# Structural Probabilistic Modeling of Fatigue Fracture for Piezoceramic Materials Under Cyclic Loading

D. Babich, O. Bezverkhyi and T. Dorodnykh

**Abstract** The aim of this paper is to develop a structural approach for the construction of statistical criterion of static and fatigue failure for the transversely isotropic piezoelectric materials. We use a probabilistic model of the mechanism of brittle microfracture. The microdamageability is considered as a process of appearance of flat elliptic or circular microcracks randomly dispersed over volume, the concentration of which increases with a load. Daniel's structural model of accumulation of microcracks is used for progressive microdamageability. Statistical criterion is convenient to use in the study of fatigue failure under cyclic loading. The reason for its applicability in such problems is experimentally established connection of fatigue failure mechanism with the phenomenon of accumulation of microdamages in the material. Statistical criterion relates macrodestruction beginning with a certain critical value of microcracks density. The model consists of derivation of constitutive equations for a damaged material, choosing the fracture criterion and the law of microdamage distribution; and determining effective electroelastic properties of the damaged medium and the model of accumulation of microdamages by the modified Eshelby method. The approach proposed makes it possible to find the residual ultimate strength of the material after $n$-fold loading and the conditional fatigue limit for the prescribed testing base $N$.

## 1 Introduction

The necessity of studying the processes of static and dynamic deformation of piezoceramic bodies is determined by continuously expanding range of application of piezoceramic materials. In frame of the mathematical theory of deforming of the piezoceramic such materials are treated as brittle and their fracture occurs at low strain level. A large body of studies reviewed in [1–5] shows that fatigue failure of

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materials is a complex multiple-stage process which includes dispersed microfailure of structural elements. This is attributed to the fact that engineering materials contain randomly scattered over a volume microdefects, which under cyclic loading initiate microcracks. Later these microdefects coalescence lead to formation of macrocracks and loss of the body integrity. Moreover, in accordance with the ideas of the mechanics of deformable solids, the main reason of fatigue failure of structural members under cyclic loading is accumulation of microdefects to the point where their concentration becomes critical due to increase in values of true stresses as a result of decrease in the effective area of a cross section with the cycles of loading. Because of this the inherent random nature of fatigue failure requires a probabilistic treatment to evaluate the life of structural components using the models describing process of simultaneous elastic deformation and dispersed fracture of materials [6, 7]. For example, some life assessment approaches based on the continuum mechanics and fracture mechanics models are outlined in [8, 9].

In the present paper, the new probabilistic structural approach for determining the service life of piezoelectric materials under multiple static or cyclic loading based on the microdamageability model [10] is proposed. In implementing this approach, the statistical fatigue failure criterion expressed in terms of damage measures (microcrack density) is employed in combination with the approximate model of microcrack accumulation under repeated loading. The criterion is identified with the statistical fracture criterion [2, 11]. The statistical nature of such criterion is attributed to the probabilistic character of microfailures in a microinhomogeneous material. The main point of the statistical criterion lies in the fact that the value of microdefect concentration, which origin under the loading kind being considered, is identified with the critical value of microdefect concentration that initiates the start of macrofailure (formation of a macrocrack) independently of the stress-state mode. It is assumed that the microdefect concentration under reversed cyclical loading increases only during the tensile half cycle when the internal stresses increase to amplitude value. At separation-like mode of microfailure, as distinct from shear-like one, the effective area of the load-bearing cross section in compression does not change due to the fact that the planes of the arising cracks are collinear with the direction in which compression acts.

We consider a mechanical failure of material and at this stage of investigation of the problem it is not essential whether such failures are caused by the mechanical, electrical, or electromechanical loading. The general procedure of the approach includes following stages. In the first phase, we derive constitutive equations for a damaged material, choose the fracture criterion and low of microdamage distribution. Such material is simulated by a solid with reduced electroelastic characteristics. In this case the type of elastic symmetry of medium being simulated depends on the pattern of microdamage distribution over the body volume as well as on the stress–strain state volume of a material. At the second phase, the method for determining effective electroelastic properties of the damaged medium and the model of accumulation of microdamages are employed. We assume that during deformation, cracks do not grow, do not interact. The volume density (concentration) of microdefects varies with increase in the level of average stresses due to features of orientation of anisotropic materials. Destruction of the structural elements occurs at different levels of stress due to random nature of the orientation and differences of the values of ultimate strength of the structural elements in different directions.

## 2 Structural Model of Accumulation of Flat Microcracks in the Elastic–Brittle Material

To describe the phenomenon of fatigue failure of materials, we use the structural model of material microdamageability. The microdamageability is considered as the process of occurence of the flat scattered microcracks. According to this model, the size and distribution of microcracks in real bodies are associated with discontinuities of structural elements. The shape and dimensions of the cracks are identified with them for ruptures in the cross sections of the structural elements of the material. To describe progressive accumulation of microdamages, the Daniels structural model is used. The main point of this model is outlined in detail in references [11, 12].

