon/l \ll 1$. Thus in the limit $K \to \infty$, we may drop the integral in (2.4.66) to obtain

$$P_u(l) = -\frac{1}{il} \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u}\,. \tag{2.4.67}$$

This expansion converges, and its summands decay like $(c_1\varepsilon/l)^n$.

Using (2.4.65), we can write the Fourier transform (2.4.59) as

$$P(s,l) = \int_{-\infty}^{\infty} P_u(l)\, e^{-ius}\, du\,. \tag{2.4.68}$$

Notice that, apart from the constraints (2.4.62), the "large" variable l can be freely chosen. We want to study the functional dependence of (2.4.68) on the parameter l. In preparation, we consider an integral of the general form

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du\,, \tag{2.4.69}$$

where we assume that $(u, \gamma(u))$ is a curve in the high energy region in the sense that $\gamma \sim 1/\varepsilon$. Furthermore, we assume that γ is monotone with $|\gamma'| \sim 1$ and that $(b-a) \sim 1/\varepsilon$. By transforming the integration variable, we can then write (2.4.69) as the Fourier integral

$$\int_{\gamma(a)}^{\gamma(b)} f\, |\gamma'|^{-1}\, e^{-i\gamma l}\, d\gamma\,. \tag{2.4.70}$$

If the function $f\,|\gamma'|^{-1}$ is smooth, its Fourier transform (2.4.70) has rapid decay in the variable l. Under the stronger assumption that $f\,|\gamma'|^{-1}$ varies on the scale $1/\varepsilon$, we conclude that the length scale for this rapid decay is of the order $l \sim \varepsilon$. As a consequence, the rapid decay can be detected even under the constraint $l < l_{\max}$ imposed by (2.4.62), and we say that (2.4.70) has *rapid decay in* l. The reader who feels uncomfortable with this informal definition can immediately make this notion mathematically precise by an integration by parts argument similar to (2.4.66) imposing for $f\,|\gamma'|^{-1}$ a condition of type (2.4.64). The precise mathematical meaning of rapid decay in l for the integral (2.4.69) is that for every integer k there should be constants $c \sim 1$ and parameters $l_{\min}, l_{\max}$ in the range $\varepsilon \ll l_{\min} \ll l_{\max} \ll 1/m$ such that for all $l \in (l_{\min}, l_{\max})$,

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du \;\leq\; c \left(\frac{\varepsilon}{l}\right)^k \int_a^b |f(u)|\, du\,.$$

We return to the analysis of the integral (2.4.68). The boundary terms in (2.4.67) at β_u yield contributions to $P(s,l)$ of the form

$$-\left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}\big(\beta_u\big)\, e^{-i\beta_u l - ius}\, du\,. \tag{2.4.71}$$

According to (2.4.61), the length scale for the oscillations of the factor $\exp(-ius)$ is $u \sim 1/\varepsilon$. Under the reasonable assumption that β_u is monotone and that the functions $|\beta'(u)|^{-1}$ and $h_u^{(n)}(\beta_u)$ vary on the scale $1/\varepsilon$, the integral (2.4.71) is of the form (2.4.70), and the above consideration yields that (2.4.71) has rapid decay in l. We conclude that it suffices to consider the boundary terms in (2.4.67) at α_u. Using this result in (2.4.68), we obtain

$$P(s,l) = \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-i\alpha_u l - ius}\, du \;+\; \text{(rapid decay in } l)\,. \tag{2.4.72}$$

The integral (2.4.72) cannot be estimated again using the "oscillation argument" after (2.4.69), because, according to (2.4.60), the function α_u tends asymptotically to zero for large u, so that the factor $\exp(-i\alpha_u l)$ is non-oscillating in this region. Instead, we expand this factor in a Taylor series,

$$P(s,l) = \sum_{n,k=0}^{\infty} \frac{1}{k!}\, (il)^{k-n-1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, (-\alpha_u)^k\, e^{-ius}\, du\,. \tag{2.4.73}$$

Let us discuss this expansion. Without regularization (2.4.60), the function $\alpha_u = m^2/u$ involves the mass. Therefore, expanding in powers of α_u corresponds precisely to the expansion in the mass expansion as considered earlier (see (2.2.8) and (2.2.113) and the explanations thereafter). With this in mind, we can regard (2.4.73) as a generalization of the *mass expansion* to the setting with regularization. This expansion is clearly justified if $\alpha_u l \ll 1$. However, as the function m^2/u has a pole at $u = 0$, the function α_u becomes large for small u, so that it is not clear whether the mass expansion is sensible. Indeed, this issue is closely related to the logarithmic mass problem which was mentioned in Sect. 2.2.6 and was resolved by working with the "regularized" distribution T_a^{reg}, (2.2.117). In the present setting, this "regularization procedure" can be understood as follows: For small momenta $u \ll 1/\varepsilon$, our oscillation argument after (2.4.69) again applies and shows that the resulting contribution to $P(s,l)$ decays rapidly in l. Therefore, disregarding contributions with rapid decay in l, we may restrict attention to the region $u \gtrsim \varepsilon$ where

$$\alpha_u < \alpha_{\max} \ll l_{\max}^{-1}\,. \tag{2.4.74}$$

Then $\alpha_u l \ll 1$, justifying the mass expansion (2.4.73).

For a fixed value of $k-n$, all summands in (2.4.73) have the same l-dependence. Let us compare the relative size of these terms. According to our regularity assumption (2.4.64), the derivatives of h scale like $h_u^{(n)} \sim \varepsilon^n$. Using the bound (2.4.74), we conclude that, for a fixed power of l, the summands in (2.4.73) decrease like $(\varepsilon \alpha_{\max})^n$. Thus it is a very good approximation to drop the summands for large n. At first sight, it might seem admissible to take into account only the first summand $n = 0$. But the situation is not quite so simple. For example, it may happen that, when restricted to the curve (u, α_u), the function $h(u,v)$ is so small that the summands for $n = 0$ in (2.4.73) are indeed not dominant. More generally, we need to know that for some $n_0 \geq 0$, the function $h_u^{(n_0)}(\alpha_u)$ is really of the order given in (2.4.64), i.e.

$$|h_u^{(n_0)}(\alpha_u)| \geq c\, \big(c_1\, \varepsilon\big)^{n_0}\, \max |h_u| \tag{2.4.75}$$

with a positive constant c which is of the order one. If this condition is satisfied, we may neglect all summands for $n > n_0$, and collecting the terms in powers of l, we conclude that

$$P(s,l) = \frac{1}{(il)^{n_0+1}} \sum_{k=0}^{\infty} (-il)^k \sum_{n=\max(n_0-k,0)}^{n_0} \frac{(-1)^{n_0-n}}{(k-n_0+n)!} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, \alpha_u^{k-n_0+n}\, e^{-ius}\, du$$

$$+ \sum_{n=n_0+1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-ius}\, du \;+\; (\text{rapid decay in } l)$$
$$+ \;(\text{higher orders in } \varepsilon\alpha_{\max})\,. \tag{2.4.76}$$

We point out that, according to (2.4.74),

$$\varepsilon\alpha_{\max} \ll \varepsilon / l_{\max}\,,$$

and this explains why we disregard the higher orders in $\varepsilon\alpha_{\max}$. In our case, the function h_u has in the low energy region according to (2.4.60) the form $h_u(\alpha_u) = m/(32\pi^3)\,\Theta(u)$. Hence it is natural to assume that (2.4.75) is satisfied for $n_0 = 0$. Introducing the shorter notation

$$h(u) := h_u(\alpha(u))\,, \quad h^{[n]}(u) := h_u^{(n)}(\alpha_u)\,, \quad \alpha(u) := \alpha_u\,, \tag{2.4.77}$$

we have thus derived the following result.

Expansion of the scalar component: *Close to the light cone* (2.4.61), (2.4.62), *the scalar component* (2.4.53) *of the fermionic projector of the vacuum has the expansion*

$$P(s,l) = \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} h\, \alpha^k\, e^{-ius}\, du \tag{2.4.78}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h^{[n]}\, e^{-ius}\, du \tag{2.4.79}$$
$$+ \;(\text{rapid decay in } l\,) \;+\; (\text{higher orders in } \varepsilon\alpha_{\max}) \tag{2.4.80}$$

with suitable regularization functions $h, h^{[n]}$ *and* α. *In the low energy region* $u \ll 1/\varepsilon$, *the regularization functions are*

$$h(u) = \frac{m}{32\pi^3}\,\Theta(u)\,, \qquad h^{[n]}(u) = 0\,, \qquad \alpha(u) = \alpha_u = \frac{m^2}{u}\,. \tag{2.4.81}$$

In this expansion, the l-dependence is written out similar to a Laurent expansion. The main simplification compared to our earlier Fourier representation is that the dependence on the regularization is now described by functions of only one variable, denoted by h, $h^{[n]}$ and α. In composite expressions in $P(s,l)$, we will typically get convolutions of these functions; such one-dimensional convolutions can be easily analyzed. The simplification to one-dimensional regularization functions became possible because many details of the regularization affect only the contribution with rapid decay in l, which we do not consider here. Notice that the summands in (2.4.78) and (2.4.79) decay like $(l\,\alpha_{\max})^k/k! \ll (l/l_{\max})^k/k!$ and $(\varepsilon/l)^n$, respectively. In the low energy limit (2.4.81), the expansion (2.4.78) goes over to a power series in m^2, and we thus refer to (2.4.78) as the *mass expansion*. In the mass expansion, the

regularization is described by only two functions h and α. The series (2.4.79), on the other hand, is a pure regularization effect and is thus called the *regularization expansion*. It involves an infinite number of regularization functions $h^{[n]}$. Accordingly, we will use the notions of mass and regularization expansions also for other expansions of type (2.4.76).

We now outline how to extend the previous analysis to the vector component. More precisely, we will analyze the Fourier integral (2.4.52) for

$$\hat{P}^\varepsilon(p) = v_j(p)\, \gamma^j\, f(p) \tag{2.4.82}$$

close to the light cone. We again choose light-cone coordinates (s, l, x_2, x_3) with $y - x = (s, l, 0, 0)$ (s and l are given by (2.4.57), while x_2 and x_3 are Cartesian coordinates in the orthogonal complement of the sl-plane). The associated momenta are denoted by $p = (u, v, p_2, p_3)$ with u and v according to (2.4.58). As in (2.4.55), we integrate out the coordinates perpendicular to u and v,

$$h_j(u, v) := \frac{1}{2\,(2\pi)^4} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3\; (v_j\, f)(u, v, p_2, p_3)\,. \tag{2.4.83}$$

We thus obtain a representation of the fermionic projector involving two-dimensional Fourier integrals

$$P(s, l) = \gamma^j\, P_j(s, l)$$

with

$$P_j(s, l) := \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h_j(u, v)\, e^{-i(us+vl)}\,. \tag{2.4.84}$$

The tensor indices in (2.4.83) and (2.4.84) refer to the coordinate system (s, l, x_2, x_3). For clarity, we denote the range of the indices by $j = s, l, 2, 3$; thus

$$\gamma^s = \frac{1}{2}\,(\gamma^0 - \gamma^1)\,, \qquad \gamma^l = \frac{1}{2}\,(\gamma^0 + \gamma^1)\,, \tag{2.4.85}$$

where $\gamma^0, \ldots, \gamma^3$ are the usual Dirac matrices of Minkowski space. Since without regularization, $\hat{P} = \not{p}\, \delta(p^2 - m^2)\, \Theta(-p^0)$, the functions h_j can be computed similar to (2.4.60) to be

$$\gamma^j\, h_j(u, v) = \frac{1}{32\pi^3}(-u\gamma^s - v\gamma^l)\, \Theta(uv - m^2)\, \Theta(u)\,. \tag{2.4.86}$$

This limiting case specifies the regularized $h_j(u, v)$ for small energy-momentum $u, v \ll 1/\varepsilon$. In order to keep the form of the functions h_j in the high energy region sufficiently general, we merely assume in what follows that the functions h_j have all the properties which se assumed for the function h above. This gives the following result.

