# Chapter 2
# Spaces $\mathbf{L}_p$, $1 \le p \le \infty$

In this chapter we study the class of $\mathbf{L}_p$ spaces, $1 \le p \le \infty$, which is one of the most important classes of symmetric spaces. We begin with the Hölder and Minkowski inequalities and prove that $\mathbf{L}_p$ is a symmetric space for all $1 \le p \le \infty$. In the case $1 \le p < \infty$, we show that $\mathbf{L}_p$ is separable and describe its dual.

## 2.1 Hölder's and Minkowski's Inequalities

We continue to study the spaces $\mathbf{L}_p$ introduced at the end of Chapter 1. Our first goal is to prove that $(\mathbf{L}_p, \|\cdot\|_{\mathbf{L}_p})$ is a complete normed space, i.e. , a Banach space.

We begin with two basic inequalities.

**Proposition 2.1.1.** *1. Let $p \ge 1$, $a > 0$, and $b > 0$. Then*

$$(a+b)^p \le 2^{p-1}(a^p + b^p). \tag{2.1.1}$$

*2. (**Young's inequality**) Let $p, q > 1$, $a > 0$, and $b > 0$. If $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.1.2}$$

*Equality is achieved if and only if $a = b^{q-1}$ or $b = a^{p-1}$.*

*Proof.* 1. Since the function $\Phi(x) = x^p$ is convex on $(0, \infty)$ for $p \ge 1$, it follows that

$$\left(\frac{a+b}{2}\right)^p \le \frac{a^p + b^p}{2}$$

for all $a > 0$, $b > 0$ (Fig. 2.1).



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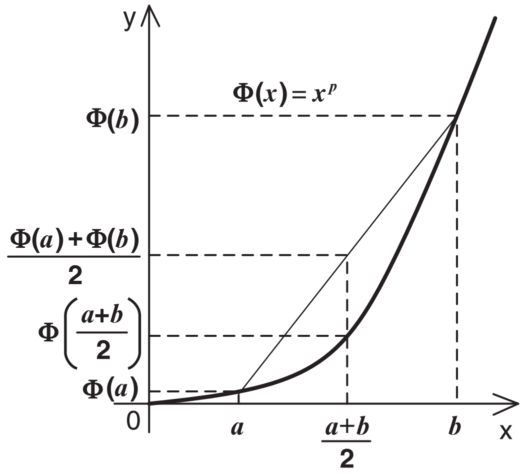


**Fig. 2.1** Inequality $\Phi\left(\frac{a+b}{2}\right) \le \frac{\Phi(a)+\Phi(b)}{2}$

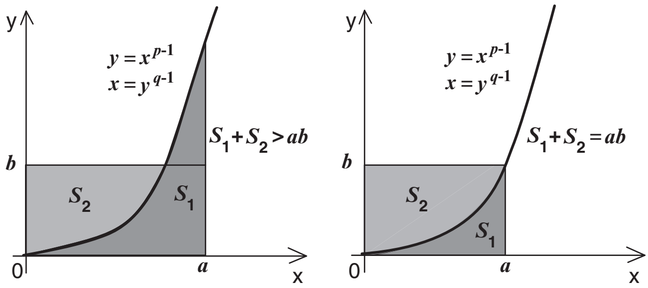


**Fig. 2.2** The functions $\frac{x^p}{p}$ and $\frac{y^q}{q}$ as areas

This implies

$$(a+b)^p \le 2^{p-1}(a^p + b^q),$$

for all $a > 0$ and $b > 0$.

2. Since $(p-1)(q-1) = 1$, the functions $y = x^{p-1}$ and $x = x^{q-1}$ are mutually inverse. Consider their graphs (Fig. 2.2).
We have

$$S_1 = \int_0^x u^{p-1} du = \frac{x^p}{p}, \quad S_2 = \int_0^y v^{q-1} du = \frac{y^q}{q},$$

and

$$xy \le S_1 + S_2 = \frac{x^p}{p} + \frac{y^q}{q}.$$

□

**Theorem 2.1.2.** *Let* $1 \le p, q \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

1. ***(Hölder's inequality)*** *Let* $f \in \mathbf{L}_p$, $g \in \mathbf{L}_q$. *Then* $fg \in \mathbf{L}_1$ *and*

$$\|fg\|_{\mathbf{L}_1} \le \|f\|_{\mathbf{L}_p} \cdot \|g\|_{\mathbf{L}_q}. \tag{2.1.3}$$

*Equality is achieved if and only if* $\alpha|f|^p = \beta|g|^q$ *for some* $\alpha \ge 0$, $\beta \ge 0$.

