Formalization of Bing's Shrinking Method in Geometric Topology

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Abstract. Bing's shrinking method is a key technique for constructing homeomorphisms between topological manifolds in geometric topology. Applications of this method include the generalized Schoenflies theorem, the double suspension theorem for homology spheres, and the 4-dimensional Poincaré conjecture. Homeomorphisms obtained in this method are sometimes counter-intuitive and may even be pathological. This makes Bing's shrinking method a good target of formalization by proof assistants. We report our formalization of this method in Coq/Ssreflect.

Keywords: Formalization · Geometric topology · Bing's shrinking method · Coq · Ssreflect

1 Introduction

The Jordan curve theorem is one of the classical theorems in topology that has been successfully formalized using proof systems (in HOL Light [10], and in Mizar [9]). These formalizations in topology show significant differences between traditional proofs based on geometric/topological intuition and computer checking of those traditional proofs. In fact, trained geometers and topologists can reproduce mathematically rigorous proof steps by reading traditional arguments appealing to geometric intuition, but this reproduction process can frequently be nontrivial to formalize. When it comes to the Schoenflies problem, which is an essential refinement to the Jordan curve theorem, formalization becomes even less trivial.

The Schoenflies problem asks whether the region bounded by a Jordan curve is topologically equal to the disk. In the category of topological spaces, we say two objects are "equal" when there exists a homeomorphism between them. However, finding a homeomorphism is a fairly non-trivial issue even when the two spaces appear similar. In fact, while the original 2 dimensional Schoenflies problem can be affirmatively answered using a fairly nontrivial topological argument or applying Carathéodory's theorem in complex analysis ([1]), the 3 dimensional analogue fails to hold due to the existence of counter-intuitive examples such as Alexander's horned sphere ([2] Fig. 1). This pathological phenomenon is a

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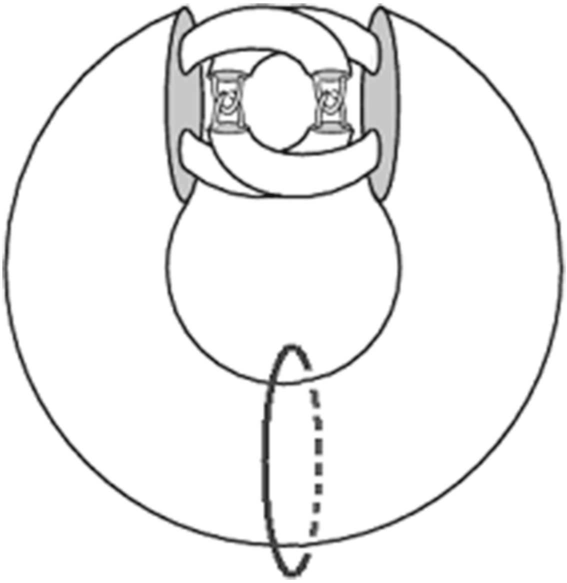

Fig. 1. Alexander's horned sphere [16] The complement of this wild sphere in the 3-dimensional sphere S^3 is not homeomorphic to the standard 3-dimensional ball. This follows from the observation that the linking circle depicted in the picture is not contractible in the complement. However, gluing two copies of the complement along the horned sphere yields the standard 3-dimensional sphere S^3. Bing constructed this counter-intuitive homeomorphism using his shrinking method in 1952 [3].

characteristic aspect of geometric topology in the topological category, and this counter-intuitive aspect makes geometric topology a good target of computer formalization.

To avoid the pathology of Alexander's horned sphere and recover the Scheonflies problem in arbitrary dimensions, a correct assumption to impose on the topological embeddings of the $n-1$ sphere in the n sphere turned out to be "local flatness": An embedding of the $n-1$ dimensional sphere S^{n-1} into the n dimensional sphere S^n, $\phi : S^{n-1} \to S^n$, is *locally flat* when there exists a topological embedding $\overline{\phi} : S^{n-1} \times \mathbb{R} \to S^n$ such that $\overline{\phi}(x, 0) = \phi(x)$ for all $x \in S^{n-1}$.

Thus in 1960 Morton Brown succeeded in proving the following theorem [4]:

Generalized Schoenflies Theorem (GST). Let $\phi : S^{n-1} \to S^n$ be a locally flat topological embedding. Then the closure of each connected component of the complement of $\phi(S^{n-1})$ is homeomorphic to the n dimensional disk D^n.