With respect to transversally isotropic material, which is simulating prepolarized piezoceramic, the Mises–Hill strength criterion can be used. Let the average stresses $\sigma_{ij}\ (i,j=1,2,3)$ be given in the laboratory (fixed) coordinate system $0x_1x_2x_3$, associated with a representative volume of the material, then this criterion can be written as

$$\frac{\sigma_{11}^2}{\sigma_{(bi)11}^2}+\frac{\sigma_{22}^2}{\sigma_{(bi)11}^2}+\frac{\sigma_{33}^2}{\sigma_{(bi)33}^2}+\frac{\sigma_{12}^2}{\sigma_{(bi)12}^2}+\frac{1}{\sigma_{(bi)13}^2}\left(\sigma_{13}^2+\sigma_{23}^2\right)-\left(\frac{2}{\sigma_{(bi)11}^2}-\frac{1}{\sigma_{(bi)33}^2}\right)\sigma_{11}\sigma_{22}$$
$$-\frac{1}{\sigma_{(bi)33}^2}\left(\sigma_{22}\sigma_{33}+\sigma_{11}\sigma_{33}\right)=1. \tag{1}$$

The main axes of symmetry of mechanical properties are directed along coordinate axes ($0x_3$—polarization axis, axes $0x_1, 0x_2$ lie in isotropic plane). According to this criterion, to determine fracture start, it is necessary to know the four constants. These constants characterize fracture under pure tension $(i=1)$ or pure compression $(i=2)$ in main direction of anisotropy $\left(\sigma_{(bi)11}=\sigma_{(bi)22},\ \sigma_{(bi)33}\right)$ and pure shear in main planes $\left(\sigma_{(bi)12},\ \sigma_{(bi)13}\right)$.

For the considered material the tensile strength (compression) and pure shear dependents on the direction determined by angle $\vartheta$—angle of rotation of coordinate system $0x_1x_2x_3$ relative to the axis $0x_2$ or axis $0x_1$. The formulas for the tensile strength (compression) $\sigma_{(bi)\vartheta}$ in a direction determined by the angle $\vartheta$, measured from the $0x_3$ axis, can be written as

$$\sigma'_{(bi)\vartheta} = \sigma'_{(bi)33} = \frac{\sigma_{(bi)33}}{\sqrt{\cos^2\vartheta + \sin^2\vartheta\,\frac{\sigma^2_{(bi)33}}{\sigma^2_{(bi)11}} + \left(\frac{\sigma^2_{(bi)33}}{\sigma^2_{(bi)13}} - 1\right)\sin^2\vartheta\cos^2\vartheta}}$$
$$= \frac{\sigma_{(bi)33}}{\sqrt{\cos^4\vartheta + \sin^2\vartheta\,\frac{\sigma^2_{(bi)33}}{\sigma^2_{(bi)11}} + \frac{\sigma^2_{(bi)33}}{\sigma^2_{(bi)13}}\sin^2\vartheta\cos^2\vartheta}}. \tag{2}$$

Let $0'x'_1x'_2x'_3$ be the local coordinate systems chosen in such a way that the $0'x'_3$ axes would be directed along the normal to the sphere (unit radius) surface. The elemental area $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\psi$ is singled out around the $0'x'_3$ axis on the surface of the random sphere. This area cuts $N$ structural elements ($\vartheta$ is the longitude; $\psi$ is the latitude). In this case, the same local true stress $\bar{\sigma}_{33}$ acts in the section of the $N$ intersected structural elements. The true stresses $\bar{\sigma}_{33}$ differ from the conditional $\sigma'_{33}$ in that the first ones refer to the areas of the damaged medium whereas the second ones refer to the areas of the continuous medium. The local conditional stresses $\sigma'_{33}$ and average stresses $\sigma_{kl}$ given in the body are connected by equation

$$\sigma'_{33} = \sigma_{kl}\alpha_{3k}\alpha_{3l},$$

where $\alpha_{3k}$, $\alpha_{3l}$ are the direction cosines of the local coordinate system with respect to the laboratory coordinate system. The relation of the first strength theory

$$\bar{\sigma}_{33} \geq \sigma(\vartheta) \tag{3}$$

Here $\sigma(\vartheta)$ is the random value, which stands for the ultimate magnitude of the true tensile or compressive normal stresses $\bar{\sigma}_{33}$ for differently oriented structural elements. When the true tensile stress $\bar{\sigma}_{33}$ reaches up the level of $\sigma(\vartheta)$ in the appropriate elemental area, the microcracks of rupture are formed with side surfaces being normal to the direction axis $0'x'_3$. When the conditional stress is compressive, the microcracks are oriented predominantly in parallel to the direction of $\bar{\sigma}_{33}$ due to the difference of Poisson's ratio of the structural elements. To approximate distributions of the microstrength properties of structural elements, the power law is used

$$F_i(\sigma_i) = \left(\frac{\sigma_i - \sigma_{0i}}{\sigma_{mi} - \sigma_{0i}}\right)^{\alpha_i} \tag{4}$$

and

$$f_i(\sigma_i) = \frac{\mathrm{d}F_i(\sigma_i)}{\mathrm{d}\sigma_i} = \alpha_i\left(\frac{1}{\sigma_{mi} - \sigma_{0i}}\right)\left(\frac{\sigma_i - \sigma_{0i}}{\sigma_{mi} - \sigma_{0i}}\right)^{\alpha_i - 1} \tag{5}$$

is the distribution density of the random value $\sigma_i$.

