

Chapter 2

Exercises on Projective Spaces

Projective spaces and projective subspaces. Projective frames and homogeneous coordinates. Projective transformations and projectivities. Linear systems of hyperplanes and duality. The projective line. Cross-ratio.

Abstract Solved problems on projective spaces and subspaces, projective transformations, the projective line and the cross-ratio, linear systems of hyperplanes and duality.

Notation: Throughout the whole chapter, the symbol \mathbb{K} denotes a subfield of \mathbb{C} .

Exercise 1. Show that the points

$$\left[\frac{1}{2}, 1, 1\right], \left[1, \frac{1}{3}, \frac{4}{3}\right], [2, -1, 2]$$

of the real projective plane are collinear, and find an equation of the line containing them.

Solution. The points $\left[\frac{1}{2}, 1, 1\right], \left[1, \frac{1}{3}, \frac{4}{3}\right]$ are distinct, so the point $[x_0, x_1, x_2]$ is collinear with them if and only if the vectors $(1, 2, 2), (3, 1, 4), (x_0, x_1, x_2)$ are linearly dependent, i.e., if and only if

$$0 = \det \begin{pmatrix} 1 & 3 & x_0 \\ 2 & 1 & x_1 \\ 2 & 4 & x_2 \end{pmatrix} = 6x_0 + 2x_1 - 5x_2.$$

Therefore, an equation of the line containing $\left[\frac{1}{2}, 1, 1\right], \left[1, \frac{1}{3}, \frac{4}{3}\right]$ is given by $6x_0 + 2x_1 - 5x_2 = 0$, and this equation is satisfied by the point $[2, -1, 2]$.

Exercise 2. Find the values $a \in \mathbb{C}$ for which the lines of equations

$$ax_1 - x_2 + 3ix_0 = 0, \quad -iax_0 + x_1 - ix_2 = 0, \quad 3ix_2 + 5x_0 + x_1 = 0$$

of $\mathbb{P}^2(\mathbb{C})$ are concurrent.

Solution (1). If

$$A = \begin{pmatrix} 3i & a & -1 \\ -ia & 1 & -i \\ 5 & 1 & 3i \end{pmatrix},$$

then the given lines all intersect if and only if the homogeneous linear system $AX = 0$ admits a non-trivial solution. This occurs if and only if $0 = \det A = -3a^2 - 4ia - 7$, i.e., if and only if either $a = i$ or $a = -\frac{7}{3}i$.

Solution (2). Via the duality correspondence (see Sect. 1.4.2), the given lines determine three points in the space $\mathbb{P}^2(\mathbb{C})^*$, and the coordinates of these points with respect to the frame induced by the standard basis of $(\mathbb{C}^2)^*$ are given by $[3i, a, -1]$, $[-ia, 1, -i]$, $[5, 1, 3i]$. An easy application of the Duality principle shows that these points are collinear if and only if the given lines are concurrent. Finally, the points $[3i, a, -1]$, $[-ia, 1, -i]$, $[5, 1, 3i]$ are collinear if and only if the determinant of the matrix A introduced above vanishes (see Exercise 1).

Exercise 3. In $\mathbb{P}^3(\mathbb{R})$ consider the points

$$P_1 = [1, 0, 1, 2], \quad P_2 = [0, 1, 1, 1], \quad P_3 = [2, 1, 2, 2], \quad P_4 = [1, 1, 2, 3].$$

- Determine whether P_1, P_2, P_3, P_4 are in general position.
- Compute the dimension of the subspace $L(P_1, P_2, P_3, P_4)$, and find Cartesian equations of $L(P_1, P_2, P_3, P_4)$.
- If possible, complete the set $\{P_1, P_2, P_3\}$ to a projective frame of $\mathbb{P}^3(\mathbb{R})$.

Solution. (a) Let $v_1 = (1, 0, 1, 2)$, $v_2 = (0, 1, 1, 1)$, $v_3 = (2, 1, 2, 2)$, $v_4 = (1, 1, 2, 3)$ be vectors in \mathbb{R}^4 such that $P_i = [v_i]$ for every i , and set

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 3 \end{pmatrix}.$$

It is easy to check that $\det A = 0$, so v_1, v_2, v_3, v_4 are linearly dependent. Therefore, the points P_1, P_2, P_3, P_4 are not in general position.

(b) The determinant of the submatrix given by the first three lines and the first three columns of A is equal to -1 , so v_1, v_2, v_3 are linearly independent. Therefore, by point (a) the dimension of the linear subspace spanned by v_1, \dots, v_4 is equal to 3, so $L(P_1, P_2, P_3, P_4) = L(P_1, P_2, P_3)$ and $\dim L(P_1, P_2, P_3, P_4) = 2$. Moreover, the

same argument as in the solution of Exercise 1 shows that a Cartesian equation of $L(P_1, P_2, P_3, P_4) = L(P_1, P_2, P_3)$ is given by

$$0 = \det \begin{pmatrix} 1 & 0 & 2 & x_0 \\ 0 & 1 & 1 & x_1 \\ 1 & 1 & 2 & x_2 \\ 2 & 1 & 2 & x_3 \end{pmatrix} = -x_0 - 2x_1 + 3x_2 - x_3.$$

(c) By point (b), if we replace the last column of A with the vector $(0, 0, 0, 1)$ we obtain an invertible matrix, so the vectors $v_1, v_2, v_3, (0, 0, 0, 1)$ provide a basis of \mathbb{R}^4 . The projective frame induced by this basis is given by the points $P_1, P_2, P_3, [0, 0, 0, 1], [3, 2, 4, 6]$. Therefore, this 5-tuple of points extends P_1, P_2, P_3 to a projective frame of $\mathbb{P}^3(\mathbb{R})$.

Exercise 4. Let $l \subset \mathbb{P}^2(\mathbb{K})$ be the line of equation $x_0 + x_1 = 0$, set $U = \mathbb{P}^2(\mathbb{K}) \setminus l$ and let $\alpha, \beta: U \rightarrow \mathbb{K}^2$ be defined as follows:

$$\alpha([x_0, x_1, x_2]) = \left(\frac{x_1}{x_0 + x_1}, \frac{x_2}{x_0 + x_1} \right),$$

$$\beta([x_0, x_1, x_2]) = \left(\frac{x_0}{x_0 + x_1}, \frac{x_2}{x_0 + x_1} \right).$$

Find an explicit formula for the composition $\alpha \circ \beta^{-1}$, and check that this map is an affinity.

Solution. Let us first determine β^{-1} . Let $\beta([x_0, x_1, x_2]) = (u, v)$. Since $x_0 + x_1 \neq 0$ on U , we may suppose $x_0 + x_1 = 1$, so that

$$u = \frac{x_0}{x_0 + x_1} = x_0, \quad v = \frac{x_2}{x_0 + x_1} = x_2, \quad x_1 = 1 - x_0 = 1 - u.$$

Therefore, $\beta^{-1}(u, v) = [u, 1 - u, v]$, hence $\alpha(\beta^{-1}(u, v)) = (1 - u, v)$, and $\alpha \circ \beta^{-1}$ is obviously an affinity.

Exercise 5. For $i = 0, 1, 2$, let $j_i: \mathbb{K}^2 \rightarrow U_i \subseteq \mathbb{P}^2(\mathbb{K})$ be the map introduced in Sect. 1.3.8.

- Find two distinct projective lines $r, s \subset \mathbb{P}^2(\mathbb{K})$ such that the affine lines $j_i^{-1}(r \cap U_i), j_i^{-1}(s \cap U_i)$ are parallel for $i = 1, 2$.
- Is it possible to find distinct lines $r, s \subset \mathbb{P}^2(\mathbb{K})$ such that the affine lines $j_i^{-1}(r \cap U_i), j_i^{-1}(s \cap U_i)$ are parallel for $i = 0, 1, 2$?

Solution. Let l_i be the projective line of equation $x_i = 0$, $i = 0, 1, 2$. If $r \subset \mathbb{P}^2(\mathbb{K})$ is a projective line, then the set $j_i^{-1}(r \cap U_i)$ is an affine line if and only if $r \neq l_i$. Moreover, for any given distinct projective lines $r \neq l_i, s \neq l_i$, the affine lines $j_i^{-1}(r \cap U_i)$ and $j_i^{-1}(s \cap U_i)$ are parallel if and only if the point $s \cap r$ belongs to l_i . Therefore, if r, s are projective lines such that $r \neq s, r \notin \{l_1, l_2\}, s \notin \{l_1, l_2\}$ and

$r \cap s = [1, 0, 0]$, then r and s satisfy the condition described in (a): for example, one may choose $r = \{x_1 + x_2 = 0\}$, $s = \{x_1 - x_2 = 0\}$.

Moreover, since $l_0 \cap l_1 \cap l_2 = \emptyset$, the condition described in (b) cannot be satisfied by any pair of distinct lines of $\mathbb{P}^2(\mathbb{K})$.

Exercise 6. Let A, B, C, D be points of $\mathbb{P}^2(\mathbb{K})$ in general position, and set

$$P = L(A, B) \cap L(C, D), \quad Q = L(A, C) \cap L(B, D), \quad R = L(A, D) \cap L(B, C).$$

Prove that P, Q, R are not collinear.

Solution. Since A, B, C, D are in general position, we can choose a system of homogeneous coordinates in $\mathbb{P}^2(\mathbb{K})$ where

$$A = [1, 0, 0], \quad B = [0, 1, 0], \quad C = [0, 0, 1], \quad D = [1, 1, 1].$$

An easy computation shows that

$$P = [1, 1, 0], \quad Q = [1, 0, 1], \quad R = [0, 1, 1].$$

Since $\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \neq 0$, the points P, Q, R are not collinear.

Exercise 7. Let $\mathcal{R} = \{P_0, \dots, P_{n+1}\}$ be a projective frame of $\mathbb{P}(V)$ and let $0 \leq k < n + 1$. Set $S = L(P_0, P_1, \dots, P_k)$, $S' = L(P_{k+1}, \dots, P_{n+1})$.

- (a) Show that there exists $W \in \mathbb{P}(V)$ such that $S \cap S' = \{W\}$.
- (b) Prove that $\{P_0, \dots, P_k, W\}$ is a projective frame of S , and that $\{P_{k+1}, \dots, P_{n+1}, W\}$ is a projective frame of S' .

Solution. (a) By the definition of projective frame, we have $\dim S = k$, $\dim S' = n - k$ and $\dim L(S, S') = n$, so Grassmann's formula implies that $\dim(S \cap S') = \dim S + \dim S' - \dim L(S, S') = 0$, and this proves (a).

(b) In order to show that $\{P_0, \dots, P_k, W\}$ is a projective frame of S it is sufficient to prove that $\dim L(A) = k$ for every subset $A \subseteq \{P_0, \dots, P_k, W\}$ containing exactly $k + 1$ points (if this is the case, then clearly $L(A) = S$).

Let A be such a subset. If $W \notin A$, then $\dim L(A) = k$, since the points of \mathcal{R} are in general position. Let us now assume that $W \in A$. The points of \mathcal{R} are in general position, so we have

$$\begin{aligned} \dim L((A \setminus \{W\}) \cup S') &= \dim L((A \setminus \{W\}) \cup \{P_{k+1}, \dots, P_{n+1}\}) = n, \\ \dim L(A \setminus \{W\}) &= k - 1. \end{aligned}$$

But $\dim S' = n - k$, so

$$\dim(L(A \setminus \{W\}) \cap S') = (k - 1) + (n - k) - n = -1,$$

i.e., $L(A \setminus \{W\}) \cap S' = \emptyset$. Since $W \in S'$, this implies that $W \notin L(A \setminus \{W\})$, so $\dim L(A) = \dim L(A \setminus \{W\}) + 1 = k$. Therefore, $\{P_0, \dots, P_k, W\}$ is a projective frame of S . In the very same way one can prove that $\{P_{k+1}, \dots, P_{n+1}, W\}$ is a projective frame of S' .

Exercise 8. Let $r, r' \subset \mathbb{P}^3(\mathbb{K})$ be skew lines, and take $P \in \mathbb{P}^3(\mathbb{K}) \setminus (r \cup r')$. Show that there exists a unique line $l \subset \mathbb{P}^3(\mathbb{K})$ that contains P and meets both r and r' . Compute Cartesian equations for l in the case when $\mathbb{K} = \mathbb{R}$, the line r has equations $x_0 - x_2 + 2x_3 = 2x_0 + x_1 = 0$, the line r' has equations $2x_1 - 3x_2 + x_3 = x_0 + x_3 = 0$, and $P = [0, 1, 0, 1]$.

Solution. Let $S = L(r, P)$, $S' = L(r', P)$. An easy application of Grassmann's formula implies that $\dim S = \dim S' = 2$. Moreover, $S \neq S'$ because otherwise r and r' would be coplanar, hence incident. It follows that $\dim(S \cap S') < 2$. On the other hand, $\dim(S \cap S') = \dim S + \dim S' - \dim L(S, S') \geq 2 + 2 - 3 = 1$, so $l = S \cap S'$ is a line. Since l and r (respectively, r') both lie on S (respectively, S'), we have that $l \cap r \neq \emptyset$ (respectively, $l \cap r' \neq \emptyset$). Therefore l satisfies the required properties.

Let now l' be any line of $\mathbb{P}^3(\mathbb{K})$ containing P and meeting both r and r' . We have $l' \subseteq L(r, P) = S$, $l' \subseteq L(r', P) = S'$, so $l' \subseteq S \cap S' = l$, and $l' = l$ since $\dim l' = \dim l$.

Let us now come to the particular case described in the statement of the exercise. The pencil of planes centred at r has parametric equations

$$\lambda(x_0 - x_2 + 2x_3) + \mu(2x_0 + x_1) = 0, \quad [\lambda, \mu] \in \mathbb{P}^1(\mathbb{R}).$$

By imposing that the generic plane of the pencil pass through P we obtain $2\lambda + \mu = 0$, so an equation of S is given by $-3x_0 - 2x_1 - x_2 + 2x_3 = 0$. In the same way one proves that S' is described by the equation $-3x_0 + 2x_1 - 3x_2 - 2x_3 = 0$. The system given by the equation of S and the equation of S' provides the required equations for l .

Exercise 9. Let W_1, W_2, W_3 be planes of $\mathbb{P}^4(\mathbb{K})$ such that $W_i \cap W_j$ is a point for every $i \neq j$, and $W_1 \cap W_2 \cap W_3 = \emptyset$. Show that there exists a unique plane W_0 such that $W_0 \cap W_i$ is a projective line for $i = 1, 2, 3$.

Solution. For $i \neq j$ set $P_{ij} = W_i \cap W_j$, and $W_0 = L(P_{12}, P_{13}, P_{23})$ (therefore, $P_{ij} = P_{ji}$ for every $i \neq j$). If P_{12}, P_{13}, P_{23} were not pairwise distinct, then $W_1 \cap W_2 \cap W_3$ would be non-empty, while if P_{12}, P_{13}, P_{23} were pairwise distinct and collinear, then the line containing them would be contained in each W_i . In any case, our hypothesis would be contradicted, so P_{12}, P_{13}, P_{23} are in general position, and W_0 is a plane. Moreover, by construction $W_0 \cap W_i$ contains the line $L(P_{ij}, P_{ik})$, $\{i, j, k\} = \{1, 2, 3\}$. On the other hand, if $\dim(W_0 \cap W_i) > 1$, then $W_0 = W_i$, so $W_i \cap W_j = W_0 \cap W_j$ contains a line for every $j \neq i$, a contradiction. Therefore, $W_0 \cap W_i$ is a line for every $i = 1, 2, 3$.

Let now W'_0 be a plane satisfying the properties described in the statement, and let $l_i = W'_0 \cap W_i$ for every $i = 1, 2, 3$. Then each l_i is a line, and $W_i \cap W_j \cap W'_0 = (W_i \cap W'_0) \cap (W_j \cap W'_0) = l_i \cap l_j \neq \emptyset$ (the lines l_i, l_j both lie on W'_0 , so they must

intersect). It follows that $P_{ij} \in W'_0$ for every $i, j = 1, 2, 3$, and $W_0 \subseteq W'_0$. Since $\dim W'_0 = \dim W_0$, we can conclude that $W'_0 = W_0$.