Here, the n dimensional sphere S^n is most often concretely defined as the unit sphere in the Euclidean $n+1$ space:

$$S^n := \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + {x_{n+1}}^2 = 1\}$$

with subspace topology. Similarly, the n dimensional disk D^n is usually defined concretely as the unit disk in the Euclidean n space:

$$D^n := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$$

However, we may also define D^n to be the cube $D^n := [0,1]^n$, the standard n simplex, or some appropriate compactification of Euclidean n space.

The choice of these concrete definitions of S^n or D^n or any space in topology is mathematically immaterial, but it does change the resulting formalization significantly and makes it less portable. Hence it is a better idea to abstract the essential property of disks and spheres required in the proof of the **GST** to a greater extent in formalization than in traditional mathematics. The essential property of these spaces needed for constructing required homeomorphisms is certain self-shrinkability of disks and spheres. We will briefly explain some abstraction of this self-shrinkability later, but the point here is this property ensures that any surjection $f : D^n \to D^n$ with finitely many nontrivial inverses is "shrinkable" in the following sense.

Definition (Bing Shrinkability). Let $f : X \to Y$ be a continuous mapping between compact metric spaces with metrics d and d' respectively. We say f is called *Bing shrinkable* if for each positive $\epsilon > 0$ there exists a homeomorphism $h_\epsilon : X \to X$ such that for all $y \in Y$, $\mathrm{diam_d} h_\epsilon(f^{-1}(y)) < \epsilon$ and for all $x \in X$, $d'(f(x), f(h_\epsilon(x))) < \epsilon$.

This property is the key property when we wish to show two topological spaces are "equal" because of the following theorem.

Bing Shrinking Theorem (BST). Suppose $f : X \to Y$ is a continuous surjection between compact metric spaces and f is Bing shrinkable. Then f can be approximated by homeomorphisms, that is, for each positive δ there exists a homeomorphism $k_\delta : X \to Y$ such that for all $x \in X$, $d'(f(x), k_\delta(x)) < \delta$. In particular X and Y are homeomorphic.

This shrinking method was first invented and used by R. H. Bing in 1952 when he produced an exotic involutive self-homeomorphism of the three sphere S^3 whose fixed point set is the Alexander horned sphere [3]. In his construction, a certain map $f : S^3 \to Y$ from the three sphere to a potentially non-manifold space Y was approximated by homeomorphisms. Hence it follows that the target space is actually a three sphere.

Other applications of this shrinking construction of non-trivial homeomorphisms include the celebrated double suspension theorem of J. Cannon and R. Edwards, and M. Freedman's proof of the 4-dimensional Poincaré conjecture. Homeomorphisms obtained in this theorem are relatively general and beyond what can be easily understood concerning equality of topological spaces, which makes this theorem an attractive target of formalization.

The purpose of this paper is to report our formalization of this method in Coq/Ssreflect. We note that our formalization is essentially non-constructive. In fact, a characteristic feature of the shrinking construction is the appearance of infinite iterations producing possibly conter-intuitive homeomorphisms.

Thus Bing's Shrinking criterion in its generality requires the Axiom of Choice in the form of the Baire Category Theorem. Our formalization also uses a topology library by Schepler [13] which assumes the classical propositional logic in the standard library of Coq.

2 Formalization

In this section we explain our formalization of the previously stated Bing Shrinking Theorem in Coq/Ssreflect. Our formalization uses the current version of Coq(Coq-8.5) with the Standard Library and its Ssreflect extension contained in Mathcomp-1.6. Our formalization also uses the Topology library in Coq by Daniel Schepler in [12] as our starting point. We also made a small library of topological lemmas, LemmasForBSC.v, and a formalization of the Baire Category Theorem, BaireSpaces.v, needed in our main formalization BingShrinkingCriterion.v. These files with complete code are in [11].