$\sigma_{0\,i,}, \sigma_{m\,i}, \alpha_i$ are the distribution parameters; $\sigma_{0\,i,}, \sigma_{m\,i}$ are minimal and maximal values of these variables, respectively; $\alpha_i$ is the microstrength scattering parameter. The distribution parameters are determined in sample quantities by the method of moments in particular. For example, it is necessary to determine, using experimental data, two selective moments: average magnitude $\bar{\sigma}_{b1}$ and dispersion of the random value $\bar{D}^2_{b1}$.

$$\bar{\sigma}_{b1} = \int_{\sigma_{01}}^{\sigma_{m11}} \sigma f_1 \mathrm{d}\sigma = \frac{\alpha_1}{1+\alpha_1}\bar{\sigma}_{m1} + \sigma_{01};$$
$$\bar{D}^2_{b1} = \int_{\sigma_{01}}^{\sigma_{m1}} (\sigma - \bar{\sigma}_{b1})^2 f_1 \mathrm{d}\sigma = \frac{\alpha_1}{(\alpha_1+1)^2(\alpha_1+2)}\bar{\sigma}^2_{m1}, \tag{6}$$

here $\bar{\sigma}_{m1} = \sigma_{m1} - \sigma_{01}$. From (6) follows

$$k_1^2 = \frac{\bar{D}^2_{b1}}{(\bar{\sigma}_{b1} - \sigma_{01})^2} = \frac{(\bar{D}_{b1}/\bar{\sigma}_{b1})^2}{(1 - \sigma_{01}/\bar{\sigma}_{b1})^2} = \frac{1}{\alpha_1(\alpha_1+2)};$$
$$\alpha_1 = -1 + \frac{1}{k_1}\sqrt{1+k_1^2}, \bar{\sigma}_{m1} = \frac{1+\alpha_1}{\alpha_1}(\bar{\sigma}_{b1} - \sigma_{01}). \tag{7}$$

Due to the small size of the structural elements it is impossible to determine $\sigma_{0i,}, \sigma_{mi}, \alpha_i$ directly. To find these values indirect methods are used. Experimental data of corresponding conditional parameters of macrostress of rupture is taken from set of macrosamples. The procedure of determining of these parameters is described in more detail in [11].

It should be noted that the element fails when the stress $\bar{\sigma}_{33}$ reaches up to the ultimate magnitude. Failure of single elements forms the population of independent random events. After some quantity of structural elements fail, redistribution of stresses between the nonfailed elements occurs.

If the conditional local tensile stress $\sigma'_{33}$ presents an independent loading parameter, then the true local stress in the sections with nonfailed structural elements can be regarded within the framework of the model being considered as the random value $\bar{\sigma}_{33} = \sigma'_{33}/\left(1 - \frac{n_1}{N}\right)$. The distribution of the true local stress $\bar{\sigma}_{33}$ depends on the number $n_1$ of the failed elements. $N$ is the total number of the elements. The expected value of the number $n_1$ has the form $\langle n_1 \rangle = NF_1(\bar{\sigma}_{33})$, and the coefficient of variation becomes $k_{w1} = \left[\frac{1 - F_1(\bar{\sigma}_{33})}{NF_1(\bar{\sigma}_{33})}\right]^{1/2}$. From the last formula, it follows that for real materials it is possible to neglect the scatter of the values $n_1$ and $\bar{\sigma}_{33}$. As a result, we have

$$\bar{\sigma}_{33} \approx \frac{\sigma'_{33}}{1 - F_1(\bar{\sigma}_{33})} \tag{8}$$

Taking into account (3) and (5), the densities of microcracks of normal rupture under tension or compression are determined by expressions

$$\varepsilon_i = F_i(\bar{\sigma}_{33}) = \left(\frac{\bar{\sigma}_{33} - \sigma_{0i}}{\sigma_{mi} - \sigma_{0i}}\right)^{\alpha_i} (i = 1, 2). \tag{9}$$

In the case of local true tensile stresses $\bar{\sigma}_{33}$, we have formula (8). In the case of compression $(i = 2)$, the cracks origin surfaces are parallel to the direction in which local normal stresses act. In this connection the effective area remains unchanged and, as a result, $\bar{\sigma}_{33} = \sigma'_{33}$.

Thereby, the average densities microcracks of structural elements, which are cut by the unit surface of the representative volume, are defined by the relations

$$\varepsilon_1 = \frac{1}{\bar{N}} \int_0^{2\pi} \int_0^{\pi} F_1(\bar{\sigma}_{33}) \mathrm{d}\Omega = \frac{1}{\bar{N}} \int_0^{2\pi} \int_0^{\pi} F_1(\bar{\sigma}_{33}) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\psi \tag{10}$$

in case of the stresses $\sigma_{ij}$ are tensile, and

$$\varepsilon_2 = \frac{1}{\bar{N}} \int_0^{2\pi} \int_0^{\pi} F_2(\sigma'_{33}) \mathrm{d}\Omega = \frac{1}{\bar{N}} \int_0^{2\pi} \int_0^{\pi} F_2(\sigma'_{33}) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\psi \tag{11}$$

in case of the stresses $\sigma_{ij}$ are compressive.

$\bar{N} = 4\pi$ is the normalizing factor, which follows from the condition

$$\frac{1}{\bar{N}} \int_0^{2\pi} \int_0^{\pi} F_i(\bar{\sigma}_{33}) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\psi = 1.$$

The physical meaning of the values $\varepsilon_i$ $(i = 1, 2)$ is that it represents the relative fraction of the unit area of the sphere surface where the normal stresses (tensile or compressive) exceed the ultimate strength $\sigma_i$ of the material of the microparticles that are cut by the surface of this sphere. The volume concentration of flat microdefects which are destroyed under tension or compression is determined by the ratio of the number of destructed microparticles $N_{0i}$ to their total number $N$ $(p_i = N_{0i}/N)$ in the representative volume. Such a result can be obtained using the technique that is common in petrography in analyzing the thin sections of sediments, so $p_i = \varepsilon_i$.

## 3 Statistical Fracture Criterion in Terms of Damage Measures of a Material

Relations (1)–(5) and (8) make it possible to determine the microcrack density allowing for loading rate and their orientation, which depends on the direction of the local conditional stresses $\sigma'_{33}(\vartheta,\psi)$ that cause microcracking. Of especial importance is allowed for the orientation in the case of complex stress state since macrocracks arise mostly in the planes normal (parallel) to the direction in which the maximum tensile (compressive) local stresses $\sigma'_{33_{\max}}(\vartheta_m,\psi_m)$ act.

