

Chapter 2. Hint(s)

1. *Hints.* 1°) A possible proof of (1) is by induction.

2°) - Apply (1) with n circular permutations σ :

$$\begin{aligned} x_1 y_1 + x_2 y_2 + \dots + x_n y_n &\leq x_1 y_1 + x_2 y_2 + \dots + x_n y_n; \\ x_1 y_2 + x_2 y_3 + \dots + x_n y_1 &\leq x_1 y_1 + x_2 y_2 + \dots + x_n y_n; \\ &\dots \\ x_1 y_n + x_2 y_1 + \dots + x_n y_{n-1} &\leq x_1 y_1 + x_2 y_2 + \dots + x_n y_n; \end{aligned}$$

and sum the inequalities to obtain:

$$(x_1 + x_2 + \dots + x_n)(y_1 + y_2 + \dots + y_n) \leq n(x_1 y_1 + x_2 y_2 + \dots + x_n y_n).$$

- One could also prove (2) without passing through (1), with the help of the following (general) expansion:

$$\sum_{i,j=1}^n (x_i - x_j)(y_i - y_j) = 2n \sum_{k=1}^n x_k y_k - 2 \left(\sum_{k=1}^n x_k \right) \left(\sum_{k=1}^n y_k \right). \quad (3)$$

With the assumption $x_1 \leq x_2 \leq \dots \leq x_n$ and $y_1 \leq y_2 \leq \dots \leq y_n$, the quantity above is nonnegative.

2. *Hint.* A possibility is to prove this inequality by induction on n .
3. *Hint.* Apply, for example, the CAUCHY-SCHWARZ inequality to $a_1 = \sqrt{x_1}, \dots, a_n = \sqrt{x_n}$ and $b_1 = 1/\sqrt{x_1}, \dots, b_n = 1/\sqrt{x_n}$.
5. *Hint.* Make use of the CAUCHY-SCHWARZ inequality.
7. *Hint.* Isolate the terms $(n-k)(k+1)$ in the expression for $(n!)^2$, then find lower bounds for $(n-k)(k+1)$.
8. *Hint.* Use the expansion of $(1-1)^n$ in parallel with that of $(1+1)^n$.
9. *Hint.* 2°) Observe that $Q_n = S_n/(S_{2n} - S_n)$.
10. *Hint.* Show that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2$.
11. *Hint.* To establish (1), use the binomial expansion of $(p+1)^{k+1}$ and sum the results from $p=1$ to $p=n$.

14. *Hints.* - The distance between home and university will not be of importance here.
 - Beware, the answer is not 30 m/h.
17. *Hints.* 1°) If n is prime, the only divisors of n are 1 and n , whence $\sigma(n) = n + 1$.
 2°) Decompose n as $n = pq$, where $1 < q \leq p < n$. Thus, $\sigma(n) \geq 1 + p + n$.
21. *Hint.* 1°) Begin by proving, by induction or by decomposing fractions into simple elements, that

$$\sum_{n=1}^N \frac{1}{n(n+1)(n+2)\dots(n+p)} = \frac{1}{p} \left(\frac{1}{p!} - \frac{1}{N(N+1)(N+2)\dots(N+p)} \right).$$

22. *Hint.* 2°) Observe that to have

$$(u, v) \in \mathbb{Q}^2 ; u > 0, v > 0 \text{ and } u^n + v^n = 1 \quad (1)$$

is equivalent to having

$$(x, y, z) \in \mathbb{N}^3 ; x > 0, y > 0, z > 0 \text{ and } x^n + y^n = z^n. \quad (2)$$

This is precisely FERMAT's equation.

23. *Hint.* It suffices to consider the case where $|x - y| \geq \frac{1}{2}$. Then use the triangle inequality on $|\cdot|$.
26. *Hint.* A possibility is to use the mean value theorem for the function

$$x \in [0, 1] \mapsto F(x) = \int_0^x f(t) dt - \frac{x}{2}.$$

30. *Hint.* A function satisfying (1) is necessarily even. For any $a > 0$, consider the sequence $(x_n)_n$ defined as follows:

$$\begin{aligned} x_0 &= a; \\ \text{for all } n &\geq 0, x_{n+1} = \sqrt{x_n}. \end{aligned}$$

We then have: $\lim_{n \rightarrow +\infty} x_n = 1$ and $f(a) = f(x_1) \dots = f(\lim_{n \rightarrow +\infty} x_n)$. Whence $f(a) = f(1)$.

31. *Hint.* Use the change of variables $u = \pi - x$ in the integral $\int_0^{\frac{\pi}{2}} x f(\sin x) dx$.
33. *Hint.* 2°) Try with $Q_\varepsilon(x) = P_n(x^2)$.
34. *Hint.* Consider the primitive (or antiderivative) function $F(x) = \int_0^x f(t) dt$, then the function $g(x) = e^{-x}F(x)$. Study the variations of the function g .

37. *Hints.* 3°) Either use the definition of the arctan function, or differential calculus ($f' = 0$ on an interval implies that f is constant on it).
38. *Hints.* 1°) Use differential calculus: the functions in the two sides of (1) are continuous on $[0, 1]$, differentiable on $(0, 1)$, and their derivatives are equal.
2°) Check that $f'(x) = 0$ for all $x \in \mathbb{R}$.
39. *Hint.* Pass via the function $\log(f)$.
46. *Hint.* Use the second-order TAYLOR–YOUNG series of f around a .
49. *Hint.* 2°) $S_n \geq 0$ for all n and the sequence (S_n) is decreasing; it is therefore convergent. Consequently

$$S_n - S_{n-1} = -\frac{2}{\pi}b_n^2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

