

Chapter 1

Introduction

Abstract We start this chapter by presenting some motivation for using nominal sets and Fraenkel-Mostowski sets in the experimental sciences. We emphasize the subdivisions of the so-called Fraenkel-Mostowski framework by mentioning the Fraenkel-Mostowski permutation model of Zermelo-Fraenkel set theory with atoms, the Fraenkel-Mostowski axiomatic set theory, the theory of nominal sets, the theory of generalized nominal sets, and Extended Fraenkel-Mostowski set theory. Finally, we present an alternative mathematics for managing infinite structures in the experimental sciences. This is called Finitely Supported Mathematics and represents, informally, Zermelo-Fraenkel mathematics rephrased in terms of finitely supported structures.

1.1 Motivation

There does not exist a full philosophical consensus on the distinction between logical and non-logical notions. This fact leads to certain doubts regarding our understanding of the nature of logic and its relationship to mathematics. Some logicians have suggested that what is distinctive about logical notions is their invariance under permutations of the domain of objects. In order to clarify this invariance under permutations, we mention that the set of numbers between 1 and 9 is invariant under the permutation of these numbers (it does not matter how we switch these numbers, we end up with the same set), but the set of prime numbers between 1 and 9 is not invariant under any permutation of these first 9 numbers (for instance, the related set formed by 2, 3, 5 and 7 is not invariant under the permutation that switches numbers 5 and 8 and maps all the other numbers to themselves).

In a logical sentence, signs for negation, conjunction, disjunction and the quantifiers should be invariant under any permutation of words, and so they count as logical notions (or logical constants), while words like “dog”, “tall” and “blue” cannot be invariant under permutations of (a larger set of) words, and so they are not logical notions. The invariance criterion seems to fit with common intuition about

logical notions. Certain technical results increase our confidence in this invariance criterion: in [152] it is proved that all of the relations definable in the language of *Principia Mathematica* are invariant under permutations, while in [111] every permutation-invariant operation can be defined in terms of logical operations such as identity, variable substitution, disjunction, negation and existential quantification, and each operation so definable is invariant under permutations.

Alfred Tarski gave a lecture in 1966 for a general audience at Bedford College in London entitled “What are logical notions?” Tarski’s answer to this question is presented in [153]. Essentially, logical notions are considered to be relations between individuals and classes, as well as relations over an arbitrary non-empty domain D of individuals. Tarski identified logical relations as exactly those invariant under arbitrary permutations of D . This thesis characterizes logical notions and logical operations by invariance under permutations.

As Tarski himself pointed out, the permutation invariance criterion for logical notions can be seen as a generalization of Felix Klein’s idea that different geometries can be distinguished by the groups of transformations under which their basic notions are invariant [96]. In his Erlangen Program, Klein classified the notions to be studied in various geometries (such as Euclidean, affine and projective geometry) according to the groups of (one-one and onto) transformations under which they are invariant. With logic thought of as the most general theory, logical notions should be those invariant under the largest group of transformations, namely the class of permutations. The “generality” argument for Tarski’s thesis is given by Bonnay in [49]:

1. The distinctive feature of logic among other theories is that it is the most general theory one can think of.
2. The bigger the group of transformations associated with a theory, the more general the theory.
3. The biggest group of transformations is the class of all permutations.

Thus, it is concluded that logical notions are those invariant under permutation.

Tarski’s thesis and related results assimilate logical notions to mathematics. From Whitehead and Russell’s *Principia Mathematica*, we know that the whole of mathematics can be formalized within set theory. In [152], set theory is described as a mathematically universal language. For Tarski, this universality gives set theory a fundamental status in mathematics and meta-mathematics, and so the whole of mathematics can be expressed in the language of an appropriate set theory. In Tarski’s words, “... we need only one non-logical constant (...) for a two-termed relation which holds between an element and a set (...). Then the only concern lies in a careful selection of the axioms. They must be weak enough to escape the antinomies, but at the same time they must be strong enough to ensure, within our universe of discourse, the existence of sets which correspond to as large a class of sentential functions as possible.”

Logical operations and notions in Tarski’s sense meet the permutation invariance criterion. If they are described set-theoretically, they should have the same meaning (independent of the set-theoretical universe). In considering an appropriate set theory for them, we take into account that if semantic concepts cannot be reduced to

logical concepts, we cannot proceed in “harmony with the postulates of the unity of science and of physicalism”, and so Tarski preferred to link mathematical universality to domain universality [152].