In [12] (or more or less in any topology library in Coq, e.g. [13–15]) topological spaces are formalized according to the axiom for the system of open subsets:

```
Record TopologicalSpace : Type := {
  point_set : Type;
  open : Ensemble point_set → Prop;
  open_family_union : ∀ F : Family point_set,
    (∀ S : Ensemble point_set, In F S → open S) →
    open (FamilyUnion F);
  open_intersection\,2: ∀ U V:Ensemble point_set,
    open U → open V → open (Intersection U V);
  open_full : open Full_set
}.
```

Here we remark on one aspect where our formalization might look different from informal arguments. That is about subspaces of Xt (we reserve X for the underlying type point_set Xt of the topological space Xt). A subset A of Xt: TopologicalSpace is of type

```
Ensemble (point_set Xt) := (point_set Xt) → Prop.
```

There is a natural topology, say At, on A induced from Xt called a relative topology or a subspace topology. Informally, a subset U of A is open in At if and only if $\mathrm{U} = \mathrm{A} \cap \mathrm{V}$ for some open set V of Xt. Formally, however, U is an element of the type

```
Ensemble {x : point_set Xt |  A x}
```

Especially when subsets are nested, $\mathrm{X} \supset \mathrm{A} \supset \mathrm{B}$, identifications of various induced topologies become non-trivial.

Another factor which complicates this situation is that we begin with metric spaces $(\mathrm{X}, \mathrm{d}), (\mathrm{Y}, \mathrm{d}')$ where d and d′ are distance functions $(\mathrm{d} : \mathrm{X} \to \mathrm{X} \to \mathrm{R})$ $(\mathrm{d}' : \mathrm{Y} \to \mathrm{Y} \to \mathrm{R})$ and R is the set of real numbers. These metrics are subject to the standard metric space axioms such as the triangle inequality. These metrics induce a metric topology on X and Y yielding topological spaces Xt and Yt. If A is a subset of X, then d restricts to a metric dA on A. Then the metric space (A, dA) induces a topology on A which is equivalent to the subspace topology from Xt. Again this identification is non-trivial.

2.1 Bing Shrinking Criterion

Let's begin with X Y: Type as underlying point sets and (d: X → X → R) (d': Y → Y→ R) as metrics on them. These metrics define topological spaces Xt and Yt. Then the Bing Shrinkability for compact spaces is formalized as

```
Hypothesis X_compact: compact Xt.
Hypothesis Y_compact: compact Yt.
Definition Bing_shrinkable (f:X→Y): Prop:=
  ∀ eps:R, eps>0 →
    ∃ h : point_set Xt → point_set Xt,
     homeomorphism h ∧
    (∀ x:X, d' (f x) (f (h x)) < eps) ∧
    (∀ x1 x2:X,  (f x1) = (f x2) → d (h x1) (h x2)  < eps).
```

Defining the conclusion as approximability by homeomorphisms, the **BST** is formalized as follows.

```
Definition approximable_by_homeos (f:X→Y): Prop:=
  ∀ eps:R, eps>0 →
    ∃ h:point_set Xt → point_set Yt,
     homeomorphism h ∧
    (∀ x:X, d' (f x) (h x) < eps).

Theorem Bing_Shrinking_Theorem:
 ∀ f: point_set Xt → point_set Yt,
continuous f → surjective f →
 (Bing_shrinkable f → approximable_by_homeos f).
```

We formalized the proof of the Bing Shrinking Theorem following the line of argument of R.D.Edwards' ICM talk in 1978 [5]. Edward's argument is different from Bing's original proof and quite succinctly outlined. This is made possible by going to the function space of continuous functions. This function space becomes a metric space with uniform topology. This formalization is a good example where formal proof becomes considerably longer than the intuitively outlined proof because of various non-trivial identifications of topologies on subspaces of function spaces with metrics, etc., which are usually unnoticed by experts in topology.

We thus consider the function space

```
Let CMap :=
  {f:X→Y | bound (Im Full_set
             (fun x:X⇒ d' (y0 x) (f x)))∧
             @continuous Xt Yt f}.
```

This CMap becomes a complete metric space with a uniform metric. Mathematically the boundedness condition is redundant as it follows immediately when the spaces are compact. However, it is added to make the definability of a uniform metric on CMap obvious.