50. *Hint.* Apply the intermediate value theorem to the continuous function $g : [0, 1 - \frac{1}{n}] \rightarrow \mathbb{R}$ defined by $g(x) = f(x + \frac{1}{n}) - f(x)$.
52. *Hint.* Use the intermediate value property or ROLLE's theorem.
53. *Hint.* 1°) Use the mean value theorem.
54. *Hint.* Apply the mean value theorem to the function $\log f$.
55. *Hint.* Apply the mean value theorem to the function $g(x) = e^{-kx}f(x)$.
58. *Hint.* Begin by reordering the a'_i 's : $a_{i_1} < a_{i_2} < \dots < a_{i_n}$. Then distinguish two cases: when n is odd and when n is even.
61. *Hint.* 1°) and 2°): show that $h'_1 = 0$ and $h'_2 = 0$ on $\mathbb{R}$.
65. *Hint.* As often in such a situation, use complex numbers: the affixes of the n vertices of the polygon are the n -th roots of 1. Then, recognize in the mean value of the lengths of the chords a RIEMANN sum associated with the integral $\int_0^1 \sin(\pi x) dx$.
66. *Hint.* Identifying the plane with $\mathbb{C}$, the n points S_k , $k = 1, 2, \dots, n$, have $r_k = \exp\left(i\frac{2\pi(k-1)}{n}\right)$ as affixes. What is asked for is the value of $\prod_{k=2}^n |1 - r_k|$.
- It is therefore useful to consider the polynomial function $P_{n-1}(z) = 1 + z + z^2 + \dots + z^{n-1}$ ($= \frac{z^n - 1}{z - 1}$ if $z \neq 1$) whose roots are $r_2, r_3, \dots, r_n$.
67. *Hint.* Check that x, y, z are necessarily the roots of a polynomial of degree 3.

70. *Hint.* Express the determinant of $M = [m_{i,j}]$ as a sum of products of random variables whose expectations are all null (because the expectation of $m_{i,j}$ is 0 and the expectation of the product of independent random variables is the product of their expectations).
72. *Hints.* 1°) Just use the two basic calculus rules: $\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u^T v$ and $\|\lambda u\|^2 = \lambda^2 \|u\|^2$.
- 2°) (b) If $A(x, y) = 0$, then

$$\begin{aligned}\|y\|^2 x - (x^T y) y &= 0 \quad (\text{from (1)}); \\ |x^T y| &= \|x\| \times \|y\| \quad (\text{from (2)}),\end{aligned}$$

whence one obtains $\frac{x}{\|x\|} = \pm \frac{y}{\|y\|}$, which means that x and y are collinear vectors. This is “the equality case” in the CAUCHY–SCHWARZ inequality.

74. *Hint.* Have the “optimization reflex” by considering $h(x) = f(x) - g(x)$. For such a function, a is a local minimizer or a local maximizer.
75. *Hints.* $[(ii) \Rightarrow (iii)]$ is clear.

Indications for proving $[(iii) \Rightarrow (i)]$. One shows firstly that $\left| \frac{b^2 + c^2 - a^2}{2bc} \right| < 1$; hence there exists an angle $\widehat{A} \in (0, \pi)$ such that $\cos \widehat{A} = \frac{b^2 + c^2 - a^2}{2bc}$. One therefore constructs a triangle based on this angle $\widehat{A}$ and with two sides AB and AC of lengths b and c , respectively. The length of the third side BC of such a triangle is necessarily a' satisfying $\cos \widehat{A} = \frac{b^2 + c^2 - a'^2}{2bc}$; consequently $a = a'$.

77. *Hint.* 1°) Consider the graph of the function $x > 0 \mapsto 1/x$ and use the fact that $\int_1^n \frac{1}{t} dt = \ln(n)$.
78. *Hint.* Write the inequality (1) for $x = n, n + 1, \dots, pn - 1$ and add up the resulting inequalities.
79. *Hint.* Use the following result:

$$\sum_{i=1}^k \frac{1}{i} - \ln k \text{ has a limit } \gamma \text{ as } k \rightarrow +\infty.$$

80. *Hint.* 2°) Consider n as “squeezed” between two consecutive powers of 2, i.e., $2^p \leq n < 2^{p+1}$. Then prove, by induction on n , that $H_n = \frac{1}{2^p} \frac{q}{r}$ with odd integers q and r .
82. *Hint.* Without loss of generality, one can assume that $x \geq y \geq z$.
86. *Hint.* The two questions 1°) and 2°) are independent.

89. *Hint.* Use the inequality

$$n! \leq \sum_{k=1}^n k! \leq 2(n-1)! + n!$$

90. *Hint.* Use the auxiliary sequence defined by $v_n = \frac{u_n}{(2n)!}$.

91. *Hint.* Use the following decomposition:

$$\text{For } n \geq 2, \quad \frac{u_n}{n} = \frac{u_1}{n} + \frac{v_1 + v_2 + \dots + v_{n-1}}{n}.$$

94. *Hint.* Only the implication $[\Leftarrow]$ asks for some effort. For that, given $\varepsilon > 0$, add the following inequalities, valid for n large enough:

$$\begin{aligned} l - \varepsilon &\leq 2u_{n+1} - u_n \leq l + \varepsilon; \\ 2(l - \varepsilon) &\leq 4u_{n+2} - 2u_{n+1} \leq 2(l + \varepsilon); \\ &\dots \\ 2^{k-1}(l - \varepsilon) &\leq 2^k u_{n+k} - 2^{k-1} u_{n+k+1} \leq 2^{k-1}(l + \varepsilon). \end{aligned}$$

A prototype of sequence (u_n) satisfying the property in the right hand-side of (1) is: $u_{n+1} = \frac{u_n + l}{2}$. This observation may be used to provide another proof of the implication $[\Leftarrow]$.

95. *Hint.* In the calculation of the sum of the series, use the following result (see Tapa 78): given an integer $p \geq 2$,

$$\text{the limit of } S_p(N) = \sum_{n=N}^{pN} \frac{1}{n} \text{ as } N \rightarrow +\infty \text{ is } \log(p).$$

96. *Hint.* 2°) The left-hand side in (1) is a RIEMANN sum for the integral $\int_0^1 \frac{dx}{1+x}$.

99. *Hint.* 2°) The following relation will be useful

$$\left[\prod_{k=1}^n \cos \left(\frac{\theta}{2^k} \right) \right] \times \sin \left(\frac{\theta}{2^n} \right) = \frac{\sin(\theta)}{2^n}. \quad (1)$$

This can be checked with the help of the rule $\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$.

109. *Hint.* A proof by induction is suggested.

110. *Hint.* The solution of $(\mathcal{C}_3)$ is defined on the whole $\mathbb{R}$. To find it, “glue together” pieces of solutions of the first two differential equations $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$.

113. *Hint.* 1°) Pass to polar coordinates.

114. *Hint.* Start with the inequality

$$2|f'(x)|\sqrt{|f(x)|} \leq |f(x)| + |f'(x)|^2,$$

and apply a change of variables to the (convergent) integral $\int_0^{+\infty} |f'(x)|\sqrt{|f(x)|} dx$.

115. *Hint.* Decompose the polynomial functions in a chosen basis consisting of LAGRANGE polynomial functions.