Expansion of the vector component: *Close to the light cone* (2.4.61), (2.4.62), *the vector component* (2.4.82) *of the fermionic projector of the vacuum has the expansion* $P = \gamma^j P_j$ *with*

$$\begin{aligned} P_s(s,l) &= \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} -u\, g_s\ \alpha^k\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} -u\, g_s^{[n]}\, e^{-ius}\, du \\ &\quad + \text{(rapid decay in } l) + \text{(higher orders in } \varepsilon\alpha_{\max}) \end{aligned} \tag{2.4.87}$$

$$\begin{aligned} P_l(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[(k-1)\, \alpha^k + k\, \frac{b}{u}\, \alpha^{k-1} \right] g_l\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} -(n+1)\, g_l^{[n]}\, e^{-ius}\, du \\ &\quad + \text{(rapid decay in } l) + \text{(higher orders in } \varepsilon\alpha_{\max}) \end{aligned} \tag{2.4.88}$$

$$\begin{aligned} P_{2\beta}(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[\alpha^k + k\, \frac{b_{2\beta}}{u}\, \alpha^{k-1} \right] g_{2\beta}\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} g_{2\beta}^{[n]}\, e^{-ius}\, du \\ &\quad + \text{(rapid decay in } l) + \text{(higher orders in } \varepsilon\alpha_{\max}) \end{aligned} \tag{2.4.89}$$

and suitable regularization functions g_j, $g_j^{[n]}$, b, $b_{2/3}$ *and the mass regularization function* α *as in* (2.4.78) *and* (2.4.81). *In the low energy region* $u \ll 1/\varepsilon$, *the regularization functions have the form*

$$g_s(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_s^{[n]}(u) = 0 \tag{2.4.90}$$

$$g_l(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_l^{[n]}(u) = b(u) = 0 \tag{2.4.91}$$

$$g_{2\beta}(u) = g_{2\beta}(u) = b_{2\beta}(u) = 0\,. \tag{2.4.92}$$

In order to explain these formulas, we consider the situation where, like in the case without regularization, the vector $v(p)$ in (2.4.82) points into the direction p. In this case, we can write the vector component as

$$\hat{P}^{\varepsilon}(p) = p_j \gamma^j\, (\phi f)(p)\,, \tag{2.4.93}$$

where (ϕf) has the form of the scalar component as considered above. Since multiplication in momentum space corresponds to differentiation in position space, we obtain for (2.4.84)

$$P(s,l) = -i \left(\gamma^s \frac{\partial}{\partial s} + \gamma^l \frac{\partial}{\partial l} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} \right) P_{\text{scalar}}(s,l) \,,$$

where P_{scalar} is the scalar component (2.4.59) with h as in (2.4.56). We now substitute for P_{scalar} the expansion on the light cone (2.4.78)–(2.4.80) and carry out the partial derivatives. For the s- and l-components, this gives exactly the expansions (2.4.87), (2.4.88) with

$$g_s = g_l = h \,, \quad g_s^{[n]} = g_l^{[n]} = h^{[n]} \,, \quad b = 0 \,. \tag{2.4.94}$$

For the components $j = 2, 3$, the calculation of the partial derivatives is not quite so straightforward because the expansion of the scalar component (2.4.78)–(2.4.80) was carried out for fixed x_2 and x_3. Nevertheless, one can deduce also the expansion (2.4.89) from (2.4.78)–(2.4.80) if one considers x_2 and x_3 as parameters of the regularization functions h, $h^{[n]}$ and α, and differentiates through, keeping in mind that differentiation yields a factor $1/l$ (to get the scaling dimensions right). In this way, the simple example (2.4.93) explains the general structure of the expansions (2.4.87)–(2.4.89). We point out that the regularization function b vanishes identically in (2.4.94). This means that b is non-zero only when the direction of the vector field v is modified by the regularization. Thinking in terms of the decomposition into the one-particle states, we refer to this regularization effect as the *shear of the surface states*.

The derivation of these formulas uses the same methods as for the scalar components. The analysis is a bit more subtle because one must carefully analyze the scaling of the different components. We refer the interested reader to [F7, Sect. 4.4].

Computing composite expressions using the above Fourier representations, one readily verifies the calculations rules stated in Sect. 2.4.4. The details can be found in [F7, Sect. 4.5].

2.5 Computation of the Local Trace

When deriving the EL equations in Sect. 1.4.1, we showed in Proposition 1.4.1 that for every minimizer of the causal action principle, the local trace is constant in spacetime. We also argued that this condition should be satisfied by the rescaling (1.4.11). In the Minkowski vacuum, the local trace is obviously constant because the kernel of the fermionic projector is translation invariant (see our ansatz (2.4.52)). But in the presence of an external potential, the local trace will in general no longer be constant, making it necessary to perform the rescaling (1.4.11). We now explain how to compute the local trace and discuss the effect of the rescaling (1.4.11).

We begin by noting that, using the abstract definition of the kernel of the fermionic projector (1.1.13), we know that the local trace can be computed by

$$\mathrm{tr}(x) = \mathrm{Tr}_{S_x}\big(P^\varepsilon(x,x)\big)\,.$$

In what follows, we usually omit the subscript S_x and regard Tr as the trace of a 4×4-matrix. In the vacuum, one can compute this trace from (2.4.52) to conclude the scaling

$$\mathrm{Tr}_{S_x}\big(P^\varepsilon(x,x)\big) = c\,\frac{m}{\varepsilon^2}\,\Big(1 + \mathcal{O}\big(m\varepsilon\big)\Big)\,, \tag{2.5.1}$$

where the constant c depends on the regularization method (for an explicit computation in the $i\varepsilon$-regularization see Exercise 2.22).

In the next proposition we specify how the local trace is affected by the external potential.

Proposition 2.5.1 *In the presence of a smooth external chiral potential (2.2.25) with the properties as in Lemma 2.1.2, the contribution ΔP to the fermionic projector to order n in perturbation theory influences the local trace only by an error term of the form*

$$\Big|\mathrm{Tr}_{S_x}\big(\Delta P^\varepsilon(x,x)\big)\Big| \leq \frac{C}{\varepsilon}\,, \tag{2.5.2}$$

where the constant C depends on m, n as well as on the potential $\mathcal{B}$ and its partial derivatives. Moreover, the function $\mathrm{Tr}_{S_x}(\Delta P^\varepsilon(x,x))$ is smooth in x.

This result implies that, when rescaling the causal fermion system according to (1.4.11), we only pick up smooth error terms of the order $\varepsilon/\ell_{\text{macro}}$. Since such error terms are neglected in the continuum limit (see (2.4.50)), we may disregard the rescaling (1.4.11). This is the reason why the rescaling (1.4.11) will not be considered further in this book.

Before coming to the proof of the above proposition, we note that for a *gravitational field*, the situation is more involved. Namely, for linear gravity as considered in Sect. 2.3, the change of the local trace is typically of the order

$$\mathrm{Tr}_{S_x}\big(\Delta P^\varepsilon(x,x)\big) \sim \frac{m}{\varepsilon^2}\,\mathcal{O}(h)\,. \tag{2.5.3}$$

Clearly, this is sufficient in order to treat a weak gravitational field. However, when constructing causal fermion systems non-perturbatively in curved space-time (as is done in [FR2, Sect. 4]), the macroscopic space-time dependence of the local trace must be taken into account, meaning that the rescaling procedure (1.4.11) will change the causal fermion system substantially. The same is true if a *scalar potential* is considered, because in this case the local trace takes the form

$$\mathrm{Tr}_{S_x}\big(P^\varepsilon(x,x)\big) = \frac{c}{\varepsilon^2}\,\frac{\mathrm{Tr}_{S_x}\big(B(x)\big)}{\dim(S_x)} + \mathcal{O}\Big(\frac{1}{\varepsilon}\Big)\,, \tag{2.5.4}$$

where the potential B again includes the mass (2.2.10) (for the derivation see Exercise 2.23).

Proof of Proposition 2.5.1 As shown in Theorem 2.2.16, to every order in perturbation theory, the non-causal high energy contribution $\tilde{p} - \tilde{p}^{\text{res}}$ is a smooth function in x and y. Therefore, it is even bounded for $x = y$, and we do not need to consider it here. Hence it suffices to consider the perturbation expansions for $\tilde{k}$ and $\tilde{p}^{\text{res}}$. These perturbation expansions must be regularized on the scale ε. The procedure for this is explained in the appendix (see Appendix F). In order to keep the presentation as simple as possible, here we shall not enter the regularized causal perturbation theory. Instead, we consider the unregularized perturbation expansion and make use of the fact that the regularization gives rise to a decay in momentum space on the scale ε^{-1}. This simplified procedure will be justified by a short remark at the end of the proof.

In view of (2.1.26) and (2.2.110), instead of $\tilde{k}$ and $\tilde{p}^{\text{res}}$ we can just as well consider the causal Green's functions $s^\wedge$ and $s^\vee$ (see (2.1.25)) as well as the Green's functions s^+ and s^- (see (2.2.108)). For the causal Green's function, we can apply the structural results on the light-cone expansion stated in Theorem 2.2.4. Using the residual argument, this theorem holds just as well for the Green's functions $s^\pm$. With this in mind, we may restrict attention to the causal Green's functions, which we again simply denote by s.

The formula (2.5.3) can also be expressed by saying that $S^{(0)} \sim \varepsilon^{-2}$. Since increasing the upper index gives a scaling factor ξ^2, which for $x = y$ is translated to a scaling factor ε^2, we have

$$S^{(h)} \sim \varepsilon^{-2+2h} . \tag{2.5.5}$$

Moreover, every factor ξ in the light-cone expansion gives rise to a scaling factor

$$\xi \sim \varepsilon . \tag{2.5.6}$$

Applying these scalings to a contribution of the light-cone expansion in Theorem 2.2.4, we find that

$$(2.2.5) \sim \varepsilon^{-2+2h+|I|} .$$

Therefore, our task is to show that all expressions of the form (2.2.5) which contribute to the local trace satisfy the inequality

$$2h + |I| > 0 . \tag{2.5.7}$$

Using the identity (2.2.34), the inequality (2.5.7) is equivalent to

$$k - 1 + \sum_{a=1}^{k} \Big(|I_a| + 2p_a \Big) > 0 .$$

Obviously, it suffices to consider the cases $k = 0$ and $k = 1$. If $k = 0$, the fermionic projector is odd (i.e. it contains an odd number of Dirac matrices), so that the local trace vanishes. In the case $k = 1$, on the other hand, the contribution involving the chiral potential is again odd and vanishes. The contribution involving the mass matrix mY, on the other hand, is precisely the term $mYS^{(0)}$ whose local trace was computed in (2.5.1). This concludes the proof, provided that the scalings (2.5.5) and (2.5.6) hold.

The scalings (2.5.5) and (2.5.6) are justified by the regularized causal perturbation theory developed in Appendix F. We here explain the reason for the scalings: In the regularized causal perturbation calculation, the "causality" is built in by *demanding* that the resulting regularized light-cone expansion again only involves integrals along the line segment $\overline{xy}$ (and not integrals along the whole straight line through x and y). In more technical terms, this is achieved by demanding that the contributions to the perturbation expansion remain bounded in the limit when the momentum of the external potential tends to zero (this method was first used in [F7, Appendix D]). This procedure ensures that a factor ξ in the unregularized light-cone expansion really gives a scaling factor ε, (2.5.6). The scaling (2.5.5), on the other hand, follows immediately from the fact that the local trace is obtained by integrating over the momentum variables (similar as in Exercise 2.22), and that the regularization gives decay in momentum space on the scale ε^{-1}. □

2.6 Spectral Analysis of the Closed Chain

In this section we explain how to analyze the EL equations corresponding to the causal action in the continuum limit. Since the Lagrangian involves the eigenvalues of the closed chain, the main task is to compute the spectral decomposition of $A^\varepsilon_{xy} = P^\varepsilon(x, y)\, P^\varepsilon(y, x)$. We first compute this spectral decomposition in the vacuum (Sect. 2.6.1). This spectral decomposition has the special properties that the eigenvalues are non-real and form complex conjugate pairs, and that the corresponding eigenvectors are null (with respect to the spin scalar product). In order to simplify the subsequent computations, it is very convenient to choose a spinor basis which reflects these special properties of the closed chain of the vacuum. This so-called *double null spinor frame* is introduced in Sect. 2.6.2. In Sect. 2.6.3 we proceed by describing the interaction perturbatively using contour integral methods. In Sect. 2.6.4 we derive a few general properties of the spectral representation of the closed chain. Finally, in Sect. 2.6.5 we use the obtained spectral representation of the closed chain to rewrite the EL equations in a form suitable for an explicit analysis.

2.6.1 Spectral Decomposition of the Regularized Vacuum

In order to analyze the causal action principle, we clearly need to know the eigenvalues λ_i^{xy} of the closed chain. Moreover, in order to bring the EL equations into a tractable form, we also need to know the corresponding eigenspaces. We now compute the spectral decomposition of the closed chain for the regularized fermionic projector of the vacuum. We first do the computation in general, and then rewrite it using the formalism of the continuum limit.

As in Sect. 1.2.5 we assume that the regularized fermionic projector of the vacuum is *homogeneous* and has a *vector-scalar structure* (1.2.44). These assumptions are reasonable and sufficiently general for our purposes. Thus we assume that $P^\varepsilon(x, y)$ can again be written again as the Fourier integral (2.4.1), where $\hat{P}^\varepsilon$ now is a distribution of the form

$$\hat{P}^\varepsilon(k) = \hat{g}_j(k)\, \gamma^j + \hat{h}(k) \tag{2.6.1}$$

with real-valued distributions $\hat{g}_j$ and $\hat{h}$. Here the parameter $\varepsilon > 0$ denotes the length scale of the regularization. Thus, expressed in momentum space, the distributions $\hat{g}_j$ and $\hat{h}$ should decay at infinity on the scale $k \sim \varepsilon^{-1}$. This means in position space that the kernel of the fermionic projector has the form

$$P^\varepsilon(x, y) = g_j(x, y)\, \gamma^j + h(x, y) \tag{2.6.2}$$

with smooth functions g_j and h whose derivatives scale at most in powers of ε^{-1}. As ε tends to zero, the regularized fermionic projectors should go over to the unregularized fermionic projector,

$$\lim_{\varepsilon \searrow 0} P^\varepsilon(x, y) = P(x, y) \qquad \text{as a distribution.} \tag{2.6.3}$$

According to (1.1.14), we introduce the corresponding closed chain by

$$A^\varepsilon_{xy} = P^\varepsilon(x, y)\, P^\varepsilon(y, x)\,. \tag{2.6.4}$$

In the next lemma we compute the roots of the characteristic polynomial of this matrix. For ease in notation we shall often omit the subscripts "xy."