We relate all these aspects to the recently developed Fraenkel-Mostowski (FM) set theory [74]. Our goal is to establish a connection between the concept of logical notions in Tarski’s sense and the sets from the Fraenkel-Mostowski universe. The main aspect is related to the interpretation of a theory as its invariance under permutations of the universe; this means that the theory does not distinguish individual objects and characterizes only those properties of a model which do not depend on its non-structural transformations.

In this book we present a start point for a future presentation of a logical notion of set, and of an appropriate theory. Inspired by FM set theory we develop a theory of *invariant sets* and *invariant algebraic structures*. In order to realize our goal we also involve and extend the theory of nominal sets [127].

The theory of nominal sets (which we call invariant sets) has its origins in an approach developed initially by Fraenkel and Mostowski in the 1930s [68, 106], in order to prove the independence of the axiom of choice and other axioms in classical Zermelo-Fraenkel (ZF) set theory. In the 2000s, the FM permutation model of Zermelo-Fraenkel set theory with atoms (ZFA) was axiomatized and presented as an independent set theory with atoms, named FM set theory [74]. The axioms of FM set theory are the ZFA axioms over an infinite set of atoms [74], together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x . Rather than using a non-standard set theory, one could alternatively work with nominal sets [127], which are defined within ZF as usual sets endowed with some group actions satisfying a finite support requirement. In some papers, the theory of nominal sets is also called Fraenkel-Mostowski (FM) set theory. While we emphasize some differences between the related theories in Section 1.2, we agree that all these theories belong to the FM framework.

In computer science, nominal sets were first used in order to properly model the syntax of formal systems involving variable-binding operations [74]; the finiteness property in the definition of nominal sets is motivated by the fact that syntax can involve only finitely many names. Nominal sets also serve as a good framework for database theory since atoms can be used as an abstraction for data values, which can appear in a relational database or in an XML document. Atoms can also be used to model sources of infinite data in other applications, such as software verification, where an atom can represent a pointer or the contents of an array cell. Atoms have the same properties as variables and names. The precise nature of names is unimportant since we focus only on their ability to identify and on their distinctness. The nominal sets approach became successful since it provides a balance between rigorous formalism and informal reasoning. This is illustrated in [126] where principles of structural recursion and induction are explained in the world of nominal sets.

Starting from the development in [74] where the FM permutative model of ZFA set theory is redefined/axiomatized for computer science, nominal sets found a lot of other applications in various areas of the experimental sciences. New nominal semantics for various process calculi were defined in [5, 6, 9, 11, 13]. The tran-

sition rules in the nominal semantics of the related process calculi are expressed compactly, using a mixing of quantifiers instead of side conditions. This means the freshness conditions in the transition rules are successfully eliminated by using a specific freshness quantifier. The nominal semantics and the usual semantics of the related process calculi have the same expressive power. Nominal sets theory was also applied to the theory of process calculi in [37], where the π -calculus was formalized in Isabelle using the nominal datatype package [157].

Nominal sets were independently rediscovered by the concurrency community, as a basis for syntax-free models of name-passing process calculi [110], and used in automata theory as a framework for describing automata on data words [45]. An extension of the notion of nominal set from [126] was used in [45] to minimize automata over infinite alphabets, such as Francez-Kaminski finite-memory automata [69]. The minimization of deterministic timed automata [27] was studied in [47] using a class of atoms represented by real numbers. History-dependent automata (HD-automata) described in [110] and [124] have been developed in order to check π -calculus expressions for bisimilarity. HD-automata are internal (in the sense of [29]) in the category of named sets [54] which is equivalent to the category of nominal sets according to [67, 75]. In [102] formal languages over infinite alphabets where words may contain binders are introduced. HD-automata are extended by adding stacks, and the recognizability of nominal languages is studied.

In [40] the monoids defined in the category of nominal sets (also called nominal monoids) are used in the study of languages (without binders) over infinite alphabets. The theory of syntactic monoids for languages of data words represents the same theory as the theory of finite monoids in the category of nominal sets, and under certain conditions, a language of data words is definable in first-order logic if and only if its syntactic monoid is aperiodic [42]. Techniques from the theory of nominal sets are used in [142] in order to implement a functional programming language incorporating facilities for manipulating syntax involving names and binding operations. Computation in nominal sets has also been defined in [44] by presenting a basic functional programming language called $N\lambda$. An imperative programming language which extends while programs and works with nominal sets is introduced in [48]. Unlike in $N\lambda$, in the programming language presented in [48] the author was able to correctly type a program for minimizing the deterministic orbit-finite automata introduced in [45]. Turing machines that operate over infinite alphabets whose letters are built of atoms that can only be tested for equality are studied in [46]. In this direction it is proved that deterministic Turing machines are strictly less expressive than nondeterministic ones.