For two-parametric approximation of the ultimate microstrength distribution, the microcrack concentration in the random volume of transversally isotropic piezoelectric material is defined by the formula

$$\varepsilon_i = F_i(\bar{\sigma}_{33}) = \left(\frac{\bar{\sigma}_{33}}{\sigma_{mi}}\right)^{\alpha_i} \quad (i=1,2), \tag{12}$$

where under tensile stress ($\sigma'_{33} > 0$) and under compression ($\sigma'_{33} < 0$) for local stress $\bar{\sigma}_{33}$ there are formulas

$$\bar{\sigma}_{33} \approx \frac{\sigma'_{33}}{1-F_1(\bar{\sigma}_{33})}, \quad \bar{\sigma}_{33} = \sigma'_{33} \tag{13}$$

The statistical fracture criterion expressed in terms of damage measures of a material is defined by the relation

$$F_i(\bar{\sigma}_{33_{\max}}) \le \varepsilon_{icr} \quad (i=1,2), \tag{14}$$

where $F_i(\bar{\sigma}_{33_{\max}}) = \varepsilon_{im}$ is the concentration of cracks in the cross section in which the normal local conditional stress reaches up to the maximum value, and $\varepsilon_{icr}$ is the critical value of the concentration of cracks.

It should be noted that the accumulation of microcracks in the material depends on the specific loading of the body (the multiplicity, the loading rate, and others.). We suppose that before the deformation in material was the initial microdamage with density $\varepsilon_{i0}$. The distribution function of the ultimate strength of the structural elements (12) in this case determines the relative proportion of structural elements not destroyed in remaining cross-sectional area of the body. The relative area of undefeated structural elements is $(1-\varepsilon_{i0})$, and the tensile strength in this area is equal to or less than a certain value $\sigma$. Then, under monotonic (static) loading, when stresses increase up to the value $\sigma'_{33}$ the microcrack concentration is defined as follows:

$$\varepsilon_i = \varepsilon_{i0} + (1 - \varepsilon_i)F_i(\bar{\sigma}_{33}) = \varepsilon_{i0} + (1 - \varepsilon_i)\left(\frac{\bar{\sigma}_{33}}{\sigma_i}\right)^{\alpha_i} \qquad (i = 1, 2) \tag{15}$$

Critical concentrations of microcracks when $\varepsilon_{i0} = 0$ are determined by the relation

$$\varepsilon_{1cr}(1 - \varepsilon_{1cr})^{\alpha_1 - 1} = \left(\frac{\sigma'_{(b1)33}}{\sigma_{m1}}\right)^{\alpha_1}, \ \varepsilon_{2cr} = (1 - \varepsilon_{2cr})\left(\frac{\sigma'_{(b2)33}}{\sigma_{m2}}\right)^{\alpha_2}. \tag{16}$$

Here, $\sigma_{(bi)33} = \sigma_{(bi)\,\vartheta}$ $(i = 1, 2)$ are the average values of the ultimate strength, which are calculated by the formula (2) under tension and compression, respectively. Samples of material are cut at an angle $\vartheta$ to the direction of the principal axis of anisotropy, which coincides with the axis of the prepolarization.

In the case of complex stress state determined by main stresses $\sigma_{ii}$ $(i = 1, 2, 3)$ in the laboratory coordinate system, the strength of statistical criterion for transversely isotropic body can be constructed on the basis of Mises–Hill strength criterion. For this purpose, the expression of Mises–Hill strength criterion (1) is represented in the main stresses

$$\frac{\sigma_{11}^2}{\sigma_{(bi)11}^2} + \frac{\sigma_{22}^2}{\sigma_{(bi)11}^2} + \frac{\sigma_{33}^2}{\sigma_{(bi)33}^2} + \left(\frac{2}{\sigma_{(bi)11}^2} - \frac{1}{\sigma_{(bi)33}^2}\right)\sigma_{11}\sigma_{22} - \frac{1}{\sigma_{(bi)33}^2}(\sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}) = 1. \tag{17}$$

Stresses $\sigma_{kk}$, $\sigma_{(bi)kk}$ $(k = 1, 2, 3)$ in (17) can be expressed by the corresponding densities of microcracks $\varepsilon_{(i)k}$. Using (12), we get following formulas:

$$\begin{aligned}
&\varepsilon_{(i)k} = F_i(\bar{\sigma}_{kk}) = \left(\frac{\bar{\sigma}_{kk}}{\sigma_{(mi)k}}\right)^{\alpha_{(i)k}} (k = 1, 2, 3), \varepsilon_{(bi)k} = F_i(\bar{\sigma}_{(bi)kk}) = \left(\frac{\bar{\sigma}_{(bi)kk}}{\sigma_{(mi)k}}\right)^{\alpha_{(i)k}};\\
&H_k = (\varepsilon_{(i)k} - \varepsilon_{(i)k0})^{\frac{1}{\alpha_{(i)k}}}(1 - \varepsilon_{(i)k})^{1 - \frac{1}{\alpha_{(i)k}}};\\
&G_k = (\varepsilon_{(bi)k})^{\frac{1}{\alpha_{(i)k}}}(1 - \varepsilon_{(bi)k})^{1 - \frac{1}{\alpha_{(i)k}}};\\
&\sigma_{kk} = H_k\sigma_{(mi)k}; \quad \sigma_{(bi)kk} = G_k\sigma_{(mi)k}; \quad \left(\frac{\sigma_{kk}}{\sigma_{(bi)kk}}\right)^2 = \left(\frac{H_k}{G_k}\right)^2,
\end{aligned} \tag{18}$$