Lemma 2.6.1 *The characteristic polynomial of the closed chain A^ε_{xy} has two roots $\lambda_\pm$. Either the $\lambda_\pm$ form a complex conjugate pair, $\overline{\lambda_+} = \lambda_-$, or else they are both real and have the same sign. The roots are given explicitly by*

$$\lambda_\pm = g\overline{g} + h\overline{h} \pm \sqrt{(g\overline{g})^2 - g^2\,\overline{g}^2 + (g\overline{h} + h\overline{g})^2}\,. \tag{2.6.5}$$

Proof We write the fermionic projector in position space as

$$P^\varepsilon(x, y) = g_j(x, y)\, \gamma^j + h(x, y)\,, \qquad P^\varepsilon(y, x) = \overline{g_j(x, y)}\, \gamma^j + \overline{h(x, y)}\,.$$

Thus, omitting the arguments x and y,

$$A^{\varepsilon}_{xy} = (\not{g} + h)(\overline{\not{g}} + \overline{h}) .$$

Omitting the superscript ε and the subscript xy, we obtain

$$A = \not{g}\, \overline{\not{g}} \;+\; h\, \overline{\not{g}} \;+\; \not{g}\, \overline{h} \;+\; h\overline{h} . \tag{2.6.6}$$

It is useful to decompose A in the form

$$A = A_1 \;+\; A_2 \;+\; \mu$$

with

$$A_1 = \frac{1}{2}\, [\not{g}, \overline{\not{g}}] , \quad A_2 = h\, \overline{\not{g}} \;+\; \not{g}\, \overline{h} , \quad \mu = g\overline{g} \;+\; h\overline{h}$$

and $g\overline{g} \equiv g_j\, \overline{g^j}$. Then the matrices A_1 and A_2 anti-commute, and thus

$$(A - \mu)^2 = A_1^2 + A_2^2 = (g\overline{g})^2 \;-\; g^2\, \overline{g}^2 \;+\; (g\overline{h} + h\overline{g})^2 . \tag{2.6.7}$$

The right side of (2.6.7) is a multiple of the identity matrix, and so (2.6.7) is a quadratic equation for A. The roots $\lambda_\pm$ of this equation as given by (2.6.5) are the zeros of the characteristic polynomial of A. If the discriminant is negative, the $\lambda_\pm$ form a complex conjugate pair. If conversely the discriminant is positive, the $\lambda_\pm$ are both real. In order to show that they have the same sign, we compute their product,

$$\begin{aligned} \lambda_+ \lambda_- &= (g\overline{g} + h\overline{h})^2 - \left[(g\overline{g})^2 - g^2\, \overline{g}^2 + (g\overline{h} + h\overline{g})^2 \right] \\ &= 2\, (g\overline{g})\, |h|^2 + |h|^4 + g^2\, \overline{g}^2 - (g\overline{h} + h\overline{g})^2 \\ &= |h|^4 + g^2\, \overline{g}^2 - g^2\, \overline{h}^2 - h^2\, \overline{g}^2 \\ &= (g^2 - h^2)(\overline{g}^2 - \overline{h}^2) \geq 0 . \end{aligned}$$

This concludes the proof. □

In the degenerate case that the two eigenvalues λ_+ and λ_- coincide, the relation (2.6.7) shows that the matrix $A - \mu$ is nilpotent. However, in this case the matrix $A - \mu$ need not vanish (as one sees from (2.6.6)), giving examples where the matrix A is *not diagonalizable*. Except for this degenerate case, the matrix A is indeed diagonalizable and has two-dimensional eigenspaces:

Lemma 2.6.2 *In the case $\lambda_+ \neq \lambda_-$, the matrix A_{xy} is diagonalizable and has two-dimensional eigenspaces. It has the spectral representation*

$$A_{xy} = \sum_{s=\pm} \lambda_s^{xy}\, F_s^{xy} , \tag{2.6.8}$$

where the spectral projections are given by

$$F_{\pm}^{xy} = \frac{\mathbb{1}}{2} \pm \frac{\frac{1}{2}\,[\not{g}, \overline{\not{g}}] + h\overline{\not{g}} + \not{g}\overline{h}}{2\sqrt{(g\overline{g})^2 - g^2\,\overline{g}^2 + (g\overline{h} + h\overline{g})^2}}\,. \tag{2.6.9}$$

Proof If we *assume* that A is diagonalizable, then $\lambda_\pm$ are the two eigenvalues of A, and the corresponding spectral projectors $F_\pm$ are given by

$$F_\pm = \frac{\mathbb{1}}{2} \pm \frac{1}{\lambda_+ - \lambda_-}\left(A - \frac{1}{2}\,(\lambda_+ + \lambda_-)\,\mathbb{1}\right). \tag{2.6.10}$$

Applying (2.6.5) gives (2.6.9). Taking their trace, one sees that the matrices F_+ and F_- both have rank two.

In order to prove that A is diagonalizable, one takes formulas (2.6.9) and shows by direct computation that (see Exercise 2.24)

$$A\,F_\pm = \lambda_\pm\,F_\pm \qquad \text{and} \qquad F_+ + F_- = \mathbb{1}\,. \tag{2.6.11}$$

This shows that the images of F_+ and F_- are indeed eigenspaces of A which span $\mathbb{C}^4$. □

Our next step is to rewrite the spectral representation using the formalism of the continuum limit. Let us compute the leading singularity on the light cone. Then

$$P(x, y) = \frac{i}{2}\,\not{\xi}\,T_{[0]}^{(-1)} + (\deg < 2)\,, \tag{2.6.12}$$

where for notational convenience we omitted the indices $\overset{-1}{}_{[0]}$ of the factor ξ, and where the bracket $(\deg < 2)$ stands for terms of degree at most one. Using this formula for the fermionic projector, the closed chain becomes

$$A_{xy} = \frac{1}{4}\,(\not{\xi} T_{[0]}^{(-1)})(\overline{\not{\xi} T_{[0]}^{(-1)}}) + \not{\xi}(\deg \leq 3) + (\deg < 3)\,, \tag{2.6.13}$$

where $\overline{\not{\xi}} := \overline{\xi_j}\gamma^j$. Its trace can be computed with the help of the contraction rules (2.4.45),

$$\mathrm{Tr}(A_{xy}) = (\xi_j \overline{\xi^j})\,T_{[0]}^{(-1)}\,\overline{T_{[0]}^{(-1)}} = \frac{1}{2}\,(z + \overline{z})\,T_{[0]}^{(-1)}\,\overline{T_{[0]}^{(-1)}} + (\deg < 3)\,.$$

We next compute the square of the trace-free part of the closed chain,

$$\left(A_{xy} - \frac{1}{4}\,\mathrm{Tr}(A_{xy})\,\mathbb{1}\right)^2 = \frac{1}{16}\left(\not{\xi}\overline{\not{\xi}} - \frac{z + \overline{z}}{2}\right)^2 \left(T_{[0]}^{(-1)}\overline{T_{[0]}^{(-1)}}\right)^2$$

$$= \frac{1}{16} \left(\not{\xi} \not{\bar{\xi}} \not{\xi} \not{\bar{\xi}} - (z+\bar{z})\, \not{\xi} \not{\bar{\xi}} + \frac{1}{4} (z+\bar{z})^2 \right) \left(T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \right)^2$$
$$= \frac{1}{64} (z-\bar{z})^2 \left(T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \right)^2 .$$

Combining these formulas, we see that to leading degree, the closed chain is a solution of the polynomial equation

$$\left(A_{xy} - \frac{1}{8} (z+\bar{z})\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \right)^2 = \left(\frac{1}{8} (z-\bar{z})\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \right)^2 . \tag{2.6.14}$$

We point out that the calculations so far are only formal, but they have a well-defined meaning in the formalism of the continuum, because to all our end formulas we will be able to apply the weak evaluation formula (2.4.48). Having this in mind, we can interpret the roots of the polynomial in (2.6.14)

$$\lambda_+ = \frac{1}{4} \left(z\, T^{(-1)}_{[0]} \right) \overline{T^{(-1)}_{[0]}} \qquad \text{and} \qquad \lambda_- = \frac{1}{4}\, T^{(-1)}_{[0]}\, \overline{\left(z\, T^{(-1)}_{[0]} \right)}$$

as the eigenvalues of the closed chain. Using the contraction rule (2.4.46), these eigenvalues simplify to (see also [F7, Eq. (5.3.20)])

$$\lambda_+ = T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} + (\deg < 3) , \qquad \lambda_- = T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}} + (\deg < 3) . \tag{2.6.15}$$

The corresponding spectral projectors become (see also [F7, Eq. (5.3.21)])

$$F_\pm = \frac{1}{2} \left(\mathbb{1} \pm \frac{[\not{\xi}, \not{\bar{\xi}}]}{z - \bar{z}} \right) + \not{\xi} (\deg \leq 0) + (\deg < 0) . \tag{2.6.16}$$

Since in the formalism of the continuum limit, the factors z and $\bar{z}$ are treated as two different functions, we do not need to worry about the possibility that the eigenvalues λ_+ and λ_- might coincide or that the denominator in (2.6.16) might vanish. Similarly, we can treat ξ and $\bar{\xi}$ simply as two different vectors. Then the methods and results of Lemma 2.6.2 apply and show that the matrices F_+ and F_- have rank two, so that the eigenvalues λ_+ and λ_- are both two-fold degenerate. By direct computation, one finds that (see Exercise 2.25)

$$F_\pm\, P(x,y) = \begin{cases} 0 & \text{for } \text{“}+\text{”} \\ \frac{i}{2}\, \not{\xi}\, T^{(-1)}_{[0]} & \text{for } \text{“}-\text{”} \end{cases} + (\deg < 2) . \tag{2.6.17}$$

From (2.6.15) and (2.6.16) one sees that the eigenvalues of the closed chain form a *complex conjugate pair* and are both *two-fold degenerate*. Using this result in (1.1.9), one comes to the important conclusion that the Lagrangian vanishes identically, implying that, using the formalism of the continuum limit, the fermionic projector

of the vacuum is a minimizer of the causal action. We will return to this point in a more general context in Sect. 2.6.5.

The lower degrees on the light cone can be computed in a straightforward way by expanding the formulas (2.6.9). To give an impression, we here list a few formulas:

$$\begin{aligned}
\lambda_\pm &= \frac{1}{4} \times \begin{cases} (z\, T^{(-1)}_{[0]})\, \overline{T^{(-1)}_{[0]}} \;+\; (z\, T^{(0)}_{[2]})\, \overline{T^{(-1)}_{[0]}} \;+\; (z\, T^{(-1)}_{[0]})\, \overline{T^{(0)}_{[2]}} & \text{for ``}+\text{''} \\ T^{(-1)}_{[0]}\, (z\, T^{(-1)}_{[0]}) \;+\; T^{(-1)}_{[0]}\, (z\, T^{(0)}_{[2]}) \;+\; T^{(0)}_{[2]}\, (z\, T^{(-1)}_{[0]}) & \text{for ``}-\text{''} \end{cases} \\
&\quad + T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} \mp \frac{T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}}}{T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}}}\, (T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} - T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}}) \\
&\quad + (\deg < 2)\,.
\end{aligned}$$

$$\begin{aligned}
F_\pm\, P(x,y) &= \frac{i}{4}\, (\not\xi\, T^{(-1)}_{[0]}) \;+\; (\deg < 2) \\
&\quad \pm \frac{i}{4}\, \frac{(\not\xi\, T^{(-1)}_{[0]})(T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}}) \;-\; 2\, (\not\xi\, \overline{T^{(-1)}_{[0]}})\, T^{(-1)}_{[0]}\, T^{(0)}_{[0]}}{T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}}}\,.
\end{aligned}$$

These formulas can be obtained more systematically with the perturbation expansion for the spectral decomposition which we now describe.