Exercise 10. Let r_1, r_2, r_3 be pairwise skew lines of $\mathbb{P}^4(\mathbb{K})$ and suppose that no hyperplane of $\mathbb{P}^4(\mathbb{K})$ contains $r_1 \cup r_2 \cup r_3$. Prove that there exists a unique line that meets r_i for every $i = 1, 2, 3$.

Solution (1). For every $i, j \in \{1, 2, 3\}, i \neq j$, let $V_{ij} = L(r_i, r_j)$. An easy application of Grassmann's formula implies that $\dim V_{ij} = 3$ for every i, j . Since the lines r_1, r_2, r_3 are not contained in a hyperplane, we have $L(V_{12} \cup V_{13}) = L(r_1, r_2, r_3) = \mathbb{P}^4(\mathbb{K})$, so $\dim(V_{12} \cap V_{13}) = 2$, again by Grassmann's formula. Moreover, if $V_{12} \cap V_{13} \subseteq V_{23}$, then the line r_1 is contained in V_{23} , and this contradicts the fact that r_1, r_2, r_3 are not contained in a hyperplane. It follows that the subspace $l = V_{12} \cap V_{13} \cap V_{23}$ has dimension one, so it is a projective line.

We now check that l meets each $r_i, i = 1, 2, 3$, and that l is the unique line of $\mathbb{P}^4(\mathbb{K})$ with this property. If $\{i, j, k\} = \{1, 2, 3\}$, by construction l lies on the plane $V_{ij} \cap V_{ik}$, which contains also r_i . Since two coplanar projective lines always intersect, we deduce that $l \cap r_i \neq \emptyset$. Moreover, if s is any line meeting each $r_i, i = 1, 2, 3$, then it is easy to show that $s \subseteq V_{ij}$ for every $i, j \in \{1, 2, 3\}$, so $s \subseteq l$. But $\dim s = \dim l = 1$, hence $s = l$.

Solution (2). Let us show how the statement of Exercise 10 can be deduced from the statement of Exercise 8.

If $V_{23} = L(r_2, r_3)$, a direct application of Grassmann's formula implies that $\dim V_{23} = 3$. Moreover, since r_1, r_2, r_3 are not contained in a hyperplane, the line r_1 is not contained in V_{23} , so $r_1 \cap V_{23} = \{P\}$ for some $P \in \mathbb{P}^4(\mathbb{K})$. Now the statement of Exercise 8 implies that there exists a unique line $l \subseteq V_{23}$ that meets r_2 and r_3 and contains P . Therefore, this line meets r_i for every $i = 1, 2, 3$. Moreover, any line that meets both r_2 and r_3 must be contained in V_{23} , so it can intersect r_1 only at P . It follows that l is the unique line of $\mathbb{P}^4(\mathbb{K})$ that satisfies the required properties.

Note. It is not difficult to show that, by duality, the statement of the exercise is equivalent to the following proposition:

Let H_1, H_2, H_3 be planes of $\mathbb{P}^4(\mathbb{K})$ such that $L(H_i, H_j) = \mathbb{P}^4(\mathbb{K})$ for every $i \neq j$, and $H_1 \cap H_2 \cap H_3 = \emptyset$. Then, there exists a unique plane H_0 such that $L(H_0, H_i) \neq \mathbb{P}^4(\mathbb{K})$ for $i = 1, 2, 3$.

On the other hand, an easy application of Grassmann's formula implies that, if S, S' are distinct planes of $\mathbb{P}^4(\mathbb{K})$, then $L(S, S') \neq \mathbb{P}^4(\mathbb{K})$ if and only if $S \cap S'$ is a line, while $L(S, S') = \mathbb{P}^4(\mathbb{K})$ if and only if $S \cap S'$ is a point. It follows that the statements of Exercises 9 and 10 are equivalent.

Exercise 11. Let r and s be distinct lines in $\mathbb{P}^3(\mathbb{K})$ and let f be a projectivity of $\mathbb{P}^3(\mathbb{K})$ such that the fixed-point set of f coincides with $r \cup s$. For every $P \in \mathbb{P}^3(\mathbb{K}) \setminus (r \cup s)$ let l_P be the line joining P and $f(P)$. Prove that the line l_P meets both r and s , for every $P \in \mathbb{P}^3(\mathbb{K}) \setminus (r \cup s)$.

Solution. We first recall that the lines r and s are skew (cf. Sect. 1.2.5). Therefore, the statement of Exercise 8 ensures that, for any given $P \in \mathbb{P}^3(\mathbb{K}) \setminus (r \cup s)$, there exists a line t_P passing through P and meeting both r and s . In order to conclude it is now sufficient to show that $t_P = l_P$.

Let $A = t_P \cap r$, $B = t_P \cap s$. Then $f(P) \in f(L(A, B)) = L(f(A), f(B)) = L(A, B) = t_P$. Therefore, t_P contains both P and $f(P)$, and so it coincides with l_P .

Exercise 12. (*Desargues' Theorem*) Let $\mathbb{P}(V)$ be a projective plane and $A_1, A_2, A_3, B_1, B_2, B_3$ points of $\mathbb{P}(V)$ in general position. Consider the triangles T_1 and T_2 of $\mathbb{P}(V)$ with vertices A_1, A_2, A_3 and B_1, B_2, B_3 ; one says that T_1 and T_2 are in central perspective if there exists a point O in the plane, distinct from the A_i and B_i , such that all lines $L(A_i, B_i)$ pass through O .

Prove that T_1 and T_2 are in central perspective if and only if the points $P_1 = L(A_2, A_3) \cap L(B_2, B_3)$, $P_2 = L(A_3, A_1) \cap L(B_3, B_1)$ and $P_3 = L(A_1, A_2) \cap L(B_1, B_2)$ are collinear. (cf. Fig. 2.1).

Solution. It is easy to check that the points P_1, P_2, P_3 satisfy the following properties: they are pairwise distinct, they are distinct from any vertex of T_1 and T_2 , and the points A_1, B_1, P_3, P_2 provide a projective frame of $\mathbb{P}(V)$. The point A_2 belongs to the line $L(A_1, P_3)$, so it has coordinates $[1, 0, a_2]$ for some $a_2 \neq 0$, since A_1 and A_2 are distinct from each other. A similar argument shows that:

$$A_3 = [a_3, 1, 1], \quad B_2 = [0, 1, b_2], \quad B_3 = [1, b_3, 1],$$

where $a_3, b_2, b_3 \in \mathbb{K}$, $b_2 \neq 0$, $a_3 \neq 1$ and $b_3 \neq 1$.

Let us set $P'_1 = L(A_2, A_3) \cap L(P_2, P_3)$ and $P''_1 = L(B_2, B_3) \cap L(P_2, P_3)$. The points P_1, P_2, P_3 are collinear if and only if $P'_1 = P''_1$ (and in this case $P_1 = P'_1 = P''_1$). The lines $L(A_2, A_3)$ and $L(B_2, B_3)$ have equations $a_2x_0 + (1 - a_2a_3)x_1 - x_2 = 0$ and $(1 - b_2b_3)x_0 + b_2x_1 - x_2 = 0$, respectively. So $P'_1 = [1, 1, 1 - a_2a_3 + a_2]$ and

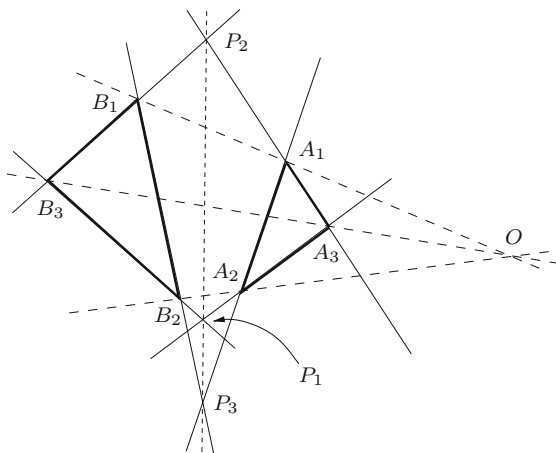


Fig. 2.1 The configuration described in Desargues' Theorem

$P_1'' = [1, 1, 1 - b_2b_3 + b_2]$. Therefore, our previous considerations imply that P_1, P_2 and P_3 are collinear if and only if the following equality holds:

$$a_2(1 - a_3) = b_2(1 - b_3). \quad (2.1)$$

Let us now analyze the condition that T_1 and T_2 are in central perspective. The line $L(A_1, B_1)$ has equation $x_2 = 0$, the line $L(A_2, B_2)$ has equation $a_2x_0 + b_2x_1 - x_2 = 0$, and the line $L(A_3, B_3)$ has equation $(1 - b_3)x_0 + (1 - a_3)x_1 + (a_3b_3 - 1)x_2 = 0$. These three lines are concurrent if and only if the corresponding points of the dual projective plane are collinear, i.e., if and only if

$$\det \begin{pmatrix} 0 & 0 & 1 \\ a_2 & b_2 & -1 \\ 1 - b_3 & 1 - a_3 & a_3b_3 - 1 \end{pmatrix} = 0. \quad (2.2)$$

We observe that, if this condition holds, then the point O belonging to the three lines has coordinates $[b_2, -a_2, 0]$. Since $a_2 \neq 0$ and $b_2 \neq 0$, the point O is distinct from any vertex of T_1 and T_2 .

Finally, the conclusion follows from the fact that conditions (2.1) and (2.2) are clearly equivalent.

Note. The solution just described directly proves that the two conditions stated in the exercise are equivalent. We have already noticed in Sect. 1.4.4 that Desargues' Theorem is an example of self-dual proposition, so, in fact, it is sufficient to prove just one implication, since the other one follows from the Duality principle.

Exercise 13. (*Pappus' Theorem*) Let $\mathbb{P}(V)$ be a projective plane, and let A_1, \dots, A_6 be pairwise distinct points such that the lines $L(A_1, A_2), L(A_2, A_3), \dots, L(A_6, A_1)$ are pairwise distinct. Consider the hexagon of $\mathbb{P}(V)$ with vertices A_1, \dots, A_6 , and suppose that there exist two distinct lines r, s such that $A_1, A_3, A_5 \in r, A_2, A_4, A_6 \in s$ and $O = r \cap s$ is distinct from each A_i .

Prove that the points where the opposite sides of the hexagon meet are collinear, i.e., that the points $P_1 = L(A_1, A_2) \cap L(A_4, A_5)$, $P_2 = L(A_2, A_3) \cap L(A_5, A_6)$ and $P_3 = L(A_3, A_4) \cap L(A_6, A_1)$ lie on a projective line (cf. Fig. 2.2).

Solution. By hypothesis we have $r = L(A_1, A_3)$ and $s = L(A_2, A_4)$. Since $r \neq s$ and the point $O = r \cap s$ is not a vertex of the hexagon, the points A_1, A_2, A_3, A_4 form a projective frame. In the corresponding system of homogeneous coordinates of $\mathbb{P}(V)$ the line r has equation $x_1 = 0$, the line s has equation $x_0 - x_2 = 0$, and the point O has coordinates $[1, 0, 1]$. The point A_5 lies on r and is distinct from O , from A_1 and from A_2 , so it has coordinates $[1, 0, a]$ for some $a \in \mathbb{K} \setminus \{0, 1\}$. In the same way, the point A_6 has coordinates $[1, b, 1]$, where $b \in \mathbb{K} \setminus \{0, 1\}$. The line $L(A_1, A_2)$ has equation $x_2 = 0$ and the line $L(A_4, A_5)$ has equation $ax_0 + (1 - a)x_1 - x_2 = 0$, so the point $P_1 = L(A_1, A_2) \cap L(A_4, A_5)$ has coordinates $[a - 1, a, 0]$. Similar computations show that P_2 has coordinates $[0, b, 1 - a]$ and P_3 has coordinates $[b, b, 1]$. It follows that the points P_1, P_2 and P_3 are collinear, since

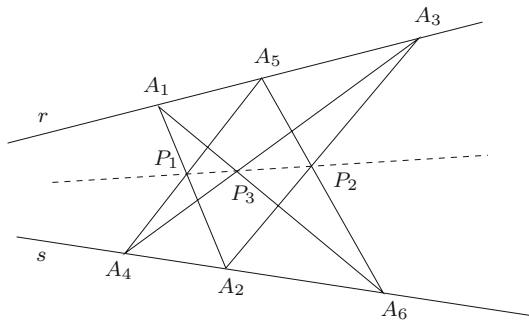


Fig. 2.2 The configuration described in Pappus' Theorem

$$\det \begin{pmatrix} a-1 & a & 0 \\ 0 & b & 1-a \\ b & b & 1 \end{pmatrix} = 0.$$

Exercise 14. Let A, A', B, B' be pairwise distinct non-collinear points of $\mathbb{P}^2(\mathbb{K})$. Prove that A, A', B, B' are in general position if and only if there exists a projectivity $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ such that $f(A) = B, f(A') = B', f^2 = \text{Id}$.

Solution. Suppose that a projectivity f as in the statement exists. We first observe that $f(B) = f(f(A)) = A, f(B') = f(f(A')) = A'$. In particular, the lines $L(A, B)$ and $L(A', B')$ are invariant under f . Moreover, since A, A', B, B' are not collinear, the lines $L(A, B)$ and $L(A', B')$ are distinct and meet at a single point O such that $f(O) = f(L(A, B) \cap L(A', B')) = L(A, B) \cap L(A', B') = O$. It is immediate to check that, if A, A', B, B' were not in general position, then we would have $O \in \{A, A', B, B'\}$, and this would contradict the fact that no point in $\{A, A', B, B'\}$ is a fixed point of f . We have thus shown that A, A', B, B' are in general position, as desired.

Let us now prove the converse implication. Assume that the points A, A', B, B' are in general position. The Fundamental theorem of projective transformations ensures that there exists a (unique) projectivity $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ such that $f(A) = B, f(A') = B', f(B) = A, f(B') = A'$. By construction, f^2 and the identity of $\mathbb{P}^2(\mathbb{K})$ coincide on A, A', B, B' , so they coincide on the whole of $\mathbb{P}^2(\mathbb{K})$, again by the Fundamental theorem of projective transformations. Therefore, f satisfies the required properties. We also observe that any projectivity satisfying the conditions described in the statement must coincide with f on A, A', B, B' , so f is uniquely determined by such conditions.

Exercise 15. Determine a projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ with the following properties: if $P = [1, 2, 1], Q = [1, 1, 1]$ and r, r', s, s' are the lines described by the equations

$$\begin{aligned} r: x_0 - x_1 &= 0, & r': x_0 + x_1 &= 0 \\ s: x_0 + x_1 + x_2 &= 0, & s': x_1 + x_2 &= 0, \end{aligned}$$

then $f(r) = r', f(s) = s'$, and $f(P) = Q$. Is such a projectivity unique?

Solution. First observe that $P \notin r \cup s$, $Q \notin r' \cup s'$. Let $P_1 = r \cap s = [1, 1, -2]$, $Q_1 = r' \cap s' = [1, -1, 1]$, and let us choose points P_2, P_3 distinct from P on r, s , respectively: for example, we set $P_2 = [0, 0, 1]$, $P_3 = [-1, 1, 0]$. It is easy to check that the points P_1, P_2, P_3, P are in general position. In the same way, if $Q_2 = [0, 0, 1]$, then $Q_2 \in r'$, and Q_1, Q_2, Q are in general position. If Q_3 is any point of s' distinct from Q_1 and from $s' \cap L(Q_2, Q) = [1, 1, -1]$, then the quadruples $\{P_1, P_2, P_3, P\}$ and $\{Q_1, Q_2, Q_3, Q\}$ provide two projective frames of $\mathbb{P}^2(\mathbb{R})$. Therefore, there exists a unique projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ such that $f(P_i) = Q_i$ for every $i = 1, 2, 3$ and $f(P) = Q$. Since $r = L(P_1, P_2)$, $s = L(P_1, P_3)$, $r' = L(Q_1, Q_2)$, $s' = L(Q_1, Q_3)$, we also have $f(r) = r'$, $f(s) = s'$. Since Q_3 may be chosen in infinitely many ways, an infinite number of projectivities satisfy the required conditions.