Thereby, formula (17) can be written in concentrations of microcracks

$$\begin{aligned}
&\sum_{k=1}^{3}\left[\frac{H_k}{G_k}\right]^2 + \left[\frac{2}{G_1^2(\sigma_{(mi)1})^2} - \frac{1}{G_3^2(\sigma_{(mi)3})^2}\right] \times H_1H_2(\sigma_{(mi)1})^2\\
&\qquad - \frac{H_3}{G_3^2(\sigma_{(mi)3})}\sum_{k=1}^{2}\sigma_{(mi)k}H_k = 1.
\end{aligned} \tag{19}$$

In (18) and (19) it is indicated by the index $i$ tension ($i = 1$) or compression ($i = 2$), the index $k$ is associated with the symbols of the principal axis of the anisotropy of the material.

## 4 Constitutive Equations of State for the Piezoelectric Ceramics with Cracks

Polarized piezoceramic is modeled as a transversely isotropic medium with the axis of isotropy coincident with the axis of polarization. In the laboratory coordinate system $0x_1x_2x_3$, ($0x_3$ is axis of polarization) state equations have the form

$$\begin{aligned}
&\varepsilon_{11} = a_{1111}\sigma_{11} + a_{1122}\sigma_{22} + a_{1133}\sigma_{33} + d_{113}E_3,\\
&\varepsilon_{22} = a_{1122}\sigma_{11} + a_{1111}\sigma_{22} + a_{1133}\sigma_{33} + d_{113}E_3,\\
&\varepsilon_{33} = a_{1133}\sigma_{11} + a_{1133}\sigma_{22} + a_{3333}\sigma_{33} + d_{333}E_3,\\
&\varepsilon_{23} = a_{2323}\sigma_{23} + d_{233}E_2,\ \varepsilon_{13} = a_{2323}\sigma_{13} + d_{233}E_1,\\
&\varepsilon_{23} = a_{1212}\sigma_{12} = 2\left(a_{1111} - a_{1122}\right)\sigma_{12,}\\
&D_1 = \ \text{э}_{11}\ E_1 + d_{232}\sigma_{13},\ \ D_2 = \ \text{э}_{11}\ E_2 + d_{232}\sigma_{23},\\
&D_3 = \ \text{э}_{33}\ E_3 + d_{113}(\sigma_{11} + \sigma_{22}) + d_{333}E_3.
\end{aligned} \tag{20}$$

Suppose that underloading in piezoelectric materials causes microdamages in the form of flat circular cracks. This type of microfracture, most unfavorable to the material because of the degree of influence of microcracks on the stiffness of the material is mainly related to the area and volume of cracks opening.

To determine the effective electroelastic constants in (20), the principle of the energy equivalence is used:

$$W = W^{(0)} + \bar{W} \tag{21}$$

Here,

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + E_jD_j \ = \ \frac{1}{2}a^{E}_{ijkl}\,\sigma_{ij}\sigma_{kl} + \frac{1}{2}\,E_i\left(d_{ikl}\sigma_{kl} + \ \text{э}^{\sigma}_{ik}\ E_k\right) \tag{22}$$

is the energy density of deformation of the continuous electroelastic medium [7] that simulates the damaged material;

$$W^{(0)} = \frac{1}{2}a^{E(0)}_{ijkl}\ \sigma_{ij}\sigma_{kl}\ \ + \frac{1}{2}E_i\left(d^{(0)}_{ikl}\sigma_{kl} + \ \text{э}^{\sigma(0)}_{ik}\ E_k\right) \tag{23}$$

is the density of the deformation energy of a solid medium; subscripts with $E$, $\sigma$ in (23) indicates the dependence of these parameters on the electric $(E)$ field and mechanical stress $(\sigma)$; $\bar{W}$ is the density of the released internal energy of the damaged medium, which can be represented as the change in mechanical and electrical energy. These changes in mechanical and electrical energy are associated with the damage of the material in the form of closed or opened flat cracks.

The effective electroelastic constants in (10) are determined from expression (11). For this purpose the terms entering in (11) should be written in terms of the components of the stress tensor $\sigma_{ij}$ and components of the electric field vector. It is assumed that $E_i$ and $\sigma_{ij}E_i$ are given in a representative volume. The coefficients of the terms $\sigma_{ij}^2$, $\sigma_{ij}E_i$, $E_iE_j$ should be equated. It makes it possible to determine the effective compliances $a_{ijkl}$, piezoelectric coefficients $d_{ikl}$, and dielectric constants $\ni_{ik}$ of fractured materials by

$$a_{ijkl} = a_{ijkl}^{(0)} + \bar{a}_{ijkl}\,, \quad d_{ijk} = d_{ijk}^{(0)} + \bar{d}_{ijk}, \\ \ni_{ij} = \ni_{ij}^{(0)} + \bar{\ni}_{ij}\ (i, j,\ k,\ l = 1,\ 2,\ 3)\,, \tag{24}$$

where the values $\bar{a}_{ijkl}$, $\bar{d}_{ijk}$, $\bar{\ni}_{ij}$ are the changes in the electroelastic parameters of a continuous medium, due to the disruption of the continuity of the material.

