Chapter 3
Singularities of Three-Dimensional Ricci Flows

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Abstract The Ricci flow is an evolution of a Riemannian metric driven by a parabolic PDEs and was introduced by Hamilton in 1982. It has been the fundamental tool for some important achievements in geometry in the early 2000s, such as Perelman's proof of the geometrization conjecture and Brendle–Schoen's proof of the differentiable sphere theorem. In these notes we provide an introduction to the Ricci flow, by giving a survey of the basic results and examples. In particular, we focus our attention on the analysis of the singularities of the flow in the three-dimensional case which is needed in the surgery construction by Hamilton and Perelman.

3.1 The Ricci Flow

Let $\mathscr{M}$ be an n-dimensional Riemannian manifold with a metric g_0. The *Ricci flow*, also called *Hamilton–Ricci flow* of $(\mathscr{M}, g_0)$ is a time-dependent family $g(t)$ (with $t \geq 0$) of metrics on $\mathscr{M}$ satisfying $g(0) = g_0$ and evolving according to the equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}$$

where $\mathrm{Ric}_{g(t)}$ is the Ricci curvature tensor associated with the metric $g(t)$. The Ricci flow is a parabolic system of partial differential equations which has a unique solution at least in some finite time interval $t \in [0, T)$ if $\mathscr{M}$ is compact.

The Ricci flow was introduced by R. Hamilton in [30]. The motivation was to define an evolution of the metric tensor analogous to the evolution of functions defined by the heat equation. An earlier example of the use of parabolic PDEs in geometric problems was the paper by Eels and Sampson [27], who considered the heat flow of a map between two Riemannian manifolds, in order to obtain a harmonic mapping as the long time limit of the solution. Hamilton expected that

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the Ricci flow should enjoy similar properties and exhibit convergence to stationary states in many cases. On the other hand, it was also clear that a complete analogy with the heat equation could not be expected. In fact, in the Ricci flow the evolution equation of the curvature contains additional reaction terms which may induce a singular behavior in finite time.

To illustrate the powerful applications of the flow, let us start by describing the first important result obtained by Hamilton in [30].

Theorem 3.1 *Any closed three-dimensional Riemannian manifold with positive Ricci curvature is diffeomorphic to a quotient of the sphere* $\mathbb{S}^3$ *under a finite group of isometries.*

To prove this result, Hamilton considered the evolution of the metric under the Ricci flow and showed that it converges to a metric of constant positive sectional curvature. More precisely, there is a finite time $T > 0$ at which the flow becomes singular and the manifold "shrinks to a point": that is, the metric tends to zero and the curvature becomes unbounded everywhere. However, by choosing an appropriate rescaling factor $\rho(t)$, the normalized metric $\rho(t)g(t)$ converges, as $t \to T$, to a metric of positive constant sectional curvature. On the other hand, it is known that a manifold with such a metric must be $\mathbb{S}^3$ or one of its quotients.

After that seminal paper, a rich variety of studies of the Ricci flow followed through the years to obtain several geometric applications. Other classes of Riemannian manifolds were found which converge under the Ricci flow to a limit of constant curvature after rescaling. For instance, this holds for all closed two-dimensional manifolds, a property which provides an alternative proof of the uniformization theorem. Moreover, manifolds with positive curvature operator also converge to space forms under the Ricci flow. This was proved by Hamilton [31] in dimension 4 and by Böhm and Wilking [7] in the general case. A further spectacular result in this direction is the recent proof of the differentiable sphere theorem by Brendle and Schoen [10], which is treated in G. Besson's contribution in this volume.

On the other hand, it was soon clear that in many cases the Ricci flow can develop local singularities where no global information on the manifold is available. During the 1990s, Hamilton proposed a strategy to study these cases, whose main goal was the proof of the *Thurston geometrization conjecture*, which provides a complete classification of the closed three-dimensional manifolds, and includes in particular the

Poincaré Conjecture Every closed simply connected three-dimensional manifold is homeomorphic to the sphere $\mathbb{S}^3$.

Hamilton's idea to handle local singularities is to define a flow with surgeries. The Ricci flow is stopped shortly before the singular time, the regions with large curvature are removed by a surgery and replaced by more regular ones, and the flow is restarted. Hamilton conjectured that, after a finite number of surgeries, each component of the manifold converges to one of the structures described by Thurston, with the consequence that the initial manifold admits the desired decomposition.

Hamilton was able to perform various important steps of his program [32, 33, 35, 36] but some crucial parts remained unsolved. Then, in 2002 and 2003 G. Perelman posted on the web three papers [49–51] which introduced several new ideas and gave a more detailed understanding of the Ricci flow. In particular, these new results allowed Perelman to finally prove the geometrization conjecture.

A central part in Hamilton's and Perelman's surgery construction is the study of the possible singular profiles of 3-manifolds under Ricci flow. This allows a description of the regions with large curvature enough detailed to perform a surgery which preserves the relevant curvature estimates and changes the topology of the manifold in a controlled way.

In these notes we will present the basic properties and techniques in the study of the Ricci flow, and the main results about the analysis of singularities which are used in the proof of the geometrization conjecture. In order to make the exposition easily accessible to non experts, the presentation will be often informal and the proofs will be omitted except in some simple and significant cases. A final bibliographical section will give to the interested reader the references for a detailed study of these topics.

These notes describe the content of the lectures given at a CIME Summer Course in 2010. The author wishes to thank CIME and the organizers of the course for the invitation and their patience while these notes were written.

3.2 Notation, Examples and Special Solutions

We consider an n-dimensional Riemannian manifold $\mathscr{M}$ and denote by $g = (g_{ij})$ its metric. We assume that the reader has some familiarity with the basic notions of Riemannian geometry, and refer to the notes by G. Besson in this volume for more details. The Riemann curvature tensor associated with the metric will be denoted by $\mathrm{Rm} = (R_{ijkl})$, the Ricci curvature by $\mathrm{Ric} = (R_{ij})$ and the scalar curvature by R. Associated to the metric there is the Levi-Civita connection, which induces a covariant differentiation ∇ on tangent vector fields and on tensor fields of arbitrary type. Because of its symmetries, the Riemann curvature tensor can also be interpreted as a symmetric bilinear map on $\Lambda^2(T_p\mathscr{M})$, the algebra of 2-forms on $T_p\mathscr{M}$; such a map is called the curvature operator of the Riemannian manifold.

In these notes, we are mainly interested in three-dimensional manifolds, where the curvature quantities admit a simpler representation than in the general case. At a given point $p \in \mathscr{M}^3$, let e_1, e_2, e_3 be an orthonormal basis of $T_p\mathscr{M}$ which diagonalizes Ric. Let λ, μ, ν be the sectional curvatures at p associated with the planes orthogonal to e_1, e_2, e_3 respectively. Then the Ricci tensor has the form

$$\mathrm{Ric}(p) = \begin{pmatrix} \mu+\nu & 0 & 0 \\ 0 & \lambda+\nu & 0 \\ 0 & 0 & \lambda+\mu \end{pmatrix}.$$

The scalar curvature is given by $R(p) = 2(\lambda + \mu + \nu)$. Thus, positive sectional curvature implies positive Ricci curvature, which in turn implies positive scalar curvature, but the reverse implications do not hold. On the other hand, it is easily seen that λ, μ, ν are also the eigenvalues of the curvature operator. Therefore, positive sectional curvature is equivalent to positive curvature operator, a property which fails in higher dimension.

As we said at the beginning, we say that a time-dependent family of metrics $g(t)$ on $\mathscr{M}$ is a solution of the Ricci flow if it satisfies

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}. \tag{3.1}$$

Usually, an initial data $g(0) = g_0$ is given and the problem is studied for $t > 0$. The choice of the sign in the right-hand side is essential to ensure the parabolic character of the equation and the well-posedness for positive times, while the factor 2 is only a matter of convenience which simplifies some later formulas.

Before giving general results about the existence and uniqueness, it is interesting to consider special solutions of the Ricci flow. There are very few cases where the solutions can be described explicitly, see e.g. [33, Sect. 2] or [17, Sect. 2]; here we present some of them. Trivial examples are Ricci-flat spaces, which are constant solutions. Other easy examples are provided by Einstein manifolds, which give rise to homothetic solutions. In fact, if g_0 satisfies $\mathrm{Ric}_{g_0} = cg_0$ for some constant c, then it is easy to check that the metric

$$g(t) = (1 - 2ct)g_0$$

is a solution to the Ricci flow. Observe that the flow is defined for $t \in (-\infty, (2c)^{-1})$ if $c > 0$ and for $t \in ((2c)^{-1}, +\infty)$ if $c < 0$.

Thus, for instance, if $(\mathscr{M}, g_0)$ is a Euclidean sphere of radius r_0 and dimension $n \geq 2$, which corresponds to $c = (n-1)r_0^{-2}$, its evolution at time t is a sphere of radius

$$r(t) = \sqrt{r_0^2 - 2(n-1)t},$$

which shrinks to a point as t approaches the maximal time $T = r_0^2/2(n-1)$. As $t \to T$, the sectional curvature blows up like $(T-t)^{-1}$. On the other hand, the flow starting from a manifold with constant negative curvature is defined for all positive times and is homothetically expanding with a curvature decay of order t^{-1}.

When we have a product metric, each factor evolves independently under Ricci flow. For example, given a cylinder of the form $\mathscr{M} = S^k_{r_0} \times \mathbb{R}^{n-k}$ where $S^k_{r_0}$ is a k-dimensional sphere of radius r_0, the spherical factor shrinks homothetically while the flat factor remains unchanged. Thus, the global evolution is given by a cylinder with shrinking radius.

At an intuitive level, it is often useful to picture the evolution of a metric under the Ricci flow as if the manifold were immersed in an Euclidean case with a shape which changes in time, contracting or expanding in the regions of positive or negative curvature respectively. This description corresponds to the exact behavior of the flow in the case of the homothetic solutions described above, but it should only be regarded as a heuristic tool in general.