Let us explicitly construct f in the case when $Q_3 = [1, 0, 0]$. A normalized basis associated to the frame $\{P_1, P_2, P_3, P\}$ is given by $\{v_1 = (3, 3, -6), v_2 = (0, 0, 8), v_3 = (-1, 1, 0)\}$, while a normalized basis associated to the frame $\{Q_1, Q_2, Q_3, Q\}$ is given by $\{w_1 = (-1, 1, -1), w_2 = (0, 0, 2), w_3 = (2, 0, 0)\}$. Therefore, f is induced by the unique linear isomorphism $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\varphi(v_i) = w_i$ for $i = 1, 2, 3$. A straightforward computation shows that this isomorphism is given, up to a non-zero scalar, by the matrix $\begin{pmatrix} 14 & -10 & 0 \\ -2 & -2 & 0 \\ -1 & -1 & -3 \end{pmatrix}$, so

the required projectivity f may be explicitly described by the following formula: $f([x_0, x_1, x_2]) = [14x_0 - 10x_1, -2x_0 - 2x_1, -x_0 - x_1 - 3x_2]$.

Exercise 16. Let r, s, r', s' be lines in $\mathbb{P}^2(\mathbb{K})$ such that $r \neq s$, $r' \neq s'$, and let $g: r \rightarrow r', h: s \rightarrow s'$ be projective isomorphisms. Find necessary and sufficient conditions for the existence of a projectivity $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ such that $f|_r = g$ and $f|_s = h$. Show that, when it exists, such an f is unique.

Solution. Let us consider the points $P = r \cap s$, $P' = r' \cap s'$ (which are uniquely defined since $r \neq s$, $r' \neq s'$). Of course, if the required projectivity exists, then we must have $g(P) = h(P)$ (so necessarily $g(P) = h(P) = P'$). Let us show that this condition is also sufficient.

Let P_1, P_2 be pairwise distinct points of $r \setminus \{P\}$, let Q_1, Q_2 be pairwise distinct points of $s \setminus \{P\}$, and set $P'_i = g(P_i)$, $Q'_i = h(Q_i)$, $i = 1, 2$. It is immediate to check that the quadruples $\mathcal{R} = \{P_1, P_2, Q_1, Q_2\}$, $\mathcal{R}' = \{P'_1, P'_2, Q'_1, Q'_2\}$ are in general position, so they define two projective frames of $\mathbb{P}^2(\mathbb{K})$. Therefore, there exists a unique projectivity $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ such that $f(P_i) = P'_i$, $f(Q_i) = Q'_i$ for $i = 1, 2$. We now show that this projectivity satisfies the required conditions.

We have $f(r) = f(L(P_1, P_2)) = L(P'_1, P'_2) = r'$, and in the same way $f(s) = s'$. Therefore, $f(P) = f(r \cap s) = r' \cap s' = P'$. So the projective transformations $f|_r$ and g coincide on three pairwise distinct points of r , hence on the whole line r . In the same way one proves that $f|_s = h$.

Finally, the fact that f is unique readily follows from the Fundamental theorem of projective transformations: any projectivity that satisfies the required conditions must coincide with f on \mathcal{R} , hence on the whole of $\mathbb{P}^2(\mathbb{K})$.

Exercise 17. Let S, S' be planes of $\mathbb{P}^3(\mathbb{K})$ and r, r' lines of $\mathbb{P}^3(\mathbb{K})$ such that $L(r, S) = L(r', S') = \mathbb{P}^3(\mathbb{K})$, and let $g: S \rightarrow S'$, $h: r \rightarrow r'$ be projective isomorphisms. Find necessary and sufficient conditions for the existence of a projectivity $f: \mathbb{P}^3(\mathbb{K}) \rightarrow \mathbb{P}^3(\mathbb{K})$ such that $f|_S = g$ and $f|_r = h$. Show that, if such an f exists, then it is unique.

Solution. An easy application of Grassmann's formula shows that there exist points $P, P' \in \mathbb{P}^3(\mathbb{K})$ such that $r \cap S = \{P\}$, $r' \cap S' = \{P'\}$. Of course, if the required projectivity exists, then we must have $g(P) = h(P) = P'$. We now show that this condition is also sufficient.

We extend P to a projective frame $\{P, P_1, P_2, P_3\}$ of S , and we choose distinct points Q_1, Q_2 on r , such that $Q_1 \neq P$, $Q_2 \neq P$. We first show that $\mathcal{R} = \{P_1, P_2, P_3, Q_1, Q_2\}$ is in general position, so it is a projective frame of $\mathbb{P}^3(\mathbb{K})$. To this aim it is sufficient to check that \mathcal{R} does not contain any quadruple of coplanar points. Since $L(P_1, P_2, P_3) = S$ and $Q_l \notin S$, the points P_1, P_2, P_3, Q_l cannot be coplanar for $l = 1, 2$. Therefore, we may assume by contradiction that P_i, P_j, Q_1, Q_2 are coplanar for some $i \neq j$. In this case, the lines $L(P_i, P_j) \subset S$, $L(Q_1, Q_2) = r$ intersect in $P = S \cap r$, hence P, P_i, P_j are collinear. But this contradicts the fact that P, P_i, P_j are in general position on S . We have thus proved that \mathcal{R} is a projective frame.

Let now $P'_i = g(P_i)$, $Q'_j = h(Q_j)$ for $i = 1, 2, 3, j = 1, 2$. The same argument as above shows that $P'_1, P'_2, P'_3, Q'_1, Q'_2$ are also in general position, so there exists a unique projectivity $f: \mathbb{P}^3(\mathbb{K}) \rightarrow \mathbb{P}^3(\mathbb{K})$ such that $f(P_i) = P'_i$, $f(Q_j) = Q'_j$ for $i = 1, 2, 3, j = 1, 2$. Let us show that this projectivity coincides with g on S and with h on r . We have

$$\begin{aligned} f(S) &= f(L(P_1, P_2, P_3)) = L(P'_1, P'_2, P'_3) = S', \\ f(r) &= f(L(Q_1, Q_2)) = L(Q'_1, Q'_2) = r', \end{aligned}$$

so $f(P) = f(r \cap S) = r' \cap S' = P'$. Therefore, $f|_S$ and g coincide on the projective frame $\{P, P_1, P_2, P_3\}$ of S , so they coincide on the whole of S . In the same way one proves that $f|_r$ and h coincide on P, Q_1, Q_2 , hence on r .

Finally, the fact that f is unique is now obvious: any projectivity satisfying the required properties must coincide with f on P_1, P_2, P_3, Q_1, Q_2 , hence on the whole projective space $\mathbb{P}^3(\mathbb{K})$, since $\{P_1, P_2, P_3, Q_1, Q_2\}$ is a projective frame of $\mathbb{P}^3(\mathbb{K})$.

Note. Exercises 16 and 17 illustrate particular cases of a general fact regarding the extension of projective transformations defined on subspaces of a projective space. In the projective space $\mathbb{P}(V)$ let us consider the projective subspaces S_1, S_2, S'_1, S'_2 , and let us fix projective isomorphisms $g_1: S_1 \rightarrow S'_1$, $g_2: S_2 \rightarrow S'_2$. Suppose also that $L(S_1, S_2) = \mathbb{P}(V)$, $g_1(S_1 \cap S_2) = g_2(S_1 \cap S_2) = S'_1 \cap S'_2$, and $g_1|_{S_1 \cap S_2} = g_2|_{S_1 \cap S_2}$. Then, there exists a projectivity $f: \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ such that $f|_{S_i} = g_i$ for $i = 1, 2$. Moreover, if $S_1 \cap S_2 \neq \emptyset$, then such a projectivity is unique.

Exercise 18. Let r, s be distinct lines of $\mathbb{P}^2(\mathbb{K})$, let A, B be distinct points of $r \setminus s$, and take $C, D \in \mathbb{P}^2(\mathbb{K}) \setminus (r \cup s)$. Show that there exists a unique projectivity $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ such that $f(A) = A, f(B) = B, f(s) = s, f(C) = D$.

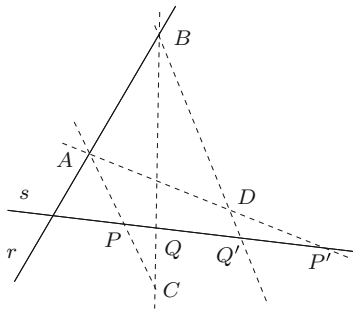


Fig. 2.3 The construction described in Exercise 18

Solution. Since $C, D \notin s$, each line $L(C, A)$, $L(C, B)$, $L(D, A)$, $L(D, B)$ meets s at exactly one point, so we can set $P = s \cap L(C, A)$, $P' = s \cap L(D, A)$, $Q = s \cap L(C, B)$, $Q' = s \cap L(D, B)$ (cf. Fig. 2.3). As $C, D \notin r$, the points P, P', Q, Q' lie on $s \setminus r$. Finally, we obviously have $P \neq Q$, $P' \neq Q'$. It follows that the quadruples $\{A, B, P, Q\}$ and $\{A, B, P', Q'\}$ define projective frames of $\mathbb{P}^2(\mathbb{K})$.

Let now f be the unique projectivity of $\mathbb{P}^2(\mathbb{K})$ such that $f(A) = A$, $f(B) = B$, $f(P) = P'$, $f(Q) = Q'$. Then we have $f(s) = f(L(P, Q)) = L(P', Q') = s$. Moreover, $f(C) = f(L(A, P) \cap L(B, Q)) = L(A, P') \cap L(B, Q') = D$, so f satisfies the required properties. Conversely, if g is a projectivity of $\mathbb{P}^2(\mathbb{K})$ satisfying the required properties, then $g(A) = A$, $g(B) = B$, $g(P) = g(L(A, C) \cap s) = L(A, D) \cap s = P'$ and $g(Q) = g(L(C, B) \cap s) = L(D, B) \cap s = Q'$, so $g = f$.



Exercise 19. In $\mathbb{P}^3(\mathbb{R})$, let r be the line of equations $x_0 - x_1 = x_2 - x_3 = 0$, let H, H' be the planes of equations $x_1 + x_2 = 0, x_1 - 2x_3 = 0$, respectively, and let $C = [1, 1, 0, 0]$. Compute the number of projectivities $f: \mathbb{P}^3(\mathbb{R}) \rightarrow \mathbb{P}^3(\mathbb{R})$ that satisfy the following conditions:

- (i) $f(r) = r, f(H) = H', f(H') = H, f(C) = C$;
- (ii) the fixed-point set of f contains a plane.

Solution. Let us first determine the incidence relations that hold among the subspaces described in the statement. It is immediate to check that $C \in r$ and $C \notin H \cup H'$. As a consequence, $r \cap H$ and $r \cap H'$ both consist of a single point: more precisely, an easy computation shows that, if $A = r \cap H$ and $B = r \cap H'$, then $A = [1, 1, -1, -1]$, $B = [2, 2, 1, 1]$. Moreover, since $H \neq H'$, the set $l = H \cap H'$ is a projective line, and the fact that $A \neq B$ readily implies that the lines r and l are skew.

Let now f be a projectivity satisfying the required conditions, and let S be a plane pointwise fixed by f . In order to understand the behaviour of f , we now determine the possible positions of S , showing first that S must contain l . We have $H \cap S = f(H \cap S) = H' \cap S$, so $H \cap S = H' \cap S$ is a projective subspace of $H \cap H' = l$. As $\dim(H \cap S) \geq 1$, we have $l = H \cap S = H' \cap S$, so in particular $l \subset S$, as desired.

Since l and r are skew, the intersection $r \cap S$ consists of one point. This point is fixed by f , so in order to determine S it is useful to study the fixed-point set

of the restriction of f to r . Recall that by hypothesis $C \in r$, while by construction $A, B \in r$. We also have $f(A) = f(r \cap H) = r \cap H' = B$, and in the same way $f(B) = A$. Since $f(C) = C$, if $g: r \rightarrow r$ is the unique projectivity such that $g(A) = B$, $g(B) = A$, $g(C) = C$, then $f|_r = g$. Moreover, an easy computation shows that, if $r = \mathbb{P}(W)$, then g is induced by the unique linear isomorphism $\varphi: W \rightarrow W$ such that $\varphi(1, 1, -1, -1) = (2, 2, 1, 1)$ and $\varphi(2, 2, 1, 1) = (1, 1, -1, -1)$. Therefore, the fixed-point set of g (hence, of $f|_r$) is given by $\{C, D\}$, where $D = [1, 1, 2, 2]$. It follows that either $r \cap S = C$ or $r \cap S = D$.

Then, let $S_1 = L(l, C)$ and $S_2 = L(l, D)$. Since $C \notin l$, $D \notin l$ and $l \subset S$, if $C \in S$ then $S = S_1$, while if $D \in S$ then $S = S_2$. Moreover, since as observed above the lines r, s are skew, we have $L(S_1, r) = L(S_2, r) = \mathbb{P}^3(\mathbb{R})$.

Take $i \in \{1, 2\}$. As g fixes both $C = S_1 \cap r$ and $D = S_2 \cap r$, we may exploit Exercise 17 to show that there exists a unique projectivity $f_i: \mathbb{P}^3(\mathbb{R}) \rightarrow \mathbb{P}^3(\mathbb{R})$ such that $f_i|_{S_i} = \text{Id}_{S_i}$ and $f_i|_r = g$. Moreover, what we have shown so far implies that, if f is a projectivity satisfying the conditions described in the statement, then f necessarily coincides either with f_1 or with f_2 . We observe that $f_1 \neq f_2$, because otherwise the fixed-point set of f_1 would contain $S_1 \cup S_2$. Being the union of pairwise skew projective subspaces (cf. Sect. 1.2.5), the fixed-point set of f_1 would then coincide with the whole of $\mathbb{P}^3(\mathbb{R})$, contradicting the fact that $f_1(A) \neq A$.

In order to conclude it is now sufficient to show that both f_1 and f_2 satisfy the required conditions. By construction, for $i = 1, 2$ we have $f_i(C) = C$, $f_i(r) = r$, and the fixed-point set of f_i contains the plane S_i . Finally, we have $f_i(H) = f_i(L(l, A)) = L(l, B) = H'$ and $f_i(H') = f_i(L(l, B)) = L(l, A) = H$, as desired.

Exercise 20. Let $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ be an injective function such that

$$\beta(P_1, P_2, P_3, P_4) = \beta(f(P_1), f(P_2), f(P_3), f(P_4))$$

for every quadruple P_1, P_2, P_3, P_4 of pairwise distinct points. Show that f is a projectivity.

Solution. Let P_1, P_2, P_3 be pairwise distinct points of $\mathbb{P}^1(\mathbb{K})$, and set $Q_i = f(P_i)$ for $i = 1, 2, 3$. Since f is injective, the sets $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are projective frames of $\mathbb{P}^1(\mathbb{K})$. Therefore, there exists a projectivity $g: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $g(P_i) = Q_i$ for $i = 1, 2, 3$. Moreover, since the cross-ratio is invariant under projectivities, for every $P \notin \{P_1, P_2, P_3\}$ we have

$$\begin{aligned} \beta(Q_1, Q_2, Q_3, g(P)) &= \beta(g(P_1), g(P_2), g(P_3), g(P)) = \beta(P_1, P_2, P_3, P) = \\ &= \beta(f(P_1), f(P_2), f(P_3), f(P)) = \beta(Q_1, Q_2, Q_3, f(P)). \end{aligned}$$

As a consequence, $f(P) = g(P)$ for every $P \neq P_1, P_2, P_3$. On the other hand, for $i = 1, 2, 3$ we have $f(P_i) = g(P_i)$ by construction, so $f = g$, and f is a projectivity.



Exercise 21. (*Modulus of a quadruple of points*) Let

$$\mathcal{A} = \{P_1, P_2, P_3, P_4\}, \quad \mathcal{A}' = \{P'_1, P'_2, P'_3, P'_4\}$$

be quadruples of pairwise distinct points of $\mathbb{P}^1(\mathbb{C})$, and set

$$k = \beta(P_1, P_2, P_3, P_4), \quad k' = \beta(P'_1, P'_2, P'_3, P'_4).$$

(a) Show that the sets $\mathcal{A}, \mathcal{A}'$ are projectively equivalent if and only if

$$\frac{(k^2 - k + 1)^3}{k^2(k-1)^2} = \frac{((k')^2 - k' + 1)^3}{(k')^2(k'-1)^2}.$$

(b) Let G be the set of projectivities $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ such that $f(\mathcal{A}) = \mathcal{A}$. For every $k \in \mathbb{C} \setminus \{0, 1\}$, compute the number $|G|$ of elements of G .

