

## Chapter 19

# Clifford and the Number of Holes

We saw that Riemann denoted the genus by “ $p$ ”, a notation which is still frequently used today, in particular for the generalizations of this notion in higher dimensions. On the other hand, Riemann did not give a name to this notion, and his definition was not the one we saw in the Introduction, in terms of holes. There is a good reason for this, namely, that the surfaces imagined by Riemann consisted of sheets which thinly cover the plane, and therefore do not admit visible holes.

It was Clifford who, a little later, had the idea of defining the genus of a surface by counting its holes. The reason being that he first proved that a Riemann surface is necessarily homeomorphic to a surface admitting holes, embedded in the usual 3-dimensional space, as in the examples of the first image of the Introduction. This is precisely stated in his paper [47] of 1877:

The object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann’s surfaces in a concise and elementary manner. [...]

If two variables  $s$  and  $z$  are connected by an equation [polynomial of degree  $n$  in  $s$  and  $m$  in  $z$ ], each is said to be an algebraic function of the other. Regarding  $z$  as a complex quantity  $x + iy$ , we represent its value by the point whose co-ordinates are  $x, y$ , on a certain plane. To every point in this plane belongs one value of  $z$ , and consequently, in general,  $n$  values of  $s$ , which are the roots of the equation [...].

We shall now go on to shew that this  $n$ -valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of  $n$  infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. [...]

Let now this  $n$ -fold plane be inverted in regard to any point outside it, so that it becomes an  $n$ -fold sphere passing through the point. [...]

We shall now prove that this  $n$ -fold spherical surface can be transformed without tearing into the surface of a body with  $p$  holes in it. [...]

A closed curve drawn on a surface is called a *circuit*. If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible*; otherwise it is *irreducible*. In general there is a finite number of irreducible circuits on a closed surface which are *independent*, that is, no one of which can be made by continuous motion to coincide with a path made out of the others. [...]

[...] on the surface of a body having  $p$  holes through it, there are  $2p$  independent irreducible circuits; one *around* each hole, and one *through* each hole.<sup>1</sup>

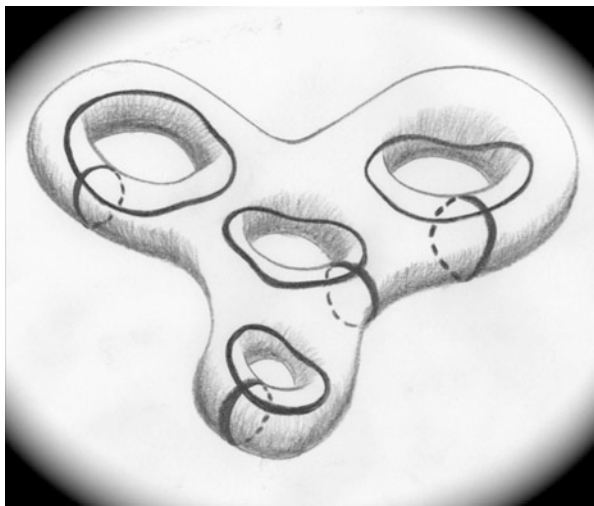
Another rather intuitive manner to define the genus of a closed orientable surface topologically is to decompose the surface as a *connected sum*.

In general, if  $S_1, S_2$  are two oriented and connected surfaces, their *connected sum*  $S_1 \# S_2$  is a new oriented surface, which is constructed by taking out a compact disk  $D_i$  from each surface  $S_i$  and by identifying the two resulting boundary circles by a diffeomorphism compatible with the orientations of the two surfaces  $S_1 \setminus D_1$  and  $S_2 \setminus D_2$ . This is illustrated in Fig. 19.2.

In this way, one gets a composition law on the set of diffeomorphism classes of closed, oriented and connected surfaces. The diffeomorphism class of the spheres is a neutral element for this law. A surface  $S$  is called *prime* if it is not a sphere and if it cannot be written non-trivially as a connected sum (that is, if one writes  $S = S_1 \# S_2$ , then necessarily one of the surfaces  $S_1$  and  $S_2$  is a sphere). One shows that *there is only one prime orientable surface, the torus*.