116. *Hint.* Begin with polynomial functions. Then use the fact that f can be approximated uniformly on $[0, 1]$ by polynomial functions (WEIERSTRASS approximation theorem).

117. *Hint.* Check firstly that $\int_0^1 \left(x - \frac{1}{2}\right)^n f(x) dx = 1$. Then prove that the assumption “ $|f(x)| < (n+1)2^n$ for all $x \in [0, 1]$ ” leads to a contradiction.

118. *Hint.* Use the following facts:

- If r_i is a root (of P) of multiplicity m_i , then $r_1^{m_1} \times r_2^{m_2} \times \dots \times r_k^{m_k} = \pm 1$.
- $P(0) = -1$, and $P(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.

123. *Hint.* 3°) The image of a connected subset of $\mathbb{R}^2$ (hence of the convex subset A) by a real-valued continuous function like g is an interval of $\mathbb{R}$.

124. *Hint.* 1°) Start with a function F_1 defined as:

$$F_1(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2°) If both f^2 and g^2 were derivative functions, that would also be the case for their sum; but $f^2 + g^2$ cannot be a derivative function since it does not satisfy the intermediate value property.

128. *Hints.* As often happens in mathematics, the inequality (1) is easy to derive if one deals with it in the right way. Firstly, the graph of f on $[a, b]$ is under the chord joining the end-points $(a, f(a))$ and $(b, f(b))$, that is:

$$\text{For all } x \in [a, b], f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a). \quad (2)$$

Secondly, decompose $\int_a^b f(x) dx$ as $\int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx$.

130. *Hint.* Begin by proving that f is of class $\mathcal{C}^\infty$. Then take derivatives with respect to x and with respect to y in the functional relation (1).

131. *Hint.* Consider $g = f'$. This new function then solves a first-order differential equation.
143. *Hint.* A proof by contradiction is suggested.
144. *Hint.* 1°) Consider the derivative of the function $x \mapsto \int_x^{x+T} f(t) dt$.
146. *Hint.* One way (but it is not the only one) is to use the CAUCHY-SCHWARZ inequality.
147. *Hint.* Use first-order and second-order TAYLOR series of f around a point.
150. *Hint.* Use the following identity: $(x * x) * (y * y) = (x * y) * (x * y)$.
151. *Hints.* 1°) Consider p such that $(e - x)^p = 0$ and expand $(e - x)^p$ by using the binomial formula.
2°) Nilpotence of $x + y$ and $x - y$. Consider p such that $x^p = y^p = 0$. Then expand $(x + y)^{2p-1}$ by using the binomial formula.
153. *Hint.* Use the expressions of the areas of the subtriangles PAB, PAC, PBC and then the arithmetic-geometric inequality.
157. *Hint.* Observe that if d is the number of elements of G , then $A^d = I_n$ for any element A in G (this is a consequence of LAGRANGE's theorem in group theory). Consequently the minimal polynomial of A is a divisor of the polynomial $X^d - 1$; hence all its roots are simple, and A is diagonalizable.
158. *Hint.* Firstly observe that $\mathcal{V}$ is contained in the kernel of the linear mapping

$$\text{trace} : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathbb{R}.$$

Secondly, find as many linearly independent matrices as possible in $\mathcal{V}$, built up from the basic matrices $E_{i,j}$ (with 1 as entry (i, j) , 0 elsewhere).

160. *Hint.* Call on matrices of the form $M - \alpha I_n$ where $\alpha \in \mathbb{C}$ is not an eigenvalue of M .
161. *Hints.* Resources either from Linear algebra or from Analysis can be used.
- Using Linear algebra. A proof by contradiction is suggested: suppose that all matrices in $\mathcal{H}$ are singular, develop an argument arriving at a contradiction. $\mathcal{H}$ is the kernel of a non-null linear form $u : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathbb{R}$. Since I_n is assumed not to belong to $\mathcal{H}$, we may assume, without loss of generality, that $u(I_n) = 1$ (if not, divide u by the non-null $u(I_n)$). The objective is to bring out two matrices in $\mathcal{H}$ (they are singular matrices) whose sum is not in $\mathcal{H}$ (it is an invertible matrix).
 - Using the Euclidean structure of $\mathcal{M}_n(\mathbb{R})$. As usual, $\mathcal{M}_n(\mathbb{R})$ is equipped with the inner product $(U, V) \mapsto \langle\langle U, V \rangle\rangle = \text{tr}(U^T V)$. Hence, the hyperplane $\mathcal{H}$ can be represented as follows: there exists a non-null $H \in \mathcal{M}_n(\mathbb{R})$ such that

$$\mathcal{H} = \{M \in \mathcal{M}_n(\mathbb{R}) : \langle\langle H, M \rangle\rangle = 0\}.$$

Given the non-null $H = [h_{ij}]$, the aim is to design an invertible $M = H = [m_{ij}]$ such that $\langle\langle H, M \rangle\rangle = \text{tr}(H^T M) = \sum_{i,j} h_{ij} m_{ij} = 0$.

- Using Analysis. A proof by contradiction is suggested: suppose that all matrices in $\mathcal{H}$ are singular, and develop an argument arriving at an impossibility. $\mathcal{H}$ is the kernel of a non-null linear form $u : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathbb{R}$. Let $\Omega \subset \mathcal{M}_n(\mathbb{R})$ denote the set of invertible matrices. Then: Ω is an open dense subset in $\mathcal{M}_n(\mathbb{R})$ (see Tapa 160); it is made of two connected components, $\Omega_1 = \{M \in \Omega : \det M > 0\}$ and $\Omega_2 = \{M \in \Omega : \det M < 0\}$. In view of the assumption made, we necessarily have (within a permutation ± 1):

$$\Omega_1 \subset \mathcal{H}_+ = \{M : u(M) > 0\} \text{ and } \Omega_2 \subset \mathcal{H}_- = \{M : u(M) < 0\}.$$

There are singular matrices in $\mathcal{H}_+$ and in $\mathcal{H}_-$ (because the set of singular matrices cannot equal the vector space $\mathcal{H} = \{M : u(M) = 0\}$). Consider, for example, a singular matrix A in $\mathcal{H}_+$ and design a sequence of matrices (B_n) in $\Omega_2 \subset \mathcal{H}_-$ converging towards A .

165. *Hint.* The expression of Δ_n in a closed form involves a power of f (denoted as f^p) and a higher order derivative of $1/f$ (denoted as $(1/f)^{(q)}$).