A more recent paper [43] studies a variant of first-order logic in the framework of nominal sets and presents a notion of model for this logic (the so-called stratified models) which admits compactness. Nominal algebraic structures are presented in terms of finitely supported objects in [4, 8, 10, 14]. The Scott recursive domain equation $D \cong (D \rightarrow D)$ has been investigated in the nominal framework in [141]. Nominal sets have also been used in game theory [2], in logic [73, 125], in topology [123] and in proof theory [157].

Although the nominal sets were introduced by Gabbay and Pitts, an earlier idea of using atoms in computer science belongs to Gandy [76]. Gandy proved that any machine satisfying four physical ‘principles’ is equivalent to some Turing machine. Gandy’s four principles define a class of computing machines, namely the ‘Gandy machines’. Gandy machines are represented by classes of ‘states’ and ‘transition operations between states’. States are represented by hereditary finite sets built up from an infinite set U of atoms, and transformations are given by restricted operations from states to states. The class HF of all hereditary finite sets over U introduced in Definition 2.1 from [76] is described quite similarly to the von Neumann cumulative hierarchy of FM sets presented in Section 2.5 of this book. The single difference between these approaches is that each HF_{n+1} is defined inductively involving ‘finite subsets of $U \cup HF_n$ ’, whilst each $FM_{\alpha+1}(A)$ is defined inductively by using ‘the disjoint union between A and the *finitely supported* subsets of $FM_\alpha(A)$ ’; HF is the union of all HF_n (with the remark that the empty set is not used in this construction), and the family of all FM sets is the union of all FM_α from which we exclude the set A of atoms. The support of an element x in HF , obtained according to Definition 2.2(1) from [76], coincides with $supp(x)$ (with notations from Theorem 2.4) if we see x as an FM set. Also, the effect of a permutation π on a structure x described in Definition 2.3 from [76] is defined analogously to the application of the S_A -action on $FM(A)$ to the element $(\pi, x) \in S_A \times FM(A)$. Obviously, Gandy’s principles can also be presented in the FM framework because any finite set is well defined in FM; however, an open problem regards the consistency of Gandy’s principles when ‘finite’ is replaced by ‘finitely supported’.

The construction of the universe of all FM sets (i.e. sets constructed according to the FM axioms) [74] is inspired by the construction of the universe of all admissible sets over an arbitrary collection of atoms [36]. The hereditary finite sets used in [76] are particular examples of admissible sets. The FM sets represent a generalization of hereditary finite sets because any FM set is an hereditary finitely supported set.

1.2 Approaches Related to the Fraenkel-Mostowski Framework

In the literature there exist some different approaches regarding the FM framework. We need to clarify the differences between these approaches.

1. **FM permutation model of ZFA set theory.** This model was introduced by Fraenkel [68] and extended by Lindenbaum and Mostowski [106]. Its original aim was to establish the independence of the axiom of choice from the other axioms of ZF set theory. There also exist several other permutations models of ZFA set theory [95] (such as Fraenkel’s basic and second model or Mostowski’s ordered model), which are defined by using countable infinite sets of atoms.
2. **FM axiomatic set theory.** This set theory was presented by Gabbay and Pitts [74] in order to provide a new formalism for managing fresh names and bindings. An advantage of modelling syntax in a model of FM set theory is that datatypes of syntax modulo α -equivalence can be modelled inductively. This is

because FM set theory provides a model of variable symbols and α -abstraction. FM axiomatic set theory is inspired by the FM permutation model of ZFA set theory. However, FM set theory, ZFA set theory and ZF set theory are independent axiomatic set theories. All of these theories are described by axioms and all of them have models. For example the Cumulative Hierarchy Fraenkel-Mostowski universe $FM(A)$ presented in Section 2.5 is a model of FM set theory, whilst some models of ZF set theory can be found in [92] and the permutation models of ZFA set theory can be found in [95]. The sets defined by using the FM axioms are called FM sets. A ZFA set is an FM set if and only if all its elements have hereditary finite supports. Note that the infinite set of atoms in FM set theory is not necessarily countable. FM set theory is consistent whether the infinite set of atoms is countable or not. Gabbay and Pitts use a countable set of atoms in order to define a model of FM set theory for computer science [74], whilst Bojanczyk describes FM sets over a set of atoms which does not represent a homogeneous structure [41]. Also, in [47] Bojanczyk and Lasota use non-countable sets of atoms (like the set of real numbers) in order to study the minimization of deterministic timed automata introduced in [27].