In addition to the exact solutions described above, there are some examples of "intuitive solutions", described in [33, Sect. 3], which are very important to understand the possible singular behavior of the flow. Consider a dumbbell-shaped manifold, of dimension at least three, consisting approximately of two big spheres S^n_R joined by a thin tube (or "neck") close to an $S^{n-1}_r \times [a,b]$. Both the spheres and the cylinder shrink under the flow; however, if r is enough small compared to R, we expect the cylinder to shrink faster. Therefore, the neck should pinch in its central part before the two spheres have become singular. Such a behavior is called *neckpinch*. It is natural to expect that a similar behavior should occur for much more general shapes, whenever there is a thin cylindrical region connecting large regions with lower curvature. A solution developing a neckpinch singularity has been later constructed rigorously in [4, 53], see also [17, Sect. 2.5].

By considering variants of the dumbbell example above, a further interesting possible behavior can be detected. We can observe that, if we take a dumbbell with r only slightly smaller than R, then the manifold has positive Ricci curvature and it will shrink with an asymptotically spherical profile as described in Theorem 3.1. Intuitively speaking, the two spheres "catch up" with the cylinder while they shrink, and the three parts merge in a shape which becomes more and more round. Therefore, there must be a threshold value of r, R where a limiting behavior occurs, and the cylinder pinches off at the same time as the two spheres collapse. Such a behavior, again conjectured in [33, Sect. 3], is called "degenerate neckpinch". A rigorous construction of solutions exhibiting these properties has been performed in [5].

An important class of solutions to the Ricci flow is provided by the so-called solitons. A *steady Ricci soliton* is a manifold $\mathscr{M}$ (not necessarily closed) with a metric $\tilde{g}$ which is a constant solution of the Ricci flow up to a diffeomorphism. By this we mean that there exists a time-dependent family of diffeomorphisms ϕ_t of $\mathscr{M}$ such that, if we set $g(t) = \phi_t^*(\tilde{g})$, i.e. the pull-back metric under ϕ_t, then $g(t)$ solves the Ricci flow.

More generally, a shrinking (resp. expanding) Ricci soliton is a homothetically shrinking (resp. expanding) solution of the flow up to a diffeomorphism. That is, the evolving metric $g(t)$ has the form $g(t) = \sigma(t)\phi_t^*(\tilde{g})$, where $\tilde{g}$ is a fixed metric, $\sigma(t)$ is a scalar function and ϕ_t a family of diffeomorphisms. It can be proved [17, Lemma 2.4] that the scaling factor is necessarily of the form $\sigma(t) = 1 + 2\rho t$ for some $\rho \in \mathbb{R}$. The case $\rho = 0$ corresponds to the steady solitons, while $\rho < 0$ and $\rho > 0$ yield a shrinking or expanding metric respectively. As in the case of Einstein metrics, such solutions are defined in an unbounded time interval of the form $(-\infty, -(2\rho)^{-1})$ or $(-(2\rho)^{-1}, +\infty)$, depending on the sign of ρ. If the family of diffeomorphisms is generated by the gradient of a function f, these solutions are called *gradient solitons*, and f is characterized by the equation

$$\mathrm{Ric}_{\tilde{g}} + \nabla^2 f = \rho \tilde{g}. \tag{3.2}$$

An explicit example of steady gradient soliton in dimension 2 is the so-called *cigar*, which is $\mathbb{R}^2$ endowed with the metric

$$g_{ij} = \frac{\delta_{ij}}{1 + x^2 + y^2},$$

see e.g. [17, Sect. 2.2] or [8, Sect. 2.1] for the details. To understand how the metric looks like, we can take a generic circle centered at the origin $\gamma_r(t) = (r\cos t, r\sin t)$; its length in the above metric is $2\pi r(1+r^2)^{-1/2}$, and thus it tends to 2π as $r \to \infty$. Intuitively speaking, the manifold looks like a one-ended infinite cylinder for r large, and it closes with a round cap for r close to zero. It is also easy to see that the curvature decays very rapidly away from the origin, while the injectivity radius is close to the value π of a cylinder. It follows in particular

$$\inf_{P \in \mathbb{R}^2} \mathrm{inj}(P) R(P) = 0. \tag{3.3}$$

These properties are important in connection to the non-collapsing property which will be introduced later in these notes.

It can be proved that the cigar is the unique rotationally symmetric nonflat steady gradient soliton on $\mathbb{R}^2$. Similarly, on any $\mathbb{R}^n$ with $n \geq 3$ there exists a unique rotationally symmetric steady gradient soliton with positive sectional curvature, as shown by Bryant [11], see also [19, Sect. 1.4]. In contrast to the cigar soliton, these higher dimensional solutions look like a paraboloid in $\mathbb{R}^{n+1}$ rather than a cylinder. In addition, they satisfy

$$\inf_{P \in \mathbb{R}^2} \mathrm{inj}(P) |Rm|(P) > 0. \tag{3.4}$$

In general, a solution which is defined on a time interval of the form $t \in (-\infty, T)$ for some finite $T > 0$ is called an *ancient solution*; if it is defined for $t \in (-\infty, \infty)$, it is called an *eternal solution*. Examples are given by the shrinking and steady Ricci solitons respectively, and other such solutions exist which are not solitons. Solutions defined for all negative times are very special, since the Ricci flow in general ill-posed backward in time. However, they are of great importance since they describe the possible profile of general solutions near a singularity, as we will see in the following.

To conclude this section, it is interesting to mention another geometric flow which has many similarities with the Ricci flow. A hypersurface in Euclidean space, or in a general Riemannian manifold, is said to evolve by the *mean curvature flow* if every point moves with normal speed given by the opposite of the mean curvature. The signs are chosen in such a way that a closed hypersurface with positive mean curvature is contracting under the flow. While the Ricci flow applies to abstract Riemannian manifolds, this evolution deals with immersed ones. In many cases, however, the two flows exhibit striking analogies: for instance, the examples of the sphere, of the cylinder, of the standard and degenerate neckpinch occur in the mean curvature flow with almost identical properties. The analogy between the two flows can be useful at a heuristic level, because results and techniques can often be exported from one problem to the other. In addition, since immersed manifolds are sometimes easier to visualize, some examples in the Ricci flow can be better understood in connection with their mean curvature flow analogue.

It should be pointed out that the two evolutions are not equivalent, and that there are some different properties as well. For example, in the mean curvature flow any closed hypersurface of the Euclidean space develops singularity in finite time. However, most of the relevant results on the Ricci flow treated on these notes have some analogues for the mean curvature flow, although possibly with some substantial difference in the hypotheses or in the method of proof. For example, a counterpart of Theorem 3.1 was obtained by G. Huisken [39], who proved that every closed convex hypersurface evolving by mean curvature flow in Euclidean space converges to a sphere after rescaling. We will mention in the following the other main correspondences and differences between the two flows.

3.3 Short Time Existence and Singularity Formation

When written in coordinates, the Ricci flow is a parabolic system of partial differential equations for the components of the metric. There exists a standard theory giving short time existence of solutions for systems which are strictly parabolic; however, the Ricci flow does not completely fit into this framework since for this system the parabolicity is not strict. Nevertheless, using the special structure of the equations, Hamilton was able to prove short time existence for the Ricci flow, as stated in the next result [30].

Theorem 3.2 *Given a closed manifold and a smooth initial metric, the Ricci flow has a unique smooth solution in a time interval* $[0, t_0)$ *for some* $t_0 > 0$.

The original proof by Hamilton [30] was rather difficult and used a sophisticated version of the implicit function theorem due to Nash and Moser. Shortly afterwards, De Turck [24] gave a simpler proof, which exploited an equivalent formulation of the flow where the parabolicity becomes strict. For more details about these matters, one can consult [30, Sects. 4–6], [33, Sect. 6], [17, Sects. 3.1–3.4].

A typical feature of parabolic problems is that boundedness of the solution implies boundedness of its derivatives of any order. A property of this kind holds also for the Ricci flow and was first proved by W.X. Shi [52].

Theorem 3.3 *Let* $g(t)$ *be a solution of the Ricci flow on a compact manifold* $\mathscr{M}$, *defined for* $t \in [0, t_0]$. *Suppose that the associated Riemann tensor* Rm *is bounded* *on $\mathscr{M}$, uniformly for $t \in [0, t_0]$. Then any derivative $\nabla^k \mathrm{Rm}$, with $k \geq 1$, is also bounded on $\mathscr{M}$ uniformly for $t \in [0, t_0]$.*

Using the above estimates, a standard continuation argument allows to prove that, whenever the maximal time of a solution is finite, the curvature necessarily becomes unbounded, as shown by the next theorem [33, Sect. 8].

Theorem 3.4 *Each solution of the Ricci flow on a compact manifold can be extended to a maximal time interval $[0, T)$, with $T \leq +\infty$. If T is finite, then necessarily*

$$\limsup_{t \to T} M(t) = +\infty,$$

where $M(t)$ is the maximum of the norm of the Riemann curvature tensor at time t.

We describe the above behavior by saying that the flow *becomes singular* at time T. Such a behavior is very frequent on compact manifolds, as it can be seen for instance from Hamilton's Theorem 3.1 in the positive Ricci case. However, there are also examples where the Ricci flow is defined for all positive times, like compact quotients of the hyperbolic space which give rise to homothetically expanding solutions.

The examples of the previous sections show that the behavior of the flow as a singularity is approached can be very different depending on the cases considered. In a shrinking soliton, and in a general compact solution with positive Ricci curvature, see Theorem 3.1, the metric tends to zero and the curvature becomes unbounded everywhere, so that we can say that the whole manifold becomes singular and collapses to a point. In the case of a neckpinch singularity, instead, the curvature becomes unbounded only in a part of the manifold, while on the rest remains regular even at the singular time.