For purely elastic materials, the density of the released elastic energy is determined on the basis of the Eshelby principle [13]. With regard to inhomogeneous electroelastic materials, the Eshelby principle is modified due to the need to take into account the electric component in the overall energy balance of the body. For this purpose, a local criterion of microfracture for electroelastic materials is used [14]. Due to the disruption of connections of the $n$th crack under rupture and opening (shear) of crack faces, the internal elastic energy is released and electric energy is loosed. The density of the released energy can be represented as the work of relative sliding and opening of crack faces induced by the action of stresses, which may arise under the given loading in the microvolumes of a continuous free of crack medium, and is determined as

$$\bar{W}^n = \frac{1}{2}\int\limits_{s_n} \sum_{i=1}^{3} \left(\sigma_{i3}^{(0)n}\bar{u}_i^n + D_i^{(0)n}\Phi_i^n\right) \mathrm{d}s_n\,, \tag{25}$$

where $\bar{u}_i^n\ (i = 1, 2, 3)$ are the discontinuities of displacements at points of the surface of the $n$th crack in the direction of the local coordinate system; $s_n$ is the half of surface area of the $n$th crack; $\sigma_{i3}^{(0)n}\ (i = 1, 2, 3)$ are the components of the tensor of the given average stress in $n$th cracks coordinate system—$0^n x_1^n x_2^n x_3^n$. In the case of elliptic cracks, the $0x_1^n$, $0x_2^n$ axes are directed along major ($a^n$) and minor ($b^n$) semi-axes, respectively, while the $0^n x_3^n$-axis directed along the normal to their planes; $D_i^{(0)n}$, are the components of the electric induction in a solid medium in $n$th cracks coordinate system, $\Phi_i^n\ (i = 1, 2, 3)$ are the discontinuities of the electric potential at the points of microcracks surfaces, which are directed along the axes in $n$th cracks coordinate system. With the use of (24) the expression for the change of the density energy of deformation due to the formation of elliptical or circular microcracks in the inhomogeneous transversely isotropic material is determined in accordance with the procedure for isotropic materials [14]. In particular, the density of the released energy under tensile could be expressed in the form

$$\bar{W} = \frac{1}{12\pi} \sum_{k=1}^{3} \int_0^{2\pi} \int_0^{\pi} F_1(\bar{\sigma}_{33})\, B_k^{'} \sin\vartheta \, d\vartheta \, d\psi. \tag{26}$$

In case of the compression, we have

$$\bar{W} = \frac{1}{12\pi} \sum_{k=1}^{2} \int_0^{2\pi} \int_0^{\pi} F_2\left(\sigma_{33}^{'}\right) B_k^{'} \sin\vartheta \, d\vartheta \, d\psi. \tag{27}$$

$B_k^{'}$ is determined by expression

$$B_k^{'}(\theta,\psi) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} S_k^{'}\left(\sigma_{k3}^{'}\right)^2 - \frac{1}{2} E_k^{'}\left(d_{kil}\sigma_{il}^{'} + Э_{ki}\right) E_{\ i}^{'}\right] d\phi \quad k = 1,\, 2,\, 3. \tag{28}$$

$S_k^{/}\left(a_{klmn}^{(0)}, \theta^n, \psi^n, \phi^n\right)$ in (28) denotes compliances of the material in the $n$th cracks plane. Definition of which is an independent task for individual crack.

Engineering elastic constants are expressed in terms of the effective compliances by relations

$$\frac{1}{E_{ii}} = a_{iiii},\ -\frac{\nu_{ij}}{E_{ii}} = a_{jjii},\ G_{ij} = \frac{1}{a_{ijij}} \quad i,j = 1,2,3. \tag{29}$$

$E_{ii},\ G_{ij},\ \nu_{ij}$ are elastic, shift moduli, Poisson's ratios accordingly.

A two-parameter distribution function of the ultimate strength of the structural elements of the material is used to determine the effective electroelastic parameters. Additionally, we rely on the continuum model of piezoelectric ceramics with progressive accumulation of damageability in the form of circular microcracks in the isotropic plane [15]. Material is subjected to uniaxial tension stresses $\sigma_{33}$ in the direction of polarization. And the component of electrical field $E_3^0$ is given. Under these assumptions, electroelastic effective parameters are determined by the expression of type (24)

$$\begin{gathered} a_{3333} = \frac{1}{E_{33}^0} + p\left[\frac{4}{\pi}\ \frac{1-\left(\nu^{'}\right)^2}{E_{11}^0}\right], \\ Э_{33} = (1-p)\ Э_{33}^{(0)},\ \ d_{333} = (1-p)\ d_{333}^{(0)}. \end{gathered} \tag{30}$$

## 5 Application of Statistical Fracture Criterion in Problems of Durability Piezoceramic Structures Submission of Electronic Version of Papers

When considering the cyclic alternating load it should be noted the difference in the mechanism of microfracture of material under the same multiple compression and multiple tensile stresses. In the first case, the concentration of microdefects in subsequent compressions (excluding first) is not changed, in the second, it increases due to the decrease in the effective cross-sectional area.

To illustrate the approach for the determination of the durability of structures such as piezoelectric transducers electromechanical power using statistical fracture criterion, we consider the problem of the durability of piezoceramic rods during longitudinal vibrations excited by the time-variable ($t$) difference of potential exerted on the end faces of the rod in the form

$$\psi_{x_3=0} = \psi_{x_3=l} = \pm V_0 e^{i\omega t}. \tag{31}$$

For solving the problem of the durability of the rods it is necessary and sufficient to have the dates of the maximum values of the axial normal stress under given parameters of external loading, as well as the critical values of concentration of microcracks under pure tensile samples of the concrete material. Procedure for determining the durability of material under more general types of electromechanical loading remains the same for the resource problems with more complex structures. The definition of parameters in (5) is independent task in each case.