2.6.2 The Double Null Spinor Frame

Before entering the perturbation calculation, it is convenient to choose a specific eigenvector basis of the closed chain of the vacuum. This basis is referred to as the *double null spinor frame* and is denoted by $(\mathfrak{f}^{L/R}_\pm)$. Performing computations in the double null spinor frame is an improvement of the method of "factorizing matrix traces" as introduced in [F7, Appendix G.2]. Following (2.6.13), we introduce the matrix

$$A^0_{xy} = \frac{1}{4}\, (\not\xi T^{(-1)}_{[0]})(\overline{\not\xi T^{(-1)}_{[0]}})\,.$$

According to (2.6.15) and (2.6.16), in the formalism of the continuum limit the corresponding eigenvalues and spectral projectors are given by

$$\lambda_+ = T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}}\,, \qquad \lambda_- = T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}} \tag{2.6.18}$$

$$F_\pm = \frac{1}{2}\Big(\mathbb{1} \pm \frac{[\not\xi, \overline{\not\xi}]}{z - \overline{z}}\Big), \tag{2.6.19}$$

and they satisfy the relations

$$F_+\, \not\xi \overline{\not\xi} = z\, F_+\,, \qquad \text{and} \qquad F_-\, \not\xi \overline{\not\xi} = \overline{z}\, F_-\,.$$

Furthermore, the matrix A^0_{xy} is invariant on the left- and right-handed components, and thus we may choose joint eigenvectors of the matrices A_0 and Γ. This leads us to introduce the four eigenvectors $\mathfrak{f}^{L/R}_\pm$ by the relations

$$\boxed{\chi_c\, F_s\, \mathfrak{f}^c_s = \mathfrak{f}^c_s} \tag{2.6.20}$$

with $c \in \{L, R\}$ and $s \in \{+, -\}$, which define each of these vectors up to a complex factor. For clarity in notation, we again write the inner product on Dirac spinors $\overline{\psi}\phi \equiv \psi^\dagger \gamma^0 \phi$ as $\prec \psi | \phi \succ$, and refer to it as the *spin scalar product*. Then the calculation

$$\prec \mathfrak{f}^L_+ \,|\, \mathfrak{f}^L_+ \succ \;=\; \prec \chi_L \mathfrak{f}^L_+ \,|\, \chi_L \mathfrak{f}^L_+ \succ \;=\; \prec \mathfrak{f}^L_+ \,|\, \chi_R\, \chi_L\, \mathfrak{f}^L_+ \succ \;=\; 0$$

(and similarly for the other eigenvectors) shows that these vectors are indeed all null with respect to the spin scalar product. Moreover, taking the adjoint of (2.6.19) with respect to the spin scalar product, one sees that

$$(F_+)^* = F_- \,. \tag{2.6.21}$$

As a consequence, the inner products vanish unless the lower indices are different, for example

$$\prec \mathfrak{f}^L_+ \,|\, \mathfrak{f}^R_+ \succ \;=\; \prec F_+ \mathfrak{f}^L_+ \,|\, F_+ \mathfrak{f}^R_+ \succ \;=\; \prec \mathfrak{f}^L_+ \,|\, F_-\, F_+\, \mathfrak{f}^R_+ \succ \;=\; 0\,.$$

We conclude that all inner products between the basis vectors vanish except for the inner products $\prec \mathfrak{f}^L_+ | \mathfrak{f}^R_- \succ$, $\prec \mathfrak{f}^R_+ | \mathfrak{f}^L_- \succ$ as well as their complex conjugates $\prec \mathfrak{f}^R_- | \mathfrak{f}^L_+ \succ$ and $\prec \mathfrak{f}^L_- | \mathfrak{f}^R_+ \succ$. We assume that all the non-vanishing inner products are equal to one,

$$\boxed{|\, \prec \mathfrak{f}^L_+ \,|\, \mathfrak{f}^R_- \succ| \;=\; 1 \;=\; |\, \prec \mathfrak{f}^R_+ \,|\, \mathfrak{f}^L_- \succ|\,.} \tag{2.6.22}$$

In order to specify the phases and relative scalings of the basis vectors, we introduce a space-like unit vector u which is orthogonal to both ξ and $\overline{\xi}$. Then the imaginary vector $v = iu$ satisfies the relations

$$\langle v, \xi \rangle = 0 = \langle v, \overline{\xi} \rangle \,, \quad \langle v, v \rangle = 1 \quad \text{and} \quad \overline{v} = -v \,. \tag{2.6.23}$$

As a consequence, the operator $\not{v}$ commutes with F_+ and F_-, and since it flips parity, we may set $\mathfrak{f}^R_+ = \not{v}\, \mathfrak{f}^L_+$. Next, a straightforward computation using (2.6.19) gives the identities

$$F_-\, \not{\xi} = \not{\xi}\, F_+ \qquad \text{and} \qquad F_-\, \overline{\not{\xi}} = \overline{\not{\xi}}\, F_+ \,. \tag{2.6.24}$$

These identities can be used as follows. The first identity implies that

$$\big(\chi_R F_-\, \not{\xi}\big)\, \mathfrak{f}^L_+ = \not{\xi}\, \chi_L F_+\, \mathfrak{f}^L_+ \sim \not{\xi}\, \mathfrak{f}^L_+ \,,$$

showing that the vectors $\not{\ell}\, \mathfrak{f}_+^L$ and $\mathfrak{f}_-^R$ are linearly dependent. The calculation

$$\prec \mathfrak{f}_+^L \,|\, \not{\ell} \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \not{\ell}\, \frac{\not{\ell}\overline{\not{\ell}}}{z}\, \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \frac{\not{\ell}\not{\ell}}{z}\, \overline{\not{\ell}}\, \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \overline{\not{\ell}} \mathfrak{f}_+^L \succ$$

(where we used (2.6.20) and (2.6.19)) shows that the vector $\not{\ell}\, \mathfrak{f}_+^L$ is in fact a real multiple of $\mathfrak{f}_-^R$. Hence by normalizing $\mathfrak{f}_+^L$ appropriately, we can arrange[3] that $\mathfrak{f}_-^R = \not{\ell}\, \mathfrak{f}_+^L$. Using the second identity in (2.6.24), we also find that $\mathfrak{f}_-^R = \overline{\not{\ell}}\, \mathfrak{f}_+^L$. Similarly, we may also set $\mathfrak{f}_-^L = \not{\ell}\, \mathfrak{f}_+^R = \overline{\not{\ell}}\, \mathfrak{f}_+^R$. The resulting relations between our basis vectors are summarized in the following diagram:

$$\begin{array}{ccc} \mathfrak{f}_+^L & \xrightarrow{\not{\psi}} & \mathfrak{f}_+^R \\ \not{\ell} \big\downarrow \overline{\not{\ell}} & & \not{\ell} \big\downarrow \overline{\not{\ell}} \\ \mathfrak{f}_-^R & \xrightarrow{-\not{\psi}} & \mathfrak{f}_-^L \end{array} \tag{2.6.25}$$

With (2.6.20), (2.6.22) and (2.6.25) we have introduced the double null spinor frame $(\mathfrak{f}_\pm^{L/R})$. The construction involves the freedom in choosing the operator $\not{\psi}$ according to (2.6.23); for given $\not{\psi}$, the basis vectors are unique up to an irrelevant common phase. The construction of the double null spinor frame is illustrated in Exercise 2.26.

We next explain how we can represent a given linear operator B on the spinors in the double null frame $(\mathfrak{f}_\pm^{L/R})$. Following the notation in [F7, Appendix G], we denote the matrix element in the column (c, s) and row (c', s') by $\mathfrak{F}_{ss'}^{cc'}(B)$. These matrix entries are obtained by acting with B on the vector $\mathfrak{f}_{s'}^{c'}$ and taking the inner product with the basis vector which is conjugate to $\mathfrak{f}_s^c$, i.e.

$$\mathfrak{F}_{ss'}^{cc'}(B) \;=\; \prec \overline{\mathfrak{f}_s^c} \,|\, B\, \mathfrak{f}_{s'}^{c'} \succ , \tag{2.6.26}$$

where the conjugation flips the indices according to $L \leftrightarrow R$ and $+ \leftrightarrow -$. Similarly, we can also express the projectors $\chi_c F_s$ in terms of the basis vectors, for example

$$\chi_L F_+ = |\mathfrak{f}_+^L \succ \prec \mathfrak{f}_-^R| \,. \tag{2.6.27}$$

For computing (2.6.26), we use the relations in (2.6.25) to express the vector $\mathfrak{f}_{s'}^{c'}$ in terms of $\mathfrak{f}_+^L$, choosing the relations which do not involve factors of $\overline{\not{\ell}}$. Similarly, we

[3] Let us explain why we do not consider the opposite sign $\mathfrak{f}_-^R = -\not{\ell}\, \mathfrak{f}_+^L$. To this end, we must show that $\prec \mathfrak{f}_+^L | \not{\ell} \mathfrak{f}_+^L \succ \;> 0$. Since for any given positive or definite spinor ζ, the vector $\chi_L F_+ \zeta$ is a multiple of $\mathfrak{f}_+^L$, it suffices to compute instead the sign of the combination $\prec \chi_L F_+ \zeta | \not{\ell} \chi_L F_+ \zeta \succ$. Applying (2.6.21) and (2.6.24), this inner product simplifies to $\prec \zeta | \chi_R F_- \not{\ell} \zeta \succ$. With the help of (2.6.17) and (2.6.12), we can treat the factor $\not{\ell}$ as an outer factor. Then our inner product simplifies to the expectation value $\prec \zeta | \chi_R \not{\ell} \zeta \succ$. This expectation value is positive if we follow the convention introduced before (2.4.48) that $\xi^0 > 0$.

express the vector $\mathfrak{f}^{\overline{c}}_{\overline{s}}$ in terms of $\mathfrak{f}^R_-$, avoiding factors of $\not{\xi}$. Applying (2.6.27), we can then rewrite the inner product as a trace involving the operator F_+. More precisely, a straightforward calculation yields

$$\left.\begin{aligned}
\mathfrak{F}^{LL}_{++}(B) &= \mathrm{Tr}(F_+ \,\chi_L\, B) \qquad &,\ \mathfrak{F}^{LR}_{++}(B) &= \mathrm{Tr}(F_+ \not{\psi}\, \chi_L\, B) \\
\mathfrak{F}^{LL}_{+-}(B) &= \mathrm{Tr}(\not{\xi}\, F_+ \not{\psi}\, \chi_L\, B) &,\ \mathfrak{F}^{LR}_{+-}(B) &= \mathrm{Tr}(\not{\xi}\, F_+ \,\chi_L\, B) \\
\mathfrak{F}^{LL}_{-+}(B) &= \frac{1}{z}\, \mathrm{Tr}(F_+ \not{\psi}\, \not{\xi} \chi_L\, B) &,\ \mathfrak{F}^{LR}_{-+}(B) &= \frac{1}{z}\, \mathrm{Tr}(F_+ \not{\xi}\, \chi_L\, B) \\
\mathfrak{F}^{LL}_{--}(B) &= \frac{1}{z}\, \mathrm{Tr}(\not{\xi}\, F_+ \not{\xi}\, \chi_L\, B) &,\ \mathfrak{F}^{LR}_{--}(B) &= \frac{1}{z}\, \mathrm{Tr}(\not{\xi}\, F_+ \not{\psi}\, \not{\xi}\, \chi_L\, B)
\end{aligned}\right\} \tag{2.6.28}$$

(see also [F7, Eq. (G.19)], where these relations are derived with a different method). Indeed, it suffices to compute the given eight matrix elements, because the other eight matrix elements are obtained by the replacements $L \leftrightarrow R$. Moreover, the matrix elements of the adjoint (with respect to the spin scalar product) are obtained by

$$\mathfrak{F}^{cc'}_{ss'}(B^*) = \prec \mathfrak{f}^{\overline{c}}_{\overline{s}} \,|\, B^*\, \mathfrak{f}^{c'}_{s'} \succ = \overline{\prec \mathfrak{f}^{c'}_{s'} \,|\, B\, \mathfrak{f}^{\overline{c}}_{\overline{s}} \succ} = \overline{\mathfrak{F}^{\overline{c'}\,c}_{s'\,s}(B)}\,.$$

A simple example for how to compute the matrix elements in the double null spinor frame is given in Exercise 2.27.

2.6.3 *Perturbing the Spectral Decomposition*

Omitting the arguments (x, y), we decompose the fermionic projector as

$$P = P_0 + \Delta P\,,$$

where P_0 is the vacuum fermionic projector (possibly modified by gauge phases). This gives rise to the decomposition of A

$$A = A_0 + \Delta A \tag{2.6.29}$$

with

$$A_0 = P_0(x, y)\, P_0(y, x) \tag{2.6.30}$$

$$\Delta A = \Delta P(x, y)\, P_0(y, x) + P_0(x, y)\, \Delta P(y, x) + \Delta P(x, y)\, \Delta P(y, x)\,. \tag{2.6.31}$$

The eigenvalues and spectral projectors of A_0 were computed explicitly in Sect. 2.6.1. In view of later generalizations, we write the obtained spectral decomposition as

$$A_0 = \sum_{k=1}^{K} \lambda_k\, F_k$$

with $K = 2$, where λ_k are distinct eigenvalues with corresponding spectral projections F_k. Since the perturbation ΔA will in general remove the degeneracies, we cannot expect that by perturbing F_k we again obtain spectral projection operators. But we can form projectors G_k on the space spanned by all eigenvectors of A whose eigenvalues are sufficiently close to λ_k. The G_k are most conveniently introduced using contour integrals. We choose $\varepsilon > 0$ such that

$$|\lambda_i - \lambda_j| \;<\; 2\varepsilon \qquad \text{for all } i, j = 1, \ldots, K \text{ and } i \neq j.$$

Then we set

$$G_k = \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (z - A)^{-1}\, dz\,, \tag{2.6.32}$$

Combining the resolvent identity with the Cauchy integral formula, one sees that G_k is indeed an idempotent operator whose image is the invariant subspace corresponding to the eigenvalues near λ_k (for details see Exercise 2.6).