2. ***(Minkowski's inequality)*** *If* $f, g \in \mathbf{L}_p$, $1 \le p \le \infty$, *then*

$$\|f + g\|_{\mathbf{L}_p} \le \|f\|_{\mathbf{L}_p} + \|g\|_{\mathbf{L}_p}. \tag{2.1.4}$$

*Proof.* 1. The inequality (2.1.3) clearly holds if $\|f\|_{\mathbf{L}_p}$ or $\|g\|_{\mathbf{L}_q}$ is equal to 0 or $\infty$. Thus we may assume that $0 < \|f\|_{\mathbf{L}_p} < \infty$ and $0 < \|g\|_{\mathbf{L}_q} < \infty$.

If $p = 1$, $q = \infty$, then

$$\|fg\|_{\mathbf{L}_1} = \int_0^\infty |fg|dm = \int_0^\infty |f|\,|g|dm \le \|g\|_{\mathbf{L}_\infty} \int_0^\infty |f|dm = \|f\|_{\mathbf{L}_1} \cdot \|g\|_{\mathbf{L}_\infty}.$$

Let $1 < p, q < \infty$. By putting in Young's inequality (2.1.2),

$$a = \frac{|f(x)|}{\|f\|_{\mathbf{L}_p}}, \quad b = \frac{|g(x)|}{\|g\|_{\mathbf{L}_q}},$$

we obtain

$$\frac{|f(x)|}{\|f\|_{\mathbf{L}_p}} \cdot \frac{|g(x)|}{\|g\|_{\mathbf{L}_q}} \le \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_{\mathbf{L}_p}}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_{\mathbf{L}_q}}\right)^q, \quad x \in [0, \infty),$$

and hence

$$\int_0^\infty \frac{|f(x)g(x)|\,dx}{\|f\|_{\mathbf{L}_p} \cdot \|g\|_{\mathbf{L}_q}} \le \frac{1}{p}\int_0^\infty \left(\frac{|f(x)|}{\|f\|_{\mathbf{L}_p}}\right)^p dx + \frac{1}{q}\int_0^\infty \left(\frac{|g(x)|}{\|g\|_{\mathbf{L}_q}}\right)^q dx = \frac{1}{p} + \frac{1}{q} = 1,$$

i.e.,

$$\|fg\|_{\mathbf{L}_1} = \int_0^\infty |f(x)g(x)|\,dx \le \|f\|_{\mathbf{L}_p} \cdot \|g\|_{\mathbf{L}_q}.$$

2. For values $p = 1$ and $p = \infty$, the inequality (2.1.4) is obvious.

Let $1 < p < \infty$. Then

$$\int_0^\infty |f + g|^p dm \le \int_0^\infty |f|\,|f + g|^{p-1}dm + \int_0^\infty |g|\,|f + g|^{p-1}dm. \tag{2.1.5}$$

If $f, g \in \mathbf{L}_p$, then by setting in (2.1.1) $a = |f|$ and $b = |g|$, we have

$$|f+g|^p \le 2^{p-1}(|f|^p + |g|^p),$$

which in turn implies $(f+g)^p \in \mathbf{L}_1$.