Thus, the first step in solving the question of the durability of structures is the solution of the problem of stress–strain state of the structure under specific operational impacts. The problem of the longitudinal vibrations piezoceramic prismatic rod with length $l$ and the axial polarization was considered in [6]. Vibrations excited by the variable potential difference were applied to the electrodes of the end of rod. External stresses on the entire surfaces of the rod are absent. Equation (20) for this case in coordinate system $0x_1x_2x_3$ have the form

$$\begin{gathered} \varepsilon_{ij} = a_{ijkl}\,\sigma_{kl} + d_{ij\,k}E_k \quad (\, i, j,\ k,\ l = 1,\ 2,\ 3); \\ D_i = d_{ijk}\sigma_{jk} + \,\mathfrak{z}_{ij}\,E_j \quad (\, i, j,\ k,\ l = 1,\ 2,\ 3), \\ \varepsilon_{33} = a^E_{3333}\sigma_{33} + d_{333}E_3;\ \ D_3 = \,\mathfrak{z}^\sigma_{33}\,E_3 + d_{333}\sigma_{33}. \end{gathered} \tag{32}$$

The problem on longitudinal vibrations of the rod is reduced to solving of the equation for axial displacements $u(x,\ t)$

$$\begin{gathered} u_{3,33} + d_{333}\psi,_{33} = \rho a^E_{3333} u_{3,tt}; \\ \frac{d_{333}}{a^E_{3333}} u_{3,33} - \,\mathfrak{z}^\sigma_{33}\,\left(1 - k^2_{333}\right)\psi,_{33} = 0, \end{gathered} \tag{33}$$

where $k_{333}^2 = d_{333}^2 / a_{3333}^E \ni_{33}^{\sigma}$ is longitudinal static electromechanical coupling factor, $\rho$ is the density of material.

Solving this task the amplitude value of axial stress in the rod is received in the form

$$\bar{\sigma}_{33} = \frac{A\lambda}{a_{3333}^E\left(1-k_{333}^2\right)}\cos\lambda\left(\frac{l}{2}-x_3\right) + \frac{d_{333}}{a_{3333}^E}B, \tag{34}$$

where

$$\begin{aligned} &A = -\frac{d_{333}\left(1-k_{333}^2\right)}{\frac{\lambda l}{2}\cos\frac{\lambda l}{2} - k_{333}^2\sin\frac{\lambda l}{2}}V_0;\ B = \frac{\frac{\lambda l}{2}\cos\frac{\lambda l}{2}}{\frac{\lambda l}{2}\cos\frac{\lambda l}{2} - k_{333}^2\sin\frac{\lambda l}{2}}\left(\frac{2V_0}{l}\right);\\ &\lambda = \omega/c,\ c = 1/\sqrt{\rho a_{3333}^E\left(1-k_{333}^2\right)}. \end{aligned} \tag{35}$$

$\omega$—is frequency.

According to (34) and (35) the maximum amplitude value of stress is in the middle of the rod

$$\bar{\sigma}_{33} = \frac{A\lambda}{a_{3333}^E\left(1-k_{333}^2\right)} + \frac{d_{333}}{a_{3333}^E}B = \frac{d_{333}\omega\left(\cos\frac{\omega l}{2c}-1\right)}{c a_{3333}^E\left(\frac{\omega l}{2c}\cos\frac{\omega l}{2c} - k_{333}^2\sin\frac{\omega l}{2c}\right)}V_0. \tag{36}$$

Let $\varepsilon_{1(0)} = 0$ and material is subjected to uniaxial cyclic tensile stress with amplitude value $\bar{\sigma}_{33}$. The first ($n = 1$) tensile half cycle of the undamaged rod leads to origin of the damage with the density

$$\varepsilon_{1(1)} = \left(1-\varepsilon_{1(1)}\right)^{1-\alpha_1}\left(\frac{\bar{\sigma}_{33(0)}}{\sigma_{m1}}\right)^{\alpha_1} \tag{37}$$

The following $n$ cycles of tensile cause breaking of structural elements in the cross section of the sample whose density is determined by

$$\varepsilon_{1(n)} = \varepsilon_{1(n-1)} + \left(1-\varepsilon_{1(n)}\right)^{1-\alpha_1}\left(\frac{\bar{\sigma}_{33(n-1)}}{\sigma_{m1}}\right)^{\alpha_1}, \tag{38}$$

where $\varepsilon_{1(n-1)}$ and $\bar{\sigma}_{33(n-1)}$ are the concentration of microdefects and amplitude value of the stress, respectively, that have appeared after the previous ($n - 1$)th cycle of tensile. The fatigue failure of the specimen begins at the $N$th cycle when the microcrack concentration becomes critical, i.e., with $\varepsilon_{1(N)} = \varepsilon_{1cr}$, where

$$\varepsilon_{1cr} = \varepsilon_{1(N-1)} + \left(1-\varepsilon_{1cr}\right)^{1-\alpha_1}\left(\frac{\bar{\sigma}_{33(N-1)}}{\sigma_{m1}}\right)^{\alpha_1} \tag{39}$$

Thus, a number of cycles $N$ determine the cyclical service life of the specimen, which is found either by solving the sequence of Eq. (38) or using an inverse calculation step based on (39).