Solution. (a) As discussed in Sect. 1.5.2, the sets $\mathcal{A}, \mathcal{A}'$ are projectively equivalent if and only if k' belongs to the set

$$\Omega(k) = \left\{ k, \frac{1}{k}, 1-k, \frac{1}{1-k}, \frac{k-1}{k}, \frac{k}{k-1} \right\}.$$

(Observe that $k, k' \in \mathbb{C} \setminus \{0, 1\}$, since $P_i \neq P_j$ and $P'_i \neq P'_j$ for every $i \neq j$.)

Consider the rational function $j(t) = \frac{(t^2 - t + 1)^3}{t^2(t-1)^2}$. In order to prove (a) it is sufficient to show that $k' \in \Omega(k)$ if and only if $j(k') = j(k)$.

Via a direct substitution, it is immediate to verify that $j(k) = j(k')$ if $k' \in \Omega(k)$.

Conversely, let us assume that $j(k) = j(k')$, and set $q(t) = (t^2 - t + 1)^3 - j(k)t^2(t-1)^2$. Of course we have $q(k') = 0$. Moreover, q is a polynomial of degree 6 admitting every element of $\Omega(k)$ as a root. Therefore, if $\Omega(k)$ contains 6 elements, then it coincides with the set of roots of q , and since $q(k') = 0$ we may deduce that $k' \in \Omega(k)$, as desired. If we set $\omega = \frac{1+i\sqrt{3}}{2}$, then a direct computation shows that $|\Omega(k)| < 6$ only in the following cases:

- when $k \in \{\omega, \bar{\omega}\}$, and in this case $\Omega(k) = \{\omega, \bar{\omega}\}$;
- when $k \in \left\{-1, 2, \frac{1}{2}\right\}$, and in this case $\Omega(k) = \left\{-1, 2, \frac{1}{2}\right\}$.

Moreover, if $k \in \left\{-1, 2, \frac{1}{2}\right\}$ then $j(k) = \frac{27}{4}$ and $q(t) = (t+1)^2(t-2)^2\left(t - \frac{1}{2}\right)^2$, while if $k \in \{\omega, \bar{\omega}\}$ then $j(k) = 0$ and $q(t) = (t-\omega)^3(t-\bar{\omega})^3$. In any case, $\Omega(k)$ coincides with the set of roots of q , and from the fact that $q(k') = 0$ we can deduce that $k' \in \Omega(k)$, as desired.

(b) Of course G is a group. If S_4 is the permutation group of $\{1, 2, 3, 4\}$, then for any given $f \in G$ there exists $\psi(f) \in S_4$ such that $f(P_i) = P_{\psi(f)(i)}$ for every $i = 1, 2, 3, 4$. Moreover, the map $\psi: G \rightarrow S_4$ thus defined is a group homomorphism. If $\psi(f) = \text{Id}$, then f coincides with the identity on 3 distinct points of $\mathbb{P}^1(\mathbb{C})$, hence on the whole of $\mathbb{P}^1(\mathbb{C})$: therefore, the homomorphism ψ is injective, so we have $|G| = |\text{Im } \psi| \leq |S_4| = 24$. Let us now investigate which permutations of the P_i are actually induced by a projectivity. To this aim, we will exploit some elementary facts about group actions on sets.

Let us consider the map $\eta: S_4 \times \Omega(k) \rightarrow \Omega(k)$ which is defined as follows: for every $h \in \Omega(k)$ and $\sigma \in S_4$, if $Q_1, Q_2, Q_3, Q_4 \in \mathbb{P}^1(\mathbb{C})$ are such that $\beta(Q_1, Q_2, Q_3, Q_4) = h$, then $\eta(\sigma, h) = \beta(Q_{\sigma(1)}, Q_{\sigma(2)}, Q_{\sigma(3)}, Q_{\sigma(4)})$. The properties of the cross-ratio described in Sect. 1.5.2 imply that $\eta(\sigma, h)$ does not depend on the choice of the Q_i ; moreover, $\eta(\sigma, h) \in \Omega(k)$, so η is indeed well defined. Finally, it is immediate to check that $\eta(\sigma \circ \tau, h) = \eta(\sigma, \eta(\tau, h))$, so η defines an action of S_4 on $\Omega(k)$.

From the fundamental property of the cross-ratio (cf. Theorem 1.5.1) we deduce that $\sigma \in \text{Im } \psi$ if and only if $\beta(P_1, P_2, P_3, P_4) = \beta(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, P_{\sigma(4)})$. In other words, $\text{Im } \psi$ coincides with the stabilizer of k with respect to the action we have just introduced; moreover, this action is transitive, since every element of $\Omega(k)$ is the cross-ratio of an ordered quadruple obtained by permuting P_1, P_2, P_3, P_4 . We thus have $|S_4| = |\text{Stab}(k)| |\Omega(k)| = |\text{Im } \psi| |\Omega(k)|$, so

$$|G| = |\text{Im } \psi| = \frac{|S_4|}{|\Omega(k)|} = \frac{24}{|\Omega(k)|}.$$

Then, it follows from (a) that $|G| = 12$ if $k \in \{\omega, \bar{\omega}\}$, $|G| = 8$ if $k \in \left\{-1, 2, \frac{1}{2}\right\}$ and $|G| = 4$ otherwise.

Exercise 22. Let $f: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$ be the projectivity defined by

$$f([x_0, x_1]) = [-x_1, 2x_0 + 3x_1].$$

- (a) Determine the fixed-point set of f .
- (b) For $P = [2, 5] \in \mathbb{P}^1(\mathbb{R})$, compute the cross-ratio $\beta(A, B, P, f(P))$, where A and B are the fixed points of f .

Solution. The projectivity f is induced by the linear map $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is represented with respect to the canonical basis of \mathbb{R}^2 by the matrix $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$. This matrix is diagonalizable, and it admits $(1, -1)$ and $(1, -2)$ as eigenvectors relative to the eigenvalues 1 and 2, respectively. It follows that $A = [1, -1]$ and $B = [1, -2]$ are the only fixed points of f .

We have seen in Sect. 1.5.4 that, since $A = [v]$ where v is an eigenvector of φ relative to the eigenvalue 1 and $B = [w]$ where w is an eigenvector of φ relative to the eigenvalue 2, the value $\beta(A, B, Q, f(Q))$ does not depend on $Q \in \mathbb{P}^1(\mathbb{R}) \setminus \{A, B\}$, and it is equal to $2/1 = 2$. Therefore, $\beta(A, B, P, f(P)) = 2$.

Exercise 23. (*Involutions of $\mathbb{P}^1(\mathbb{K})$*) Let f be a projectivity of $\mathbb{P}^1(\mathbb{K})$. Recall that f is an involution if $f^2 = \text{Id}$, and that an involution f is non-trivial if $f \neq \text{Id}$.

- (a) If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix associated to f , show that f is a non-trivial involution if and only if $a + d = 0$.
- (b) Show that f is a non-trivial involution if and only if there exist two distinct points Q_1, Q_2 that are switched by f , i.e., such that $f(Q_1) = Q_2$ and $f(Q_2) = Q_1$.

- (c) Suppose that f is a non-trivial involution. Show that f has exactly either 0 or 2 fixed points, and that it has exactly 2 fixed points if $\mathbb{K} = \mathbb{C}$.
- (d) Suppose that f fixes the points A, B . Show that f is a non-trivial involution if and only if, for any given point $P \in \mathbb{P}^1(\mathbb{K}) \setminus \{A, B\}$, the equality $\beta(A, B, P, f(P)) = -1$ holds (i.e., the characteristic of f is -1 , cf. Sect. 1.5.4).
- (e) Show that f is the composition of two involutions.

Solution. (a) First observe that, if $f \neq \text{Id}$, then the minimal polynomial of M cannot have degree 1 (so it is equal to the characteristic polynomial of M). Moreover, we have $f^2 = \text{Id}$ if and only if there exists $\lambda \in \mathbb{K}^*$ such that $M^2 = \lambda I$. It follows that f is a non-trivial involution if and only if the minimal polynomial and the characteristic polynomial of M coincide and are equal to $t^2 - \lambda$. Now the conclusion follows since the coefficient of t in the characteristic polynomial of M is equal to $-a - d$.

(b) If $f \neq \text{Id}$ is an involution, it is sufficient to choose any point Q_1 not fixed by f and set $Q_2 = f(Q_1)$. Conversely, let Q_1, Q_2 be points that are exchanged by f , let P be any point of $\mathbb{P}^1(\mathbb{K}) \setminus \{Q_1, Q_2\}$ and set $P' = f(P)$. Then

$$\begin{aligned} \beta(Q_1, Q_2, P', P) &= \beta(f(Q_1), f(Q_2), f(P'), f(P)) = \\ &= \beta(Q_2, Q_1, f(P'), P') = \beta(Q_1, Q_2, P', f(P')), \end{aligned}$$

where the first equality follows from the invariance of the cross-ratio under projectivities (cf. Sect. 1.5.1), while the second and the third one follow from the symmetries of the cross-ratio (cf. Sect. 1.5.2). Then $f^2(P) = f(P') = P$ and f is an involution.

(c) Point (a) implies that, if M is a matrix associated to f , then the characteristic polynomial of M is equal to $t^2 + \det M$, so either M does not admit any eigenvalue (when $-\det M$ is not a square in \mathbb{K}) or M admits two distinct eigenvalues relative to one-dimensional eigenspaces (when $-\det M$ is a square in \mathbb{K}). Therefore, f has exactly either 0 or 2 fixed points. Moreover, if \mathbb{K} is algebraically closed, then M admits two distinct eigenvalues, so f has exactly two fixed points.

(d) As observed in Sect. 1.5.4, the cross-ratio $\beta(A, B, P, f(P))$ is independent of the choice of P . More precisely, in a projective frame \mathcal{R} having A and B as fundamental points, f is represented by a matrix $N = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \neq 0$. If $P \in \mathbb{P}^1(\mathbb{K}) \setminus \{A, B\}$ and $[P]_{\mathcal{R}} = [a, b]$, then $[f(P)]_{\mathcal{R}} = [\lambda a, \mu b]$, so $\beta(A, B, P, f(P)) = \frac{\mu}{\lambda}$. Therefore, $\beta(A, B, P, f(P)) = -1$ if and only if $\mu = -\lambda$, i.e., if and only if $N = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. This condition is equivalent to the fact that N^2 is a multiple of the identity, i.e., to the fact that $f^2 = \text{Id}$.

(e) If $f = \text{Id}$ there is nothing to prove, so we suppose that there exists $A \in \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A' \neq A$, and we set $A'' = f(A')$. If $A'' = A$, then f switches A and A' , so it is an involution by point (b). Therefore, we can suppose that $A'' \neq A$, so that necessarily $A' \neq A''$, because otherwise we would have $f(A') = A' = f(A)$, which contradicts the fact that f is injective. As a consequence, being pairwise distinct, the points A, A', A'' define a projective frame of $\mathbb{P}^1(\mathbb{K})$, and by the Fundamental theorem of projective transformations there exists a projectivity $g: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$

such that $g(A) = A''$, $g(A') = A'$, $g(A'') = A$. Since g switches A and A'' , g is an involution. Moreover, $f \circ g$ switches A' and A'' , so it is an involution too. Therefore, $f = f \circ (g \circ g) = (f \circ g) \circ g$ is the composition of two involutions.

Exercise 24. Let A, B be distinct points of $\mathbb{P}^1(\mathbb{K})$. Show that there exists a unique non-trivial involution of $\mathbb{P}^1(\mathbb{K})$ having A and B as fixed points.

Solution (1). Take $P \in \mathbb{P}^1(\mathbb{K}) \setminus \{A, B\}$. If $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ is a projectivity such that $f(A) = A$ and $f(B) = B$, then it follows from point (d) of Exercise 23 that f is a non-trivial involution if and only if $\beta(A, B, P, f(P)) = -1$, i.e., if and only if $f(P)$ is the unique point such that $A, B, P, f(P)$ is a harmonic quadruple. Since a projectivity of $\mathbb{P}^1(\mathbb{K})$ is uniquely determined by the values it takes on A, B, P , this concludes the proof.

Solution (2). If we fix a projective frame of $\mathbb{P}^1(\mathbb{K})$ having A and B as fundamental points, every projectivity $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A$ and $f(B) = B$ is represented in the induced coordinate system by a matrix $N = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{K}^*$. It is easily seen that N^2 is a multiple of the identity if and only if $\lambda = \pm 1$. Therefore, the unique non-trivial involution having A and B as fixed points is the projectivity represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exercise 25. Let $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ be a projectivity, and let $A, B, C \in \mathbb{P}^1(\mathbb{K})$ be pairwise distinct points such that $f(A) = A, f(B) = C$. Show that A is the unique fixed point of f (i.e., f is parabolic, cf. Sect. 1.5.3) if and only if $\beta(A, C, B, f(C)) = -1$.

Solution. We endow $\mathbb{P}^1(\mathbb{K})$ with the homogeneous coordinates induced by the projective frame $\{A, B, C\}$. With respect to these coordinates, f is represented by a matrix M of the form $\begin{pmatrix} 1 & \lambda \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{K}^*$. Now, if $\lambda = 1$ the matrix M has exactly one eigenvalue, and this eigenvalue has geometric multiplicity one, so f is parabolic; otherwise, M has two distinct eigenvalues, and it is hyperbolic. So we need to show that $\beta(A, C, B, f(C)) = -1$ if and only if $\lambda = 1$. But the coordinates of $f(C)$ are $[1 + \lambda, \lambda]$, so

$$\beta(A, C, B, f(C)) = \beta([1, 0], [1, 1], [0, 1], [1 + \lambda, \lambda]) = -\lambda,$$

and this concludes the proof.

Exercise 26. Let A_1, A_2, A_3, A_4 be points of $\mathbb{P}^2(\mathbb{K})$ in general position, and set

$$\begin{aligned} P_1 &= L(A_1, A_2) \cap L(A_3, A_4), & P_2 &= L(A_2, A_3) \cap L(A_1, A_4), & r &= L(P_1, P_2), \\ P_3 &= L(A_2, A_4) \cap r, & P_4 &= L(A_1, A_3) \cap r. \end{aligned}$$

Compute $\beta(P_1, P_2, P_3, P_4)$.

Solution (1). If we endow $\mathbb{P}^2(\mathbb{K})$ with homogeneous coordinates such that $A_1 = [1, 0, 0]$, $A_2 = [0, 1, 0]$, $A_3 = [0, 0, 1]$, $A_4 = [1, 1, 1]$, we easily obtain that $P_1 = [1, 1, 0]$ and $P_2 = [0, 1, 1]$. So $r = \{x_0 - x_1 + x_2 = 0\}$, hence $P_3 = [1, 2, 1]$ and $P_4 = [1, 0, -1]$. It follows that, if $r = \mathbb{P}(W)$, then a normalized basis of W associated to the projective frame $\{P_1, P_2, P_3\}$ is given by $(1, 1, 0)$, $(0, 1, 1)$. Since $(1, 0, -1) = (1, 1, 0) - (0, 1, 1)$, it follows that the required cross-ratio is equal to -1 .

Solution (2). We set $t = L(A_1, A_3)$ and $Q = t \cap L(A_2, A_4)$ (cf. Fig. 2.4). Of course $A_2 \notin r \cup t$, $A_4 \notin r \cup t$, so the perspectivity $f: r \rightarrow t$ centred at A_2 and the perspectivity $g: t \rightarrow r$ centred at A_4 are well defined. By construction we have $f(P_1) = A_1$, $f(P_2) = A_3$, $f(P_3) = Q$, $f(P_4) = P_4$, so, being A_1, A_3, Q, P_4 pairwise distinct, the points P_1, P_2, P_3, P_4 are pairwise distinct too. Moreover, again by construction we have $g(A_1) = P_2$, $g(A_3) = P_1$, $g(Q) = P_3$, $g(P_4) = P_4$. Being the composition of two projectivities, the map $g \circ f: r \rightarrow r$ is a projectivity, so

$$\begin{aligned} \beta(P_1, P_2, P_3, P_4) &= \beta(g(f(P_1)), g(f(P_2)), g(f(P_3)), g(f(P_4))) = \\ &= \beta(P_2, P_1, P_3, P_4) = \frac{1}{\beta(P_1, P_2, P_3, P_4)}, \end{aligned}$$

where the first equality is due to the invariance of the cross-ratio under projectivities (cf. Sect. 1.5.1), while the last one is due to the symmetries of the cross-ratio (cf. Sect. 1.5.2). So $\beta(P_1, P_2, P_3, P_4)^2 = 1$, hence $\beta(P_1, P_2, P_3, P_4) = -1$ since the P_i are pairwise distinct, as observed above.