Taking into account the equality $(p-1)q = p$, we obtain

$$(|f+g|^{p-1})^q = |f+g|^{(p-1)q} = |f+g|^p \in \mathbf{L}_1,$$

i.e., $|f+g|^{p-1} \in \mathbf{L}_q$. Now we apply Hölder's inequality (2.1.3) to the functions $|f| \in \mathbf{L}_p$ and $|f+g|^{p-1} \in \mathbf{L}_q$:

$$\int_0^\infty |f|\,|f+g|^{p-1} dm \le \|f\|_{\mathbf{L}_p} \cdot \|\,|f+g|^{p-1}\,\|_{\mathbf{L}_q} = \|f\|_{\mathbf{L}_p} \left( \int_0^\infty |f+g|^{(p-1)q} dm \right)^{\frac{1}{q}}$$

$$= \|f\|_{\mathbf{L}_p} \left( \int_0^\infty |f+g|^p dm \right)^{\frac{1}{q}} = \|f\|_{\mathbf{L}_p} \cdot \left( \|f+g\|_{\mathbf{L}_p} \right)^{\frac{p}{q}}.$$

In a similar way, for $|g| \in \mathbf{L}_p$ and $|f+g|^{p-1} \in \mathbf{L}_q$, we have

$$\int_0^\infty |g|\,|f+g|^{p-1} dm \le \|g\|_{\mathbf{L}_p} \cdot \left( \|f+g\|_{\mathbf{L}_p} \right)^{\frac{p}{q}}.$$

By combining these two inequalities with (2.1.5), we obtain

$$\|f+g\|_{\mathbf{L}_p}^p = \int_0^\infty |f+g|^p dm \le (\|f\|_{\mathbf{L}_p} + \|g\|_{\mathbf{L}_p}) \cdot \left( \|f+g\|_{\mathbf{L}_p} \right)^{\frac{p}{q}}.$$

Since $p - \dfrac{p}{q} = 1$, we obtain also (2.1.4). □

Minkowski's inequality is just the triangle inequality in the space $\mathbf{L}_p$. The equality

$$\|cf\|_{\mathbf{L}_p} = |c| \cdot \|f\|_{\mathbf{L}_p}, \quad c \in \mathbb{R},\ f \in \mathbf{L}_p,$$

also holds, and $\|f\|_{\mathbf{L}_p} = 0$ if and only if $f = 0$.

Thus, for $1 \le p \le \infty$, $(\mathbf{L}_p, \|\cdot\|_{\mathbf{L}_p})$ is a normed space.

## 2.2 Completeness of $\mathbf{L}_p$

Let $1 \le p < \infty$ and $\{f_n\}$ be a fundamental sequence in $\mathbf{L}_p$, i.e.,

$$\lim_{n,m\to\infty} \|f_n - f_m\|_{\mathbf{L}_p} = 0.$$

By passing, if necessary, to a subsequence, we may assume that

$$f_0 = 0, \ \|f_k - f_{k-1}\|_{\mathbf{L}_p} < \frac{1}{2^k}, \ \ k = 1, 2, \dots. \tag{2.2.1}$$

We show that the sequence $\{f_n\}$ converges almost everywhere to a function $f \in \mathbf{L}_p$ and

$$\lim_{n\to\infty} \|f_n - f\|_{\mathbf{L}_p} = 0.$$

To prove this, consider the series $\sum\limits_{k=1}^{\infty}(f_k - f_{k-1})$, which has partial sums $f_n = \sum\limits_{k=1}^{n}(f_k - f_{k-1})$, and set

$$g_n = \sum_{k=1}^{n} |f_k - f_{k-1}| \ \text{ and } \ g = \sum_{k=1}^{\infty} |f_k - f_{k-1}|.$$

Then $g_n \uparrow g$.[1] The integrals

$$\int_0^{\infty} g_n^p dm = \|g_n\|_{\mathbf{L}_p}^p$$

are uniformly bounded, since by Minkowski's inequality (2.1.4),

$$\|g_n\|_{\mathbf{L}_p} \le \sum_{k=1}^{n} \|f_k - f_{k-1}\|_{\mathbf{L}_p} \le \sum_{k=1}^{n} \frac{1}{2^k} \le 1.$$

Since $g_n^p \uparrow g^p$, Levi's theorem implies that the function $g^p$ is integrable (and hence finite almost everywhere) and

$$1 \ge \lim_{n\to\infty} \int_0^{\infty} g_n^p dm = \int_0^{\infty} g^p dm = \|g\|_{\mathbf{L}_p}^p,$$

i.e., $g \in \mathbf{L}_p$.