The integral formula (2.6.32) is very useful for a perturbation expansion. To this end, we substitute (2.6.29) into (2.6.32) and compute the inverse with the Neumann series,

$$\begin{aligned} G_k &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \big(z - A_0 - \Delta A\big)^{-1}\, dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \big(\mathbb{1} - (z - A_0)^{-1}\, \Delta A\big)^{-1}\, (z - A_0)^{-1}\, dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \sum_{n=0}^{\infty} \big((z - A_0)^{-1}\, \Delta A\big)^n\, (z - A_0)^{-1}\, dz\,. \end{aligned}$$

Interchanging the integral with the infinite sum gives the perturbation expansion,

$$G_k = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \big((z - A_0)^{-1}\, \Delta A\big)^n\, (z - A_0)^{-1}\, dz\,, \tag{2.6.33}$$

where n is the order in perturbation theory. After substituting in the spectral representation for $(z - A_0)^{-1}$,

$$(z - A_0)^{-1} = \sum_{l=1}^{K} \frac{F_l}{z - \lambda_l}\,, \tag{2.6.34}$$

the contour integral in (2.6.33) can be carried out with residues. For example, we obtain to second order,

$$G_k = F_k + \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} (F_k \,\Delta A\, F_l + F_l \,\Delta A\, F_k) + \mathcal{O}((\Delta A)^3)$$
$$+ \sum_{l,m \neq k} \frac{1}{(\lambda_k - \lambda_l)(\lambda_k - \lambda_m)}$$
$$\times (F_k \,\Delta A\, F_l \,\Delta A\, F_m + F_l \,\Delta A\, F_k \,\Delta A\, F_m + F_l \,\Delta A\, F_m \,\Delta A\, F_k)$$
$$- \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2}$$
$$\times (F_k \,\Delta A\, F_k \,\Delta A\, F_l + F_k \,\Delta A\, F_l \,\Delta A\, F_k + F_l \,\Delta A\, F_k \,\Delta A\, F_k)\,. \tag{2.6.35}$$

To order $n > 2$, the corresponding formulas are clearly more complicated, but even then they involve matrix products which are all of the form

$$F_{k_1} \,\Delta A\, F_{k_2} \,\Delta A \cdots F_{k_n} \,\Delta A\, F_{k_{n+1}}\,. \tag{2.6.36}$$

An example of a first order perturbation computation is given in Exercise 2.28.

2.6.4 General Properties of the Spectral Decomposition

We now derive a few general properties of the spectral decomposition of the closed chain.

Lemma 2.6.3 *Assume that for a one-parameter family of fermionic projectors $P(\tau)$ and fixed $x, y \in M$, the matrices A_{xy} and A_{yx} are diagonalizable for all τ in a neighborhood of $\tau = 0$, and that the eigenvalues of the matrix $A_{xy}|_{\tau=0}$ are all non-real. Then the unperturbed closed chain A_{xy} has a spectral representation*

$$A_{xy}\big|_{\tau=0} = \sum_{k=1}^{4} \lambda_k^{xy} F_k^{xy} \tag{2.6.37}$$

with the following properties. The last two eigenvalues and spectral projectors are related to the first two by

$$\lambda_3^{xy} = \overline{\lambda_1^{xy}}\,, \quad F_3^{xy} = (F_1^{xy})^* \quad \text{and} \quad \lambda_4^{xy} = \overline{\lambda_2^{xy}}\,, \quad F_4^{xy} = (F_2^{xy})^*\,. \tag{2.6.38}$$

The first order perturbation $\delta A_{xy} = \partial_\tau A_{xy}|_{\tau=0}$ of the closed chain is diagonal in the bases of the non-trivial degenerate subspaces, i.e.

$$F_k^{xy} (\delta A_{xy}) F_l^{xy} = 0 \qquad \text{if } k \neq l \text{ and } \lambda_k^{xy} = \lambda_l^{xy}. \tag{2.6.39}$$

The closed chain A_{yx} has a corresponding spectral representation satisfying (2.6.37)–(2.6.39) with all indices 'xy' are replaced by 'yx'. The spectral representations of A_{xy} and A_{yx} are related to each other by

$$\lambda_k^{xy} = \lambda_k^{yx} \qquad \text{and} \qquad F_k^{xy}\, P(x,y) = P(x,y)\, F_k^{yx}\,. \tag{2.6.40}$$

Proof By continuity, the eigenvalues of the matrix A_{xy} are non-real in a neighborhood of $\tau = 0$. Moreover, by direct computation one sees that the matrix A_{xy} is symmetric in the sense that $A_{xy} = A_{xy}^* = \gamma^0 A_{xy}^\dagger \gamma^0$. Hence, using the idempotence of the matrix γ^0 together with the multiplicity of the determinant, we find that

$$\det(A_{xy} - \lambda) = \det(\gamma^0 (A_{xy}^\dagger - \lambda)\gamma^0) = \det(A_{xy}^\dagger - \lambda) = \overline{\det(A_{xy} - \overline{\lambda})}\,.$$

Hence if λ is an eigenvalue of the matrix A_{xy}, so is $\overline{\lambda}$. Thus the eigenvalues must form complex conjugate pairs.

We first complete the proof in the case that there are no degeneracies. For any eigenvalue λ of A_{xy} we choose a polynomial $p_\lambda(z)$ with $p_\lambda(\lambda) = 1$ and $p_\lambda(\mu) = 0$ for all other spectral points μ. Then the spectral projector on the eigenspace corresponding to λ, denoted by F_λ^{xy}, is given by

$$F_\lambda^{xy} = p_\lambda(A_{xy})\,. \tag{2.6.41}$$

Taking the adjoint and possibly after reordering the indices k, we obtain the relations (2.6.37) and (2.6.38). The general matrix relation $\det(BC - \lambda) = \det(CB - \lambda)$ (see for example [F10, Sect. 3]) shows that the closed chains A_{xy} and A_{yx} have the same spectrum. Multiplying (2.6.41) by $P(x,y)$ and iteratively applying the relation

$$A_{xy}\, P(x,y) = P(x,y)\, P(y,x)\, P(x,y) = P(x,y)\, A_{yx}\,,$$

we find that $F_\lambda^{xy}\, P(x,y) = P(x,y)\, F_\lambda^{yx}$. Thus we can label the eigenvalues of the matrix A_{yx} such that (2.6.40) holds.

In the case with degeneracies, the assumption that A_{xy} is diagonalizable in a neighborhood of $\tau = 0$ allows us to diagonalize δA_{xy} on the degenerate subspaces (see for example [Ba] or the similar method for self-adjoint operators in [S2, Sect. 11.1.2]). This yields (2.6.39), whereas (2.6.38) can be arranged by a suitable ordering of the spectral projectors F_k^{xy}. In the degenerate subspaces of A_{yx} we can choose the bases such that (2.6.37) and (2.6.38) hold (with 'xy' replaced by 'yx') and that (2.6.40) is satisfied. It remains to prove that (2.6.39) also holds for A_{yx}: From (2.6.39) we know that for any pair l, k with $\lambda_l^{xy} = \lambda_k^{xy}$,

$$\begin{aligned} 0 &= F_k^{xy}(\delta A_{xy}) F_l^{xy} = F_k^{xy}\Big(\delta P(x,y)\, P(y,x) + P(x,y)\, \delta P(y,x)\Big) F_l^{xy} \\ &= F_k^{xy}(\delta P(x,y)) F_l^{yx}\, P(y,x) + P(x,y) F_k^{yx}(\delta P(y,x)) F_l^{xy}\,, \end{aligned}$$

where in the last line we applied the second equation in (2.6.40). Multiplying by $P(y,x)$ from the left and by $P(x,y)$ on the right, we find

$$0 = P(y,x) F_k^{xy} (\delta P(x,y)) F_l^{yx} \lambda_l^{yx} + \lambda_k^{yx} F_k^{yx} (\delta P(y,x)) F_l^{xy} P(x,y) .$$

After dividing by $\lambda_l^{yx} = \lambda_k^{yx}$ (note that the eigenvalues are non-zero because they are assumed to form complex conjugate pairs), we can again use the second equation in (2.6.40) to obtain

$$\begin{aligned} 0 &= P(y,x) F_k^{xy} (\delta P(x,y)) F_l^{yx} + F_k^{yx} (\delta P(y,x)) F_l^{xy} P(x,y) \\ &= F_k^{yx} \Big(P(y,x)\, \delta P(x,y) + \delta P(y,x)\, P(x,y) \Big) F_l^{yx} = F_k^{yx} (\delta A_{yx}) F_l^{yx} , \end{aligned}$$

concluding the proof. □

2.6.5 *Spectral Analysis of the Euler-Lagrange Equations*

We now explain how the spectral decomposition of the closed chain can be used to analyze the causal action principle introduced in Sect. 1.1.1 as well as the corresponding EL equations as worked out in Sect. 1.4.1. For the regularized Dirac sea vacuum as considered in Sect. 2.6.1, the situation is quite simple. Namely, according to Lemma 2.6.1 (or more explicitly in (2.6.18)), the closed chain has two eigenvalues which form a complex conjugate pair. As a consequence, the eigenvalues all have the same absolute value. Writing the Lagrangian in the form (1.1.9), one sees that the Lagrangian vanishes identically. We come to the following conclusion:

> In the formalism of the continuum limit, the regularized Dirac sea vacuum is a minimizer of the causal action. (2.6.42)

If the fermionic projector of the vacuum is perturbed (for example by an external potential or by additional particle or antiparticle states), the degeneracy of the eigenvalues will in general disappear, so that the spectrum will consist of two complex conjugate pairs. As a consequence, the causal action will no longer vanish. In order to analyze whether we still have a critical point of the causal action, one needs to analyze the corresponding EL equations in Proposition 1.4.3. To this end, it is very convenient to rewrite these EL equations using the spectral decomposition of the closed chain, as we now explain.

For simplicity, we again restrict attention to Dirac spinors and spin dimension two. Moreover, we only consider the case that the Lagrange multipliers κ and λ in Proposition 1.4.3 are both equal to zero. The generalization to higher spin dimension and to non-trivial κ and λ are straightforward and will be carried out later on (see Lemmas 3.6.2, 3.7.1 and the similar results in Sect. 4.4.1). Writing the Lagrangian in the form (1.1.9), we have

$$\mathcal{L}(x,y) = \frac{1}{8} \sum_{i,j=1}^{4} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 . \tag{2.6.43}$$

The relation (2.6.39) allows us to compute the variation of the eigenvalues by a standard first order perturbation calculation without degeneracies,

$$\delta \lambda_k^{xy} = \mathrm{Tr}(F_k^{xy}\, \delta A_{xy}) . \tag{2.6.44}$$

Using that that $\delta|\lambda| = \mathrm{Re}(\overline{\lambda}\, \delta\lambda/|\lambda|)$, we can compute the first variation of (2.6.43) by

$$\delta\mathcal{L}(x,y) = \frac{1}{2}\, \mathrm{Re} \sum_{j,k=1}^{4} \left(|\lambda_k^{xy}| - |\lambda_j^{xy}| \right) \frac{\overline{\lambda_k^{xy}}}{|\lambda_k^{xy}|}\, \mathrm{Tr}(F_k^{xy}\, \delta A_{xy}) . \tag{2.6.45}$$

We now insert the identity

$$\delta A_{xy} = \delta P(x,y)\, P(y,x) + P(x,y)\, \delta P(y,x) .$$

Cyclically commuting the arguments of the trace, we obtain

$$\begin{aligned} \delta\mathcal{L}(x,y) = \frac{1}{2} \sum_{j,k=1}^{4} & \left(|\lambda_k^{xy}| - |\lambda_j^{xy}| \right) \\ & \times\, \mathrm{Re}\; \mathrm{Tr} \left[\frac{\overline{\lambda_k^{xy}}}{|\lambda_k^{xy}|}\, P(y,x)\, F_k^{xy}\, \delta P(x,y) + \frac{\overline{\lambda_k^{xy}}}{|\lambda_k^{xy}|}\, F_k^{xy} P(x,y)\, \delta P(y,x) \right] . \end{aligned}$$

Using (2.6.38) and (2.6.40), one sees that the first summand in the square bracket is the adjoint of the second summand. Therefore, the trace of the square bracket is real-valued, so that it is unnecessary take the real part. Comparing with (1.4.16), we conclude that

$$Q(x,y) = \frac{1}{2} \sum_{j,k=1}^{4} \left(|\lambda_k^{xy}| - |\lambda_j^{xy}| \right) \frac{\overline{\lambda_k^{xy}}}{|\lambda_k^{xy}|}\, F_k^{xy}\, P(x,y) \tag{2.6.46}$$

(where we again used (2.6.40)). In the vacuum, when the eigenvalues of the closed chain form a complex conjugate pair (2.6.18), the kernel $Q(x,y)$ vanishes identically in the formalism of the continuum limit. If the fermionic projector of the vacuum is perturbed, the first order perturbation of $Q(x,y)$ can be computed easily with the help of (2.6.44). The higher orders in perturbation theory can be treated systematically by using the contour method in Sect. 2.6.3 and by evaluating the resulting expressions in the formalism of the continuum limit.