168. *Hints.* 2°) - Check that if M_σ and M_τ are two permutations matrices, then $M_\sigma M_\tau = M_{\sigma \circ \tau}$.

- Consequence: since any permutation is the composition of a finite number of transpositions, it suffices that A commutes with the (specific) permutation matrices associated with transpositions.

169. *Hints.* Use calculus rules governing products of matrices which are triangular by blocks.

1°) (a) Multiply M by $\begin{bmatrix} D^T & 0 \\ C^T & D^{-1} \end{bmatrix}$, and use calculus rules on determinants.

171. *Hint.* Apply the CAYLEY–HAMILTON theorem in $\mathcal{M}_2(\mathbb{C})$, so that in our specific case:

$$A^{-1} = -A + \text{tr}(A) I_2; \quad B^{-1} = -B + \text{tr}(B) I_2; \quad (AB)^2 = \text{tr}(AB) AB - I_2.$$

Then, use the decomposition of BA as

$$BA = A^{-1}(AB)^2 B^{-1}.$$

172. *Hints.* 1°) Show that the unique solution x of $(A^{-1} + I_n)x = 0$ is $x = 0$.

2°) Begin, because it is easier, with the case where A is diagonalizable.

173. *Hint.* In the third question, the uniqueness result is a bit more difficult to prove than the existence.

174. *Hints.* There are (at least) two ways of proving (1):

- First proof: show that $A^{2n+1} = (-r^2)^n A$ for all integers n ; then $A^{2n+2} = A(A^{2n+1})$ for all n ; finally, use the definition of $\exp(A)$ as the sum of a series.
- Second proof: diagonalize A in $\mathcal{M}_3(\mathbb{C})$, its eigenvalues are $0, ir, -ir$; then use the fact that $\exp(P^{-1}AP) = P^{-1}\exp(A)P$.

176. *Hint.* The inertia of q_1 (or of the associated symmetric matrix Q_1) is a triple of integers: the number of positive eigenvalues, the number of negative eigenvalues, and the number of null eigenvalues.

179. *Hint.* Let $M_\lambda = \lambda A + (1 - \lambda)B$. Consider the following sets

$$\begin{aligned} I_A &= \{ \lambda \in [0, 1] : x^T M_\lambda x \geq 0 \text{ for all } x \in S \}, \\ I_B &= \{ \lambda \in [0, 1] : x^T M_\lambda x \geq 0 \text{ for all } x \in T \}. \end{aligned}$$

Check that I_A and I_B are nonempty closed intervals. Then prove that $I_A \cup I_B = [0, 1]$; this will imply that $I_A \cap I_B$ is nonempty, which is the aimed conclusion.

180. *Hint.* $[(a) \implies (b)]$. Prove it by contradiction. Assuming the contrary of (b), for every positive integer n , there exists an $x_n \neq 0$ such that

$$x_n^T Q x_n + n \|A x_n\|^2 \leq 0,$$

or, by putting $u_n = x_n / \|x_n\|$,

$$\frac{u_n^T Q u_n}{n} + \|A u_n\|^2 \leq 0.$$

One can then extract from (u_n) a convergent subsequence $(u_{n_k})_k$.

182. *Hints.* 1°) $\hat{q}$ is the “homogenized” version of q , that is:

$$\hat{q}(x, t) = t^2 q\left(\frac{x}{t}\right).$$

2°) $\hat{A} \in \mathcal{S}_{n+1}(\mathbb{R})$, the “augmented” version of A , is the matrix representing the quadratic form $\hat{q}$ on $\mathbb{R}^{n+1}$.

184. *Hints.* Guiding thread for a proof. For t in $\mathbb{R}$, let $A(t) = A + tB$. Then:

- For $t \geq 0$, the symmetrized product of $A(t)$ and B is positive definite.
- There exists a threshold $t_0 \geq 0$ such that $A(t)$ is positive definite for all $t > t_0$.
- If one supposes that $A(0) = A$ is not positive definite, show that there exists a $t_1 \in [0, t_0]$ such that $A(t_1)$ is positive semidefinite but not positive definite.
- Obtain a contradiction by using a non-null vector in the kernel of $A(t_1)$.

189. *Hints.* - For any nonsingular matrix P , the matrix PP^T is symmetric positive definite.
 - A symmetric positive definite S can be factorized as $S = P^T P$, where P is a nonsingular matrix. A real symmetric matrix B is always diagonalizable.
190. *Hints.* For $x \neq 0$ in $\mathbb{R}^n$, consider the vector $y = (A^T)^{-1}x$ and the matrix $M = xy^T$. This particular matrix M is of rank 0 or 1. This allows us to determine the eigenvalues of M . Then observe that:

$$\begin{aligned}\det(M + I_n) &= 1 + x^T y; \\ M + I_n &= (xx^T + A)A^{-1}; \\ x^T y &= x^T A^{-1}x.\end{aligned}$$

191. *Hints.* Consider $(A^{-1})_x$ built up like A_x but with A^{-1} instead of A . Then calculate the product $A_x \cdot (A^{-1})_x$ and use the rule $\det[A_x \cdot (A^{-1})_x] = \det(A_x) \det[(A^{-1})_x]$.
192. *Hints.* 1°) Proof by contradiction, for example. Suppose that the rank of A is $r \geq 1$. There exist nonsingular matrices P and Q such that

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

Then choose $A = P \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q$ in order to contradict the relation (1).

2°) The right-hand side of (2) equivalently says that

$$\det(A - B + Y) = \det(Y) \text{ for all } Y \text{ in } \mathcal{M}_n(\mathbb{R}).$$

193. *Hints.* - Various proofs are possible: one can use resources either of Linear algebra or of Analysis.
 - One (indirect) way would consist in using formula (1) in Tapa 202, followed by the characterization of $AB = 0$ in Tapa 309.
196. *Hints.* - Make use of the characterization of positive semidefinite matrices A in terms of minors of A .
 - More specifically, for answering the first question, show that $a_{i,j} \in [-1, 1]$ for all i, j .
197. *Hint.* The results have to do with the largest and the smallest eigenvalues of A .
198. *Hint.* The results have to do with the largest and the smallest eigenvalues of A .
203. *Hint.* Proof by induction on n .