3.4 Evolution of Curvature, Preservation of Positivity

As the metric on a manifold evolves by Ricci flow, the Riemann curvature tensor also evolves and satisfies an equation which can be computed explicitly and has the form

$$\frac{\partial}{\partial t} \mathrm{Rm} = \Delta \, \mathrm{Rm} + Q(\mathrm{Rm}). \tag{3.5}$$

Here $\Delta = \Delta_{g(t)}$ is the Laplace operator associated to the evolving metric $g(t)$, while $Q(\mathrm{Rm})$ is a tensor which is a quadratic function of Rm. Its explicit expression can be found for instance in [30] or in the notes of G. Besson in this volume. From the evolution equation for the Riemann tensor one can easily derive equations satisfied by the Ricci tensor and other quantities. The equation satisfied by the scalar curvature has the following particularly simple expression

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2. \tag{3.6}$$

Using the parabolic maximum principle one can obtain several invariance results for the positivity of curvature under Ricci flow, see [17, Sect. 6], [33, Sect. 5]. For example, Eq. (3.6) immediately implies that the minimum of the scalar curvature on a compact manifold evolving by Ricci flow is nondecreasing in time. As a consequence, if the initial metric has positive scalar curvature, then the same holds at all positive times. More precisely, the strong maximum principle says that, if the initial metric has nonnegative scalar curvature and is not Ricci flat, then the solution has strictly positive scalar curvature everywhere for all positive times.

In addition to the usual maximum principle for scalar functions evolving by parabolic equations, see e.g. [28, Sect. 7.1.4], there are maximum principles for systems of reaction-diffusion equations which ensure the invariance of sets satisfying suitable conditions. Some of these statements, which are due to Hamilton, are particularly useful for the study of geometric flows. A first version, see [30, Theorem 9.1], gave a criterion for the preservation of the positivity of a 2-tensor. We state here a more general formulation which was proved in [31], see also [20, Chap. 10].

Theorem 3.5 *Let $(\mathscr{M}, g(t))$ be a Riemannian manifold evolving by Ricci flow and let F be a time dependent section of a tensor bundle $\mathscr{E}$ on $\mathscr{M}$. Suppose that F satisfies the system*

$$\frac{\partial F}{\partial t} = \Delta F + \Phi(F) \tag{3.7}$$

for some function Φ mapping each fiber of $\mathscr{E}$ into itself. Let Z be a closed subset of $\mathscr{E}$ which is invariant under parallel translation and such that its intersection with each fiber is convex. If Z is invariant in each fiber under the ordinary differential system $dZ/dt = \Phi(Z)$, then Z is also invariant for system (3.7). That is, if F belongs to Z at a given time, it also belongs to Z for all later times.

The above maximum principle for tensors is a fundamental tool for the analysis of the Ricci flow, and its application relies on the study of the ordinary differential equation $\frac{d}{dt}\mathrm{Rm} = Q(\mathrm{Rm})$ associated to (3.5). In the three dimensional case, it is enough to study the corresponding equation for the Ricci tensor, which can be obtained by taking the trace of (3.5). It can be proved, see [30] or [8, Chap. 6], that if the Ricci tensor is diagonal at the initial time with respect to a given basis, it remains so along the evolution by the ODE. In addition, the sectional curvatures

λ, μ, ν in the principal directions evolve according to the system

$$\begin{cases} \dfrac{d\lambda}{dt} = \lambda^2 + \mu\nu \\ \dfrac{d\mu}{dt} = \mu^2 + \lambda\nu \\ \dfrac{d\nu}{dt} = \nu^2 + \lambda\mu. \end{cases} \tag{3.8}$$

The following result then follows as an easy application of Theorem 3.5.

Theorem 3.6 *Let $g(t)$ be a solution to the Ricci flow on a closed three-dimensional manifold $\mathscr{M}$.*

(i) If the initial metric has positive sectional curvature, then the same holds at any positive time.
(ii) If the initial metric has positive Ricci curvature, then the same holds at any positive time.

Proof Both properties describe a convex cone in the space of 2-tensors which is invariant under parallel translations. Positive sectional curvature corresponds to $\lambda > 0, \mu > 0, \nu > 0$, a property which is preserved by the ODE (3.8) since the expressions in the right hand side are positive. Positive Ricci means instead $\lambda + \mu > 0, \lambda + \nu > 0, \mu + \nu > 0$. From (3.8) we deduce

$$\frac{d}{dt}(\lambda + \mu) = \lambda^2 + \mu^2 + (\lambda + \mu)\nu \geq (\lambda + \mu)\nu.$$

Thus, if $\lambda + \mu$ is positive, it remains so for all later times. The other two expressions are treated similarly. Thus the maximum principle for tensors can be applied to deduce the desired properties.

It should be pointed out that the above invariance properties are peculiar of the three-dimensional case. In higher dimensions, neither positive Ricci nor positive sectional curvature is preserved. Properties valid in any dimension are the preservation of the positivity for the scalar curvature and for the curvature operator.

It is important to study the limiting cases of Theorem 3.6. Using a strong version of the maximum principle for tensors [31, Sects. 8–9], Hamilton proves the following statement:

Theorem 3.7 *Let $g(t)$, with $t \in [0, T)$, be a solution to the Ricci flow on a complete, connected, not necessarily compact, three-dimensional manifold $\mathscr{M}$. If $(\mathscr{M}, g(t))$ has nonnegative sectional curvature, and there is at least a point $P \in \mathscr{M}$ where one sectional curvature vanishes at a positive time t, then $\mathscr{M}$ splits as a product with a flat factor, that is $(\mathscr{M}, g(t)) = (\mathscr{M}_0 \times (\tilde{\mathscr{M}}, \tilde{g}(t))$, where $\mathscr{M}_0$ is a flat one-dimensional factor and $(\tilde{\mathscr{M}}, \tilde{g}(t))$ is a two-dimensional solution to the Ricci flow with strictly positive sectional curvature for all $t > 0$. The same result holds if $(\mathscr{M}, g(t))$ has* *nonnegative Ricci curvature, and the Ricci curvature vanishes at some point and positive time.*

3.5 Curvature Properties Which Improve

In addition to the invariance properties described in the previous section, there are some remarkable results showing that when the flow becomes singular some curvature properties improve. A property of this kind is the crucial step in the proof of Hamilton's first result [30] on manifolds with positive Ricci curvature. To show this, we follow the simpler approach of the paper [31, Sect. 5], see also [8, Chap. 6].

Theorem 3.8 *Let $g(t)$ be a solution to the Ricci flow on a closed three-dimensional manifold $\mathscr{M}$ with positive Ricci curvature. Denote by $\lambda \le \mu \le \nu$ the sectional curvatures in an orthonormal frame which diagonalizes Ricci, and let δ, C be positive constants such that the initial metric satisfies at every point*

$$\lambda + \mu \ge 2\delta\nu, \qquad (\nu - \lambda)^{1+\delta} \le C(\lambda + \mu). \tag{3.9}$$

Then the same inequalities hold on $(\mathscr{M}, g(t))$ for all $t > 0$ such that the flow exists.

Proof Let us check that the two inequalities in (3.9) define a set which satisfies the hypotheses of the maximum principle for tensors. In general, any set defined by inequalities on the sectional curvatures λ, μ, ν is invariant under parallel translation. We therefore only need to check the convexity and the invariance with respect to the ODE (3.8).

Recall that, if A is a symmetric matrix, then its smallest and largest eigenvalues λ, ν are given by

$$\lambda = \min_{||v||=1} \langle Av, v\rangle, \qquad \nu = \max_{||v||=1} \langle Av, v\rangle.$$

It follows that λ is a concave function of A, being the infimum of linear functions. Similarly, ν is a convex function of A. In addition, the trace $\lambda + \mu + \nu$ is a linear function of A since it coincides with the sum of the elements on the diagonal. Thus, we also obtain that $\lambda + \mu = (\lambda + \mu + \nu) - \nu$ is a concave function of A. These properties show that each of the two inequalities in (3.9) defines a convex set, and so do the two together.

Let us now show that the first inequality $\lambda + \mu \ge 2\delta\nu$, with $\delta > 0$, defines an invariant set under the ODE. This is equivalent to the property

$$\lambda + \mu = 2\delta\nu \quad \Longrightarrow \quad \frac{d}{dt}(\lambda + \mu) \ge \frac{d}{dt} 2\delta\nu.$$

We therefore suppose that $\lambda + \mu = 2\delta\nu$. Observe that this is only possible if $\delta \le 1$. We compute from (3.8)

$$\frac{d}{dt}(\lambda + \mu - 2\delta\nu) = \lambda^2 + \mu^2 - 2\delta\lambda\mu + (\lambda + \mu - 2\delta\nu)\nu$$
$$= \lambda^2 + \mu^2 - 2\delta\lambda\mu = (1-\delta)(\lambda^2 + \mu^2) + \delta(\mu - \lambda)^2 \ge 0.$$

Thus, the set defined by the first inequality in (3.9) is invariant. Again from (3.8), we find

$$\frac{d}{dt}\ln(\lambda + \mu) = \frac{\lambda^2 + \mu^2}{\lambda + \mu} + \nu \ge \frac{1}{2}(\lambda + \mu) + \nu \ge (1+\delta)\nu.$$

In addition,

$$\frac{d}{dt}\ln(\nu - \lambda) = \frac{\nu^2 + \lambda\mu - \lambda^2 - \mu\nu}{\nu - \lambda} = \nu + \lambda - \mu \le \nu.$$

From this we deduce

$$\frac{d}{dt}\left[(1+\delta)\ln(\nu - \lambda) - \ln(\lambda + \mu)\right] \le 0.$$

This shows that the ratio $(\nu - \lambda)^{1+\delta}/(\lambda + \mu)$ is decreasing in time, and that the set defined by (3.9) is invariant under (3.8). The assertion follows.

Observe that condition (3.9) is always satisfied on the initial metric for suitable constants δ, C, by the positivity of the Ricci tensor and by compactness. However, when the singular time is approached and the curvature becomes unbounded, the second inequality in (3.9) has important consequences. We see that the difference $\nu - \lambda$ is only allowed to grow at a lower rate than the sum $\lambda + \mu$. This implies that, if the three curvatures become unbounded, their ratio must tend to one. After justifying that the curvatures blow up everywhere on $\mathscr{M}$ as the singular time is approached, and that a smooth limit of the rescaled flow exists, Hamilton obtained in this way that the limit has $\lambda = \mu = \nu$ at each point, a property which implies constant curvature on $\mathscr{M}$. This gives an outline of the strategy of the proof of Theorem 3.1.