3. **Nominal sets.** The theory of nominal sets represents a ZF alternative to FM set theory. These sets can be defined both in the ZF framework [127] and in the FM framework [74]. In ZF, fix an infinite set A , and call it the set of names. A nominal set is defined as a usual ZF set endowed with a particular group action of the group of permutations of A that satisfies a certain finite support requirement. There exists also an alternative definition for nominal sets in the FM framework (when the set of names is related to the set of atoms in FM). They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). These two ways of defining nominal sets finally lead to similar properties. According to the previous remark, we will use the terminology “invariant” for “nominal”, in order to establish a connection between the approaches in the FM framework and in the ZF framework, respectively. Another reason for choosing the terminology “invariant” instead of “nominal” is presented in Section 2.6. Moreover, we can say that any set defined according to the FM axioms (any FM set) can be seen as a subset of the nominal set $FM(A)$. However, an FM set is itself a nominal set only if it has an empty support. The theory of nominal sets makes sense even if the set of atoms is infinite and not countable. Informally, since ZFA set theory collapses into ZF set theory when the set of atoms is empty, we can say that the nominal sets represent a natural extension of sets. In computer science, nominal sets offer an elegant formalism for describing λ -terms modulo α -conversion [74]. Informally, we can think of the elements of a nominal set as having a finite set of free names. The action of a permutation on such an element actually represents the renaming of the free names. Nominal sets are also used in algebra [14, 18], in semantics [5, 6, 11, 13], in logic [125], in topology [123], in proof theory [157], in programming [142], and in domain theory [141], [156]. A model of predicate logic defined by using nominal sets is presented in [62]. A survey of the applications of nominal sets in computer science can be found in [15].

4. **Generalized nominal sets.** The classical theory of nominal sets over a fixed set A of atoms is generalized in [45] to a new theory of nominal sets over arbitrary un-fixed sets of data values (which we call generalized nominal sets). The notion of ' S_A -set' (in Definition 2.2) is replaced by the notion of 'set endowed with an action of a subgroup of the symmetric group of \mathbb{D} ' for an arbitrary set of data values \mathbb{D} , and the notion of 'finite set' is replaced by the notion of 'set with a finite number of orbits according to the previous group action (orbit-finite set)'. This approach is useful for studying automata on data words [45], languages over infinite alphabets [42], or Turing machines that operate over infinite alphabets [46]. Computations in these generalized nominal sets is defined in [48], [44].
5. **Extended Fraenkel-Mostowski axiomatic set theory (EFM).** This set theory is introduced in [7] and represents an extension of FM set theory obtained by replacing the finite support axiom with a consequence of it which states that any subset of the set A of atoms is either finite or cofinite. Thus, in EFM, the finite support axiom is replaced by requiring only an amorphous structure on A . Even if the finite support axiom from FM set theory is relaxed in EFM set theory, many properties of the group of all bijections of A , such as torsioness or locally finiteness, are preserved [7]. EFM set theory has been used in [16] in order to generalize the notion of permutative renaming introduced in Section 2 from [74].

1.3 Finitely Supported Mathematics

Since the experimental sciences are mainly interested in quantitative aspects, and since there exists no evidence for the presence of infinite structures, it becomes more useful to develop a mathematics which deals with a more relaxed notion of infiniteness. We present our attempt to build the necessary concepts and structures for a finitely supported mathematics.

Finitely Supported Mathematics (FSM) is introduced in this book in order to prove that many ZF finiteness properties still remain valid if we replace the term 'finite' with 'infinite, but with finite support'. Some results of this type have already been presented in [18], where we proved that a class of multisets over infinite alphabets (interpreted in the framework of nominal sets) has similar properties to the classical multisets over finite alphabets. The main aim of FSM is to characterize infinite algebraic structures using their finite supports.

As their name says, nominal sets are used especially in order to manage notions like renaming, binding or fresh name. We see a new possibility of using the theory of nominal sets in order to characterize some infinite structures in terms of finitely supported objects.