206. *Hint.* Prove that the second differential of F is null everywhere, whence F is an affine function.
207. *Hint.* Equip $\mathcal{M}_n(\mathbb{R})$ with the inner product $\langle U, V \rangle = \text{tr}(U^T V)$ and the resulting norm $\|M\| = \sqrt{\langle M, M \rangle}$. Expand $\|A - B\|^2$.
208. *Hint.* 2°) $\|\cdot\|_{(k)}$ compares easily with $\|\cdot\|_\infty$ and $\|\cdot\|_1$; beware, however, that the string of inequalities between these norms and the string of inclusions on the corresponding unit balls are not in the same order.

209. *Hint.* Use the relation

$$\alpha\beta = \frac{1}{4} [(\alpha + \beta)^2 - (\alpha - \beta)^2]$$

with $\alpha = u_i$ and $\beta = v_i$ in order to bound $(u_i v_i)^2$. Furthermore, the assumption $u^T v \geq 0$ means that $\|u - v\| \leq \|u + v\|$.

210. *Hint.* A proof by induction on n .

212. *Hint.* To prove it, use the so-called divergence theorem (or GREEN–OSTROGRADSKY, or OSTROGRADSKY–GAUSS theorem) relating the flow of a vector field $\vec{F}$ through the boundary of S (thus, a surface integral) to a triple integral of its divergence function $\text{div} \vec{F}$ on S . Here, the vector-valued function to be considered is simply $\vec{F}(x, y, z) = \frac{1}{3}(x, y, z)$.

214. *Hints.* Note that $Q - Q' = P$. Prove that $\lim_{|x| \rightarrow +\infty} Q(x) = +\infty$ and $\inf_{x \in \mathbb{R}} Q(x) \geq 0$.

215. *Hint.* For question 3°), distinguish the roots of P' which are roots of P (nothing to prove in that case) and those which are only roots of P' . The key then is to decompose the fractional expression P'/P into simple elements:

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^k \frac{b_i}{x - a_i},$$

and write it for x a root of P' which is not a root of P .

216. *Hint.* With P standing for the point with coordinates (x, y) , consider the function g defined on S by $g(P) = f(P) - f(-P)$.
217. *Hint.* [$\Leftarrow$] Prove this implication by contradiction. The key property of f to be used is its continuity.
220. *Hints.* 2°) Use the basic definitions of minimizers of functions and the triangle inequality on $\|\cdot\|$.
- 3°) Start with an ε^2 -minimizer of f and choose $\lambda = \varepsilon$; then apply the results of question 2°).

221. *Hints.* Use the original definition of $\operatorname{arctanh}$ as the inverse of the hyperbolic tangent function ($\tanh$), or the expression of $\operatorname{arctanh}(x)$ in terms of the logarithm function. Other idea: use differential calculus, prove that the gradient vectors of the functions of (x, y, z) appearing in the two sides of (1) are equal.

222. *Hint.* Use the basic definitions of eigenvalues and eigenvectors: $\lambda = u + iv \in \mathbb{C}, z = x + iy \in \mathbb{C}^n$ (where $u \in \mathbb{R}, v \in \mathbb{R}, x \in \mathbb{R}^n, y \in \mathbb{R}^n$),

$$(A + iB)(x + iy) = (u + iv)(x + iy).$$

Then express (u, v) as a convex combination of two vectors which lie in K .

224. *Hint.* Play with φ_{e_i, e_j} , where $\{e_1, e_2, \dots, e_p\}$ is the canonical basis of $\mathbb{R}^p$.

227. *Hint.* Begin by proving that $f(\mathbb{R}^n)$ is a closed interval of $\mathbb{R}$.

232. *Hints.* 1°) Integrate by parts $\int_0^1 t(1-t)\varphi''(t) dt$.

4°) Consider the polynomial function $f(x, y) = xy(1-x)(1-y)$, precisely the one appearing under the integral sign in (3). For such a function,

$$\begin{aligned} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y) &= 4 \text{ for all } (x, y) \in [0, 1] \times [0, 1], \\ \int_0^1 \int_0^1 f(x, y) dx dy &= \frac{1}{36}. \end{aligned}$$

Hence, the inequality (1) becomes an equality: $\frac{1}{36} = \frac{1}{144} \times 4$.

233. *Hint.* Describe Σ as the graph of a function $(x, y) \in D \mapsto z = f(x, y) = \sqrt{a^2 - (x^2 + y^2)}$, and observe that the term to be integrated over D is

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \frac{a}{f(x, y)}.$$

234. *Hint.* 2°) Due to the form of the solid S , a good idea is to make use of spherical coordinates (ρ, φ, θ) ; the solid S is then described as: $0 \leq \rho \leq 3, 0 \leq \varphi \leq \frac{\pi}{4}, 0 \leq \theta \leq 2\pi$.

The integral to be calculated is

$$\operatorname{vol}(S) = \iiint_S dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 (\rho^2 \sin \varphi) d\rho d\varphi d\theta.$$

236. *Hints.* To visualize and sketch Σ as well as S , it is helpful to determine the curves obtained by intersecting the two cylinders; they are ellipses lying in the planes with equations $y = z$ and $y = -z$.

The roof of S_0 is Σ_0 ; its lateral surface is of the same kind.

238. *Hint.* 2°) The form of the domain of integration and the shape of the function to be integrated suggest the changes of variables that it would be appropriate to perform.
240. *Hint.* Beware that, in S , the cylindric part is of radius 3, while the spherical part is of radius 5.
242. *Hints.* 1°)– 3°). To sketch the arcs Γ_α is helpful. The larger $\alpha \geq 2$ is, the more “curved inwards” the arc Γ_α is. Examples: Γ_1 is a quarter of the unit circle; Γ_2 is the line-segment joining the points $(1, 0)$ and $(0, 1)$.

An example of a Cartesian equation is, for Γ_4 ,

$$\sqrt{x} + \sqrt{y} = 1.$$

4°) Rotate the initial orthonormal basis $(O; \vec{i}, \vec{j})$ to a new one $(O; \vec{u}, \vec{v})$, with $\vec{u} = \frac{\sqrt{2}}{2}(\vec{i} - \vec{j})$ and $\vec{v} = \frac{\sqrt{2}}{2}(\vec{i} + \vec{j})$.