Using the maximum principle one can prove further estimates which yield the improvement of some curvature properties near the singular time. An important example is given by the next result, usually called *Hamilton-Ivey pinching estimate* and which was proved in [33, Theorem 24.4] and [43]. We follow here the presentation of [8, Sect. 6.2].

Theorem 3.9 *Let $g(t)$ be a solution of the Ricci flow on a closed three-manifold $\mathscr{M}$ and let R_0 be the minimum of the scalar curvature at time* 0. *Then there exists a function $\phi : [R_0, +\infty) \to (0, \infty)$ such that $\phi(r)/r \to 0$ as $r \to +\infty$ and such that* *the smallest sectional curvature* λ *at any point and time satisfies*

$$\lambda \geq -\phi(R). \tag{3.10}$$

Proof Up to a homothety in our solution, we can assume that the initial metric satisfies $R = 2(\lambda + \mu + \nu) \geq -1$. Then this inequality also holds for positive times, as it follows from (3.6) and the maximum principle. Let us now consider the function $f(x) = x \ln x - x$, whose derivative is $f'(x) = \ln x$. We have that f is convex and strictly increasing for $x \in (1, +\infty)$, and therefore one-to-one from $(1, +\infty)$ to $(-1, +\infty)$. Let us denote by $\phi : (-1, +\infty) \to (1, +\infty)$ its inverse. Then ϕ is increasing, concave and satisfies

$$\phi'(y) = \frac{1}{\ln(\phi(y))}. \tag{3.11}$$

In addition,

$$\lim_{y\to\infty} \frac{\phi(y)}{y} = \lim_{x\to\infty} \frac{x}{f(x)} = 0.$$

Let us consider the set of 3×3 symmetric matrices defined by the inequalities

$$\left\{\lambda + \mu + \nu \geq -\frac{1}{2},\ \lambda \geq -\phi(\lambda + \mu + \nu)\right\} \tag{3.12}$$

on their eigenvalues $\lambda \leq \mu \leq \nu$. As observed in the proof of the previous theorem, λ is a concave function while $\lambda + \mu + \nu$ is linear. From the concavity of ϕ we deduce that the set defined above is convex. We claim that it is also invariant under the ODE (3.8). The first condition corresponds to $R \geq -1$, which we already know to hold. To check the invariance of the second inequality, assume that

$$\lambda + \mu + \nu \geq -\frac{1}{2}, \qquad \lambda = -\phi(\lambda + \mu + \nu).$$

Then $\lambda < -1$ and

$$\mu + \nu = f(-\lambda) - \lambda = -\lambda \ln(-\lambda) > 0.$$

We also have

$$\ln(\phi(\lambda + \mu + \nu)) = \ln(-\lambda) = -\frac{\mu + \nu}{\lambda}.$$

It follows, using (3.8), (3.11)

$$\begin{aligned}\frac{d}{dt}\left[\lambda+\phi(\lambda+\mu+\nu)\right] &= \frac{d}{dt}\lambda-\frac{\lambda}{\mu+\nu}\frac{d}{dt}(\lambda+\mu+\nu)\\ &= \lambda^2+\mu\nu-\frac{\lambda(\lambda^2+\mu^2+\nu^2+\lambda\mu+\mu\nu+\lambda\nu)}{\mu+\nu}\\ &= (\mu-\lambda)\nu-\lambda\frac{\lambda^2+\mu^2}{\mu+\nu}.\end{aligned}$$

We recall that $\lambda < -1$ and $\mu+\nu > 0$, which also implies $\nu \geq \frac{1}{2}(\mu+\nu) > 0$. Thus the above expression is positive, showing the invariance of the set defined by (3.12).

Intuitively speaking, the previous theorem says that when the scalar curvature becomes large (that is, when the singular time is approached) the negative sectional curvatures, if there are any, become negligible compared to the other ones. In fact, if we consider a sequence of points approaching the singular time such that $\lambda < 0$ and $R = 2(\lambda+\mu+\nu) \to +\infty$, we have

$$\frac{|\lambda|}{\nu} = -\frac{2\lambda}{2\nu} \leq -\frac{2\lambda}{\lambda+\mu+\nu} \leq 4\,\frac{\phi(R)}{R} \to 0.$$

Thus, even if the sign of the curvature at the initial time is completely arbitrary, the asymptotic profile near the singularity necessarily has nonnegative curvature. This property will be stated in a more precise way later in these notes, when we introduce the rescaling of a solution near a singularity.

Invariance properties for positive curvature as the ones of the previous section also hold for the mean curvature flow described in Sect. 3.2. For instance, convexity or positive mean curvature are invariant properties in all dimensions. There is also an analogue of the Hamilton-Ivey estimate for the mean curvature flow, which was proved in [41] and [56]. Unlike the Ricci flow case, the result for the mean curvature flow holds for general dimension, but requires the positivity of the mean curvature. Under this hypothesis, the smallest principal curvature satisfies a lower bound similar to (3.10), which implies that its negative part becomes negligible near a singularity.

3.6 Differential Harnack Inequality

The classical Harnack inequality for elliptic equations is an estimate controlling the oscillation of positive solutions. We recall the statement in the case of the Laplace equation, see e.g. [29, Theorem 2.5].

Theorem 3.10 *Let $A \subset \mathbb{R}^n$ be open and let Ω be any bounded set such that $\overline{\Omega} \subset A$. Then there exists a constant $C > 0$, depending only of Ω and A, with the following property: given any nonnegative function $u \in C^2(A)$ such that $\Delta u = 0$ in A, we have*

$$\sup_{\Omega} \leq C \inf_{\Omega} u.$$

Similar estimates hold for parabolic equations; in this case, however, the supremum of the solution in a spatial domain at a given time is estimated above by a multiple of the infimum in the same domain at a later time, (see e.g. [28, Sect. 7.1.4b]). In [45] P. Li and S.-T. Yau introduced an alternative approach to Harnack inequalities in the parabolic case, showing that in certain cases they can be obtained from suitable estimates involving derivatives. Since this approach has been of fundamental importance in the study of the Ricci flow afterwards, we illustrate the main ideas in the "toy model" provided by the heat equation in $\mathbb{R}^n$; to avoid technicalities, we assume some a priori bound on the derivatives of the solutions which are stronger than needed for the validity of the result. Our exposition follows [13, Chap. 2].

Proposition 3.11 *Let $w \in C^2(\mathbb{R}^n \times [0,T])$ satisfy*

$$\frac{\partial w}{\partial t} = \Delta w.$$

Suppose in addition that $w \geq 0$ and that its first and second derivatives are bounded. Then w satisfies

$$D^2 w + \frac{w}{2t} I - \frac{Dw \otimes Dw}{w} \geq 0; \tag{3.13}$$

$$\frac{\partial w}{\partial t} + \frac{nw}{2t} - \frac{|Dw|^2}{w} \geq 0. \tag{3.14}$$

Here $D^2 w$ denotes the Hessian matrix of w with respect to the space variables and I the identity matrix; inequality (3.13) means that the matrix at the left-hand side is positive semi-definite.

Proof It is not restrictive to assume that w is greater than some positive constant; if this is not the case, we can replace w by $w + \varepsilon$ and then let $\varepsilon \to 0^+$. Let us set $u(x,t) = -\ln(w(x,t))$. Then it is easily checked that u is a solution of equation

$$\frac{\partial u}{\partial t} + |Du|^2 = \Delta u.$$

In addition, u is bounded together with its first and second derivatives. Given any unit vector $\nu \in \mathbb{R}^n$, let us then set $h(x,t) = t\frac{\partial^2 u}{\partial \nu^2}(x,t)$. We have

$$\begin{aligned}\frac{\partial h}{\partial t} &= \frac{h}{t} + \Delta h - 2\langle D(|Du|^2), Dh\rangle - 2t\left|D\left(\frac{\partial u}{\partial \nu}\right)\right|^2 \\ &\leq \frac{h}{t}(1-2h) + \Delta h - 2\langle D(|Du|^2), Dh\rangle.\end{aligned}$$

Since $h(\cdot,t) \to 0$ uniformly as $t \to 0$, the maximum principle implies that $2h(x,t) \leq 1$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. By the definition of h and the arbitrariness of ν, this means

$$D^2 u \leq \frac{1}{2t} I.$$

On the other hand, an easy computation shows that

$$D^2 u = -\frac{D^2 w}{w} + \frac{Dw \otimes Dw}{w^2}$$

and this proves (3.13). Taking the trace of the left-hand side of (3.13), we obtain

$$\Delta w + \frac{nw}{2t} - \frac{|Dw|^2}{w} \geq 0,$$

which implies (3.14), since w solves the heat equation.

Inequalities of the form (3.13)–(3.14) are often called *differential Harnack inequalities*. The connection with the classical Harnack inequality is explained by the following result.

Corollary 3.12 *Let w be as in the previous proposition. Then w satisfies*

$$w(y,t_2) \geq w(x,t_1)\left(\frac{t_1}{t_2}\right)^{n/2} \exp\left(-\frac{|y-x|^2}{4(t_2-t_1)}\right)$$

for all $x, y \in \mathbb{R}^n$, $t_2 > t_1 > 0$. Therefore, given any bounded set $\Omega \subset \mathbb{R}^n$ and $t_2 > t_1 > 0$, we have

$$\max_{\Omega} w(\cdot,t_1) \leq C \min_{\Omega} w(t_2,\cdot) \tag{3.15}$$

for some constant $C > 0$ depending on t_1, t_2 and the diameter of Ω but not on w.

Proof Let us set

$$\gamma(s) = x + \frac{s - t_1}{t_2 - t_1}(y - x), \qquad s \in [t_1, t_2].$$

Then we have, using (3.14),

$$\begin{aligned} \frac{d}{ds} w(\gamma(s), s) &= \partial_t w + Dw \cdot \dot{\gamma} \\ &\geq \partial_t w - \frac{|Dw|^2}{w} - \frac{w|\dot{\gamma}|^2}{4} \\ &\geq -\frac{nw}{2t} - \frac{w|\dot{\gamma}|^2}{4}. \end{aligned}$$

It follows that

$$\begin{aligned} \ln\left(\frac{w(y, t_2)}{w(x, t_1)}\right) &= \int_{t_1}^{t_2} \frac{d}{ds} \ln w(s, \gamma(s))\, ds \\ &\geq \int_{t_1}^{t_2} \left(-\frac{n}{2t} - \frac{|y - x|^2}{4(t_2 - t_1)^2}\right) ds \\ &= -\frac{n}{2} \ln\left(\frac{t_2}{t_1}\right) - \frac{|y - x|^2}{4(t_2 - t_1)}, \end{aligned}$$

which proves our statement.