We then suppose f : CMap satisfies the Bing shrinking criterion: Bing_shrinkable f.

```
set fH : Ensemble (point_set CMapt) :=
  fun gP: CMap ⇒ ∃ hx: point_set Xt → point_set Xt,
            homeomorphism hx ∧
            ∀ x: point_set Xt, (proj1_sig gP) x = f (hx x).

set CfH := closure fH.
set CfHt := SubspaceTopology CfH.
```

Then we can check that CfHt becomes a complete metric space and hence by applying the Baire Category Theorem this space is a Baire space:

```
have CfHt_baire: baire_space CfHt.
apply BaireCategoryTheorem
          with um_restriction um_restriction_metric.
```

We construct a sequence of open dense subsets of CfHt by setting:

```
Let W (eps:R):
 Ensemble (point_set CMapt) :=
 fun g:CMap ⇒  ∀ (x1 x2:X),
  (proj1_sig g x1) = (proj1_sig g x2) → d x1 x2 < eps.
```

From the reasons mentioned above the openness of W is not as straightforward as it might look:

```
Lemma W_is_open: ∀ (eps:R),
                       eps > 0 → open (W eps).
```

The point of the Bing shrinkability is that this property amounts to saying each such W is dense.

Then

```
set Wn: IndexedFamily nat (point_set CfHt) := fun n:nat ⇒
   inverse_image (subspace_inc CfH)  (W (/INR (S n))).
have WnOD: ∀ n:nat, open (Wn n) ∧ dense (Wn n).
```

Then applying the Baire property, the intersection of W_n's is dense.

```
have IWn_dense: dense (IndexedIntersection Wn).
apply CfHt_baire.
by apply WnOD.
```

This intersection consists of the desired homeomorphisms, which completes the formalization.

2.2 Baire Category Theorem

As we needed the Baire Category Theorem (**BCT**) in our formalization of the **BST**, we formalized the **BCT** for compact metric spaces.

```
Variable T: Topological_space.
Definition baire_space : Prop :=
 ∀ V : IndexedFamily nat (point_set T),
  (∀ n: nat, (open (V n)) ∧ (dense (V n))) →
   dense (IndexedIntersection V).
```

Let X be a point set (Type) and d a metric (X → X → R) defining a metric topology on X. Then

```
Theorem BaireCategoryTheorem :
                      complete d d_metric → baire_space.
```

Our formalization follows a more or less straightforward argument of choosing an appropriate convergent sequence. One point we need to mention here is that to choose this sequence it is inevitable to assume the Axiom of Choice. In this sense, our formalization is essentially non-constructive. Explicitly we used the following form of the Axiom of Choice:

```
Axiom FDC : FunctionalDependentChoice_on
                 (point_set X * {r:R | r > 0} * nat).
```

(In our code in BaireCategory.v [11], it is a lemma derived from the Coq standard library.)

3 Relation to Some Theorems in Geometric Topology

In this section, to give an idea how Bing's shrinking method is used in more concrete geometric situations, we first sketch a traditional proof of the Generalized Schoenflies Theorem using this method (Details can be found in [4,7]).

3.1 Proof Sketch of the GST Using Bing's Shrinking Method

Consider a map $f : S^n \to \Sigma S^{n-1}$, where ΣS^{n-1} is the suspension of S^{n-1}, i.e., the compact space obtained from the infinite cylinder $S^{n-1} \times \mathbb{R}$ by adding two ideal points $+\infty$ and $-\infty$. Define f to be the map collapsing two bounded regions of $S^n - \overline{\phi}(S^{n-1} \times \mathbb{R})$ and mapping them to the corresponding ideal points $\pm\infty$. By drilling a small ball from the image of $\overline{\phi}$, the **GST** becomes the $k = 2$ case of the following 'Disk to Disk Theorem'.

Theorem (Disk to Disk Theorem). Suppose $f : D^n \to D^n$ is a map such that there are only finitely many points (say k points) p_i with the property $\mathrm{Card}(f^{-1}(p_i)) > 1$. If all p_i are in the interior of $\mathrm{Im}(f)$ then f is Bing shrinkable.

For the sake of simplicity we assume there is only one such point $p_i = p$. To shrink the point inverse $f^{-1}(p)$, consider the homeomorphism $h : D^n \to D^n$ obtained by applying the Relative Annulus Property of (D^n, N, B) where N is a standard thin collar neighborhood of ∂D^n and B is a small ball around p.

Then the map $\sigma : D^n \to D^n$ defined by $\sigma(x) = x$ for $x \in f^{-1}(p)$ and $\sigma(x) = f^{-1}(h(f(x)))$ for $x \in D^n - f^{-1}(p)$ turns out to be a homeomorphism. This σ is used to produce nested neighborhoods of $f^{-1}(p)$, which then shows the Bing Shrinkability of f.