244. *Hint.* Due to the shape of S , it is advisable to conduct the following change of variables: $x = u^2v$, $y = uv^2$.
246. *Hints.* Write $\|\vec{s}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\cos(\angle(\vec{u}, \vec{v})) + \dots$. But, since $\angle(\vec{u}, \vec{v})$ and $\angle(\vec{CB}, \vec{CA}) = \hat{C}$ add up to π , we have $\cos(\angle(\vec{u}, \vec{v})) = -\cos \hat{C}$. Thus,

$$\|\vec{s}\|^2 = a^2 + b^2 - 2ab \cos \hat{C} + \dots$$

Now, use the so-called law of cosines in a triangle, or theorem of AL-KASHI, which states that $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$, and so on. Whence one obtains $\|\vec{s}\|^2 = 0$.

249. *Hint.* 2°) (a). Consider a new orthonormal basis $(O; \vec{u}, \vec{v}, \vec{w})$ built up from the points A_1, A_2, A_3 as follows:

$$\vec{u} = \frac{\overrightarrow{OA_1}}{\|\overrightarrow{OA_1}\|}, \vec{v} = \frac{\overrightarrow{OA_2}}{\|\overrightarrow{OA_2}\|}, \vec{w} = \frac{\overrightarrow{OA_3}}{\|\overrightarrow{OA_3}\|}.$$

Then express the (matrix of the) quadratic form $q(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ in the new basis $(O; \vec{u}, \vec{v}, \vec{w})$.

250. *Hint.* Several proofs, including elementary ones, are possible. Use, for example, properties of various rectangular triangles in the tetrahedron, or vector calculus (the area of the hypotenuse-face is $\frac{1}{2} \|\vec{AB} \wedge \vec{AC}\|$), or pass via the calculations of the volume of $OABC$ and of the distance from O to the plane passing through the points A, B, C .

251. *Hints.* 2°) In the calculations and answers, one has to distinguish a_0, a_{2p} (terms of even index, with $p \geq 1$), a_{2p+1} (terms of odd index, with $p \geq 0$), b_1, b_n (for $n \geq 2$).

Trigonometric formulas which are useful here:

$$\begin{aligned}\cos a \cos b &= \frac{1}{2} [\cos(a-b) + \cos(a+b)] \\ \sin a \sin b &= \frac{1}{2} [\cos(a-b) - \cos(a+b)] \\ \sin a \cos b &= \frac{1}{2} [\sin(a+b) + \sin(a-b)] \\ \cos(2a) &= 2 \cos^2 a - 1 = 1 - 2 \sin^2 a.\end{aligned}$$

252. *Hint.* 2°) The following results are useful:

$$\begin{aligned}& - \text{An antiderivative of the function } x^2 \cos(nx) \text{ on } \mathbb{R} \\ & \text{is } \frac{x^2 \sin(nx)}{n} + 2 \frac{x \cos(nx)}{n^2} - 2 \frac{\sin(nx)}{n^3}; \\ & - \text{An antiderivative of the function } x \cos(nx) \text{ on } \mathbb{R} \\ & \text{is } \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}.\end{aligned}$$

253. *Hint.* 2°) In the calculations and answers, one has to distinguish a_0, a_{2p} (terms of even index, with $p \geq 1$), a_{2p+1} (terms of odd index, with $p \geq 0$), b_n (for $n \geq 1$).

254. *Hints.* 2°) In the calculations and answers, one has to distinguish a_0, a_{2p} (terms of even index, with $p \geq 1$), a_{2p+1} (terms of odd index, with $p \geq 0$), b_n (for $n \geq 1$).

4°) Consider $x = 0$ (or π) and $x = \frac{\pi}{4}$.

Note that, for integers p , $(-1)^{\lfloor (p+1)/2 \rfloor}$ is periodic of period 4; the first values are 1, -1, -1, 1, 1, -1, -1, 1, etc.

257. *Hint.* With $t = \tan(\frac{\theta}{2})$, we have $y = x \tan(\theta) = \frac{2tx}{1-t^2}$. Then, use the fact that t, θ and y are of the same sign.

260. *Hint.* 3°) Starting from the expression of $E'(t)$, change $\frac{\partial u}{\partial t}(t, x)$ into $\frac{\partial^2 u}{\partial x^2}(t, x)$, and conduct an integration by parts.

261. *Hint.* 2°) A family of solutions of the partial differential equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is provided by

$$u(x, t) = f_1(x + ct) + f_2(x - ct),$$

where f_1 and f_2 are twice continuously differentiable functions of a real variable.

262. *Hint.* 1°) There are several ways of calculating $\mathcal{A}(a, b)$ exactly: with a simple integral $2 \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} dx$, via a curvilinear integral $\frac{1}{2} \oint_{\mathcal{E}} (x dy - y dx)$, by performing a change of variables in $\iint_C dx dy$, and so on.

263. *Hints.* - The sought locus Γ of the center of the ellipse $\mathcal{E}$ is a piece of a simple curve in the plane. To support intuition, start with the two limiting cases: when $a = b$, in which case $\mathcal{E}$ is a circle and Γ is reduced to a point (a, a) ; when $b = 0$, in which case $\mathcal{E}$ degenerates to a line-segment and Γ is a quarter circle centered at O and of radius a .

- Calculate the distance $O\Omega$ from the origin O of the axes to the center Ω of the ellipse.

264. *Hint.* Use FOURIER series techniques and results

265. *Hints.* Firstly, check that f takes nonnegative values and satisfies

$$f(x) \geq [f(x^{1/n})]^n \text{ for all positive integer } n.$$

Then, develop f around the point 1, on the right with the help of the right-derivative p of f at 1, and on the left with the help of the left-derivative q of f at 1.

Additional work consists in comparing $f(x)$ with x^q on $[0, 1]$ and with x^p on $[1, +\infty)$.

266. *Hint.* Prove (1) by induction on *card* $\mathcal{F}$. At some point of the proof, the inequality $(\sqrt{ab} + \sqrt{cd})^2 \leq (a + c)(b + d)$ for positive a, b, c, d is helpful.

267. *Hint.* One can stay stuck for a long time if one does not deal with mathematical calculations in the right way... Set $a = \sqrt{11 + 2\sqrt{29}}$ and $b = \sqrt{11 - 2\sqrt{29}}$; then calculate $a + 2b\sqrt{5} + 5 + b^2$ and $(a + b)^2$.

268. *Hints.* 1°) $S + T$ and U are integers with 44 digits; emphasize the first 10.

2°) ε_r is very small, suggesting that $S + T$ and U are “numerically equal”.