The above methods give a mathematical meaning to $Q(x,y)$ in the formalism of the continuum limit. The remaining difficulty is that in the EL equations worked out

in Proposition 1.4.3, the kernel $Q(x, y)$ appears inside an integral (1.4.18), and one must control the error terms (2.4.50) and (2.4.51) inside this integral. The method is to choose a vector $u \in \mathcal{H}$ such that its physical wave function ψ^u is supported away from x, up to a small error. This method is referred to as *testing on null lines*. In a more physical picture, one chooses ψ^u as an *ultrarelativistic wave packet* localized near a null curve which does not meet the space-time point x. Applying this method to (1.4.18), the left side is evaluated weakly on the light cone, whereas the right side vanishes. In this way, the EL equations in the continuum limit reduce to

$$Q(x, y) = 0 \qquad \text{evaluated weakly on the light cone .}$$

We refer for details to Sect. 3.5.2. The estimates of all the error terms are worked out in Appendix A.

Exercises

Exercise 2.1 *(external field problem)* In physics textbooks, the notions of a "particle" and "anti-particle" are often associated to the frequency (or equivalently the energy) of the solutions: solutions of positive frequency are called particles, whereas the negative-frequency solutions are reinterpreted as describing anti-particle states. The aim of this exercise is to explain why these notions are ill-defined in the presence of a time-dependent potential. To this end, we consider the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\psi = 0 \,, \tag{2.6.47}$$

where $\mathcal{B}$ is a "step potential in time" i.e.

$$\mathcal{B}(t, \vec{x}) = V\gamma^0\, \Theta(t)\, \Theta(1 - t)$$

with a real parameter V.

(a) Separate out the spatial dependence for any given $\vec{k} \in \mathbb{R}^3$ with the plane-wave ansatz

$$\psi(t, \vec{x}) = e^{i\vec{k}\vec{x}}\, \phi(t)$$

(where ϕ is a spinor-valued function). Derive the resulting ordinary differential equation for $\phi(t)$.

(b) Clearly, the potential has discontinuities at $t = 0$ and $t = 1$. Show that there are two fundamental solutions $\phi_1, \phi_2 \in C^0(\mathbb{R}, \mathbb{C}^4)$ which are smooth solutions of the ODE except at the points $t = 0$ and $t = 1$. *Remark:* This procedure is familiar to physics students from quantum mechanics textbooks where wave functions are "glued together" at discontinuities of step potentials. From the mathematical point of view, the "glueing" of the solutions can be justified by saying that ϕ_1 and ϕ_2 are a fundamental system of *weak solutions* of the ODE. To the reader who is not familiar with these concepts, it might be instructive to verify that the notion of "weak solution" really gives rise to a continuity condition for ϕ. (Likewise,

for a second order equation like the Schrödinger equation, the notion of "weak solution" gives rise to C^1-solutions whose second derivatives are discontinuous.)

(c) Consider a "scattering process" where for negative times the solution is of the form

$$\phi(t) = e^{-i\omega t}\, \chi\,,$$

where χ is a constant spinor and $\omega := \sqrt{\vec{k}^2 - m^2}$. Show that for time $t > 1$, this solution can be written as

$$\phi(t) = e^{-i\omega t}\, \chi_+ + e^{+i\omega t}\, \chi_-$$

with constant spinors χ_+ and χ_-. Compute χ_+ and χ_- explicitly as functions of χ and V. Verify in particular that χ_- in general does not vanish.

(d) What does this mean for the interpretation of the solution in terms of "particles" and "anti-particles"? Why can the frequency of the solutions not be used for a global concept of particles and anti-particles? How can a pair creation/annihilation process be understood in our example? *Remark:* In order to avoid misunderstandings, we point out that the above arguments only show that the *frequency* cannot be used to obtain a global particle interpretation. They do not rule out the possibility that there may be a well-defined global particle interpretation using other properties of the solutions. In fact, such a global particle interpretation is provided by the causal perturbation expansion (or the corresponding functional analytic constructions in [FR2, FR3, FMR]). However, this global particle interpretation in general does not coincide with the "particles" and "anti-particles" as experienced by a local observer.

Exercise 2.2 This exercise is devoted to the *advanced Green's function* $s_m^\vee$ (for a more computational exercise on the advanced Green's function see Exercise 2.12).

(a) Assume that $m > 0$. Show that the limit $\nu \searrow 0$ in (2.1.9) exist in the distributional sense.

(b) Show that the limit $\nu \searrow 0$ in (2.1.9) also exists in the massless case $m = 0$ and that

$$\lim_{m \searrow 0} s_m^\vee(k) = s_0^\vee(k) \qquad \text{as a distribution}\,.$$

Hint: Proceed similar as in Exercise 1.21.

(c) Consider the Fourier integral in the q^0-variable

$$\int_{-\infty}^{\infty} \frac{1}{q^2 - m^2 - i\nu q^0}\, e^{iq^0 t}\, dq^0\,.$$

Show with residues that this integral vanishes for sufficiently small ν if $t < 0$.

(d) Argue with Lorentz invariance to prove the left side of (2.1.12).

Exercise 2.3 Modifying the location of the poles in (2.1.9) gives rise to the distribution

$$s_m^F(k) := \lim_{\nu \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + i\nu} .$$

This is the well-known *Feynman propagator*, which is often described intuitively by saying that "positive frequencies move to the future and negative frequencies move to the past." Make this sentence precise by a computation similar to that in Exercise 2.2 (c).

Exercise 2.4 (a) Assume that $m > 0$. Give a detailed proof of the distributional relation (2.1.14). *Hint*: Argue similar as in Exercise 1.21.

(b) Prove that (2.1.14) also holds in the case $m = 0$. *Hint*: The subtle point is to analyze the behavior at $q = 0$. To this end, apply Lebesgue's dominated convergence theorem.

Exercise 2.5 *(probability integral and current conservation)* Let ψ, ϕ be two solutions of the Dirac equation (2.1.5) with a smooth potential $\mathcal{B}$ which is symmetric (2.1.20). Moreover, assume that ψ and ϕ are smooth and have spatially compact support.

(a) Show that the integral (2.1.19) is independent of t_0.

(b) More generally, let $\mathcal{N}$ be a Cauchy surface in Minkowski space with future-directed normal ν. Show that the integral

$$\int_{\mathcal{N}} \overline{\psi}(\psi\phi)\, d\mu_{\mathcal{N}}$$

is independent of the choice of the Cauchy surface (where $d\mu_{\mathcal{N}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathcal{N}$). *Hint*: Show that the vector field $\overline{\psi}\gamma^j\phi$ is divergence-free and apply the Gauß divergence theorem.

Exercise 2.6 *(resolvent and contour integrals)* The aim of this exercise is to make the reader familiar with the notion of the resolvent and the contour integral representation of spectral projectors in the finite-dimensional setting. More details and generalizations to infinite dimensions can be found in the book by Kato [Ka].

(a) Let $A \in \mathrm{L}(\mathbb{C}^k)$ be a $k \times k$-matrix. The *resolvent set* is the set of all $\lambda \in \mathbb{C}$ for which the matrix $(A - \lambda)$ is invertible. The *spectrum* is the complement of the resolvent set. For any λ in the resolvent set, we define the *resolvent* R_λ by

$$R_\lambda = (A - \lambda \mathbb{1})^{-1}$$

(we use this sign convention consistently, although some authors use the opposite sign convention). Prove the *resolvent identity*

$$R_\lambda\, R_{\lambda'} = \frac{1}{\lambda - \lambda'} \left(R_\lambda - R_{\lambda'}\right),$$

valid for any λ, λ' in the resolvent set. *Hint*: Multiply the identity $\lambda' - \lambda = (A - \lambda) - (A - \lambda')$ from the left and right by a resolvent.

(b) Assume that A is a Hermitian matrix. Let Γ be a contour which encloses only one eigenvalue λ_0 with winding number one. Show that the contour integral

$$-\frac{1}{2\pi i} \oint_\Gamma R_\lambda \, d\lambda \tag{2.6.48}$$

is an orthogonal projection onto the corresponding eigenspace. *Hint*: Choose an eigenvector basis and apply the Cauchy integral formula.

(c) Now let A be any matrix. Let Γ be a contour which encloses a point λ_0 in the spectrum with winding number one. Show that the contour integral (2.6.48) is an idempotent operator whose image is the corresponding invariant subspace. *Hint*: Choose a Jordan representation of the matrix. Restrict attention to one Jordan block. Then the resolvent can be written as a Neumann series, which reduces to a finite sum. The resulting integral can be computed with residues.

(d) Derive the idempotence relation in (c) directly from the resolvent identity. *Hint*: A very similar computation is given in the proof of Theorem 2.1.6.

Exercise 2.7 In this exercise we explore an alternative and more computational proof of Lemma 2.1.8.

(a) Show by direct computation in momentum space that $k_m |_{t_0} k_m = k_m$. *Hint*: Proceed similarly as in the derivation of (1.2.24) in the proof of Lemma 1.2.8.

(b) Show that due to current conservation (see Exercise 2.5 above), the operator $\tilde{k}_m |_{t_0} \tilde{k}_m$ is independent of t_0. Therefore, it suffices to compute the limit $t_0 \to -\infty$. In order to study this limit, assume for technical simplicity that $\mathcal{B}$ has compact support. Show with the help of (2.1.14), (2.1.25) and (2.1.26) that for sufficiently small $t_0 < 0$,

$$\begin{aligned} \tilde{k}_m |_{t_0} \tilde{k}_m &= \frac{1}{4\pi^2} \sum_{n,n'=0}^{\infty} (-s_m^\wedge \mathcal{B})^n \, s_m^\wedge |_{t_0} s_m^\vee \, (-\mathcal{B} s_m^\vee)^{n'} \\ &= \sum_{n,n'=0}^{\infty} (-s_m^\wedge \mathcal{B})^n \, k_m |_{t_0} k_m \, (-\mathcal{B} s_m^\vee)^{n'} \,. \end{aligned}$$

(c) Apply the result of (a) together with (2.1.14) to conclude that $\tilde{k}_m |_{t_0} \tilde{k}_m = \tilde{k}_m$.

Exercise 2.8 *(causal perturbation expansion to second order)*

(a) Compute P^{sea} to second order in $\mathcal{B}$. *Hint*: Use (2.1.64) as well as the perturbation series for $\tilde{k}$. The resulting formulas are also listed in [FT2, Appendix A].

(b) The so-called *residual fermionic projector* is defined by modifying the integrand in (2.1.64) to

$$P^{\text{sea}}_{\text{res}} = -\frac{1}{2\pi i} \oint_{\Gamma_-} \tilde{R}_\lambda \, d\lambda \,.$$

Show that to first order in $\mathcal{B}$, the operators P^{sea} and $P^{\text{sea}}_{\text{res}}$ coincide. However, there is a difference to second order in $\mathcal{B}$. Compute it. *Hint*: In order to simplify the computation, it is helpful to write the difference as

$$P^{\text{sea}} - P^{\text{sea}}_{\text{res}} = \frac{1}{2\pi i} \oint_{\Gamma_-} (\lambda + 1)\, \tilde{R}_\lambda \, d\lambda \,,$$

and to use that the factor $\lambda + 1$ decreases the order of the pole at $\lambda = -1$.

Exercise 2.9 *(the fundamental solution $\tilde{p}$)*

(a) Show that the operator $\tilde{k}$ has the contour integral representation

$$\tilde{k} = -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} \lambda\, \tilde{R}_\lambda \, d\lambda \,.$$

Hint: Use (2.1.54) or the functional calculus of Theorem 2.1.6.

(b) Conclude that the fermionic projector P^{sea}, (2.1.64), can be represented as

$$P^{\text{sea}} = \frac{1}{2} \left(\tilde{p} - \tilde{k} \right),$$

where $\tilde{p}$ is defined by

$$\tilde{p} := -\frac{1}{2\pi i} \left(\oint_{\Gamma_+} - \oint_{\Gamma_-} \right) \lambda\, \tilde{R}_\lambda \, d\lambda \,. \tag{2.6.49}$$

Exercise 2.10 *(structural properties of $\tilde{p}$: even number of factors k)* The goal of this exercise is to show that every contribution to the perturbation expansion of $\tilde{p}$ contains an even number of factors k.