[1]From now on, we shall use the notation $g_n \uparrow g$ if $\{g_n\}$ is increasing and $g = \sup\limits_n g_n$ almost everywhere.

In a similar way, $g_n \downarrow g$ means that $\{g_n\}$ is decreasing and $g = \inf\limits_n g_n$ almost everywhere.

Since

$$\sum_{k=1}^{\infty} |f_k(x) - f_{k-1}(x)| = g(x) < \infty$$

for almost all $x \in [0,\infty)$, the original series $\sum_{k=1}^{\infty}(f_k - f_{k-1})$ converges almost everywhere to a function $f$. Then $f_n = \sum_{k=1}^{n}(f_k - f_{k-1})$ also tends to $f$ almost everywhere. Finally,

$$\|f\|_{\mathbf{L}_p}^p = \int_0^\infty |f|^p dm \le \int_0^\infty g^p dm = \|g\|_{\mathbf{L}_p}^p < \infty,$$

i.e., $f \in \mathbf{L}_p$, and by (2.2.1), we have

$$\|f - f_n\|_{\mathbf{L}_p} \le \sum_{k=n+1}^{\infty} \|f_k - f_{k-1}\|_{\mathbf{L}_p} \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} \to 0,\ n \to \infty.$$

Hence, $\mathbf{L}_p$, $1 \le p < \infty$, is a Banach space.

For $p = \infty$, convergence in $\mathbf{L}_\infty$ is *essentially uniform* convergence. This means that

$$\|f_n - f\|_{\mathbf{L}_\infty} \to 0,\ \ n \to \infty,$$

if and only if there exists a set $N$ of measure 0 such that $f_n(x) \to f(x)$ uniformly on $\mathbb{R}^+ \setminus N$.

This implies in turn the completeness of the space $\mathbf{L}_\infty$. Indeed, if $\{f_n\}$ is a fundamental sequence in $\mathbf{L}_\infty$, then it is a uniformly fundamental sequence on $\mathbb{R}^+ \setminus N$ for a suitable set $N$ of measure 0,

$$\lim_{m,n\to\infty} \sup_{x\in\mathbb{R}^+\setminus N} |f_m(x) - f_n(x)| = 0.$$

Hence $f_n(x) \to f(x)$ on $\mathbb{R}^+ \setminus N$ for a function $f \in \mathbf{L}_\infty$ and $\| f_n - f \|_{\mathbf{L}_\infty} \to 0$, $n \to \infty$.

Thus, for all $p \in [1,\infty]$, $(\mathbf{L}_p, \|\cdot\|_{\mathbf{L}_p})$ is a Banach space.

## 2.3 Separability of $\mathbf{L}_p$, $1 \le p < \infty$

Denote by $\mathbf{F}_1$ the set of all simple[2] integrable functions, and by $\mathbf{F}_0$ the set of all simple functions with bounded support. Functions from $\mathbf{F}_1$ have the form

$$f = \sum_{i=0}^{n} a_i \cdot 1_{A_i},\ a_i \in \mathbb{R}, \quad mA_i < \infty,$$

and if $f \in \mathbf{F}_0$, the sets $A_i$ are bounded, i.e., there exists $a > 0$ such that $A_i \subseteq [0, a]$ for all $i$.

**Theorem 2.3.1.** *Let* $1 \le p < \infty$. *Then*

1. $\mathbf{F}_0$ *is dense in* $\mathbf{L}_p$ *in norm* $\|\cdot\|_{\mathbf{L}_p}$.
2. $\mathbf{L}_p$ *is separable.*

*Proof.* 1. Let $0 \le f \in \mathbf{L}_p$. Consider step functions $f^{(n)}$ that approximate the function $f$ from below. Let

$$f^{(n)}(x) = \frac{i-1}{2^n},\ \text{ if }\ \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n},\ \ i \in \mathbb{N},$$

and

$$f_n = \min(n, f^{(n)}) \cdot 1_{[0,n]}.$$

Then $f_n \in \mathbf{F}_0$ and

$$\|f - f_n\|^p_{\mathbf{L}_p} = \int_0^\infty |f - f_n|^p dm \le \int_{\{f \ge n\}} f^p dm + \int_n^\infty f^p dm + \frac{n}{2^n}. \tag{2.3.1}$$