After the work [45] by Li and Yau, Hamilton developed extensively this approach for various geometric evolution equations. In particular, for the Ricci flow he obtained the following result [32].

Theorem 3.13 *Let $(\mathscr{M}, g(t))$ be a solution to the Ricci flow, defined for $t \in [0, T)$, which is either closed or complete with bounded curvature, and has nonnegative curvature operator. Then, for any vector field V and any time $t \in [0, T)$ we have*

$$\frac{\partial R}{\partial t} + \frac{1}{t} R + 2\langle \nabla R, V\rangle + 2\mathrm{Ric}(V, V) \geq 0. \tag{3.16}$$

The above result is sometimes called "trace differential Harnack inequality" for the Ricci flow, because it is obtained taking the trace of a more general tensor inequality, similarly to what we have described above for the heat equation in $\mathbb{R}^n$. Integrating along a suitable path in space time as in Corollary 3.12, one obtains the following result.

Corollary 3.14 *Under the same hypotheses as in the previous theorem, given any* $P_1, P_2 \in \mathscr{M}$ *and* $0 < t_1 < t_2$*, we have*

$$R(P_2, t_2) \geq \frac{t_1}{t_2} R(P_1, t_1)\, e^{-\frac{d^2}{2(t_2-t_1)}}$$

where d *is the distance between* P_1 *and* P_2 *at time* t_1.

Differential Harnack estimates for geometric flows are elegant and mysterious at the same time. They are strictly related to special solutions of the flow, where they hold identically as equalities, similarly to the case of the heat equation in $\mathbb{R}^n$, where they hold as equalities on the heat kernel. Some geometric interpretations that give a deeper insight in these inequalities can be found in [14]. A detailed general exposition can be found in [48]. Harnack estimates have also been obtained for the mean curvature flow [34] and for more general curvature flows of immersed manifolds [1].

3.7 The Intuitive Picture of the Flow with Surgeries

To motivate the analysis of the following sections, we give here an intuitive description of the strategy of proof of the Poincaré and Thurston conjecture using the Ricci flow with surgeries. We follow Hamilton's original approach [31], where surgeries are performed shortly before the singular time, which in our opinion is slightly easier to picture than Perelman's one [50], where surgeries are performed exactly at the singular time.

Let us consider an arbitrary closed three-dimensional manifold $\mathscr{M}$, and suppose that we want to study its possible topology. It is not restrictive to assume that $\mathscr{M}$ has a differentiable structure, and some smooth Riemannian metric g_0 chosen arbitrarily. Then, we let the metric g_0 evolve by the Ricci flow and study the behavior of the solution. The aim is to show that the metric $g(t)$ eventually converges, up to rescaling, to some limit that can be explicitly described. We then obtain that the initial manifold $\mathscr{M}$ has to be diffeomorphic to one of the possible limits of the flow.

This strategy works very well in the case of Hamilton's first result [30]. Here we have the additional assumption that the initial metric has positive Ricci curvature, and the only possible limits are the sphere and its quotients. However, for more general initial metrics, neckpinch singularities can occur and the smooth Ricci flow does no longer give a global information on the manifold. In a neckpinch singularity, in fact, we can hope to describe the structure of the region were the curvature becomes unbounded, but we have no knowledge of the remaining part of the manifold, which can have arbitrary topology.

To overcome the difficulty, Hamilton proposed a way to continue the flow after singularities using a surgery procedure. Consider a three-manifold which develops a neckpinch singularity, where the region with the largest curvature looks like a portion of a cylinder with radius shrinking to zero as the singular time approaches. Such a cylindrical region is usually called a *neck*. The strategy is to stop the flow shortly before the singular time, remove the neck and fill smoothly the two remaining holes with two 3-balls. Such a modification is called a *surgery*, and in general it can change the topological type of the manifold. However, the surgery defined in this way is the reverse of a standard operation in algebraic topology called *connected sum*. In particular, the possible topological changes of our manifold under surgery can be precisely described.

After the surgery, the flow can be restarted until the next singularity occurs, and then other surgeries are performed. At each surgery time, we are allowed to discard connected components of the manifold of known topology. Now suppose that we can show that, after a finite number of surgeries, all the remaining components of the manifold are diffeomorphic to one of the eight model manifolds allowed by Thurston conjecture. This implies that the initial manifold can be obtained performing a finite number of connected sums on these model geometries, and proves the validity of the conjecture. In particular the Poincaré conjecture is obtained, since the only simply connected manifold allowed by Thurston conjecture is the sphere.

In order to implement this program, one needs to prove that the singular behavior of the Ricci flow in three dimension must be, roughly speaking, of one of two kinds described above, namely:

(i) Either the curvature becomes large on the whole manifold, and the manifold converges to a sphere up to rescaling, or
(ii) the curvature becomes unbounded only in a part of the manifold, which becomes asymptotically close to a portion of a shrinking cylinder.

In case (i) the flow is stopped, while the other connected components of the manifold, if any, continue their evolution. In case (ii) the flow is stopped shortly before the singular time, the neck is removed by a surgery, and the flow is restarted.

Therefore, a crucial part of the implementation of the program is the analysis of the possible asymptotic profile of the singularities. This will be the object of the remainder of these notes. The other part of the program, consisting of showing that after finitely many surgeries we are left with components which satisfy Thurston conjecture, is outlined in M. Boileau's notes in this volume. Before passing to more precise statements in the next sections, let us add two important details to the intuitive picture given above.

A first observation is that, in order to really increase the lifespan of the solution, we must perform the surgery in such a way that the maximum of the curvature on the manifold is substantially decreased after removing the necks. However, if our necks are close to a portion of a cylinder, as in the crude description above, the curvature on the boundary of the neck is comparable to the one in the interior, and we cannot claim that the surgery decreases the curvature. Instead, our necks should be only diffeomorphic to a cylinder, but the curvature on the boundary should be much smaller than the one in the middle part. Intuitively speaking, we should think of them as long tubes, with a very small radius in the middle region which becomes slowly larger as we move towards the ends. To obtain such a picture, one needs to prove that the points where the curvature is large enough, but not necessarily close to the maximum value, possess a neighborhood almost isometric to a portion o f a cylinder. By gluing together all these neighborhoods, we obtain a neck with the desired properties which covers a whole region with large curvature but has much smaller curvature on its boundary.

The other caveat is that there is a third possible singular behavior in addition to (i) and (ii) described above. In fact, in the case of the degenerate neckpinch described in Sect. 3.2, the region with the largest curvature is not cylindrical, but it is instead diffeomorphic to a ball. However, the surgery procedure can be adapted to this case too. In fact, it is possible to show that the spherical region with the largest curvature is surrounded by a neck along which the curvature gradually decays. Then we can remove these two regions, which together are diffeomorphic to a ball, and fill the remaining hole with another ball with smaller curvature. In this case the surgery is topologically trivial, but it again reduces the maximum of the curvature and it allows to restart the flow to obtain a solution defined in a longer time interval.

The above description should be kept in mind in the following sections, to understand the goal of the analysis of the singularities.

3.8 Rescaling Around Singularities

To study the behavior of the solutions of the Ricci flow when the curvature becomes unbounded one can use rescaling procedures which are common also for other kinds of PDEs. We will describe the technique in an informal way because the rigorous statements are rather technical, see [33, Sect. 16].

Let us first observe that the Ricci flow is invariant under parabolic rescalings, that is, if we dilate a solution by a factor $\lambda > 0$ in space and λ^2 in time, we obtain another solution of the flow, which has the norm of the curvature $|\mathrm{Rm}|$ reduced by a factor λ^2. Suppose now that we have a solution $(\mathscr{M}, g(t))$ of the Ricci flow which becomes singular as $t \to T$. We can consider a sequence of rescalings with larger and larger factors near the singular time and then take a limit which describes, intuitively speaking, the singular profile of the original solution. More precisely, let us take a sequence of points $P_j \in \mathscr{M}$ and times t_j such that $t_j \uparrow T$ and in addition

$$|\mathrm{Rm}(P,t)| \leq C|\mathrm{Rm}(P_j,t_j)| \qquad \forall P \in \mathscr{M},\ t \in [0,t_j]$$

for some constant $C \geq 1$ independent of j. For any $j \geq 1$ we now rescale our flow by a factor λ_j, where $\lambda_j = \sqrt{|\mathrm{Rm}(P_j,t_j)|}$. In addition, we take P_j to be the origin of the rescaled flow and we translate the time so that t_j becomes zero. Then the j-th flow is defined for $t \in [-\lambda_j^2 t_j, (T-t_j)\lambda_j^2]$. Observe that the initial endpoint of the time interval tends to $-\infty$ at $j \to \infty$; the final endpoint is positive, and it can be proved that it stays bounded away from zero for all j. By construction, each rescaled flow satisfies $|\mathrm{Rm}| \leq C$ everywhere at all times $t \leq 0$. It is possible to show that this curvature bound ensures the existence of a converging subsequence, provided the rescaled flows also satisfy an injectivity radius bound.