(a) Use the multiplication rules (2.1.52) and (2.1.69) to show that the last summand $-(\mathbb{1} - p)/\lambda$ in (2.1.57) drops out of the perturbation expansion for $\tilde{p}$ as defined by (2.6.49). Conclude that instead of (2.1.57), we may work with the formula

$$R_\lambda = R_\lambda^p + R_\lambda^k \qquad \text{with} \qquad R_\lambda^p = p\, \frac{\lambda}{1-\lambda^2}\,, \quad R_\lambda^k = k\, \frac{1}{1-\lambda^2}\,. \tag{2.6.50}$$

(b) Use the perturbation series for $\tilde{R}_\lambda$ in (2.1.56) and restrict attention to a contribution for fixed n. Insert (2.6.50) and multiply out. Analyze the symmetry of the contour integral under the transformation $\lambda \to -\lambda$. Show that all contributions to $\tilde{p}$ which involve an even number of factors R_λ^p vanish.

(c) Deduce from (2.1.53) and (2.1.55) that every contribution to Δk involves an even numbers of factors k.

(d) Show that every contribution to the perturbation expansion of $\tilde{p}$ contains an even number of factors k. *Hint*: Combine the results of (b) and (c) and use the multiplication rules (2.1.52) and (2.1.69).

Exercise 2.11 *(structural properties of $\tilde{p}$: replacing k by p)* In this exercise we compute what one gets if in the perturbation series for $\tilde{p}$ one replaces all factors p by k.

(a) Show that replacing all factors p by k, the formulas (2.1.53), (2.1.55) and (2.1.57) simplify to

$$\Delta k \to -p + \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\, b^{\prec}\, p\, (bp)^{2\beta}\, b^{\succ}\,, \qquad R_\lambda \to p\, \frac{1}{1-\lambda}\,.$$

Hint: See also Exercise 2.10 (a).

(b) Show that, using the formulas of part (a) in (2.1.56), the contour integral (2.6.49) simplifies to

$$\tilde{p} \to -\frac{1}{2\pi i} \oint_{\Gamma_+} \lambda \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}}\, (-p \cdot \Delta k)^n \cdot p\, d\lambda\,.$$

Compute the contour integral with residues to obtain

$$\tilde{p} \to \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\, b^{\prec}\, p\, (bp)^{2\beta}\, b^{\succ}\,.$$

Hint: Again use the multiplication rules (2.1.52) and (2.1.69).

Exercise 2.12 This exercise explains the notion of the *light-cone expansion* in simple examples.

(a) What is the light-cone expansion for a smooth function on $M \times M$? In which sense is it trivial? In which sense is it non-unique?

(b) Show that $A(x, y) = \log\left(|y-x|^2\right)$ is a well-defined distribution on $M \times M$. What is the order on the light cone? Write down a light-cone expansion.

(c) Now consider the distributional derivatives

$$\left(\frac{\partial}{\partial x^0}\right)^p A(x, y) \qquad \text{with} \qquad p \in \mathbb{N}$$

and $A(x, y)$ as in part (b). What is the order on the light cone? Write down a light-cone expansion.

(d) Consider the function

$$E(x, y) = \sin\left((y-x)^2\right)\, \log\left(|y-x|^2\right)\,.$$

Determine the order on the light cone and give a light cone expansion.

(e) Consider the function

$$E(x,y) = \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Determine the order on the light cone and give a light cone expansion.

(f) Show that the expression

$$\lim_{\varepsilon \searrow 0} \frac{\log\left(|y-x|^2\right)}{(y-x)^4 + i\varepsilon}$$

is a well-defined distribution on $M \times M$. Derive its light-cone expansion.

Exercise 2.13 This exercise is devoted to computing the Fourier transform of the *advanced Green's function* (2.2.5) and deriving the series expansion (2.2.7).

(a) As in Lemma 1.2.9, we set $\xi = y - x$ and $\xi = (t, \vec{\xi})$ with $t > 0$. Moreover, we choose polar coordinates $r = (|\vec{\xi}|, \vartheta, \varphi)$. Carry out the ω-integration with residues and compute the angular integrals to obtain

$$S^{\vee}_{m^2}(x,y) = \frac{i}{8\pi r} \int_0^\infty \frac{p}{\omega(p)} \left(e^{-ipr} - e^{ipr}\right)\left(e^{i\omega(p)\,t} - e^{-i\omega(p)\,t}\right) dp \,,$$

where $p = |\vec{p}|$ and $\omega(p) := \sqrt{|\vec{p}^2| + m^2}$. Justify this integral as the Fourier transform of a distribution and show that

$$S^{\vee}_{m^2}(x,y) = \frac{i}{8\pi r} \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon p} \frac{p}{\omega(p)} \left(e^{-ipr} - e^{ipr}\right)\left(e^{i\omega(p)\,t} - e^{-i\omega(p)\,t}\right) dp$$

with convergence as a distribution.

(b) Verify (2.2.6) in the case $m = 0$ by setting $\omega(p) = p$ and using (1.2.33).

(c) In order to analyze the behavior away from the light cone, it is most convenient to take the limit $r \searrow 0$ and use Lorentz invariance. Show that in this limit,

$$S^{\vee}_{m^2}(x,y) = \frac{1}{4\pi} \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon p} \frac{p^2}{\omega(p)} \left(e^{i\omega(p)\,t} - e^{-i\omega(p)\,t}\right) dp \tag{2.6.51}$$

$$= \frac{1}{4\pi} \lim_{\varepsilon \searrow \omega} \int_m^\infty e^{-\varepsilon p} \sqrt{\omega^2 - m^2} \left(e^{i\omega t} - e^{-i\omega t}\right) d\omega \,. \tag{2.6.52}$$

Compute this integral using [GR, formula (3.961.1)] (similar as in the proof of Lemma 1.2.9. Use the relations between Bessel functions [OLBC, (10.27.6), (10.27.11)] to obtain (2.2.6) away from the light cone.

As an alternative method for computing the Fourier integral, one can begin from the integral representation for J_0 in [OLBC, (10.9.12)], differentiate with respect to x and use [OLBC, (10.6.3)].

(d) Combine the results of (b) and (c) to prove (2.2.6). Why is there no additional contribution at $\xi = 0$?

(e) Use the series expansion [OLBC, (10.2.2)] to derive (2.2.7).
(f) The series expansion (2.2.7) can also be derived without using Bessel functions. To this end, one expands (2.6.51) in powers of m^2 and computes the Fourier transform term by term. Verify explicitly that this procedure really gives (2.2.7).

Exercise 2.14 This exercise is devoted to the proof of Lemma 2.2.3 as given in [F6, Lemma 2.2].

(a) Use (2.2.24) to derive the identity

$$\int d^4z\, S^{(l)}(x,z)\, V(z)\, (y-z)_k\, S^{(-1)}(z,y) = -2\, \frac{\partial}{\partial y^k} (S^{(l)}\, V\, S^{(0)})(x,y)\,. \tag{2.6.53}$$

(b) Apply Lemma 2.2.2 and carry out the y-derivative in (2.6.53) to obtain the formula in Lemma 2.2.3. *Hint*: Use the identity

$$\partial_k \Box^n V(z) = -\frac{1}{2(n+1)}\, \Box_z^{n+1} \Big(V(z)\, (y-z)_k \Big) + \frac{1}{2(n+1)}\, \Big(\Box_z^{n+1} V(z) \Big)\, (y-z)_k$$

and shift the summation index.

Exercise 2.15 In this exercise we collect elementary properties of the *ordered exponential*.

(a) Assume that the matrix-valued function F in Definition 2.2.5 is commutative in the sense that

$$\big[F(\alpha), F(\beta)\big] = 0 \qquad \text{for all } \alpha, \beta \in [a,b]\,.$$

Show that the ordered exponential reduces to the ordinary exponential,

$$\text{Pexp} \left(\int_a^b F(\alpha)\, d\alpha \right) = \exp \left(\int_a^b F(\alpha)\, d\alpha \right).$$

Hint: Show inductively that

$$\int_a^b dt_0\, F(t_0) \int_{t_0}^b dt_1\, F(t_1) \cdots \int_{t_{n-1}}^b dt_n\, F(t_n) = \frac{1}{(n+1)!} \left(\int_a^b F(t)\, dt \right)^{n+1}.$$

(b) Assume that F is continuous on $[a,b]$. Show that the Dyson series converges absolutely and that

$$\left\| \text{Pexp} \left(\int_a^b F(\alpha)\, d\alpha \right) \right\| \leq \exp \left(\int_a^b \big\| F(\alpha) \big\|\, d\alpha \right).$$

Hint: Estimate the integrals and apply (a).
(c) Show by direct computation that the ordered exponential satisfies the equations

$$\frac{d}{da}\,\mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) = -F(a)\;\mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) \tag{2.6.54}$$

$$\mathrm{Pexp}\left(\int_a^a F(\alpha)\,d\alpha\right) = \mathbb{1}\,. \tag{2.6.55}$$

Use the uniqueness theorem for solutions of ordinary differential equations to give an alternative definition in terms of the solution of an initial-value problem. Use this reformulation to show the group property

$$\mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right)\mathrm{Pexp}\left(\int_b^c F(\alpha)\,d\alpha\right) = \mathrm{Pexp}\left(\int_a^c F(\alpha)\,d\alpha\right). \tag{2.6.56}$$

(d) Show that

$$\frac{d}{db}\,\mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) = \mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) F(b)\,. \tag{2.6.57}$$

Hint: Differentiate the identity (2.6.56) in the case $c = a$ and use the group properties (2.6.55) and (2.6.56).

(e) Show that

$$\mathrm{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right)^* = \mathrm{Pexp}\left(\int_b^a \big(-F(\alpha)^*\big)\,d\alpha\right).$$

Deduce that if $F(\alpha)$ is an anti-Hermitian matrix, then the ordered exponential is a unitary matrix. *Hint*: There are two alternative methods. One method is to argue using the differential equations (2.6.54) and (2.6.57) or with the group property. A more computational approach is to take the adjoint of the Dyson series and reparametrize the integrals.

Exercise 2.16 This exercise recalls the concept of *local gauge transformations* and gets the connection to the *ordered exponential*.

(a) An electromagnetic potential A of the form $A_j = \partial_j \Lambda$ with a real-valued function Λ is called a pure gauge potential. Show that $(i\partial\!\!\!/ + A\!\!\!/ - m) = U(i\partial\!\!\!/ - m)U^{-1}$, where U is the phase factor $U = e^{i\Lambda}$. Conclude that every solution of the Dirac equation $(i\partial\!\!\!/ + A\!\!\!/ - m)\tilde{\psi} = 0$ can be written in the form $\tilde{\psi} = U\psi$, where ψ is a solution of the vacuum Dirac equation. In other words, pure gauge potentials merely describe local phase transformations of the wave functions.

(b) Generalize the argument of (a) to the case of non-abelian gauge fields and an additional gauge potential using the relation

$$U(i\partial\!\!\!/ + A\!\!\!/ - m\,\mathbb{1})U^{-1} = i\partial\!\!\!/ + UA\!\!\!/U^{-1} + iU\big(\partial\!\!\!/ U^{-1}\big) - m\,\mathbb{1}\,,$$

where now $U(x)$ is a unitary matrix (the mass matrix was left out for simplicity). How does the gauge potential transform under local unitary transformations of the spinors?

(c) Prove that for a pure gauge potential $A = iU(\partial\!\!\!/ U^{-1})$ the ordered exponential of Definition 2.2.5 simplifies to

$$\mathrm{Pexp}\left(-i\int_x^y A^j\,(y-x)_j\right) = U(x)\,U(y)^{-1}\,.$$

Hint: Apply the integration-by-parts method of Exercise 2.17 to the Dyson series. Alternatively, one can make use of the differential equation (2.6.54) with initial conditions (2.6.55).

Exercise 2.17 This exercise illustrates the handling of the *tangential derivatives* mentioned before Proposition 2.2.6. Let $z = \beta y + (1-\beta)x$ be a point on the line segment $\overline{xy}$. Show that

$$\int_z^y [p,q|0]\,f(z')\,dz' = \int_0^1 \alpha^p\,(1-\alpha)^q\,f\Big(\alpha\,(1-\beta)(y-x)+z\Big)$$

Deduce the identity

$$\begin{aligned}(y-x)^j &\int_z^y [p,q|0]\,(\partial_j f)(z')\,dz' \\ &= \frac{1}{1-\beta}\int_0^1 \alpha^p\,(1-\alpha)^q\,\frac{d}{d\alpha} f\Big(\alpha\,(1-\beta)(y-x)+z\Big)\,d\alpha\,.\end{aligned}$$

In the case $p, q > 0$, integrate by parts to derive the computation rule

$$(y-x)^j\int_z^y [p,q|0]\,(\partial_j f)(z')\,dz' = -\frac{1}{1-\beta}\int_z^y \Big(p\,[p-1,q\,|\,0] - q\,[p,q-1\,|\,0]\Big)\,f\,.$$

What is the analogous computation rule in the cases $p = 0$ and/or $q = 0$?