Note. (*Construction of the harmonic conjugate*) The previous exercise suggests a way to explicitly construct the *harmonic conjugate* of three pairwise distinct points P_1, P_2, P_3 of a projective line $r \subseteq \mathbb{P}^2(\mathbb{K})$, i.e., the point $P_4 \in r$ such that $\beta(P_1, P_2, P_3, P_4) = -1$. To this aim, let us consider a line $s \neq r$ passing through P and choose distinct points A_1, A_2 on $s \setminus \{P\}$. Let $A_4 = L(A_2, P_3) \cap L(A_1, P_2)$ and

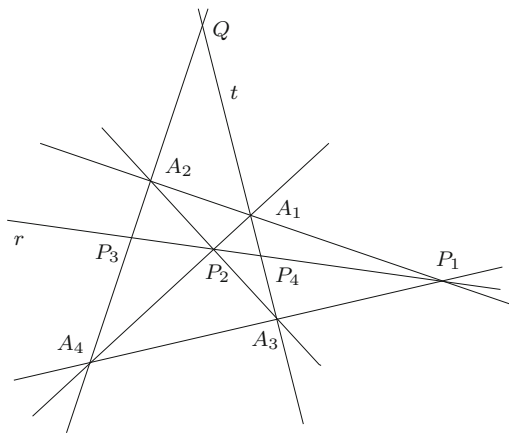


Fig. 2.4 Constructing a harmonic quadruple

$A_3 = L(P_1, A_4) \cap L(A_2, P_2)$, and let $P_4 = r \cap L(A_1, A_3)$ (observe that A_3, A_4, Q_4 are actually well defined). It is immediate to check that A_1, A_2, A_3, A_4 are in general position, and it is proven in Exercise 26 that $\beta(P_1, P_2, P_3, P_4) = -1$.

Exercise 27. Let P, Q, R, S be points of $\mathbb{P}^2(\mathbb{K})$ in general position, let $l_P = L(Q, R)$, $l_Q = L(P, R)$, $l_R = L(P, Q)$ and set

$$P' = L(P, S) \cap l_P, \quad Q' = L(Q, S) \cap l_Q, \quad R' = L(R, S) \cap l_R.$$

Also let $P'' \in l_P, Q'' \in l_Q, R'' \in l_R$ be the points that are uniquely determined by the following conditions:

$$\beta(Q, R, P', P'') = \beta(R, P, Q', Q'') = \beta(P, Q, R', R'') = -1.$$

Show that P'', Q'', R'' are collinear.



Solution (1). Let $T = L(Q', R') \cap l_P$, $W = L(T, S) \cap l_R$, $Z = L(T, S) \cap l_Q$. By applying Exercise 26 to the case when $A_1 = R, A_2 = R', A_3 = Q', A_4 = Q$, it is easy to verify that $\beta(S, T, W, Z) = -1$ (cf. Figs. 2.4 and 2.5). Thanks to the symmetries of the cross-ratio (cf. Sect. 1.5.2) we thus have $\beta(W, Z, S, T) = \beta(S, T, W, Z) = -1$. Moreover, the perspectivity $f: L(S, T) \rightarrow l_P$ centred at P maps W, Z, S, T to Q, R, P', T , respectively, so $\beta(Q, R, P', T) = -1$. Since by hypothesis $\beta(Q, R, P', P'') = -1$, we can conclude that $T = P''$.

Let now $g: l_Q \rightarrow l_R$ be the perspectivity centred at P'' . By construction we have $g(P) = P, g(R) = Q$, and moreover $g(Q') = R'$ since $P'' = T$. Since the cross-ratio is invariant under projectivities, we then have

$$\begin{aligned} \beta(P, Q, R', g(Q'')) &= \beta(g(P), g(R), g(Q'), g(Q'')) = \\ &= \beta(P, R, Q', Q'') = \frac{1}{\beta(R, P, Q', Q'')} = -1, \end{aligned}$$

so $g(Q'') = R''$. The definition of perspectivity implies now that $R'' = L(P'', Q'') \cap l_R$: in particular, the points P'', Q'', R'' are collinear.

Solution (2). By hypothesis the points P, Q, R, S define a projective frame of $\mathbb{P}^2(\mathbb{K})$, so we can choose homogeneous coordinates such that $P = [1, 0, 0]$, $Q = [0, 1, 0]$, $R = [0, 0, 1]$, $S = [1, 1, 1]$. We thus have $l_P = \{x_0 = 0\}$, $l_Q = \{x_1 = 0\}$, $l_R = \{x_2 = 0\}$, $L(P, S) = \{x_1 = x_2\}$, $L(Q, S) = \{x_0 = x_2\}$, $L(R, S) = \{x_0 = x_1\}$, hence $P' = [0, 1, 1]$, $Q' = [1, 0, 1]$, $R' = [1, 1, 0]$.

It is now easy to determine P'', Q'', R'' : a normalized basis induced by the projective frame $\{Q, R, P'\}$ of l_P is given by $v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$, so $P'' = [v_1 - v_2] = [0, 1, -1]$. In the same way one obtains $Q'' = [1, 0, -1]$, $R'' = [1, -1, 0]$. It follows that P'', R'', Q'' all belong to the line of equation $x_0 + x_1 + x_2 = 0$.

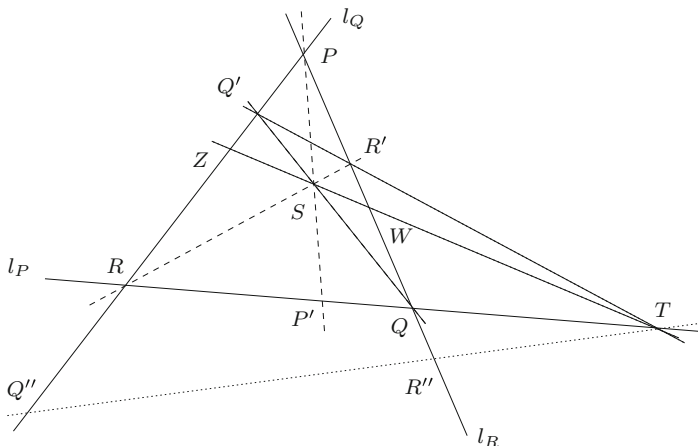


Fig. 2.5 The configuration described in Exercise 27

Exercise 28. Let $\mathbb{P}(V), \mathbb{P}(W)$ be projective spaces over the field \mathbb{K} , let H, K be projective subspaces of $\mathbb{P}(V)$, and let $f: \mathbb{P}(V) \setminus H \rightarrow \mathbb{P}(W)$ be a degenerate projective transformation. Show that $f(K \setminus H)$ is a projective subspace of $\mathbb{P}(W)$ of dimension $\dim K - \dim(K \cap H) - 1$.

Solution. Let S, T be the linear subspaces of V such that $H = \mathbb{P}(S), K = \mathbb{P}(T)$. The degenerate projective transformation f is induced by a linear map $\varphi: V \rightarrow W$ such that $\ker \varphi = S$, and it is immediate to check that $f(K \setminus H) = \mathbb{P}(\varphi(T))$. On the other hand, the dimension of the linear subspace $\varphi(T)$ is given by

$$\dim T - \dim(T \cap S) = (\dim K + 1) - (\dim(K \cap H) + 1) = \dim K - \dim(K \cap H),$$

so we finally have $\dim f(K \setminus H) = \dim K - \dim(K \cap H) - 1$.

Exercise 29. Let S, H be projective subspaces of the projective space $\mathbb{P}(V)$ such that $S \cap H = \emptyset$ and $L(S, H) = \mathbb{P}(V)$, and let $\pi_H: \mathbb{P}(V) \setminus H \rightarrow S$ be the projection onto S centred at H (cf. Sect. 1.2.7). Show that π_H is a degenerate projective transformation.

Solution (1). Let $n = \dim \mathbb{P}(V)$, $k = \dim S$, $h = \dim H$. An easy application of Grassmann's formula shows that $k + h = n - 1$. Moreover, it is easy to check that, if P_0, \dots, P_k are independent points of $S = \mathbb{P}(U)$ and $P_{k+1}, \dots, P_{h+k+1}$ are independent points of H , then the set $\{P_0, \dots, P_{h+k+1}\}$ is in general position in $\mathbb{P}(V)$, so it can be extended to a projective frame $\mathcal{R} = \{P_0, \dots, P_{h+k+1}, Q\}$ of $\mathbb{P}(V)$.

Let us fix the system of homogeneous coordinates x_0, \dots, x_n induced by \mathcal{R} . The Cartesian equations of H and S are given by $x_0 = \dots = x_k = 0$ and $x_{k+1} = \dots = x_n = 0$, respectively (cf. Sects. 1.3.5 and 1.3.7). For any given $P = [y_0, \dots, y_n] \notin H$ it is easy to check that the subspace $L(H, P)$ is the set of points $[\lambda_0 y_0, \dots, \lambda_0 y_k, \lambda_0 y_{k+1} + \lambda_1, \dots, \lambda_0 y_n + \lambda_{h+1}]$, where $[\lambda_0, \dots, \lambda_{h+1}] \in \mathbb{P}^{h+1}(\mathbb{K})$. So $L(H, P) \cap S$ is the point $[y_0, \dots, y_k, 0, \dots, 0]$ and π_H is the degenerate projective transformation induced by

the linear map $\varphi: V \rightarrow U$ described in coordinates by the formula $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_k, 0, \dots, 0)$.

Solution (2). Let W, U be linear subspaces of V such that $H = \mathbb{P}(W)$ and $S = \mathbb{P}(U)$. It is easy to check that the conditions $S \cap H = \emptyset$, $L(S, H) = \mathbb{P}(V)$ imply that $W \cap U = \{0\}$, $W + U = V$, respectively. So $V = W \oplus U$, and the projection $p_U: V \rightarrow U$ mapping every $v \in V$ to the unique vector $p_U(v) \in U$ such that $v - p_U(v) \in W$ is well defined. The map p_U is linear, and we have $\ker p_U = W$. Therefore, for every $v \in V \setminus W$, the line spanned by the vector $p_U(v)$ coincides with the intersection of U with the subspace spanned by $W \cup \{v\}$. Hence for every $v \in V \setminus W$ we have $\pi_H([v]) = [p_U(v)]$, so π_H is the degenerate projective transformation induced by p_U .

Exercise 30. In $\mathbb{P}^3(\mathbb{R})$, let us consider the plane T_1 of equation $x_3 = 0$, the plane T_2 of equation $x_0 + 2x_1 - 3x_2 = 0$, and the point $Q = [0, 1, -1, 1]$, and let $f: T_1 \rightarrow T_2$ be the perspectivity centred at Q . Find Cartesian equations for the image through f of the line r obtained by intersecting T_1 with the plane $x_0 + x_1 = 0$.

Solution. By definition of perspectivity we have $f(r) = L(Q, r) \cap T_2$, so the Cartesian equations of $f(r)$ are given by the union of an equation of $L(Q, r)$ and an equation of T_2 . Moreover, r is defined by the equations $x_0 + x_1 = x_3 = 0$, so the pencil of planes \mathcal{F}_r centred at r has parametric equations

$$\lambda(x_0 + x_1) + \mu x_3 = 0, \quad [\lambda, \mu] \in \mathbb{P}^1(\mathbb{R}).$$

By imposing that the generic plane of the pencil pass through Q one obtains $[\lambda, \mu] = [1, -1]$, so $L(Q, r)$ has equation $x_0 + x_1 - x_3 = 0$. Therefore, $f(r)$ is defined by the equations $x_0 + 2x_1 - 3x_2 = x_0 + x_1 - x_3 = 0$.

Exercise 31. Let $r, s \subset \mathbb{P}^2(\mathbb{K})$ be distinct lines, set $A = r \cap s$ and let $f: r \rightarrow s$ be a projective isomorphism. Prove that:

- (a) f is a perspectivity if and only if $f(A) = A$.
- (b) If $f(A) \neq A$, then there exist a line t of $\mathbb{P}^2(\mathbb{K})$ and two perspectivities $g: r \rightarrow t$, $h: t \rightarrow s$ such that $f = h \circ g$.
- (c) Every projectivity $p: r \rightarrow r$ of r is the composition of at most three perspectivities.

Solution. (a) Every perspectivity between r and s fixes the point A (cf. Sect. 1.2.8). Conversely, if the isomorphism $f: r \rightarrow s$ fixes A , then we choose distinct points $P_1, P_2 \in r \setminus \{A\}$ and we set $Q_1 = f(P_1)$, $Q_2 = f(P_2)$ (cf. Fig. 2.6). The lines $L(P_1, Q_1)$ and $L(P_2, Q_2)$ are distinct and meet at $O \notin r \cup s$. If $g: r \rightarrow s$ is the perspectivity centred at O , we have $g(A) = A$, $g(P_1) = Q_1$, $g(P_2) = Q_2$. The Fundamental theorem of projective transformations now implies that $f = g$, so f is a perspectivity.

(b) Choose pairwise distinct points $P_1, P_2, P_3 \in r \setminus \{A\}$ in such a way that the points $Q_1 = f(P_1)$, $Q_2 = f(P_2)$, $Q_3 = f(P_3)$ are distinct from A . Denote by M the point of intersection of the lines $L(P_1, Q_2)$ and $L(P_2, Q_1)$, by N the point of intersection of the lines $L(P_1, Q_3)$ and $L(P_3, Q_1)$, and by t the line $L(M, N)$ (cf. Fig. 2.6). It

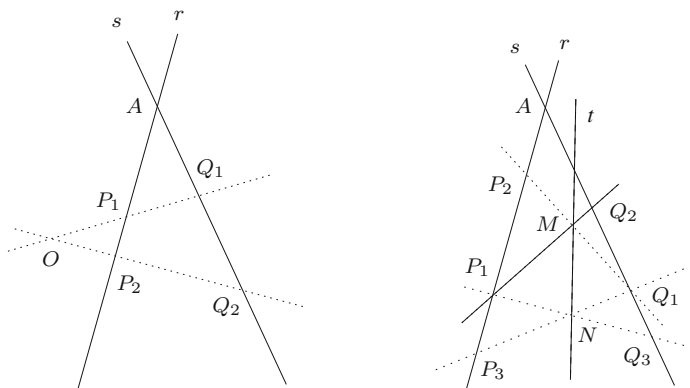


Fig. 2.6 Statements **a** on the *left* and **b** on the *right* of Exercise 31

is easy to check that the line t is distinct both from r and from s and does not contain the points P_1 and Q_1 . Let us denote by $g: r \rightarrow t$ the perspectivity centred at Q_1 and by $h: t \rightarrow s$ the perspectivity centred at P_1 . Since $(h \circ g)(P_i) = Q_i$ for $i = 1, 2, 3$, as in the previous point the Fundamental theorem of projective transformations implies that $f = h \circ g$.

When considering a projectivity $p: r \rightarrow r$, in order to prove (c) it is sufficient to apply (b) to the composition of p with any perspectivity $h: r \rightarrow t$ between r and any line $t \neq r$.

Exercise 32. (*Parametrization of a pencil of hyperplanes*) Let $\mathbb{P}(V)$ be a projective space of dimension n and let $H \subset \mathbb{P}(V)$ be a subspace of codimension 2. Denote by \mathcal{F}_H the pencil of hyperplanes centred at H . If t is a line transverse to \mathcal{F}_H , i.e., if $t \subset \mathbb{P}(V)$ is a line disjoint from H , then denote by $f_t: t \rightarrow \mathcal{F}_H$ the map sending any point $P \in t$ to the hyperplane $L(P, H) \in \mathcal{F}_H$. Prove that f_t is a projective isomorphism (which is called the *parametrization of \mathcal{F}_H via the transverse line t*).