Two-sided approximate estimation of the durability of the sample can be obtained by identifying the increment of the concentration of microdefects after any act of loading with minimum and maximum increments, respectively.

$$\begin{gathered}\Delta_{\min} = \varepsilon_{1(2)} - \varepsilon_{1(1)}, \\ \Delta_{\max} = \varepsilon_{1cr} - \varepsilon_{1(N-1)} = \left(1 - \varepsilon_{1cr}\right)^{1-\alpha_1} \left(\frac{\bar{\sigma}_{33(N-1)}}{\sigma_{m1}}\right)^{\alpha_1}.\end{gathered} \tag{40}$$

According this approach, we get durability $N$

$$\varepsilon_{1cr}/\Delta_{\max} < N < \varepsilon_{1cr}/\Delta_{\min} \tag{41}$$

Another approximate determination of the service life $N$ is attributed to calculation by (38) the sequence of the $n$ values of increments of the microcrack density $\Delta_n \varepsilon_1$ for sampling acts of tension along the loading path, which is accompanied by the following averaging. Such approach yields

$$N = \varepsilon_{1cr} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{\bar{\sigma}_{33(i)}}{\sigma_{m1}}\right)^{\alpha_1} \left(1 - \varepsilon_{1(i)}\right)^{1-\alpha_1}\right]^{-1}, \tag{42}$$

where $\varepsilon_{1(i)} = i\frac{\varepsilon_{1cr}}{n}$ is the microdefect concentration within the range $\left[\varepsilon_{1(1)}, \varepsilon_{1cr}\right]$.

The approach proposed makes it possible to find the residual ultimate strength of the material $\sigma_{(b1)oc}$ after n-fold loading and the conditional fatigue limit $\sigma_{1yc}$ for the prescribed testing base $N$. The unknown values are determined by

$$\begin{gathered}\sigma_{(b1)oc} = \sigma_{m1} \left(\varepsilon_{1cr} - \varepsilon_{1(n)}\right)^{1/\alpha_1} \left(1 - \varepsilon_{1cr}\right)^{1-1/\alpha_1}; \\ \sigma_{1y} = \sigma_{m1} \varepsilon_{1cr^{1/\alpha_1}} \left(1 - \varepsilon_{1cr}\right)^{1-1/\alpha_1} / N_0^{1/\alpha_1},\end{gathered} \tag{43}$$

where $\varepsilon_{1(n)}$ is the microdefect concentration caused by the n-fold loadings. In relations (38)–(41) index in brackets show the dependence of the amplitude value of stress $\bar{\sigma}_{33(n-1)}$ on the number of half cycles of tension. Such dependence, according to (36), is associated with the change compliance $a^E_{3333}$, piezoelectric $d_{333}$ and dielectric $\text{э}^{\sigma}_{33}$ constants with increasing concentration of microdefects, which increases with the half cycles of tension.

Half cycles of compression in this model does not affect on the constructions resource at the same compressive loading. However, fatigue failure is possible under compression due to increasing the stress amplitude with increasing the compression cycles without changing the effective area.

## 6 Numerical Example

To illustrate the approach for determining the durability of structures such as piezoceramic transducer of the electromechanical energy using a statistical fracture criterion, the problem of the durability of piezoceramic rods at the longitudinal vibrations is considered. For the piezoelectric ceramic CTBS-3 rod value of cyclical durability $N$ is calculated. Rod has length equal to $l = 0.2\,\mathrm{m}$ and parameters

$$E_{11}^{(0)} = E_{22}^{(0)} = 1.12 \times 10^{11}\,\mathrm{Pa}, \quad E_{33}^{(0)} = 1.19 \times 10^{11}\,\mathrm{Pa}$$

$$\nu_{12}^{(0)} = \nu_{21}^{(0)} = 0.30,\ \rho = 7.10 \times 10^{3}\,\frac{\mathrm{kg}}{\mathrm{m}^3},\ \sigma_{(b3)0} = 0.36 \times 10^{8}\,\mathrm{Pa},\ D_{(b3)0} = 0.14 \times 10^{7}\,\mathrm{Pa},$$

$$k_2 = 0.04,\ \ \text{э}_{33}^{(0)} = 0.21 \times 10^{-7}\,\frac{\Phi}{\mathrm{m}},\ e_{333}^{(0)} = 0.43 \times 10^{2}\,\frac{\mathrm{Kl}}{\mathrm{m}^2},\ d_{333}^{(0)} = 0.36 \times 10^{-9}\,\frac{\mathrm{Kl}}{\mathrm{N}}.$$

Longitudinal vibrations excited by the time-variable difference of potential is exerted on the end faces

$$V_0 = 2 \times 10^4 e^{i\omega t},\ \omega = 2 \times 10^4\,\mathrm{Hz}.$$

The parameters of (38) and (39), determining the concentration of microdefects under cyclic loading, in accordance with (7), have values

$$\alpha_1 = 24.660\,,\ \sigma_{m1} = 0.427 \times 10^{8}\,\mathrm{Pa}.$$

Critical concentrations of microcracks in accordance with (16) is

$$\varepsilon_{1cr} = 0.305 \times 10^{-1}$$

Assessment of the durability of piezoceramic rod under cyclic tension which is caused by potential difference accordance of (40) and (41) gives the result

$$0.859 \times 10^{39} > N > 0.305 \times 10^{37}.$$

Using (42), we obtain more specific result: $N = 0.240 \times 10^{38}$.

As it follows from the fatigue theory, such results are well admissible. It should be noted that the service life of the rod is minimal when the exciting frequency coincides with the main frequency of the natural vibrations of the rod, i.e., under conditions of resonance.

## 7 Conclusions

In the paper, the statistical fracture criterion under static and cyclic loadings has been proposed based on modern ideas about the macrodestruction mechanism of brittle materials. This criterion can be used in the assessment of durability, residual strength for piezoceramic products at electroelasticity loading.

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