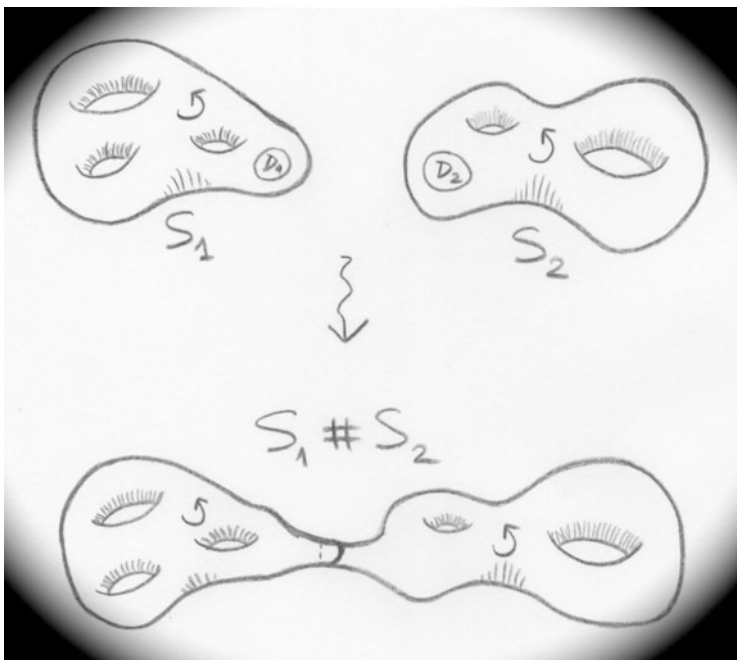
Here is the announced interpretation of the genus:

**Theorem 19.1** *If one decomposes an orientable closed surface of genus  $p$  as a connected sum of prime surfaces (that is, of tori), then there are exactly  $p$  of them.*



**Fig. 19.1** Clifford's circuits *around* and *through* the holes

<sup>1</sup>The curves *around* each hole are for instance those represented in Fig. 4 of the Introduction. In Fig. 19.1 is also represented a curve *through* each hole.



**Fig. 19.2** The connected sum of two surfaces

An analogous vision was developed in the twentieth century for the oriented 3-dimensional closed manifolds. More precisely, the definition of *prime 3-manifolds* mimicking that of prime surfaces, one has:

**Theorem 19.2** *Each closed, connected and oriented 3-dimensional manifold  $M$  may be written as a connected sum of a finite number of prime manifolds. Moreover, up to permutations, those are independent of the chosen decomposition.*

The first statement was proved in 1929 by Kneser in [119] and the second one in 1962 by Milnor in [132]. By analogy with Theorem 19.1, one could define the *genus* of a 3-dimensional manifold as the number of prime factors which appear in such a decomposition. But this denomination is not used.

An illustration of the explosion of the complexity of the topological structure when one passes from surfaces to 3-dimensional manifolds is given by the fact that there is only one prime surface (the torus), but there are infinitely many prime 3-dimensional manifolds. One still does not have a complete system of invariants to differentiate them.

At the end of his article [132], Milnor gave an example which shows that there is no analog of Theorem 19.2 in dimension 4. I will explain his example, because it uses two notions introduced in the sequel, the *blow ups of points* (see Chaps. 22 and 29) and the *stereographical projection* of a smooth quadric  $S$  of the complex projective space onto a complex projective plane  $P$  (see Chap. 29).

This stereographical projection is birational. It may be obtained by composing the blow up of the center of projection  $O \in S$ , which produces a complex projective surface  $M$ , and the contraction of the strict transforms  $E_1, E_2$  on  $M$  of the two straight lines  $L_1, L_2$  of  $S$  which pass through  $O$ . These two strict transforms are disjoint, as the blow-up of  $O$  separates  $L_1$  and  $L_2$ . Denote by  $O_1$  and  $O_2$  the points of  $P$  obtained by contracting  $E_1$  and  $E_2$ .