269. *Hints.* Perform calculations in the Euclidean (hence factorial) ring $\mathbb{Z}[i\sqrt{2}]$, in order to prove that:

- the equation $y^2 = x^3 - 2$ has infinitely many rational solutions but only one solution involving positive integers, $(x, y) = (3, 5)$;

- the equation $y^2 = x^3 + 2$ has no solution involving positive integers x and y .

275. *Hint.* A key-ingredient is to study the sequence $(\|x_k - y\|)_k$, where y is arbitrarily chosen in S . Such a sequence is decreasing, and the sequence $(x_k)_k$ is bounded.
278. *Hint.* The solution of $(\mathcal{C}_3)$ is defined on the whole $\mathbb{R}$. To get it, “glue together” the solutions of $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ when they cross the Ox axis.
A discussion on the values of b is necessary; three situations have to be considered: when $|b| \leq 1$, when $b > 1$, when $b < -1$.
280. *Hint.* A proof by contradiction is suggested. Use simple techniques and results on differentiable functions of a real variable.
281. *Hint.* Decompose $P(x)$, and also $P'(x)$, as products of monomials $(x - \alpha_i)^{m_i}$, and study the behavior of $\frac{P'(x)}{P(x)}$ as x approaches α_i .
284. *Hint.* One can try to answer by considering progressively larger and larger classes of functions: differentiable functions (the simplest case!), locally LIPSCHITZ or absolutely continuous functions, etc., but the question is posed for functions enjoying no further properties than the one posed at the beginning.
285. *Hint.* The result is known for $g(x) = \sin(\pi x)$ and $g(x) = \cos(\pi x)$. This is the so-called LEBESGUE–RIEMANN lemma (see Tapa 49). It therefore holds true for the functions $g(x) = \sin(k\pi x)$, $g(x) = \cos(k\pi x)$, and

$$g(x) = \sum_{k=1}^n [a_k \cos(k\pi x) + b_k \sin(k\pi x)]. \quad (1)$$

Then, any continuous and 1-periodic function g can be uniformly approximated by functions like the ones displayed in (1).

286. *Hint.* Prove the nonnegativity of Δ_n on $\mathbb{R}$ by induction on n . Key: if r is a real root of P , then $P(x) = (x - r)Q(x)$, where Q now has $n - 1$ real roots.
287. *Hint.* Begin by proving that f is of class $\mathcal{C}^\infty$. Then derive with respect to x and with respect to y the functional relations (1) and (2).
288. *Hint.* Since the Hessian (matrix) $\nabla^2 f$ of f is a continuous mapping from $\mathbb{R}^n$ into $\mathcal{S}_n(\mathbb{R})$, start with local “diff-convex” decompositions of f around x , $f = r \|\cdot\|^2 - (r \|\cdot\|^2 - f)$, and then “glue” them.
289. *Hint.* 1°) Try affine functions $x \mapsto f(x) = ax + b$.
290. *Hint.* Begin by considering uniformly continuous functions f .

293. *Hints.* 1°) - We have $s = u + v$, $t = u - v$. Use trigonometric formulas to obtain (1).
 - Since $v \in (0, \pi]$, we have $0 \leq \frac{\sin v}{v} < 1$. Derive from the expression (1) that $\|q\| < 1$.

2°) The set Q is a bizarre non-convex subset contained in the open unit ball $B(0, 1)$ in $\mathbb{R}^2$; it is shaped like the complementary set in $B(0, 1)$ of a cardioid; hence the intersection of Q and $f'([0, 2\pi])$ is empty. However, by close-convexifying it, we obtain the closed unit ball of $\mathbb{R}^2$.

295. *Hint.* Let $M = \sum_{i=1}^d M_i$. Calculate $M_i M$ for all i , show that M is nilpotent (with the help of the trace), calculate M^k .

296. *Hint.* Use the characteristic polynomial or the minimal polynomial of M .

298. *Hint.* To answer the second question, use the fact (to be proved) that any nilpotent matrix is similar to a matrix with zeros on the diagonal.

304. *Hints.* 1°) Relation (1) is a consequence of the classical CAUCHY-SCHWARZ inequality in $\mathbb{R}^n$ by observing that:

$$\begin{aligned} |x^T y| &\leq \sum_{i=1}^n |x_i| \times |y_i| = \sum_{i=1}^n \max(|x_i|, |y_i|) \times \min(|x_i|, |y_i|); \\ (\max(|x_i|, |y_i|))^2 &= \max(x_i^2, y_i^2), \quad (\min(|x_i|, |y_i|))^2 = \min(x_i^2, y_i^2). \end{aligned}$$

2°) As above, relation (3) is a consequence of the classical CAUCHY-SCHWARZ inequality in $\mathcal{C}([a, b], \mathbb{R})$ by observing that:

$$\begin{aligned} \left| \int_a^b f(x)g(x) dx \right| &\leq \int_a^b |f(x)g(x)| dx = \int_a^b \max(|f(x)|, |g(x)|) \times \min(|f(x)|, |g(x)|) dx; \\ (\max(|f(x)|, |g(x)|))^2 &= \max(f^2(x), g^2(x)), \quad (\min(|f(x)|, |g(x)|))^2 = \min(f^2(x), g^2(x)). \end{aligned}$$

305. *Hints.* - The property (1) implies that A is positive semidefinite.

- Begin with a positive definite A . Consider $a \neq 0$, $b \neq 0$ and the positive real number $\alpha = \left(\frac{a^T A a}{b^T A b} \right)^{1/4}$.

Then use the following expansions and their squares:

$$\begin{aligned} a^T M b &= \frac{1}{4} [(a+b)^T M (a+b) - (a-b)^T M (a-b)]; \\ a^T M b &= \left(\frac{a}{\alpha} \right)^T M (\alpha b). \end{aligned}$$

306. *Hint.* Clearly, positive definite A and negative definite A satisfy (1) and (2). But there are others... To determine them, use the spectral decomposition of A (and thus that of A^{-1}).

307. *Hints.* We have:

$$\begin{aligned}\dim(\ker \varphi) + \dim(\operatorname{Im} \varphi) &= \dim(\mathcal{S}_n(\mathbb{R})) = \frac{n(n+1)}{2}; \\ \dim(\operatorname{Im} \varphi) &\leq \dim(\mathcal{AS}_n(\mathbb{R})) = \frac{n(n-1)}{2}.\end{aligned}$$

Observe that (1) is equivalent to $\varphi(S) = 0$.

308. *Hints.* Inequality (1) is just the “homogenized” form of inequality (2). For the implication $[(2) \Rightarrow (1)]$, use tA and tB for $t > 0$, factor out t^4 and let t tend to 0.