FSM is ZF mathematics rephrased in terms of finitely supported structures, where the set of atoms has to be infinite (countable or not countable). ZF mathematics is actually Empty Supported Mathematics. In FSM, we use either 'nominal sets' (which from now on will be called 'invariant sets') or 'finitely supported sets' instead of 'sets'. FSM is not at all the theory of nominal sets from [127] presented in a differ-

ent manner. However, the theory of nominal sets [127] could be considered as a tool for defining FSM which is, informally, a theory of ‘invariant algebraic structures’.

We do not employ axioms in order to describe FSM because FSM is already consistent with the ZF axioms. However, we describe FSM by using principles. The principles of constructing FSM (presented also in [17]) have historical roots in the definition of ‘logical notions’ in Tarski’s view [153]. The general principle of constructing FSM is that all the structures have to be invariant or finitely supported. So, as a general rule, we are not allowed to use in the proofs of the results of FSM any construction that does not preserve the property of finite support. This means we cannot obtain a property in FSM only by employing a ZF result without an appropriate proof presented according to the finite support requirement.

It is worth noting that every ZF set is a particular invariant set equipped with a trivial permutation action (Example 2.1(2)). Therefore, the general properties of invariant sets lead to valid properties of ZF sets. The converse is not always valid, namely not every ZF result can be directly rephrased in the world of invariant sets, in terms of finitely supported objects according to arbitrary permutation actions. This is because, given an invariant set X , there could exist some subsets of X (and also some relations or functions involving subsets of X) which fail to be finitely supported. Therefore, the remark that everything that can be done in ZF can also be done in FSM is not valid. This means there may exist some valid results depending on several ZF structures which fail to be valid in FSM if we simply replace “ZF structure” with “FSM structure” in their statement.

Since invariant sets can be defined in the ZFA framework similarly as in the ZF framework (see the first paragraph in Section 2.3), the construction of FSM also makes sense over the ZFA axioms. Due to the connections between axiomatic FM set theory and the framework of nominal (invariant) sets, we find it convenient to say that FSM is the mathematics developed in the “FM framework”. Thus, the meaning of the term “FM framework” is “the framework used in order to construct a mathematics in which all the structures are finitely supported”. Obviously, anything that is constructed according to the FM axioms also makes sense in FSM.

1.4 Outline

The book is organized as follows:

Chapter 2: In this chapter we recall the basics of FM sets and invariant (nominal) sets as studied by Gabbay and Pitts in [71, 74, 127]. We review some basic notions from the FM framework like S_A -action, invariant (nominal) set, FM set, freshness (nominal) quantifier, support, finiteness, fresh element and abstraction. These notions are used in the following chapters. We also present some original results such as the relationship between the axioms of FM set theory and various forms of choice, and we prove some properties of invariant sets obtained by comparing various choice principles. Another goal of this chapter is to establish a connection between the theory of FM sets and the concept of logical notion presented by Tarski [153]. Concretely, we prove that any nominal set defined in the FM framework, i.e. any equivariant FM set, is a logical notion according to Tarski's definition. Moreover, the freshness quantifier \mathbb{I} is also a logical symbol.

Chapter 3: Our goal is to answer the question “Do we obtain valid results if we replace the notion of *infinite set* with the notion of *invariant/finitely supported set* in the classical ZF results?” In order to answer this question, we translate into FSM several algebraic concepts which were initially described using the axioms of ZF set theory. We focus on multisets, generalized multisets, partially ordered sets, Galois connections and groups because these are particularly relevant to the experimental sciences. The FSM properties of these algebraic structures are compared with their related ZF properties. We also develop a theory of abstract interpretation for programming languages within invariant sets, and we present some calculability results within finitely supported structures.

Chapter 4: We generalize FM set theory by giving a new set of axioms which defines Extended Fraenkel-Mostowski (EFM) set theory. The finite support axiom in FM set theory is replaced by a consequence of it which states only that each subset of the set A of atoms is either finite or cofinite. Many algebraic and topological properties of sets which are valid in the FM framework remain valid in the EFM framework. Permutative renamings are defined and studied in the EFM framework.

Chapter 5: We describe an algorithm to define an FSM semantics for a certain process calculus. We present in detail the case of the fusion calculus. The transition rules of the FSM semantics of the fusion calculus are expressed compactly using the quantifier \forall and the freshness quantifier \mathbb{I} instead of the additional freshness conditions. Atoms are used to represent “names”, and FSM abstraction is used to replace the usual “binding operation”. According to the finite support requirement in FSM, we prove a complete equivalence between the new FSM semantics of the fusion calculus and the usual semantics of this process calculus.



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