In  $M$ , one gets three smooth rational curves:  $E, E_1$  and  $E_2$ . Each one of the three curves contracts to a smooth point of one of the surfaces  $S$  and  $P$ . Therefore, they have neighborhoods bounded by 3-dimensional spheres  $\Sigma, \Sigma_1, \Sigma_2$ : one may simply take preimages of balls centered at the smooth points  $O, O_1$  and  $O_2$ . This allows one to decompose  $M$  in different ways as a connected sum, by cutting either along the sphere  $\Sigma$  or along the sphere  $\Sigma_1$  and then by filling the resulting boundary components by 4-dimensional balls.

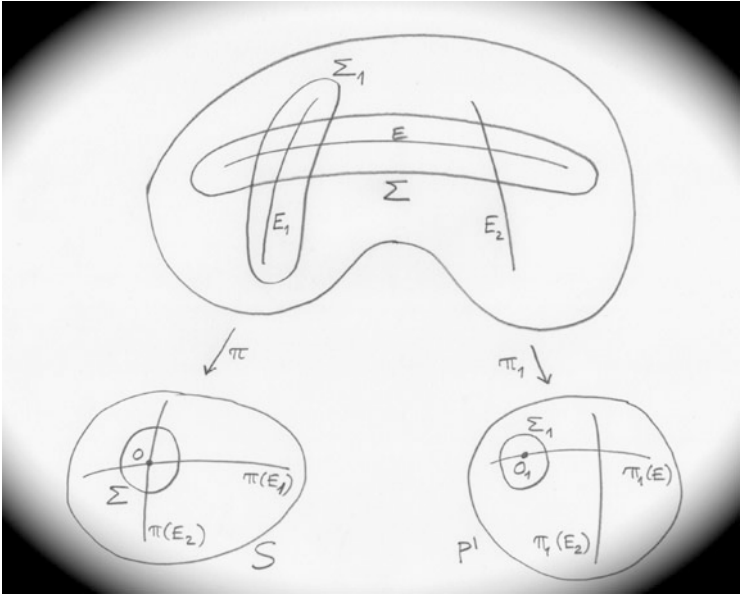
One may show that the closed 4-dimensional manifolds resulting from the neighborhoods of the complex rational curves  $E$  and  $E_1$  are orientation-preserving diffeomorphic to the complex projective plane with the opposite orientation from the orientation induced by its complex structure, denoted  $\overline{\mathbb{P}^2}$ .

The two other closed 4-dimensional manifolds, obtained by filling with balls the complements of the neighborhoods of  $E$  and  $E_1$ , are diffeomorphic with the complex surfaces obtained by contracting  $E$  and  $E_1$  respectively in  $M$ . That is, they are diffeomorphic with  $S$  and with the blow-up  $P'$  of  $P$  at  $O_2$ . The two previous contractions as well as the objects involved in the above explanation are represented schematically in Fig. 19.3.

Therefore, the two decompositions along the spheres  $\Sigma$  and  $\Sigma_1$  allow one to write  $M$  in two different ways as a connected sum:

$$M = S \natural \overline{\mathbb{P}^2} = P' \natural \overline{\mathbb{P}^2}.$$

The point is that the 4-dimensional manifolds  $S$  and  $P'$  are not even homeomorphic, which shows that they cannot be decomposed into the same prime factors. In fact, in the language explained in Chap. 39, their intersection forms on their second homology groups  $H_2(S, \mathbb{Z})$  and  $H_2(P', \mathbb{Z})$  are not isomorphic. The first one takes only even values, which may be shown by using the fact that the smooth quadric  $S$  is doubly ruled, that is, algebraically isomorphic (therefore diffeomorphic) to  $\mathbb{P}^1 \times \mathbb{P}^1$ . This is not true for the second intersection form, which takes the value  $-1$  on the homology class of the oriented surface  $E_2$  in  $H_2(P', \mathbb{Z})$ .



**Fig. 19.3** Two different contractions

This shows that the two previous decompositions of  $M$  as connected sums cannot lead to decompositions into prime factors which coincide up to permutations.

Is there a notion of “ideal factors” in 4-dimensional topology, giving back the uniqueness of the decomposition?

After this brief incursion into higher dimensions, let us come back to algebraic curves and their associated Riemann surfaces.

What is the Genus?

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