Inequality (3) is a rewriting of (1) using only traces. To prove (3), one can make use of the symmetric bilinear form b on $\mathcal{M}_2(\mathbb{R})$ defined by $b(A, B) = \operatorname{tr}(AB) - \frac{(\operatorname{tr} A)(\operatorname{tr} B)}{2}$, which is nonnegative when restricted to the subspace of symmetric matrices.

309. *Hints.* The proposed equivalences are tougher to establish than what one might think at first glance... $[(1) \Leftrightarrow (2)]$ (sometimes called OGAWA’s theorem) is more involved than $[(1) \Leftrightarrow (3)]$ (sometimes called the CRAIG–SAKAMOTO theorem). Things would have been easier under the hypothesis that A and B commute... but this is not assumed (that will be a consequence of the equivalences).

One can use resources either from Linear algebra (properties of traces and determinants of matrices) or from Analysis (expanding $\det(I_n + zA)$ as the sum of power series (see Tapa 202), or differential calculus).

311. *Hints.* 1°) Since both $(1-t)I_n + tS_1$ and S_2 are positive definite,

$$d(t) = \det M(t) = \det[(1-t)I_n + tS_1] \times \det S_2 > 0 \text{ for all } 0 \leq t \leq 1.$$

Hence the symmetric matrix $M(t)$ is always nonsingular for $0 \leq t \leq 1$.

2°) The “variational” expression of $\lambda(t)$ as the minimum of the quadratic form $x^T M(t)x$ over the set of unit vectors x of $\mathbb{R}^n$ allows us to show easily that λ is a continuous function. Now, the continuous function λ goes from $\lambda(0) =$ smallest eigenvalue of $M(0) = S_2$ to $\lambda(1) =$ smallest eigenvalue of $M(1) = S_1 S_2$. It never crosses 0 because $M(t)$ is always nonsingular for $0 \leq t \leq 1$. Hence $\lambda(1)$ is of the same sign as $\lambda(0)$, that is to say: $\lambda(1) > 0$. Consequently, $S_1 S_2$ is positive definite.

313. *Hint.* The n nonnegative real numbers $\sigma_i(M)$ are called the *singular values* of M . The number of non-null singular values of M is exactly the rank of M . The key-tool here is the so-called singular value decomposition of matrices: for M in $\mathcal{M}_n(\mathbb{R})$, there exist orthogonal matrices U and V such that

$$M = U \Sigma V^T, \text{ where } \Sigma = \operatorname{diag}(\sigma_1(M), \sigma_2(M), \dots, \sigma_n(M)).$$

315. *Hint.* Begin by proving the results for a square.

316. *Hint.* 1°) Proof by induction on n .

324. *Hint.* 3°) Prove it by contradiction.

325. *Hints.* The answer is Yes and this is somehow surprising. Here are hints for a proof.

Since the function f is positively homogeneous (*i.e.*, $f(\lambda A, \lambda x) = \lambda f(A, x)$ for all $\lambda > 0$ and (A, x) in $\mathcal{P}_n \times \mathbb{R}^n$), it suffices, to ensure the convexity of f , to prove its subadditivity, *i.e.*,

$$f(A_0 + A, x_0 + x) \leq f(A_0, x_0) + f(A, x)$$

for (A_0, x) and (A, x) in $\mathcal{P}_n \times \mathbb{R}^n$. Consider therefore

$$I = x_0^T A_0^{-1} x_0 + x^T A^{-1} x - (x_0 + x)^T (A_0 + A)^{-1} (x_0 + x).$$

We intend to prove that $I \geq 0$.

Then: decompose A_0 as CC^T , where C is an invertible matrix; there exists a diagonal matrix D with positive diagonal terms d_i and an orthogonal matrix U such that $A = (CU)D(U^T C^T)$; change to new variables $y = (CU)^{-1}x$ and $y_0 = (CU)^{-1}x_0$; reformulate I and conclude that $I \geq 0$.

326. *Hints.* This is an excellent opportunity to use integral formulas on surfaces of revolution:

- The volume of a solid of revolution obtained by rotating the smooth curve $y = f(x) > 0$ for $x = a$ to $x = b$ around the x -axis is $\mathcal{V} = \pi \int_a^b [f(x)]^2 dx$.

- The area of the surface of revolution obtained by rotating the smooth curve $y = f(x) > 0$ for $x = a$ to $x = b$ around the x -axis is $\mathcal{A} = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$.

2°) Beware, the result is counterintuitive.

329. *Hint.* The four vectors are $\vec{v}_1 = \frac{1}{2} \vec{BC} \wedge \vec{BA}$, $\vec{v}_2 = \frac{1}{2} \vec{BA} \wedge \vec{BD}$, $\vec{v}_3 = \frac{1}{2} \vec{BD} \wedge \vec{BC}$ and $\vec{v}_4 = \frac{1}{2} \vec{DA} \wedge \vec{DC}$. Indeed, the lengths of these vectors are equal to the areas of the corresponding faces; the vector-product of two vectors is orthogonal to the plane determined by these vectors, and it points outward because of the right-hand rule. It remains to prove that the four vectors $\vec{v}_i$ add up to the zero vector.

331. *Hints.* 1°) Beware, here and for the calculations coming afterwards, that $f(x)$ is not e^x for x between -2π and 0 .

2°) The following results are useful:

an antiderivative of $e^x \cos(nx)$ on $\mathbb{R}$ is $\frac{e^x}{n^2 + 1} [\cos(nx) + n \sin(nx)]$;

an antiderivative of $e^x \sin(nx)$ on $\mathbb{R}$ is $\frac{e^x}{n^2 + 1} [\sin(nx) - n \cos(nx)]$.

4°) Choose successively $x = 1$ and $x = -1$.

5°) Two ways of obtaining this sum: either choose $x = 0$ in (1), or use PARSEVAL's theorem.

332. *Hints.* As is often the case in this context, use FOURIER series techniques and results. The function f is extended to an odd function on $[-\pi, \pi]$ and then to a 2π -periodic function on $\mathbb{R}$. Since $\mathcal{I}_1(\sin) = 0$, what has to be proved is:

$$\int_0^\pi [f'(x)]^2 dx \leq \int_0^\pi [f(x)]^2 dx,$$

with equality only when $f = \sin$. Then use the relations linking FOURIER coefficients of f and f' , and PARSEVAL relations.

Mathematical Tapas

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