3.2 Abstract Property of Disks Necessary for Shrinking Arguments

A problem which arises as soon as we try to formalize concrete theorems in Geometry/Topology is that geometric objects are usually given concretely in terms of some specific representation using coordinates. However, formalizations based on these specific representations make them less portable. Thus we state our example using an abstraction of disks from our on-going formalization of a key property of subsets called cellularity.

Suppose D is the standard disk D^n in the Euclidean space formalized using coordinates in some specific way. Given a point x in the interior of D, an open neighborhood U of x, and a closed subset K in the interior of D, we can construct a concrete homeomorphism h of the Euclidean space which moves interior points of D towards x until K is contained in U while at the same time fixing exterior points of D. This is the property abstracted in the following definition. (Here, a coercion from TopologicalSpace to Sortclass is defined).

```
(* Definition of abstract cell *)
Definition abstract_cell (D : Ensemble X) : Prop :=
  Inhabited (interior D) ∧ closed D ∧
  ∀ (x : X) (U K : Ensemble X),  open U → In U x
       → Included U D → closed K → Included K (interior D) →
  ∃ h : X → X, homeomorphism h ∧ h x = x ∧
         Included (Im K h) U ∧
         (∀ y : X, In (Complement D) y → h y = y).
```

This abstract property of disks is sufficient for us to formulate a property of subsets called cellularity, and it makes formalizations of the main body of the proofs using cellularity independent of concrete coordinate representations of disks.

Also, in this definition, abstract_cell is defined not as a property of topological spaces but of subsets of a fixed ambient type X, and each homeomorphism h is defined all over X. This kind of flattening is important in practice, especially for dealing with infinitely nested structures. Cellularity, abstractly defined below, is a typical example of such an infinite nesting. Concretely, a subset of a manifold (such as the n-disk or the n-sphere) is cellular if it is the intersection of some decreasing sequence of disks in the manifold. When a subset is cellular, we can shrink it explicitly without using the Baire Category Theorem. Our abstract cellularity is:

```
(* Definition of abstract cellularity *)
Definition abstract_cellular (K : Ensemble X) : Prop :=
  ∃ D : IndexedFamily nat X,
    (∀ n : nat, abstract_cell (D n) ∧
    Included (D (S n)) (interior (D n)) ) ∧
    IndexedIntersection D = K.
```

If a surjective map between spheres (of the same dimension) has only finitely many non-trivial point-inverses, then these point-inverses are inevitably cellular and that map can be approximated arbitrarily closely by homeomorphisms. The Generalized Schoenflies Theorem corresponds to the case of two non-trivial point-inverses. The Sphere to Sphere theorem in the next section corresponds to the case of countablely many nontrivial point-inverses:

3.3 Future Plan

M. Freedman's proof of the 4-dimensional Poincaré Conjecture is a triumph of geometric topology. The Bing shrinking construction of homeomorphisms we have discussed in this article plays an essential role in this proof. In fact the core theorem in [6] asserts certain topological objects, called the Casson handles, are homeomorphic to $D^2 \times \mathbb{R}^2$. This result is obtained by placing a Casson handle CH in the 4 dimensional sphere S^4 and applying the following theorem to construct a homeomorphism $CH \approx S^2 \times \mathbb{R}^2$.

Theorem (Sphere to Sphere Theorem)) [6,8]. Suppose $f : S^n \to S^n$ is a surjective map such that there are only countably many points p_i, $(i \in \mathbb{N})$ with the property $\mathrm{Card}(f^{-1}(p_i)) > 1$. We assume $\lim_{i\to\infty} \mathrm{diam}(f^{-1}(p_i)) = 0$ and the subset $\{p_i \mid i \in \mathbb{N}\} \subset S^n$ is nowhere dense. Then f is Bing shrinkable.

This theorem is the Disk to Disk Theorem for $k = \infty$ and its proof can be restated in terms of abstract_cells defined in Sect. 3. Thus formalization of this theorem may not be far from our formalization of the Bing Shrinking method.

4 Conclusion

We formalized the Bing shrinking theorem, a basic method of constructing possibly wild and even pathological homeomorphisms in geometric topology. We also extracted an abstract property of the disk to further facilitate formalization of arguments using the shrinking method.

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