Since $\int_0^\infty f^p dm < \infty$, the right side of (2.3.1) tends to 0 as $n \to \infty$, and hence

$$\|f - f_n\|_{\mathbf{L}_p} \to 0,\ \ n \to \infty.$$

For arbitrary functions $f \in \mathbf{L}_p$, we can use the decomposition

$$f = f^+ - f^-,\ \ f^+ = \max(f, 0),\ \ f^- = -\min(f, 0),$$

reducing to the case $f \ge 0$.

[2]A function is simple if it is a measurable step function with finitely many values.

2. Let $A \in \mathscr{F}_m$, $mA < \infty$, and $\varepsilon > 0$. Then there exists an open set $G$ such that $m(A \bigtriangleup G) < \varepsilon$. The set $G$ has the form

$$G = \bigcup_{i=1}^{\infty}(a_i, b_i),\ 0 \le a_i < b_i,\ i \ge 1.$$

Since $mG < \infty$, we can find $n$ large enough that $m(G \bigtriangleup G_n) < \varepsilon$, where $G_n = \bigcup_{i=1}^{n}(a_i, b_i)$. We additionally may assume that $a_i$ and $b_i$ are rational numbers.

Consider the subset $\mathbf{F}^{(0)} \subset \mathbf{F}_0$ consisting of all functions $g$ of the form

$$g = \sum_{i=0}^{n} c_i \cdot 1_{[a_i,b_i]}$$

with rational $c_i$, $a_i$, and $b_i$.

Since for every pair $A,\ B \in \mathscr{F}_m$, we have

$$m(A \bigtriangleup B) < \varepsilon \Longrightarrow \ \|1_A - 1_B\|_{\mathbf{L}_p} \le \varepsilon^{1/p},$$

the countable set $\mathbf{F}^{(0)}$ is dense in $\mathbf{F}_0$, and hence in $\mathbf{L}_p$ by part 1 of the proposition. □

## 2.4 Duality

Let $\mathbf{X}$ be a symmetric space and $\mathbf{X}^*$ the dual Banach space. The space $\mathbf{X}^*$ consists of all linear continuous (bounded) functionals $u : \mathbf{X} \to \mathbb{R}$ equipped with the norm

$$\|u\|_{\mathbf{X}^*} = \sup\{|u(f)| : \|f\|_{\mathbf{X}} \le 1\} < \infty.$$

Let $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in \mathbf{L}_q$ we define a linear functional $u_g$ on $\mathbf{L}_p$ by setting

$$u_g(f) = \int_0^{\infty} fgdm,\ \ f \in \mathbf{L}_p.$$

**Theorem 2.4.1.** *Let $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $u_g \in \mathbf{L}_p^*$ for all $g \in \mathbf{L}_q$, and the mapping*

$$\upsilon : \mathbf{L}_q \ni g \to u_g \in \mathbf{L}_p^*$$

*is an isometric isomorphism of $\mathbf{L}_q$ into the dual space $\mathbf{L}_p^*$ of the space $\mathbf{L}_p$. If $1 \le p < \infty$, then $\upsilon(\mathbf{L}_q) = \mathbf{L}_p^*$, and for $p = \infty$, the embedding $\upsilon(\mathbf{L}_1) \subset \mathbf{L}_\infty^*$ is strict.*

*Proof.* Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's inequality (2.1.3) implies that the functional $u_g$ is continuous and

$$|u_g(f)| = \left| \int_0^\infty fgdm \right| \le \|f\|_{\mathbf{L}_p} \|g\|_{\mathbf{L}_q},$$

i.e.,

$$\|u_g\|_{\mathbf{L}_p^*} \le \|g\|_{\mathbf{L}_q}. \tag{2.4.1}$$

Suppose that $u \in \mathbf{L}_p^*$. Since $1_A \in \mathbf{L}_p$ for $mA < \infty$, the equality $\mu(A) = u(1_A)$ defines a set function

$$\mu : A \to \mu(A) = u(1_A), \;\; A \in \mathscr{F}_m, \; mA < \infty. \tag{2.4.2}$$