Exercise 2.18 This exercise explains how the Maxwell field tensor and the Maxwell current arise in the light cone expansion. To this end, we consider the first order perturbation of the massless Green's function by an electromagnetic potential A,

$$\Delta s_0 := -s_0\,A\!\!\!/\,s_0\,.$$

(a) Show that the leading contributions to the light-cone expansion of Δs_m have the form

$$(\Delta s_0)(x,y) = \frac{1}{2}\int_x^y A_i(z)\,\xi^i\,\not{\xi}\,S^{(-1)}(x,y) \tag{2.6.58}$$

$$+ \int_x^y dz\, [0, 1 \,|\, 0]\, (\not\partial A_i)(z)\, \xi^i\; S^{(0)}(x, y) \tag{2.6.59}$$

$$- \int_x^y dz\, [0, 0 \,|\, 0]\, \not A(z)\; S^{(0)}(x, y) \tag{2.6.60}$$

$$+ \not A(x)\; S^{(0)}(x, y) + \not\xi\, \mathcal{O}\big(\xi^{-2}\big) + \mathcal{O}\big(\xi^0\big)\,, \tag{2.6.61}$$

where $\xi := y - x$. *Hint*: First compute s_0 using (2.2.20) and (2.2.24). Then perform the light-cone expansion of the first order perturbation by using Lemma 2.2.3 and then by differentiating similar as done in the displayed computation before (2.2.24). Finally, the resulting formulas can be simplified by using (2.2.18) and by integrating the tangential derivatives by part (see Exercise 2.17 or the proof of Proposition 2.2.6).

(b) Which of the above contributions are phase-free? Show that the contribution which is *not* phase-free can be understood as the first-order contribution to the gauge phase in (2.2.57).

(c) Rewrite the phase-free contributions in an explicitly gauge-invariant way. *Hint*: In (2.6.59) use the identity $\not\partial A_i)(z)\, \xi^i = \gamma^j F_{ji} \xi^i - \xi^j \partial_j \not A$. Note that this generates a tangential derivative (see (2.2.58)). Integrate it by parts as explained in Exercise 2.17 or in the proof of Proposition 2.2.6.

(d) Compute the contributions to the above light-cone expansion of the form $\sim \not\xi \cdots S^{(0)}$. There is a term involving $\Box A$. Rewrite it in an explicitly gauge-invariant way using the Maxwell current $j_i := \partial_{ik} A^k - \Box A_i$.

(e) The reader who wants to get more computational practice may find it instructive to carry out the light-cone expansion up to the order $\mathcal{O}(\xi^2)$. In particular, there is a term $\sim (\Box \not A)\, S^{(1)}$. Rewriting the contributions again an explicitly gauge-invariant form, one thus obtains a contribution $\sim \gamma^k j_k\, S^{(1)}$. In fact, this contribution gives rise to the Maxwell current in the field equations in the continuum limit.

We note that all these computations are explained in more detail in [F6, Appendix A].

Exercise 2.19 *(contour integral representation of the residual fermionic projector)* In Exercise 2.9 (a) we derived a contour integral representation for the operator $\tilde{k}$ in (2.2.111). Thus it remains to derive a contour integral representation for the operator $\tilde{p}^{\text{res}}$ as defined by defined by (2.2.110). Verify to second order in perturbation theory (see Exercise 2.8) that $\tilde{p}^{\text{res}}$ has the contour integral representation

$$\tilde{p}^{\text{res}} = -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} \tilde{R}_\lambda\, d\lambda\,.$$

Remark: This equation indeed holds to every order in perturbation theory. This is a consequence of an underlying symmetry of the perturbation expansions with mass and spatial normalizations as explained in [FT2, Sect. 3.4].

Exercise 2.20 The goal of this exercise is to explore *weak evaluation on the light cone* in the example of the massless closed chain of the vacuum (2.4.11). Thus in view of (2.4.17), we want to analyze the integral

$$\int_{-\infty}^{\infty} \eta(t)\, \frac{(t^2 - r^2) - i\varepsilon[\gamma^0, \not{\xi}] + \varepsilon^2}{\big|(t - i\varepsilon)^2 - r^2\big|^4}\, dt \tag{2.6.62}$$

for a test function $\eta \in C_0^\infty(\mathbb{R})$ asymptotically as $\varepsilon \searrow 0$.

(a) Choose $r > 0$. Show that, changing the integral only by contributions which are bounded uniformly in ε, we may replace $\eta(t)$ by a test function supported in the interval $(r/2, 2r)$ around the upper light cone.

(b) Use the identity

$$\frac{1}{(t - i\varepsilon)^2 - r^2} = \frac{1}{(t - i\varepsilon - r)(t - i\varepsilon + r)} = \frac{1}{2r}\left(\frac{1}{t - i\varepsilon - r} - \frac{1}{t - i\varepsilon + r}\right)$$

to rewrite the integrand in (2.6.62) in the form

$$\sum_{p,q=0}^{2} \frac{\eta_{p,q}(t, r, \varepsilon)}{(t - i\varepsilon - r)^p\, (t + i\varepsilon - r)^q}\,,$$

with functions $\eta_{p,q}(t, r, \varepsilon)$ which in the limit $\varepsilon \searrow 0$ converge in C^∞ to smooth functions $\eta_{p,q}(t, r)$, i.e.

$$\lim_{\varepsilon \searrow 0} \partial_t^\alpha \partial_r^\beta \eta_{p,q}(t, r, \varepsilon) = \partial_t^\alpha \partial_r^\beta \eta_{p,q}(t, r) \qquad \text{for all } \alpha, \beta \geq 0\,.$$

Compute the functions $\eta_{p,q}$. Verify that the contribution for $p = q = 2$ agrees with the approximation (2.4.16).

(c) We now compute the leading contributions and specify what we mean by "leading." First compute the following integrals with residues:

$$I_0(\varepsilon) := \int_{-\infty}^{\infty} \frac{1}{(t - i\varepsilon - r)^2\, (t + i\varepsilon - r)^2}\, dt$$
$$I_1(\varepsilon) := \int_{-\infty}^{\infty} \frac{t - r}{(t - i\varepsilon - r)^2\, (t + i\varepsilon - r)^2}\, dt\,.$$

Show that

$$\int_{-\infty}^{\infty} \frac{\eta_{2,2}(t, r)}{(t - i\varepsilon - r)^2\, (t + i\varepsilon - r)^2}\, dt$$
$$= I_0(\varepsilon)\, \eta_{2,2}(r, r) + I_1(\varepsilon)\big(\partial_t \eta_{2,2}\big)(r, r) + \mathcal{O}(\varepsilon)\,.$$

Hint: To estimate the error term, proceed similar as in Exercise 1.10 (a).

(d) We now analyze the dependence of the resulting terms on r. To this end, first compute $\eta_{2,2}(r, r)$ and $(\partial_t \eta_{2,2})(r, r)$. Verify the rules (2.4.20). Verify the scaling of the error terms (2.4.18) and (2.4.19), where we use the convention that every derivative of η gives rise to a factor $1/\ell_{\text{macro}}$.
(e) Show that the integrals for $p < 2$ or $q < 2$ can be absorbed into the error terms. Also show that the term $\sim\varepsilon^2$ in (2.4.11) can be absorbed into the error terms.
(f) So far we analyzed the integrals with the simplified test functions $\eta_{p,q}(t, r)$. Show that replacing them by $\eta_{p,q}(t, r, \varepsilon)$ changes the integrals only by error terms of the form (2.4.18) and (2.4.19).

Exercise 2.21 This exercise explains how the identities (2.4.24) and (2.4.26) can be derived by explicit computation.

(a) Use (2.2.118) together with (2.2.117) and the series expansion (2.2.97) to derive explicit formulas for $T^{(l)}$ for all $l \geq 0$. Use the relation (2.4.4) in the case $l = 0$ to also compute $T^{(-1)}$.
(b) Show that for all $n \geq 0$,

$$\xi^2\, T^{(l)}(x, y) = -4\, T^{(l-1)} + (\text{smooth contributions})\,. \tag{2.6.63}$$

Why do the "smooth contributions" arise?
(c) Verify that the relation (2.6.63) remains valid for the $i\varepsilon$-regularization. *Hint*: One can argue without computations directly with a meromorphic extension using (2.4.3).
(d) Verify the identities (2.4.26) by explicit computation. What are the "smooth contributions"? Show that these identities remain valid for the $i\varepsilon$-regularization.

Exercise 2.22 *(computation of the local trace)* Compute $P^\varepsilon(x, x)$ in the Minkowski vacuum with $i\varepsilon$-regularization (see (2.4.1) and (2.4.2)). How do the vector and scalar components scale in m and ε? Verify the scaling of the local trace (2.5.1).

Exercise 2.23 *(scalar potentials and the local trace)* Consider a potential $\mathcal{B}$ composed of chiral potentials and a scalar potential, i.e. in generalization of (2.2.25),

$$\mathcal{B} = \chi_L \mathcal{A}_R + \chi_R \mathcal{A}_L + \Phi(x)\,.$$

(a) Show that the scalar potential can be combined with the mass terms to obtain a Dirac equation of the form (2.2.10) with B as in (2.2.29), but now with $Y(x)$ depending on x. We remark that this so-called *dynamical mass matrix* was first introduced in [F6, Sect. 2] (also including a pseudoscalar potential); see also [F7, Sect. 2.5].
(b) Go through the proof of Theorem 2.2.4 and convince yourself that the statement of the theorem remains valid in the presence of a scalar potential if in (2.2.33) the matrix Y is replaced by $Y(x)$. *Remark:* This generalization of Theorem 2.2.4 is given in [F6, Theorem 2.3].
(c) Use this generalization of Theorem 2.2.4 together with the scaling argument in the proof of Proposition 2.5.1 to derive the formula for the local trace (2.5.4).

Exercise 2.24 *(spectral representation of A_{xy})* Derive the formulas (2.6.11) by a straightforward computation using (2.6.6) and (2.6.9).

Exercise 2.25 *(spectral representation in the continuum limit)* Derive (2.6.17) by using (2.6.16), (2.6.12) as well as the contraction rules (2.4.44)–(2.4.46).

Exercise 2.26 In this exercise we consider the *double null spinor frame* in the example of the $i\varepsilon$-regularization.

(a) Consider a point $(t, \vec{\xi})$ on the upper light cone, i.e. $t = |\vec{\xi}|$ (more specifically one may choose $\vec{x} = (t, 0, 0)$). Use (2.4.6) to and compute z (up to errors of the form (2.4.18)). Compute the spectral projectors (2.6.19). Verify the relations (2.6.24).
(b) Compute the solutions $\mathfrak{f}^c_s$ of the eigenvector equations (2.6.20). Normalize them according to (2.6.22). What is the remaining freedom to modify the eigenvectors.
(c) Choose a space-like unit vector u which is orthogonal to ξ and $\bar{\xi}$. What is the freedom in choosing this vector? Show that by suitably choosing the phases of the eigenvectors $\mathfrak{f}^c_s$ one can arrange that the relations in (2.6.25) hold. What is the remaining freedom in choosing the frame $(\mathfrak{f}^c_s)$?
(d) The diagram (2.6.25) implies in particular that $\not{\xi}\, \mathfrak{f}^L_+ = \overline{\not{\xi}}\, \mathfrak{f}^L_+$. Explain how this identity can be understood in view of the error terms (2.4.18).

Exercise 2.27 *(matrix elements in the double null spinor frame)* Compute the matrix elements $\mathfrak{F}^{LL}_{++}(B)$, $\mathfrak{F}^{LL}_{+-}(B)$, $\mathfrak{F}^{LR}_{+-}(B)$ and $\mathfrak{F}^{RR}_{+-}(B)$ for B given by

$$B = \frac{i}{2}\, \chi_L \not{\xi}\, T^{(-1)}_{[0]}\,.$$

Simplify the expression as far as possible. *Hint*: Use the cyclic property of the trace, the anti-commutation relations of the Dirac matrices and the contraction rules.

Exercise 2.28 *(Perturbation of the eigenvalues of the closed chain)* The light-cone expansion can be understood as giving corrections to the fermionic projector of lower order on the light cone. We now explore how these corrections affect the eigenvalues of the closed chain, and which of them are compatible with the EL equations. In order to work in a specific example, we assume that the unperturbed fermionic projector is

$$P(x, y) = \frac{i}{2}\, \not{\xi}\, T^{(-1)}_{[0]}$$

(similar as considered in Exercise 2.27), whereas the perturbation has a left- and right-handed component,

$$\Delta P(x, y) = \chi_L \not{\psi}_L + \chi_R \not{\psi}_R\,,$$

where ν_L and ν_R are given vectors in Minkowski space.

(a) Compute the corresponding perturbation $\Delta\lambda_k^{xy}$ to leading order in the degree on the light cone. What is the leading degree? Which eigenvalues change, which remain the same? *Hint*: Use the usual formula for first order perturbations (see (2.6.44)) and rewrite it in the double null spinor frame.

(b) For which vectors ν_L and ν_R does the relation $|\lambda_k^{xy}| = |\lambda_l^{xy}|$ hold for all $k, l \in \{1, \ldots, 4\}$? Show that these relations are a sufficient condition for the EL equations to be satisfied. What would one need to verify in order to conclude that these relations are necessary? *Hint*: Consider (2.6.44) and (2.6.46). Keep in mind that the EL equations are evaluated weakly on the light cone.

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