Theorem 3.15 *Let $(\mathscr{M}, g(t))$ a solution of the Ricci flow which becomes singular as $t \to T$, and let us consider a family of rescaled flows defined as above. Suppose in addition that the injectivity radius of our manifold satisfies the estimate*

$$\mathrm{inj}(P,t) \geq \frac{c}{\sqrt{\max_{\mathscr{M}} |\mathrm{Rm}|(\cdot,t)}}, \qquad \forall P \in \mathscr{M},\ t \in [0,T) \tag{3.17}$$

for some $c > 0$. Then a subsequence of the rescaled flows converges uniformly on compact sets to a limit $(\hat{\mathscr{M}}, \hat{g}(t))$, which is a solution to the Ricci flow and is defined in an interval of the form $(-\infty, T^*)$, with $T^* > 0$ (possibly infinite). If $n = 3$ then the limit flow has nonnegative sectional curvature at every point and satisfies the improved differential Harnack estimate

$$\frac{\partial R}{\partial t} + 2\langle DR, V\rangle + 2\mathrm{Ric}(V,V) \geq 0. \tag{3.18}$$

For the proof of the first part of this statement, see [33, Sect. 16]. The assertion concerning the sectional curvature can be obtained from Theorem 3.9; in fact, the right-hand side of (3.10) disappears in the rescaling procedure due to the sublinearity of ϕ. Observe also that in three dimensions positive sectional curvature is equivalent to positive curvature operator. Thus the limit flow satisfies the Harnack inequality (3.16) where the R/t term can be replaced by $R/(t-t_0)$ with t_0 arbitrarily small since the solution is defined in $(-\infty, T^*)$. Thus, letting $t_0 \to -\infty$, this term vanishes and we obtain the improved inequality (3.18).

In [33, Sect. 16] one can also find a precise definition of the "convergence on compact sets" mentioned in the statement. In particular, even if the rescaled flows are all compact, their diameter can go to $+\infty$, and thus the limit flow can be noncompact. The typical example is the neckpinch of Sect. 3.2, where the limit flow is an infinite cylinder $\mathbb{S}^2 \times \mathbb{R}$.

Hamilton then proved the following classification results of the possible structure of the limit flow in dimension 3.

Theorem 3.16 *Let $g(t)$ be a solution of the Ricci flow on a closed three-manifold $\mathscr{M}$. Suppose that the flow becomes singular as $t \to T$ and that we have an injectivity radius estimate of the form (3.17). Then it is possible to choose the sequence (P_j, t_j) in the above construction in such a way that the limit flow is one of the following (or a quotient under a finite group of isometries)*

(i) the shrinking sphere $\mathbb{S}^3$, or
(ii) the shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$, or
(iii) $\Sigma \times \mathbb{R}$, where Σ is the "cigar" soliton described in Sect. 3.2.

The above theorem is given at the end of the paper [33] and the proof uses all the properties of the limit flow which we have mentioned before. Although such a result already gave strong restrictions on the possible structure of the singularities, there remained two unsatisfactory aspects. One was the lack of a general argument which could provide the injectivity radius estimate needed in the theorem. The second problem regarded case (iii): if such a limit could occur then it would represent a fatal obstruction to Hamilton's program, because there is no clear way to do surgery on a singularity which exhibits such a profile. Hamilton conjectured that case (iii) cannot occur, but did not succeed in proving this. We will see in the next sections how Perelman's new results have solved both of these difficulties.

3.9 Perelman's Monotonicity Formula

In [49, Sect. 3] Perelman introduced the following functional. Let $\mathcal{M}$ be a closed n-dimensional manifold. Given a metric g on $\mathcal{M}$, a function $f : \mathcal{M} \to \mathbb{R}$ and a positive number τ, consider

$$\mathcal{W}(g,f,\tau) = \int_{\mathcal{M}} [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f} d\mu_g,$$

known as *Perelman's $\mathcal{W}$-entropy functional.* Define also, for fixed g and τ,

$$\mu(g,\tau) = \inf \left\{ \mathcal{W}(g,f,\tau) \;:\; f \text{ such that } \int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-f} d\mu_g = 1 \right\}.$$

Then the following result holds.

Theorem 3.17 *Let $g(t)$ be a solution of the Ricci flow for $t \in [t_0, t_1]$ on a closed manifold $\mathcal{M}$, and let $\tau(t) = \bar{t} - t$ for some $\bar{t} > t_1$. Let $f : \mathcal{M} \times [t_1, t_2] \to \mathbb{R}$ satisfy*

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}.$$

Then

$$\frac{d\mathcal{W}}{dt} = \int_{\mathcal{M}} 2\tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu. \tag{3.19}$$

In addition, the quantity $\mu(g(t), \tau(t))$ is nondecreasing in t for $t \in [t_0, t_1]$.

The above result, known as Perelman's entropy monotonicity formula, has important applications to the analysis of singularities of the Ricci flow. Let us introduce the notion of local collapsing.

Definition 3.18 Let $(\mathcal{M}, g(t))$ be a solution of the Ricci flow for $t \in [0, T)$, with T finite. We say that the solution is *locally collapsing* at time T if there exists a sequence of times $t_k \uparrow T$, of points $P_k \in \mathcal{M}$ and of radii $r_k > 0$ such that $\{r_k\}$ is bounded and such that, if we denote by B_k the ball of center P_k and radius r_k with respect to the metric $g(t_k)$, we have that $|\mathrm{Rm}|(P, t_k) \leq r_k^{-2}$ for all $P \in B_k$ and that $\mathrm{Vol}\,(B_k)/r_k^n \to 0$ as $k \to \infty$.

Roughly speaking, collapsing means that we can find metric balls where the volume ratio is arbitrarily small. This is due to an increasingly smaller injectivity radius, which implies that the balls actually cover many times the same tiny portion of the manifold. The hypothesis that $|\mathrm{Rm}|(P, t_k) \leq r_k^{-2}$ on B_k shows that the smallness of the injectivity radius is not due to the size of the curvature. Therefore, for instance, a family of shrinking n-dimensional spheres, with $n > 1$, is not collapsing. Instead, a family of manifolds of the form $\mathscr{M} = \mathbb{S}^1_{r(t)} \times \mathscr{M}'$, where $\mathbb{S}^1_{r(t)}$ is a one-dimensional circle with radius $r(t) \to 0$ and $\mathscr{M}'$ is any fixed $(n-1)$-dimensional manifold, is collapsing: the shrinking one-dimensional factor is flat and does not influence the curvature, but the injectivity radius and the volume ratio go to zero as $r(t) \to 0$. The intuitive expectation is that such a behavior should not occur in the Ricci flow, since a flat factor would stay constant and not shrink. Indeed, Perelman shows that the monotonicity of $\mathscr{W}$ prevents the collapsing described above, and he obtains from Theorem 3.17 the following crucial result [49, Sect. 4].

Theorem 3.19 *If $g(t)$ is a solution of the Ricci flow for $t \in [0, T)$ on a closed manifold $\mathscr{M}$, then $(\mathscr{M}, g(t))$ is not locally collapsing at time T.*

To prove this result, Perelman shows that if the flow is collapsing at time T, then $\mu(g(t_k), r_k^2) \to -\infty$, by plugging suitable functions f in the functional $\mathscr{W}$. On the other hand, by Theorem 3.17, $\mu(g(t_k), r_k^2) \geq \mu(g(0), t_k + r_k^2)$, which cannot be arbitrarily small, and this gives a contradiction.

As mentioned before, the collapsing behavior is related to the smallness of the injectivity radius at the points (P_k, t_k), see e.g. [55, Sect. 8.4]. In particular, if the solution is not locally collapsing, then it also satisfies the injectivity radius estimate required in Theorem 3.16. Thus, Perelman's result ensures that the injectivity radius estimate is always satisfied.

Theorem 3.19 also allows to exclude that the cigar $\Sigma \times \mathbb{R}$ is obtained as limit of rescaled flows. In fact, using (3.3) one can check that the metric on the cigar Σ is locally collapsing. Since the collapsing property is invariant under rescaling, $\Sigma \times \mathbb{R}$ cannot occur as the limit of the rescalings of a noncollapsed solution.

The noncollapsing property of the Ricci flow allows to exclude case (iii) of Hamilton's Theorem 3.16. This result, however, does not suffice yet to define a flow with surgeries. For this purpose, we need to know that, when the singular time is approached, all points of the manifold with curvature larger than a certain threshold lie in regions that can be removed by the surgeries. In this way, we know that the manifold after surgeries has bounded curvature, so that the flow can be restarted and exists for some given time before possible new singularities occur. We therefore need to obtain a more detailed description of the singular regions than the one provided by Theorem 3.16, which only describes the behavior around suitable sequences of points with large curvature. We will see this in the remaining part of these notes.

3.10 $\mathscr{L}$-Distance and Reduced Volume

In [49, Sect. 7] Perelman introduces some geometric quantities which provide further powerful results for the analysis of the singularities of the Ricci flow.

Let $(\mathscr{M}, g(t))$ be a solution of the Ricci flow, for $t \in [0, T]$. For the purposes of this section, it is convenient to reverse the time direction, and consider the variable $\tau := T - t$. The metric then satisfies the backward Ricci flow $\frac{d}{d\tau} g(\tau) = 2\,\mathrm{Ric}\, g(\tau)$. Given a curve $\gamma : [\tau_1, \tau_2] \to \mathscr{M}$, with $0 \le \tau_1 < \tau_2 \le T$, we define the $\mathscr{L}$-length of γ as

$$\mathscr{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}[R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2]\, d\tau. \tag{3.20}$$

Here $R(\gamma(\tau))$ and $|\dot{\gamma}(\tau)|^2$ are computed with respect to the metric $g(\tau)$. Functionals of this type are classical in Calculus of Variations, see e.g. [13, Chap. 6] and the references therein. The $\mathscr{L}$-length can be regarded as a generalization of the usual energy functional for curves on a Riemannian manifold, see e.g. [25, Sect. 9.2], whose minimizers are geodesics, and it shares some basic properties. Given any pair of points $p, q \in \mathscr{M}$, we can minimize the $\mathscr{L}$-length over all the curves such that $\gamma(\tau_1) = p$, $\gamma(\tau_2) = q$. By standard methods, it can be proved that the minimum exists, and it is called the $\mathscr{L}$-distance between (p, τ_1) and (q, τ_2). The curve γ achieving the minimum is not necessarily unique; any such curve is called an $\mathscr{L}$-geodesic and satisfies a suitable ordinary differential equation. It should be noted that the $\mathscr{L}$-distance is not necessarily positive unless the evolving metric has positive scalar curvature.