Solution. Choose in $\mathbb{P}(V)$ a system of homogeneous coordinates such that H has equations $x_0 = x_1 = 0$ and the line t has equations $x_2 = \dots = x_n = 0$; in this way x_0, x_1 is a system of homogeneous coordinates on t . In the dual system of homogeneous coordinates a_0, \dots, a_n on $\mathbb{P}(V)^*$, the pencil \mathcal{F}_H has equations $a_2 = \dots = a_n = 0$, so a_0, a_1 is a system of homogeneous coordinates on \mathcal{F}_H . With respect to these coordinates, the map $f_t: t \rightarrow \mathcal{F}_H$ is given by $[x_0, x_1] \mapsto [x_1, -x_0]$, so it is a projective isomorphism.

Note. One can use the fact that the parametrization f_t is a projective isomorphism in order to provide an alternative definition of the cross-ratio of four hyperplanes S_1, S_2, S_3, S_4 belonging to a pencil \mathcal{F}_H , without referring to the notion of dual projective space (cf. Sect. 1.5.1). If t is a transverse line, it is sufficient to set $\beta(S_1, S_2, S_3, S_4) = \beta(P_1, P_2, P_3, P_4)$, where $P_i = t \cap S_i$, $i = 1, \dots, 4$: in fact, the invariance of the cross-ratio under projective isomorphisms and the fact that $f_t(P_i) = S_i$ ensure that the value $\beta(P_1, P_2, P_3, P_4)$ does not depend on the choice of t .

We also observe that one can prove that the cross-ratio $\beta(S_1 \cap t, S_2 \cap t, S_3 \cap t, S_4 \cap t)$ is independent of t without bringing the pencil \mathcal{F}_H into play. To this aim it is sufficient to note that, if t' is any other transverse line, and $P'_i = t' \cap S_i$, $i = 1, 2, 3, 4$, then the points P'_1, P'_2, P'_3, P'_4 are the images of the points P_1, P_2, P_3, P_4 via the perspectivity between t and t' centred at H (cf. Sect. 1.2.8), so $\beta(P'_1, P'_2, P'_3, P'_4) = \beta(P_1, P_2, P_3, P_4)$.

When $n = 2$ we have thus proved that, if \mathcal{F}_O is the pencil of lines of $\mathbb{P}^2(\mathbb{K})$ centred at O , then the parametrization of \mathcal{F}_O obtained via the transverse line t is a projective isomorphism. Moreover, if the lines t_1, t_2 do not contain O , then the perspectivity between t_1 and t_2 centred at O is the composition $f_{t_2}^{-1} \circ f_{t_1}$. Since the composition of projective isomorphisms is a projective isomorphism, this argument provides an alternative proof of the fact that every perspectivity between two lines of the projective plane is a projective isomorphism.

Exercise 33. Let r and H be a line and a plane of $\mathbb{P}^3(\mathbb{K})$, respectively; suppose that $r \not\subseteq H$, and let $P = r \cap H$. Let \mathcal{F}_r be the pencil of planes of $\mathbb{P}^3(\mathbb{K})$ centred at r , and let \mathcal{F}_P be the pencil of lines of H centred at P . Prove that the map $\beta: \mathcal{F}_r \rightarrow \mathcal{F}_P$ defined by $\beta(K) = K \cap H$ is a well-defined projective isomorphism.

Solution. Let $s \subseteq H$ be a line such that $P \notin s$, and let $f_r: s \rightarrow \mathcal{F}_r, f_P: s \rightarrow \mathcal{F}_P$ be the maps defined by $f_r(Q) = L(r, Q), f_P(Q) = L(P, Q)$ for every $Q \in s$. By construction, s does not contain P and is skew to r , so Exercise 32 ensures that f_r and f_P are well-defined projective isomorphisms. It is now immediate to check that the map β coincides with the composition $f_P \circ f_r^{-1}$, so β is a well-defined projective isomorphism.

Exercise 34. Let $r, s \subset \mathbb{P}^2(\mathbb{K})$ be distinct lines, let $A = r \cap s$, and let $f: r \rightarrow s$ be a projective isomorphism such that $f(A) = A$. Let also

$$W(f) = \{L(P_1, f(P_2)) \cap L(P_2, f(P_1)) \mid P_1, P_2 \in r, P_1 \neq P_2\}.$$

Prove that $W(f)$ is a projective line containing A .



Solution (1). We have seen in Exercise 31 that f is a perspectivity centred at $O \in \mathbb{P}^2(\mathbb{K}) \setminus (r \cup s)$. Let $l = L(A, O)$, and let \mathcal{F}_A be the pencil of lines of $\mathbb{P}^2(\mathbb{K})$ centred at A : by construction $r, s, l \in \mathcal{F}_A$. In order to show that $W(f)$ is contained in a line passing through A it is sufficient to show that, as M varies in $W(f) \setminus \{A\}$, the cross-ratio $\beta(s, r, l, L(A, M))$ does not depend on M : namely, if this is the case and $k = \beta(s, r, l, L(A, M))$, then we necessarily have $W(f) \subseteq t$, where t is the unique line of \mathcal{F}_A such that $\beta(s, r, l, t) = k$.

So let P_1, P_2 be distinct points of r , and let

$$M = L(P_1, f(P_2)) \cap L(P_2, f(P_1)) = L(P_1, s \cap L(O, P_2)) \cap L(P_2, s \cap L(O, P_1)).$$

Of course, if $P_1 = A$ or $P_2 = A$ then $M = A$, so we can suppose that P_1, P_2 are distinct from A . Then it is easy to check that $M \neq A$. We now show that $\beta(s, r, l, L(A, M)) = -1$. As already observed, this implies that $W(f)$ is contained in the line $t \in \mathcal{F}_A$ such that $\beta(s, r, l, t) = -1$.

Let $w = L(O, M)$, and observe that $A \notin w$, so w is transverse to the pencil \mathcal{F}_A . If $N = s \cap w$ and $Z = r \cap w$, then by the Note following Exercise 32 we have

$$\beta(s, r, l, L(A, M)) = \beta(s \cap w, r \cap w, l \cap w, L(A, M) \cap w) = \beta(N, Z, O, M).$$

On the other hand it is easy to check that, if we apply the construction described in the statement of Exercise 26 to the case when $A_1 = P_1$, $A_2 = f(P_1)$, $A_3 = P_2$, $A_4 = f(P_2)$, then we get $\beta(O, M, N, Z) = -1$ (cf. Figs. 2.4 and 2.7). Thanks to the symmetries of the cross-ratio (cf. Sect. 1.5.2) we then have $\beta(N, Z, O, M) = -1$, so $\beta(s, r, l, L(A, M)) = -1$, as desired. We have thus proved that the inclusion $W(f) \subseteq t$ holds.

Let us check that also the converse inclusion holds. For every $P \in r \setminus \{A\}$ we have $L(P, f(A)) \cap L(f(P), A) = A$, so $A \in W(f)$. So let $R \in t \setminus \{A\}$ and let v be any line passing through R and distinct both from t and from $L(O, R)$. Let $P_1 = v \cap r$, $P_2 = f^{-1}(v \cap s)$. Since $R \neq A$, $v \neq t$ and $v \neq L(O, R)$, the points P_1, P_2 are distinct and they are also distinct from A . Moreover $L(P_1, f(P_2)) = v$, so what has been proved above implies that $L(P_1, f(P_2)) \cap L(P_2, f(P_1)) \in v \cap t = \{R\}$, and $R \in W(f)$. We have thus shown that $t \subseteq W(f)$, hence $W(f) = t$, as desired.

⚡ Solution (2). Let us fix a point $P \in r \setminus \{A\}$, take $P_0 \in r \setminus \{A, P\}$ and let $M = L(P, f(P_0)) \cap L(P_0, f(P))$. We will show that, if $t = L(A, M)$, then $W(f) = t$.

We first prove the inclusion $W(f) \subseteq t$. Observe that, if $g: r \rightarrow t$ is the perspectivity centred at $f(P)$ and $h: t \rightarrow s$ is the perspectivity centred at P , then f and $h \circ g$ coincide on A, P, P_0 , so $f = h \circ g$ (cf. Fig. 2.8). We easily deduce that for every $P_1 \in r \setminus \{P\}$ we have $L(P_1, f(P)) \cap L(P, f(P_1)) = g(P_1) \in t$.

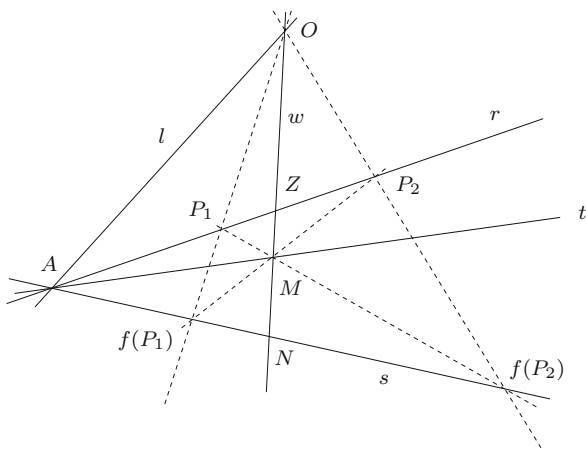


Fig. 2.7 The configurations described in Solution (1) of Exercise 34

Let us now prove that indeed $L(P_1, f(P_2)) \cap L(P_2, f(P_1)) \in t$ for every $P_1, P_2 \in r$, $P_1 \neq P_2$. This obviously holds if $P_1 = A$ or $P_2 = A$, and also holds if $P_1 = P$ or $P_2 = P$ thanks to the previous considerations. Therefore, we may assume $P_1, P_2 \in r \setminus \{A, P\}$. We consider the hexagon with vertices

$$P_1, Q_2 = f(P_2), P, Q_1 = f(P_1), P_2, Q = f(P).$$

By Pappus' Theorem (cf. Exercise 13 and Figs. 2.2, 2.8) the points

$$L(P_1, Q_2) \cap L(P_2, Q_1), \quad L(P, Q_2) \cap L(P_2, Q), \quad L(P, Q_1) \cap L(P_1, Q)$$

are collinear. We have observed above that the second and the third of these points are distinct and lie on the line t , so also the point $L(P_1, Q_2) \cap L(P_2, Q_1) = L(P_1, f(P_2)) \cap L(P_2, f(P_1))$ lies on t . We have thus proved that $W(f) \subseteq t$.

We now come to the opposite inclusion. We observe that $A \in W(f)$ because for every $P \in r \setminus \{A\}$ we have $L(P, f(A)) \cap L(A, f(P)) = A$. So let $Q \in t \setminus \{A\}$. We first show that f cannot be a perspectivity centred at Q . In fact, it is immediate to check that, if P_1, P_2 are distinct points of $r \setminus \{A\}$, then the points $P_1, P_2, f(P_1), f(P_2)$ are in general position, so the points $A = L(P_1, P_2) \cap L(f(P_1), f(P_2))$, $B = L(P_1, f(P_2)) \cap L(P_2, f(P_1))$, $C = L(P_1, f(P_1)) \cap L(P_2, f(P_2))$ are not collinear (cf. Exercise 6). But we have proved above that $A \in t$, $B \in t$, so if f were a perspectivity centred at Q , then we would have $L(P_1, f(P_1)) \cap L(P_2, f(P_2)) = Q \in t$, and the points A, B, Q would be collinear, a contradiction. So there exists a line $v \neq t$ containing Q and such that the points $R_1 = v \cap r$ and $R_2 = f^{-1}(v \cap s)$ are distinct. We thus have $L(R_1, f(R_2)) = v$, hence $L(R_1, f(R_2)) \cap L(R_2, f(R_1)) \in t$ by the considerations above. Therefore $L(R_1, f(R_2)) \cap L(R_2, f(R_1)) = t \cap v = Q$, so $Q \in W(f)$. We have thus proved that $t \subseteq W(f)$.

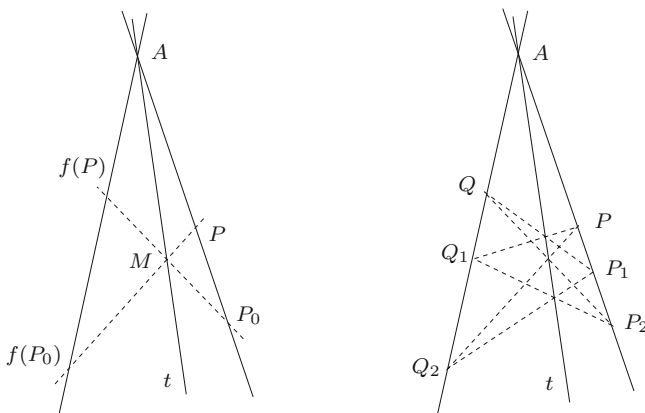


Fig. 2.8 Solution (2) of Exercise 34: on the *left*, f is described as the composition of two perspectivities; on the *right*, the inclusion $W(f) \subseteq t$ as a consequence of Pappus' Theorem

Solution (3). By Exercise 31, f is a perspectivity centred at $O \in \mathbb{P}^2(\mathbb{K}) \setminus (r \cup s)$. Take points $B \in r$, $C \in s$ such that A, B, C, O are in general position, and endow $\mathbb{P}^2(\mathbb{K})$ with the system of homogeneous coordinates induced by the projective frame $\{A, B, C, O\}$. Then we have $r = \{x_2 = 0\}$, $s = \{x_1 = 0\}$. Let now P, P' be distinct points in $r \setminus \{A\}$. We have $P = [a, 1, 0]$, $P' = [a', 1, 0]$ for some $a, a' \in \mathbb{K}$. Using that

$$f(P) = L(O, P) \cap s = L([1, 1, 1], [a, 1, 0]) \cap \{x_1 = 0\},$$

one easily proves that $f(P) = [1 - a, 0, 1]$. In a similar way one gets $f(P') = [1 - a', 0, 1]$. We thus have $L(P, f(P')) = \{x_0 - ax_1 + (a' - 1)x_2 = 0\}$ and $L(P', f(P)) = \{x_0 - a'x_1 + (a - 1)x_2 = 0\}$, so $L(P, f(P')) \cap L(P', f(P)) = [1 - a - a', -1, 1]$. It follows that, if P, P' are distinct points in $r \setminus \{A\}$, then the point $L(P, f(P')) \cap L(P', f(P))$ lies on $t \setminus \{A\}$, where t is the line (through A) of equation $x_1 + x_2 = 0$, and every point of $t \setminus \{A\}$ arises in this way. On the other hand, for every $P \in r \setminus \{A\}$ one has $L(P, f(A)) \cap L(A, f(P)) = A$, so $W(f) = t$, as desired.

Exercise 35. Let $r, s \subset \mathbb{P}^2(\mathbb{K})$ be distinct lines, set $A = r \cap s$, and let $f: r \rightarrow s$ be a projective isomorphism such that $f(A) \neq A$. Let also $B = f^{-1}(A) \in r$ and set

$$W(f) = \{L(P, f(P')) \cap L(P', f(P)) \mid P, P' \in r, P \neq P', \{P, P'\} \neq \{A, B\}\}.$$

Prove that $W(f)$ is a projective line.



Solution (1). Let us fix a point $P_1 \in r \setminus \{A, B\}$. If P_2, P_3 are distinct points of $r \setminus \{A, B, P_1\}$ and we set $M = L(P_1, f(P_2)) \cap L(P_2, f(P_1))$, $N = L(P_1, f(P_3)) \cap L(P_3, f(P_1))$, $t = L(M, N)$ (cf. Fig. 2.9), then it is easy to check that the points M, N and the line t are well defined and that $f = h \circ g$, where $g: r \rightarrow t$ is the perspectivity centred at $f(P_1)$ and $h: t \rightarrow s$ is the perspectivity centred at P_1 . It readily follows that

$$L(f(P_1), P) \cap L(P_1, f(P)) = g(P) \in t \quad \forall P \in r \setminus \{P_1\}. \quad (2.3)$$

In order to prove that $W(f) \subseteq t$, let us first show that t does not depend on the choice of P_1 . If $C = f(A) \in s$, since $f(A) \neq A$ the points A, B, C are in general position. Together with our previous considerations, this implies that

$$\begin{aligned} B &= L(P_1, A) \cap L(f(P_1), B) = L(P_1, f(B)) \cap L(f(P_1), B) \in t \\ C &= L(f(P_1), A) \cap L(P_1, C) = L(f(P_1), A) \cap L(P_1, f(A)) \in t. \end{aligned} \quad (2.4)$$

Therefore, the line t contains both B and C , so that $t = L(B, C)$. In particular, t does not depend on the choice of P_1 .