For every finite segment $[0, n]$, the restriction of $\mu$ to the $\sigma$-algebra

$$\mathscr{F}_m(0, n) = \{A \in \mathscr{F}_m : A \subseteq [0, n]\}$$

is a $\sigma$-additive set function, which is absolutely continuous with respect to the measure $m$. The Radon–Nikodym theorem states the existence of a unique function $h_n$ integrable on $[0, n]$ such that

$$\mu(A) = \int_A h_n dm, \;\; A \subseteq [0, n], \; A \in \mathscr{F}_m. \tag{2.4.3}$$

Since $\mathbb{R}^+ = \bigcup_{n=1}^{\infty} [0, n)$, we obtain a unique measurable function $g$ such that $g|_{[0,n]} = h_n$ for all $n = 1, 2, \dots$ and

$$u(1_A) = \mu(A) = \int_A gdm \tag{2.4.4}$$

for all $A \in \mathscr{F}_m$ with $m(A) < \infty$.

The linearity of the functional $u$ yields that

$$u(f) = \int_0^\infty fgdm = u_g(f), \;\; f \in \mathbf{F}_0. \tag{2.4.5}$$

We choose now a sequence $g_n \in \mathbf{F}_0$ such that $g_n \uparrow |g|$. Such a sequence can be constructed by setting

$$g_n = \min(n, g^{(n)}) \cdot 1_{[0,n]},$$

where for all $i = 1, 2, \ldots,$

$$g^{(n)}(x) = \frac{i-1}{2^n}, \ \text{ if } \ \frac{i-1}{2^n} \le |g(x)| < \frac{i}{2^n}.$$

We define now

$$f_n = (g_n)^{q-1} \cdot \operatorname{sign}(g), \ \ n = 1, 2, \ldots.$$

Then $f_n \in \mathbf{F}_0$ and

$$u(f_n) = \int_0^\infty f_n g dm = \int_0^\infty g_n^q dm = \|g_n\|_{\mathbf{L}_q}^q.$$

On the other hand,

$$u(f_n) \le \|u\|_{\mathbf{L}_p^*} \|f_n\|_{\mathbf{L}_p},$$

where $|f_n|^p = g_n^{(q-1)p} = g_n^q$, by the equality $(q-1)p = q$.

This implies

$$\|f_n\|_{\mathbf{L}_p} = \left( \int_0^\infty g_n^q dm \right)^{1/p} = \|g_n\|_{\mathbf{L}_q}^{q/p}$$

and

$$\|g_n\|_{\mathbf{L}_q}^q \le \|u\|_{\mathbf{L}_p^*} \|g_n\|_{\mathbf{L}_q}^{q/p},$$

i.e.,

$$\|g_n\|_{\mathbf{L}_q} \le \|u\|_{\mathbf{L}_p^*}.$$

Thus, $g_n^q \uparrow |g|^q$ and

$$\int_0^\infty g_n^q dm \le \|u\|_{\mathbf{L}_p^*}^q.$$

By Levi's theorem,

$$\lim_{n\to\infty} \int_0^\infty g_n^q dm = \int_0^\infty |g|^q dm \le \|u\|_{\mathbf{L}_p^*}^q,$$

i.e., $g \in \mathbf{L}_q$ and $\|g\|_{\mathbf{L}_q} \le \|u\|_{\mathbf{L}_p^*}$.

The functional $u_g$ coincides with $u \in \mathbf{L}_p^*$ on $\mathbf{F}_0$, and $\mathbf{F}_0$ is dense in $\mathbf{L}_q$. Hence $u_g = u$.

The equality $\upsilon(\mathbf{L}_\infty) = \mathbf{L}_1^*$ and the fact that the embedding $\upsilon(\mathbf{L}_1) \subset \mathbf{L}_\infty^*$ is strict will be shown later in Examples 7.1.1 and 7.1.2. □

**Corollary 2.4.2.** *The spaces* $\mathbf{L}_p$ *are reflexive for* $1 < p < \infty$.

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