It is convenient to fix the initial endpoint $p \in \mathscr{M}$ and $\tau_1 = 0$ and analyze the properties of the $\mathscr{L}$-distance as a function of the final endpoint. Namely, we define $L(q, \tau)$ to be the $\mathscr{L}$-distance between $(p, 0)$ and (q, τ) for any given $q \in \mathscr{M}$ and $\tau > 0$. In [49, Sect. 7], several identities and inequalities relating the derivatives of L and the minimizers are derived, which are inspired by the corresponding first and second order conditions satisfied by the classical geodesics. As in the case of the ordinary distance from a given point, the function L is in general Lipschitz continuous but not differentiable everywhere; more precisely, it is not differentiable at those points (q, τ) for which the minimizing geodesic from $(p, 0)$ is not unique, see e.g. [13, Corollary 6.4.10]. Therefore, the differential inequalities satisfied by L must be understood in a suitable weak sense, for instance in the sense of barriers introduced by Calabi in [12], or the viscosity sense [23]. We recall two important inequalities derived by Perelman, see formulas (7.13) and (7.15) in [49].

Theorem 3.20

(i) *If we set* $l(q,\tau) := \frac{1}{2\sqrt{\tau}}L(q,\tau)$, *then* l *satisfies*

$$\frac{\partial l}{\partial \tau} \geq \Delta l - |\nabla l|^2 + R - \frac{n}{2\tau}. \tag{3.21}$$

(ii) *If we set* $\bar{L}(q,\tau) := 2\sqrt{\tau}L(q,\tau)$, *then* $\bar{L}$ *satisfies*

$$\frac{\partial \bar{L}}{\partial \tau} + \Delta \bar{L} \leq 2n. \tag{3.22}$$

As an immediate consequence, we obtain

Corollary 3.21

(i) *Let us define*

$$V(\tau) := \int_{\mathscr{M}} \tau^{-n/2} e^{-l(q,\tau)} d\mu_{g(\tau)}. \tag{3.23}$$

Then V *is a nonincreasing function of* τ.

(ii) *We have* $\min_{\mathscr{M}} l(\cdot,\tau) \leq \frac{n}{2}$ *for all* $\tau > 0$.

Proof We argue as if the function L were smooth, but the computations can be justified also in the case where the inequalities of the previous theorem only hold in a weak sense. To prove (i), let us set $w(q,t) = \tau^{-n/2}e^{-l(q,\tau)}$. We find, using (3.21),

$$\frac{\partial w}{\partial \tau} - \Delta w + wR = w\left(-\frac{n}{2\tau} - |\nabla l|^2 + \Delta l - \frac{\partial l}{\partial \tau} + R\right) \leq 0.$$

When we compute the derivative of $V(\tau)$, we must take into account that the volume element on a solution of the backward Ricci flow evolves according to

$$\frac{\partial}{\partial \tau} d\mu_{g(\tau)} = R\, d\mu_{g(\tau)},$$

see e.g. [17, Lemma 3.9]. We conclude

$$V'(\tau) = \int_{\mathscr{M}} \left(\frac{\partial w}{\partial \tau} + Rw\right) d\mu_{g(\tau)} \leq \int_{\mathscr{M}} \Delta w\, d\mu_{g(\tau)} = 0,$$

which proves the monotonicity of V.

To obtain (ii), let us first estimate $l(q,\tau)$ in the case where τ is small and $q = p$. If we compute the $\mathscr{L}$-distance using the constant path $\gamma \equiv p$ and use the local boundedness of R, we easily obtain an upper bound of the form $L(p,\tau) \leq C\tau^{3/2}$ for a suitable $C > 0$. It follows that $\min_{\mathscr{M}} l(\cdot,\tau) < \frac{n}{2}$ for τ enough small. Now, if apply the maximum principle backward in time to (3.22), we deduce that the minimum of $\bar{L}(\cdot,\tau) - 2n\tau$ is a nonincreasing function of τ. On the other hand, by definition,

$$\min_{\mathcal{M}} \left\{ \bar{L}(\cdot,\tau) - 2n\tau \right\} = 4\tau \min_{\mathcal{M}} \left\{ l(\cdot,\tau) - \frac{n}{2} \right\},$$

which is negative for small $\tau > 0$. Therefore, the above minimum stays negative for all $\tau > 0$, which proves (ii).

The function V defined in (3.23) is called the *reduced volume* by Perelman. The monotonicity of the reduced volume gives an alternative argument to prove the noncollapsing property for the solutions of the Ricci flow. Roughly speaking, one can consider a sequence of points as in Definition 3.18 and consider for each k the reduced volume V_k obtained setting $p = p_k$ and $\tau = \tau_k - t$. Then, using the smallness of the standard volume of suitable balls around p_k given by the collapsing assumption, it can be proved that $V_k(\tau)$ becomes close to zero for suitably small τ. On the other hand, one can show that $V_k(t_k)$ is bounded away from zero, since $\tau = t_k$ corresponds to $t = 0$ and thus $V_k(t_k)$ can be estimated in terms of the behavior of the manifold in a neighborhood of a fixed regular time. The two properties together are in contradiction with the monotonicity of the reduced volume proved above, and this allows to prove a result similar to Theorem 3.19.

It is interesting to remark that the monotonicity of the reduced volume stated in Corollary 3.21 has some similarity with a well-known result for the mean curvature flow, called *Huisken's monotonicity formula* [40]. The proof of the two results, however, are not related, and the applications to the analysis of singularities are rather different in the two cases. The entropy monotonicity and the noncollapsing property described in the previous section have also some analogues in the mean curvature flow, see [2, 26].

3.11 Properties of κ-Solutions

In the rest of these notes, we restrict ourselves three-dimensional manifolds. In Sect. 3.8 we have seen that the profile of the solution of the Ricci flow near a singularity can be studied by rescaling techniques. In particular, by taking the limit of a sequence of flows rescaled around points where the curvature becomes unbounded, one obtains an ancient solution which describes the singular profile and enjoys some special properties. The study of such ancient solutions is the first step to a more detailed analysis of the singularities which will enable the surgery construction. The results in this section are taken from [49, Sects. 11 and 12], [50, Sects. 1 and 3] We start with a definition.

Definition 3.22 Given $\kappa > 0$, a nonflat solution of the Ricci flow on a (possibly noncompact) three-dimensional manifold $\mathcal{M}$ is called a *κ-solution* if it satisfies the following properties

(i) It is ancient, i.e., it is defined for $t \in (-\infty, T)$ for some $T > 0$.
(ii) It has bounded curvature and nonnegative curvature operator at each fixed time t.
(iii) The solution is κ-noncollapsed, in the sense that for any time t and any ball B of radius r in $(\mathscr{M}, g(t))$ which satisfies $|\mathrm{Rm}|(p,t) \le r^{-2}$ for all $p \in B$, we have that $\mathrm{Vol}\,(B) \ge \kappa r^3$.

As we have observed in Sect. 3.8, any limit obtained by rescaling a given solution near a singularity satisfies properties (i) and (ii) above. Property (iii) follows from Theorem 3.19 and from the fact that the noncollapsing property is scale-invariant. In addition, a κ-solution satisfies the stronger form of Hamilton's Harnack inequality (3.18). Choosing $V = 0$ in that inequality, we obtain in particular that $\frac{\partial R}{\partial t} \ge 0$. This means that the scalar curvature is pointwise nondecreasing, and therefore any bound on R at a certain time holds also for all previous times.

Examples of κ-solutions are the shrinking sphere, the shrinking cylinder or the Bryant soliton mentioned in Sect. 3.2. The product $\Sigma \times \mathbb{R}$, where Σ is the cigar soliton, is not a κ-solution because it does not satisfy the noncollapsing property (iii). A product $\mathbb{S}^2 \times \mathbb{S}^1$, where $\mathbb{S}^2$ is homothetically shrinking while $\mathbb{S}^1$ remains constant because it has no curvature, also violates property (iii), as it is seen by considering arbitrarily large negative times.

There exist also more elaborate κ-solutions which are not solitons. In [50, Sect. 1.4], Perelman describes a compact κ-solution for $t \in (-\infty, 0)$, which is close to a shrinking sphere as $t \to 0$, while as $t \to -\infty$ it resembles a more and more eccentric oval. However, the analysis is simplified by the next result, see [49, Sect. 11.2], which associates to any κ-solutions a gradient shrinking soliton.

Theorem 3.23 *Let $(\mathscr{M}, g(t))$ be a κ-solution of the Ricci flow, let $p \in \mathscr{M}$, $t_0 \in (-\infty, T)$ be fixed arbitrarily, and let $l(q,\tau)$ be the reduced length centered at (p, t_0) defined in Sect. 3.10. For any $\tau > 0$, let $q(\tau) \in \mathscr{M}$ be a point such that $l(q,\tau) \le n/2$, whose existence follows from Corollary 3.21. Then the rescalings of the metrics $g(t_0 - \tau)$ around the point $q(\tau)$ with factor τ^{-1} converge along a subsequence $\tau_k \to \infty$ to a nonflat gradient shrinking soliton.*

The soliton obtained from the above theorem is called an *asymptotic soliton* of the κ-solution. It can be proved that the only possibilities for such a soliton are the following, up to quotients:

(i) the asymptotic soliton is a shrinking sphere;
(ii) the asymptotic soliton is a shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$.

In addition, case (i) only occurs if the κ-solution is itself a sphere. This also shows that the asymptotic soliton is unique, because the two possibilities are incompatible.

An interesting property of noncompact κ-solutions concerns the *asymptotic volume ratio*, defined by

$$\mathscr{V} = \lim_{r \to +\infty} \frac{\mathrm{Vol}\, B(p_0, r)}{r^3},$$

where p_0 is any fixed point in $\mathscr{M}$ and $B(p_0, r)$ is the metric ball of radius r around p_0. Then, it can be proved that on a κ-solution we have $\mathscr{V} = 0$ at each time. Such a result is not in contrast with the noncollapsing property; in fact, for a fixed p_0 and arbitrarily large r, we cannot have the property $|\mathrm{Rm}| \le r^{-2}$ in the ball $B(p_0, r)$, and therefore the lower bound on the volume ratio in the noncollapsing property does not apply on such a ball. Intuitively speaking, the property that $\mathscr{V} = 0$ implies that a noncompact κ-solutions cannot open up asymptotically like a cone, but rather like a paraboloid, as in the case of the Bryant soliton.