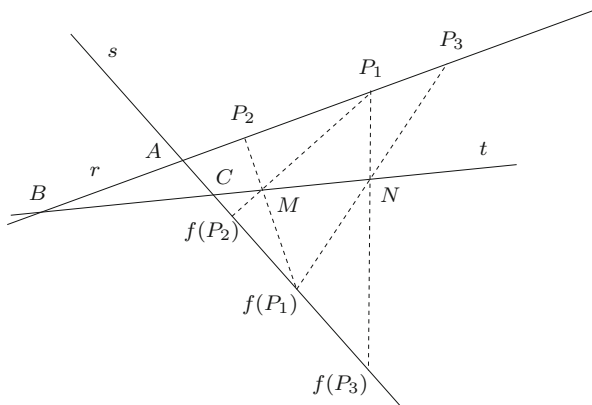


Fig. 2.9 The construction of the set $W(f)$ described in Solution (1) of Exercise 35

Combining (2.3) with the fact that t is independent of P_1 we deduce that

$$L(P, f(P')) \cap L(P', f(P)) \in t \quad \forall P \in r \setminus \{A, B\}, P' \in r \setminus \{P\}.$$

Now the inclusion $W(f) \subseteq t$ readily follows from the fact that the expression $L(P, f(P')) \cap L(P', f(P))$ is symmetric in P, P' .

Let us now check that $t \subseteq W(f)$. We first observe that, by (2.4), the points $B = t \cap r$, $C = t \cap s$ belong to $W(f)$. Let then $Q \in t \setminus \{B, C\}$. If $f(P) = L(P, Q) \cap s$ for every $P \in r \setminus \{A, B\}$, then f is the perspectivity centred at Q , so $f(A) = A$, against the hypothesis. Therefore, there exists $P_1 \in r \setminus \{A, B\}$ such that $f(P_1) \neq L(P_1, Q) \cap s$. Let $P_2 = f^{-1}(L(P_1, Q) \cap s)$. By construction $P_2 \neq P_1$. Let now $Q' = L(P_1, f(P_2)) \cap L(P_2, f(P_1))$. By construction $Q' \in L(P_1, f(P_2)) = L(P_1, Q)$, and our previous considerations imply that $Q' \in W(f) \subseteq t$. It follows that $Q' = t \cap L(P_1, Q) = Q$, so $Q \in W(f)$. We have thus shown that $t \subseteq W(f)$, as desired.

Solution (2). Let us set $C = f(A) \in s$. The points A, B, C are not collinear, so we may choose a system of homogeneous coordinates x_0, x_1, x_2 such that $A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [0, 0, 1]$. We then have $r = \{x_2 = 0\}$, $s = \{x_1 = 0\}$, and we can endow r and s with the systems of homogeneous coordinates induced by x_0, x_1, x_2 . Since $f(B) = A$ and $f(A) = C$, with respect to these coordinates the isomorphism f is represented by the matrix $\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$, for some $\lambda \in \mathbb{K}^*$.

If $P = [a, b, 0]$, $P' = [a', b', 0]$ are distinct points of r such that $\{P, P'\} \neq \{A, B\}$, we thus have $f(P) = [\lambda b, 0, a]$, $f(P') = [\lambda b', 0, a']$; so $L(P, f(P'))$ and $L(P', f(P))$ have equations $ba'x_0 - aa'x_1 - \lambda bb'x_2 = 0$ and $ab'x_0 - aa'x_1 - \lambda bb'x_2 = 0$, respectively. Since $P \neq P'$ we have $ab' - ba' \neq 0$, so the lines $L(P, f(P'))$ and $L(P', f(P))$ meet at $[0, \lambda bb', -aa']$ (observe that we cannot have $\lambda bb' = aa' = 0$ because $\{P, P'\} \neq \{A, B\}$). If t is the line of equation $x_0 = 0$, we can conclude that $W(f) \subseteq t$.

On the other hand, take $P = [\lambda, 1, 0]$ and $P' = [a', b', 0]$ with $a' \neq \lambda b'$. Then $P \neq P'$ and $\{P, P'\} \neq \{A, B\}$, and the previous computation shows that


$$L(P, f(P')) \cap L(P', f(P)) = [0, b', -a'].$$

This proves that every point of t , except possibly $[0, 1, -\lambda]$, belongs to $W(f)$. On the other hand, of course we can choose distinct elements $a, a' \in \mathbb{K}$ such that $aa' = \lambda^2$. If $P = [a, 1, 0]$, $P' = [a', 1, 0]$, then it is immediate to check that $P \neq P'$ and $\{P, P'\} \neq \{A, B\}$, and

$$L(P, f(P')) \cap L(P', f(P)) = [0, 1, -\lambda].$$

We thus have $t \subseteq W(f)$, hence $W(f) = t$, as desired.

Note. In Solution (1), after proving (2.3) one can conclude that $W(f) = t$ by exploiting Pappus' Theorem in the same spirit as in Solution (2) of Exercise 34.

-  **Exercise 36.** (a) Let A, A', C, C' be points of $\mathbb{P}^1(\mathbb{K})$ such that $A \notin \{C, C'\}$ and $A' \notin \{C, C'\}$. Prove that there exists a unique non-trivial involution $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A', f(C) = C'$.
- (b) Let A, B, C, A', B', C' be points of $\mathbb{P}^1(\mathbb{K})$ such that each quadruple A, B, C, C' and A', B', C', C contains only pairwise distinct points. Prove that there exists a unique involution $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A', f(B) = B', f(C) = C'$ if and only if

$$\beta(A, B, C, C') = \beta(A', B', C', C).$$

- (c) Let $r \subseteq \mathbb{P}^2(\mathbb{K})$ be a projective line and let P_1, P_2, P_3, P_4 be points of $\mathbb{P}^2(\mathbb{K}) \setminus r$ in general position. For every $i \neq j$ let $s_{ij} = L(P_i, P_j)$, and set

$$\begin{aligned} A &= r \cap s_{12}, & B &= r \cap s_{13}, & C &= r \cap s_{14} \\ A' &= r \cap s_{34}, & B' &= r \cap s_{24}, & C' &= r \cap s_{23} \end{aligned}$$

(cf. Fig. 2.10). Prove that there exists a unique involution f of r such that $f(A) = A', f(B) = B', f(C) = C'$.

Solution. (a) The case when $A = A'$ and $C = C'$ has already been settled in Exercise 24.

Let us now suppose $A = A', C \neq C'$ (the case when $A \neq A', C = C'$ being similar). If f satisfies the conditions of the statement, then necessarily $f(A) = A' = A$, $f(C) = C', f(C') = f(f(C)) = C$. Moreover, by the Fundamental theorem of projective transformations, there exists a unique projectivity mapping A, C, C' to A, C', C , respectively. By point (b) of Exercise 23, this projectivity is an (obviously non-trivial) involution, and this concludes the proof in the case when $A = A', C \neq C'$.

We finally suppose $A \neq A', C \neq C'$. Our hypothesis implies that the triples A, A', C and A, A', C' define two projective frames of $\mathbb{P}^1(\mathbb{K})$, so there exists a unique

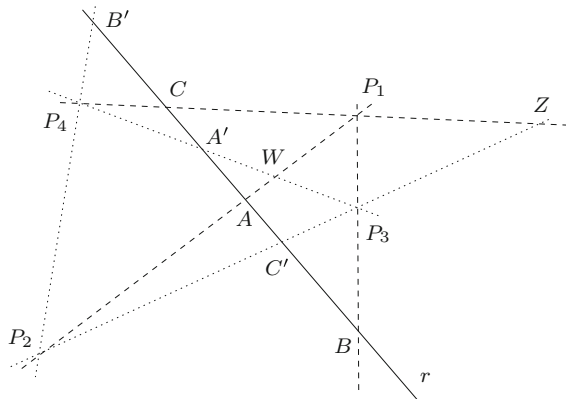


Fig. 2.10 Exercise 36, point (c): the case when $Z \notin r$

projectivity $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A', f(A') = A, f(C) = C'$. Moreover, by point (b) of Exercise 23, the projectivity f is a non-trivial involution. On the other hand, any projectivity satisfying the required conditions necessarily maps A, A', C to A', A, C' , respectively, so it must coincide with f .

(b) If an involution f with the required properties exists, then $f(C') = f(f(C)) = C$, so

$$\beta(A, B, C, C') = \beta(f(A), f(B), f(C), f(C')) = \beta(A', B', C', C)$$

thanks to the invariance of the cross-ratio with respect to projective transformations.

We then suppose that $\beta(A, B, C, C') = \beta(A', B', C', C)$, and we show that the required involution exists (the fact that such involution is unique readily follows from the Fundamental theorem of projective transformations). By point (a), there exists a non-trivial involution $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(A) = A', f(C) = C'$. Since $f(C') = C$, we have $\beta(A', B', C', C) = \beta(A, B, C, C') = \beta(f(A), f(B), f(C), f(C')) = \beta(A', f(B), C', C)$, where the first equality follows from the hypothesis, while the second one is due to the invariance of the cross-ratio with respect to projective transformations. Since the points A', B', C', C are pairwise distinct, the condition $\beta(A', B', C', C) = \beta(A', f(B), C', C)$ implies that $f(B) = B'$, as desired.

(c) We first observe that the points A, B, C are pairwise distinct, because otherwise the line r would contain P_1 , against the hypothesis. In a similar way one proves that the points A', B', C' are also pairwise distinct, because otherwise the line r would pass through P_2, P_3 or P_4 , against the hypothesis.

Let us set $W = s_{12} \cap s_{34}$, $T = s_{13} \cap s_{24}$, $Z = s_{23} \cap s_{14}$. Since P_1, P_2, P_3, P_4 are in general position, the points W, T, Z are not collinear (cf. Exercise 6).

We first consider the case when r does not contain Z . Since $C = r \cap s_{14}$ and $C' = r \cap s_{23}$, we have that $C \neq C'$. Also observe that $C' \neq A$ (since $P_2 \notin r$), and $C' \neq B$ (since $P_3 \notin r$). In a similar way, since $P_4 \notin r$ we have $C \neq A'$ and $C \neq B'$. Therefore, the quadruples A, B, C, C' and A', B', C', C contain only pairwise distinct points, so by point (b) we are left to show that $\beta(A, B, C, C') = \beta(A', B', C', C)$.

Let $g: r \rightarrow s_{23}$ be the perspectivity centred at P_1 and $h: s_{23} \rightarrow r$ the perspectivity centred at P_4 . By construction, g maps the points A, B, C, C' to the points P_2, P_3, Z, C' , respectively, and h maps P_2, P_3, Z, C' to B', A', C, C' , respectively. We thus have

$$\begin{aligned}\beta(A, B, C, C') &= \beta(h(g(A)), h(g(B)), h(g(C)), h(g(C'))) = \\ &= \beta(B', A', C, C') = \beta(A', B', C, C),\end{aligned}$$

where the first equality is due to the invariance of the cross-ratio under projective isomorphisms, while the last one is due to the symmetries of the cross-ratio (cf. Sect. 1.5.2). This concludes the proof in the case when r does not contain Z .

If $Z \in r$ but $T \notin r$, then we have $B \neq B'$; a similar argument as above shows that the quadruples A, C, B, B' and A', C', B', B contain only pairwise distinct points. By point (b), in order to conclude it is sufficient to show that $\beta(A, C, B, B') = \beta(A', C', B', B)$. This equality can be proved as above by considering first the perspectivity between r and $L(P_2, P_4)$ centred at P_1 and then the perspectivity between $L(P_2, P_4)$ and r centred at P_3 .

Finally, if r passes both through Z and through T , then necessarily $W \notin r$, because otherwise W, T, Z would be collinear. Then, a suitable variation of the previous argument allows one to conclude the proof also in this case.

Note. A *quadrilateral* of $\mathbb{P}^2(\mathbb{K})$ is an unordered set of 4 points in general position (called *vertices*) of the projective plane. A *pair of opposite sides* of a quadrilateral Q is a pair of lines whose union contains the vertices of Q (observe that any quadrilateral has exactly three pairs of opposite sides).

Let r be a line that does not contain any vertex of Q , and observe that the union of each pair of opposite sides of Q meets r at exactly two points. Point (c) of Exercise 36 can be reformulated as follows: there exists an involution of r that switches the points of each pair determined on r by a pair of opposite sides of Q .

Exercise 37. Let $\mathbb{P}(V)$ be a projective space of dimension n and let S_1, S_2 be distinct hyperplanes of $\mathbb{P}(V)$. Show that a projective isomorphism $f: S_1 \rightarrow S_2$ is a perspectivity if and only if $f(A) = A$ for every $A \in S_1 \cap S_2$.

Solution (1). It is sufficient to show that, if $f: S_1 \rightarrow S_2$ is a projective isomorphism that is the identity on $S_1 \cap S_2$, then f is a perspectivity. Let us choose points $P_1, P_2 \in S_1 \setminus (S_1 \cap S_2)$, set $Q_1 = f(P_1)$, $Q_2 = f(P_2)$ and denote by A the point at which the line $L(P_1, P_2)$ meets the subspace $S_1 \cap S_2$. Since f sends collinear points to collinear points and $f(A) = A$ by hypothesis, the point A is collinear with Q_1 and Q_2 . So the subspace $L(P_1, P_2, Q_1, Q_2)$ is a plane, and the lines $L(P_1, Q_1)$ and $L(P_2, Q_2)$ meet at a point O . It is immediate to check that $O \notin S_1 \cup S_2$; therefore, we can consider the perspectivity $g: S_1 \rightarrow S_2$ centred at O . The projective isomorphisms f and g coincide on P_1, P_2 and on the whole of $S_1 \cap S_2$. Therefore, the Fundamental theorem of projective transformations ensures that $f = g$.

Solution (2). We now give an analytic proof of the statement, by carrying out some computations in a suitably chosen system of homogeneous coordinates x_0, \dots, x_n

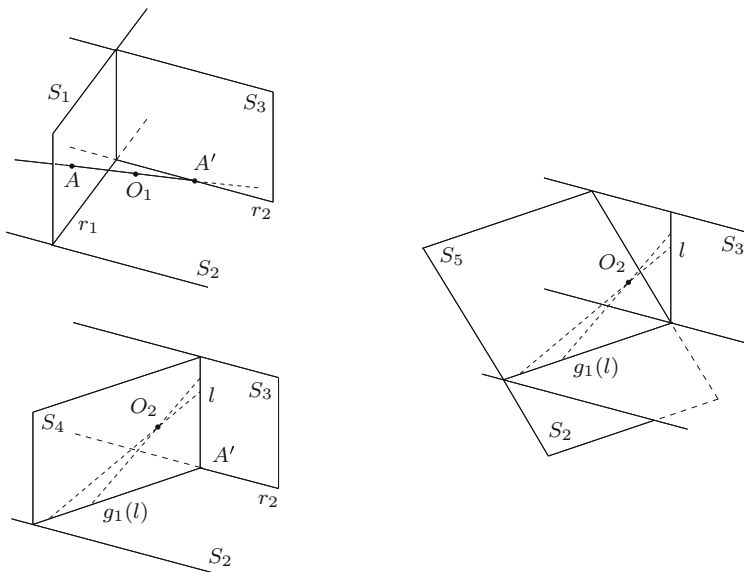



Fig. 2.11 On the *left*: on the *top*, the construction of g_1 ; on the *bottom*, the choice of l and the construction of S_4 . On the *right*: the choice of S_5 and the conclusion of the proof

such that S_1 has equation $x_n = 0$ and S_2 has equation $x_{n-1} = 0$. Using the fact that f is the identity on the subspace of equations $x_{n-1} = x_n = 0$, it is not difficult to show that the map f is described by the formula $[x_0, \dots, x_{n-1}, 0] \mapsto [x_0 + a_0x_{n-1}, \dots, x_{n-2} + a_{n-2}x_{n-1}, 0, a_nx_{n-1}]$, where $a_0, \dots, a_{n-2} \in \mathbb{K}$ and $a_n \in \mathbb{K}^*$. Then, one can check directly that f is the perspectivity centred at $O = [a_0, \dots, a_{n-2}, -1, a_n]$.