A fundamental step in Perelman's analysis of the singularities is the following compactness result modulo scaling for κ-solutions [49, Sect. 11.7].

Theorem 3.24

(i) For any $r > 0$, there exists a universal constant $M = M(r)$ such that, given any point and time (p, t) in a κ-solution such that $R(p, t) = 1$ and any other point q such that $d_{g(t)}(p, q) < r$, we have $R(q, t) \le M$.

(ii) Given a sequence of κ-solutions $\{(\mathscr{M}_k, g_k(t))\}$ and points $p_k \in \mathscr{M}_k$ such that $R(p_k, 0) = 1$, there exists a subsequence centered at (p_k, t) which converges smoothly to a limit which is also κ-solution.

(iii) There exists a constant $C > 0$ such that, on any κ solution, we have the following derivative estimates:

$$|\nabla R(p,t)|^2 \le C|R(p,t)|^3, \quad |\partial_t R(p,t)| \le C|R(p,t)|^2, \quad p \in \mathscr{M}, t \in (-\infty, T).$$

The above result is the main step towards a precise description of the structure of κ-solutions. We first give a formal definition of the notion of "almost cylindrical region" inside a manifold evolving by Ricci flow.

Definition 3.25 Given a solution of the Ricci flow and $\varepsilon > 0$, we say that (p_0, t_0) is the center of an ε-neck if, after setting $Q_0 = R(p_0, t_0)$, the parabolic neighborhood

$$\left\{ (p, t) \ : \ t_0 - \frac{1}{\varepsilon Q_0} \le t \le t_0, \ d^2_{g(t_0)}(p, p_0) \le \frac{1}{\varepsilon Q_0} \right\}$$

is ε-close, after scaling with the factor Q, to a subset of a shrinking round cylinder. Here "ε-close" means that the metric, and its derivatives up to a suitable order, differ from the corresponding ones of the cylinder by no more than ε.

The following result [49, 11.8], see also [44, Sect. 48] or [46, Theorem 9.93], states that on any κ-solution the points either lie at the center of a neck, or they belong to a compact region whose diameter satisfies an apriori bound.

Theorem 3.26 *For any $\varepsilon > 0$ enough small, there exists $C = C(\varepsilon) > 0$ with the following property. Take any κ-solution of the Ricci flow and denote by $\mathscr{M}_\varepsilon$ the points which are not at the center of an ε-neck at some given time t_0. Then $\mathscr{M}_\varepsilon$ is compact. In addition, $\mathscr{M}_\varepsilon$ can be written as the union of at most two components M_1, M_2 which have the following properties, after setting $Q_i = R(p_i, t_0)$ for an arbitrary $p_i \in M_i$:*

(i) the diameter of M_i is at most $CQ_i^{-1/2}$,
(ii) we have $C^{-1}Q_i \le R(q, t_0) \le CQ_i$, for any $q \in M_i$.

Examples of the possible structure of M_ε can be obtained by looking at the κ-solutions described at the beginning of the section. The set M_ε is clearly empty on a shrinking cylinder. On a sphere, M_ε is instead the whole manifold if ε is small enough, and thus it consists of single component, which satisfies (i) and (ii). The Bryant soliton is a more interesting example. As mentioned in Sect. 3.2, it consists of a rotationally symmetric metric on $\mathbb{R}^3$ where the curvature decreases as the distance from the origin increases. It turns out that any point sufficiently far from the origin lies at the center of an ε-neck, so that M_ε is a ball of radius R_ε, with R_ε becoming large if ε becomes small. Therefore, M_ε consists of a single compact component, which satisfies properties (i) and (ii). The other κ-solution described at the beginning, which is compact and becomes more and more oval as $t \to -\infty$, gives instead an example where M_ε consists of two components: they are the two opposite ends of the solution, while the central part consists of points which are all centers of a neck.

3.12 Canonical Neighborhoods and the Structure of Singularities

The results of the previous section give an accurate description of the structure of κ-solutions. The next fundamental result [49, Sect. 12.1] shows that the same description extends to the regions with large curvature of an arbitrary solution of the Ricci flow in three dimensions.

Theorem 3.27 *Let $(\mathscr{M}, g(t))$, with $t \in [0, T)$, be a solution of the Ricci flow on a closed three-dimensional manifold $\mathscr{M}$. For any $\varepsilon > 0$ there exists $r_0 > 0$, only depending on ε and the initial data, with the following property. Let (p_0, t_0) be any point with $t_0 \ge 1$ and $R(p_0, t_0) \ge r_0^{-2}$. Then, if we set $Q = R(p_0, t_0)$, we have that the parabolic neighborhood*

$$\left\{(p,t) \;:\; t_0 - (\varepsilon Q)^{-1} \le t \le t_0,\ d^2_{g(t_0)}(p, p_0) \le (\varepsilon Q)^{-1}\right\},$$

rescaled by a factor Q, is ε-close to a suitable space-time subset of a κ-solution.

The power of the above result lies in the fact that the curvature $R(p_0, t_0)$ is only required to be larger than some given threshold, but we do not need to assume, for instance, that $R(p_0, t_0)$ is the maximum of R at time t_0 or is comparable with the maximum. Thus we have no apriori bound on the curvature in the parabolic neighborhood under consideration, which would easily yield a proof by compactness.

Combining Theorems 3.26 and 3.27 one obtains a precise description of the possible structure of the solution near the points with large curvature. We introduce a further terminology. We say that a parabolic neighborhood inside a solution of the Ricci flow is an *ε-cap* if each point lies at the center of an ε-neck outside of a compact set satisfying properties (i) and (ii) of Theorem 3.26. Then we have the following result.

Theorem 3.28 *Let $(\mathscr{M}, g(t))$ be a three dimensional solution of the Ricci flow. Then, for any $\varepsilon > 0$ there exists $r_0 > 0$, only depending on ε and the initial data, such that each point (p_0, t_0) with $R(p_0, t_0) \geq r_0^{-2}$ has a parabolic neighborhood $\mathscr{P}$ satisfying one of the following properties*

(i) $\mathscr{P}$ is an ε-neck.
(ii) $\mathscr{P}$ is an ε-cap.
(iii) $\mathscr{P}$ has positive curvature and coincides with the whole manifold.

A neighborhood satisfying one of the properties (i)–(iii) above is called a *canonical neighborhood*. The above result allows to define surgeries at the first singular time, as outlined in Sect. 3.7. After restarting the flow, in order to do the following surgeries, one must then repeat the previous analysis at the subsequent singularities. This is a highly nontrivial part of the procedure, because the estimates needed in this study are derived in the case of a smooth solution and one should justify them also in the case of a flow which has been modified by surgeries. Sections 4 and 5 of [50] are devoted to this delicate issue. The rest of [50] then studies the long time behavior of the flow with surgeries, leading to the proof of the Thurston conjecture. Some important aspects of this part are described in the notes of M. Boileau in this volume.

Let us finally mention that a surgery procedure has also been defined for the mean curvature flow of suitable classes of hypersurfaces. More precisely, in [42] a flow with surgeries was constructed for hypersurfaces in $\mathbb{R}^{n+1}$, with $n \geq 3$, which are 2-convex, i.e. the sum of the two smallest principal curvatures is positive everywhere. The procedure of [42] is inspired by Hamilton's original approach [35] and the main part consists of the analysis of the singularities, which yields a similar picture to the one described here for the three-dimensional Ricci flow. The flow with surgeries allows to prove that any closed 2-convex hypersurface is diffeomorphic to a sphere or to a connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$. More recently, the procedure was extended to the case of surfaces in $\mathbb{R}^3$ with positive mean curvature [9]. An alternative derivation of these results, which uses techniques closer to Perelman's ones for the Ricci flow, has been given in [37, 38].

3.13 Bibliographical Notes

A landmark date in the literature on the Ricci flow are the years 2002–2003 when Perelman's papers appeared. While now many expository books and survey articles can be found, before that date the only references in the field were the original papers, especially Hamilton's ones. A notable exception was the survey [14] which gave a nice overview of Hamilton's program for geometrization and its progress at that time. Despite the vast literature appeared in the last decade, Hamilton's original works still represent a fundamental reference in the field. In particular, the long paper [33], which gives a survey of the previous results and presents many original ones, can be recommended to anyone interested in the Ricci flow for the richness of ideas and the beauty of the exposition. A useful reference for the results on the Ricci flow before 2002 is the volume [16], which collects all the most relevant papers appeared until that time.

Perelman's papers [49–51] are famous not only for the historical relevance of the results, but also for the difficulty of their mathematical content. Even the experts in the field have required a long time of careful analysis before working out all the details of the arguments. After some time, three different detailed expositions of Perelman's papers have appeared, namely the notes by Kleiner and Lott [44], the paper by Cao and Zhu [15], and the book by Morgan and Tian [46]. In particular, the notes [44] have had a great influence on the understanding of Perelman's papers, because preliminary versions were posted on the web while the work was in progress. The references [15, 46] are self-contained, while the notes [44] are meant as a complement to the Perelman's papers, to be read along with them. All these three references are a valuable source for Perelman's results, and anyone who wants to learn these topics in detail is strongly advised to look at least at one of them. Nevertheless, interested readers should also absolutely read the original papers by Perelman; although the proofs are in most of the cases very difficult to understand, the main ideas are often easy to follow and the beauty of the results is stunning even without following all the details.

While the three above mentioned references are mainly focussed on the proof of Poincaré conjecture, the two later books [6] and [47] treat in more detail the long time behavior of the flow with surgery and the proof of the full Thurston conjecture. We also mention the interesting commentary on Perelman's proof by T. Tao [54].

In addition, there are other books on the Ricci flow which are not aimed at a presentation of the full Hamilton-Perelman's theory, but which are excellent references for the basic results as well as some parts of that theory. A particularly rich and detailed source is provided by the series of books by B. Chow and many coauthors [17–22]. On the other opposite, the books by S. Brendle [8], by P. Topping [55] give short and clear expositions of some basic important aspects of the theory and are definitely recommendable for a beginner. The book by B. Andrews and C. Hopper [3] is another interesting reference for the various developments of the Ricci flow.

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