Note. Exercise 37 extends point (a) of Exercise 31 (that provides a characterization of perspectivities between lines in a projective plane) to the case of any dimension.

 **Exercise 38.** Let S_1, S_2 be distinct planes of $\mathbb{P}^3(\mathbb{K})$. Prove that any projective isomorphism $f: S_1 \rightarrow S_2$ is the composition of at most three perspectivities.

Solution. Let $r_1 = S_1 \cap S_2$ and take a point $A \in S_1 \setminus r_1$ such that the point $A' = f(A)$ does not belong to r_1 (cf. Fig. 2.11). Let $S_3 \neq S_2$ be a plane passing through A' and not containing A , and choose a point $O_1 \in L(A, A') \setminus \{A, A'\}$. If $\pi_1: S_3 \rightarrow S_1$ is the perspectivity centred at O_1 , then $g_1 = f \circ \pi_1: S_3 \rightarrow S_2$ is a projective isomorphism. Moreover, $A' \in r_2 = S_3 \cap S_2$ and $g_1(A') = A'$.

Let $l \neq r_2$ be a line contained in S_3 and passing through A' . Since $g_1(A') = A'$, the map g_1 transforms l into a line passing through A' . Let now S_4 be the plane containing l and $g_1(l)$, so that $l = S_3 \cap S_4$. Then Exercise 31 implies that $g_1|_l$ is a perspectivity centred at a point $O_2 \in S_4$ such that $O_2 \notin S_3$.

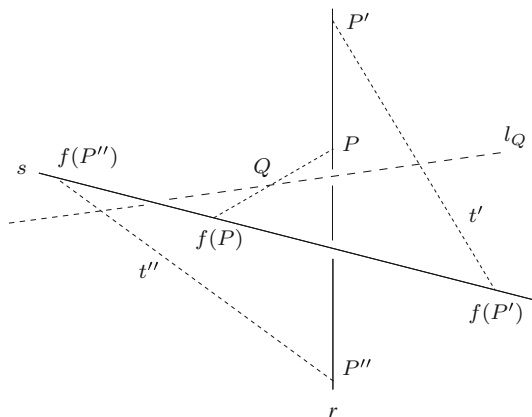


Fig. 2.12 The construction described in the solution of Exercise 39

Let now S_5 be a plane containing $g_1(l)$ and distinct both from S_2 and from S_4 , and let us consider the perspectivity $\pi_2: S_5 \rightarrow S_3$ centred at O_2 . Then $g_1 \circ \pi_2: S_5 \rightarrow S_2$ is a projective isomorphism that fixes the line $g_1(l) = S_5 \cap S_2$ pointwise. Now Exercise 37 implies that there exists a perspectivity π_3 such that $g_1 \circ \pi_2 = \pi_3$, so that $f \circ \pi_1 \circ \pi_2 = \pi_3$. This implies the conclusion, since the inverse of a perspectivity is again a perspectivity.

Exercise 39. Let $r, s \subset \mathbb{P}^3(\mathbb{K})$ be skew lines, and let $f: r \rightarrow s$ be a projective isomorphism. Prove that there exist infinitely many lines l such that f coincides with the perspectivity centred at l .

Solution. Take $P \in r$, and let $t = L(P, f(P))$. We will show that, for every $Q \in t \setminus \{P, f(P)\}$, there exists a line l_Q with the following properties: l_Q passes through Q , l_Q is skew both to r and to s , and f coincides with the perspectivity centred at l_Q . When Q varies in $t \setminus \{P, f(P)\}$, the lines l_Q are pairwise distinct, and this implies the conclusion.

Let us fix $Q \in t \setminus \{P, f(P)\}$, and choose $P', P'' \in r \setminus \{P\}$, such that $P' \neq P''$. We set $t' = L(P', f(P'))$, $t'' = L(P'', f(P''))$ (cf. Fig. 2.12). If t', t'' were not skew, then $P', P'', f(P'), f(P'')$ would be coplanar, so r and s would also be coplanar, a contradiction. In the same way one proves that t is skew both to t' and to t'' , so $Q \notin t' \cup t''$. Therefore, Exercise 8 implies that there exists a unique line l_Q that contains Q and meets both t' and t'' . Observe that no plane S can contain $l_Q \cup r$ (or $l_Q \cup s$), because otherwise each of the lines t', t'' would contain at least two points of S , thus t' and t'' would be coplanar, a contradiction. So l_Q is skew both to r and to s . By construction, the perspectivity from r to s centred at l_Q maps P to $f(P)$, P' to $f(P')$ and P'' to $f(P'')$. By the Fundamental theorem of projective transformations, we deduce that this perspectivity coincides with f , whence the conclusion.

Note. An alternative solution of Exercise 39 can be obtained by following the strategy described in the Note following Exercise 178.

Exercise 40. Let P_1 and P_2 be distinct points of $\mathbb{P}^2(\mathbb{K})$ and let $\mathcal{F}_1, \mathcal{F}_2$ be the pencils of lines centred at P_1 and P_2 , respectively. Let $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be any function. Show that the following facts are equivalent:

- (i) f is a projective isomorphism such that $f(L(P_1, P_2)) = L(P_1, P_2)$,
- (ii) there exists a line r disjoint from $\{P_1, P_2\}$ and such that $f(s) = L(s \cap r, P_2)$ for every $s \in \mathcal{F}_1$.

Solution. (i) \Rightarrow (ii). Via the duality correspondence (cf. Sect. 1.4.2), the pencils $\mathcal{F}_1, \mathcal{F}_2$ correspond to lines l_1, l_2 of the dual projective plane $\mathbb{P}^2(\mathbb{K})^*$. Moreover, the projective isomorphism $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ induces the dual projective isomorphism $f_*: l_1 \rightarrow l_2$. The condition $f(L(P_1, P_2)) = L(P_1, P_2)$ translates into the fact that f_* fixes the point at which l_1 and l_2 meet, and this implies in turn that f_* is a perspectivity (cf. Exercise 31). Let $R \in \mathbb{P}^2(\mathbb{K})^*$ be the centre of this perspectivity, and let $r \subset \mathbb{P}^2(\mathbb{K})$ be the line corresponding to R via duality. Since $R \notin l_1 \cup l_2$ we have that $P_1 \notin r$ and $P_2 \notin r$, and the fact that $f_*(Q) = L(R, Q) \cap l_2$ for every $Q \in l_1$ implies that $f(s) = L(P_2, r \cap s)$ for every $s \in \mathcal{F}_1$, as desired.

(ii) \Rightarrow (i). For $i = 1, 2$, let $g_i: \mathcal{F}_i \rightarrow r$ be the map defined by $g_i(s) = s \cap r$. We have shown in Exercise 32 and in the Note following it that the map g_i is a well-defined projective isomorphism. It follows that $f = g_2^{-1} \circ g_1$ is a projective isomorphism. The fact that $f(L(P_1, P_2)) = L(P_1, P_2)$ readily follows from the definition of f .

Exercise 41. In $\mathbb{P}^2(\mathbb{R})$ consider the point $P = [1, 2, 1]$ and the lines

$$\begin{aligned} l_1 &= \{x_0 + x_1 = 0\}, & m_1 &= \{x_0 + 3x_2 = 0\}, \\ l_2 &= \{x_0 - x_1 = 0\}, & m_2 &= \{x_2 = 0\}, \\ l_3 &= \{x_0 + 2x_1 = 0\}, & m_3 &= \{3x_0 + x_2 = 0\}. \end{aligned}$$

Determine the points $Q \in \mathbb{P}^2(\mathbb{R})$ for which there exists a projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ such that $f(P) = Q$ and $f(l_i) = m_i$, $i = 1, 2, 3$.

Solution. First observe that the lines l_1, l_2, l_3 (m_1, m_2, m_3 , respectively) belong to the pencil \mathcal{F}_O centred at $O = [0, 0, 1]$ (to the pencil $\mathcal{F}_{O'}$ centred at $O' = [0, 1, 0]$, respectively). Let $r = L(O, P) = \{2x_0 - x_1 = 0\}$.

Suppose now that f is a projectivity such that $f(l_i) = m_i$ for every $i = 1, 2, 3$. Then $f(O) = O'$, and the dual projectivity associated to f induces a projective isomorphism between \mathcal{F}_O and $\mathcal{F}_{O'}$. This isomorphism maps l_i to m_i for $i = 1, 2, 3$. Therefore, if $r' = f(r)$, then $\beta(l_1, l_2, l_3, r) = \beta(f(l_1), f(l_2), f(l_3), f(r)) = \beta(m_1, m_2, m_3, r')$ (cf. Sect. 1.5.1 for the definition and the basic properties of the cross-ratio of concurrent lines). We now fix on \mathcal{F}_O (on $\mathcal{F}_{O'}$, respectively) a projective frame with respect to which the line of equation $ax_0 + bx_1 = 0$ (the line of equation $ax_0 + bx_2 = 0$, respectively) has homogeneous coordinates equal to $[a, b]$. With this choice, the lines $l_1, l_2, l_3, r \in \mathcal{F}_O$ have coordinates $[1, 1], [1, -1], [1, 2], [2, -1]$, respectively, so $\beta(l_1, l_2, l_3, r) = -9$. On the other hand, the lines $m_1, m_2, m_3 \in \mathcal{F}_{O'}$ have coordinates $[1, 3], [0, 1], [3, 1]$, respectively, so if $[a_0, b_0]$ are the coordinates of r' in $\mathcal{F}_{O'}$, then we have

$$-9 = \beta([1, 3], [0, 1], [3, 1], [a_0, b_0]) = \frac{1 \cdot b_0 - 3 \cdot a_0}{1 \cdot 1 - 3 \cdot 3} \cdot \frac{3 \cdot 1 - 0 \cdot 3}{a_0 \cdot 1 - b_0 \cdot 0},$$

hence $[a_0, b_0] = [1, 27]$ and $r' = \{x_0 + 27x_2 = 0\}$. We deduce that $f(P) \in f(r) = r'$. Moreover, since f is injective, we have $f(P) \neq f(O)$, so $f(P) \in r' \setminus \{O'\}$.

Let now $Q \in r' \setminus \{O'\}$, and let us show that there exists a projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ such that $f(l_i) = m_i$ for $i = 1, 2, 3$, and $f(P) = Q$. Let $A_1 \in l_1 \setminus \{O\}$ and $A_2 \in l_2 \setminus (\{O\} \cup (l_2 \cap L(A_1, P)))$. By construction, the points O, A_1, A_2, P define a projective frame of $\mathbb{P}^2(\mathbb{R})$. In a similar way, if $B_1 \in m_1 \setminus \{O'\}$ and $B_2 \in m_2 \setminus (\{O'\} \cup L(B_1, Q))$, then the points O', B_1, B_2, Q provide a projective frame of $\mathbb{P}^2(\mathbb{R})$. Let $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ be the unique projectivity such that $f(O) = O', f(A_1) = B_1, f(A_2) = B_2, f(P) = Q$. Since f transforms lines into lines, we obviously have $f(l_1) = m_1, f(l_2) = m_2$, so we are left to show that $f(l_3) = m_3$.

The dual projectivity f_* associated to f maps \mathcal{F}_O to $\mathcal{F}_{O'}$ and satisfies $f_*(l_1) = m_1, f_*(l_2) = m_2, f_*(r) = r'$, so thanks to the invariance of the cross-ratio under projective isomorphisms we get

$$\begin{aligned} \beta(m_1, m_2, m_3, r') &= \beta(l_1, l_2, l_3, r) = \\ &= \beta(f_*(l_1), f_*(l_2), f_*(l_3), f_*(r)) = \beta(m_1, m_2, f_*(l_3), r') \end{aligned}$$

and $f(l_3) = m_3$, as desired.

Exercise 42. In $\mathbb{P}^2(\mathbb{R})$ consider the lines l_1, l_2, l_3 having equations $x_2 = 0, x_2 - x_1 = 0, x_2 - 2x_1 = 0$, respectively, and the line l_4 having equation $\alpha(x_0 - x_1) + x_2 - 4x_1 = 0$, where $\alpha \in \mathbb{R}$. Consider also the lines m_1, m_2, m_3, m_4 having equations $x_1 = 0, x_1 - x_0 = 0, x_0 = 0, x_1 - \gamma x_0 = 0$, respectively, where $\gamma \in \mathbb{R}$.

Find the values of α and γ for which there exists a projectivity f of $\mathbb{P}^2(\mathbb{R})$ such that $f(l_i) = m_i$ for $i = 1, \dots, 4$, and that transforms the line $x_0 = 0$ into the line $x_2 = 0$.

Choose a pair of such values, and describe explicitly a projectivity satisfying the required properties.

Solution (1). It is easy to check that the lines l_1, l_2, l_3 all intersect at $R = [1, 0, 0]$, while the lines m_1, m_2, m_3, m_4 all intersect at $S = [0, 0, 1]$. Therefore, a necessary condition for the existence of f is that also the line l_4 passes through R , i.e., that $\alpha = 0$. Moreover, if f exists, then the restriction of f to the line $r = \{x_0 = 0\}$ is a projective isomorphism between r and the line $s = \{x_2 = 0\}$. We observe that r intersects the lines $l_i, i = 1, \dots, 4$ at the points $P_1 = [0, 1, 0], P_2 = [0, 1, 1], P_3 = [0, 1, 2], P_4 = [0, 1, 4]$, respectively, while s intersects the lines $m_i, i = 1, \dots, 4$, at the points $Q_1 = [1, 0, 0], Q_2 = [1, 1, 0], Q_3 = [0, 1, 0], Q_4 = [1, \gamma, 0]$, respectively. Therefore, if f exists, then the restriction $f|_r: r \rightarrow s$ is a projective isomorphism such that $f(P_i) = Q_i$ for $i = 1, \dots, 4$, and the cross-ratio $\beta(P_1, P_2, P_3, P_4)$ must coincide with the cross-ratio $\beta(Q_1, Q_2, Q_3, Q_4)$. An easy computation now shows that $\beta(P_1, P_2, P_3, P_4) = \frac{2}{3}$ and $\beta(Q_1, Q_2, Q_3, Q_4) = \frac{\gamma}{\gamma - 1}$. Therefore, by imposing $\frac{\gamma}{\gamma - 1} = \frac{2}{3}$, we obtain a second necessary condition for the existence of f , namely $\gamma = -2$.

We now prove that the conditions $\alpha = 0$ and $\gamma = -2$ are also sufficient for the existence of a projectivity f with the required properties. In fact, if $\gamma = -2$ then there exists a (unique) projective isomorphism $g: r \rightarrow s$ such that $g(P_i) = Q_i$ for $i = 1, \dots, 4$. Let us endow r, s with the systems of homogeneous coordinates $[x_1, x_2]$, $[x_0, x_1]$, respectively, and let B be a 2×2 invertible matrix representing g with respect to these coordinate systems. Then the matrix $A = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \\ 1 & 0 \end{array} \right)$ represents (with respect to the standard homogeneous coordinates of $\mathbb{P}^2(\mathbb{R})$) a projectivity f satisfying the required properties.

Let us now construct g so that $g([1, 0]) = [1, 0]$, $g([1, 1]) = [1, 1]$, $g([1, 2]) = [0, 1]$. In this way $g(P_i) = Q_i$ for $i = 1, \dots, 3$, so also $g(P_4) = Q_4$, due to the condition imposed on the cross-ratios. An easy computation shows that g is induced e.g. by the matrix $B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$. Therefore, the projectivity $\mathbb{P}^2(\mathbb{R})$ induced by the matrix $A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ satisfies the required properties.

We observe that, once the necessary conditions $\alpha = 0$ and $\gamma = -2$ have been established, one could also directly construct f as follows. Let U be a point in $l_3 \setminus \{R, P_3\}$, e.g. $U = [1, 1, 2]$, and let V be a point in $m_3 \setminus \{S, Q_3\}$, e.g. $V = [0, 1, 1]$. Since $\{R, P_1, P_2, U\}$ and $\{S, Q_1, Q_2, V\}$ are both projective frames of $\mathbb{P}^2(\mathbb{R})$, there exists a unique projectivity f of $\mathbb{P}^2(\mathbb{R})$ that maps R, P_1, P_2, U to S, Q_1, Q_2, V , respectively. As $r = L(P_1, P_2)$ and $l_3 = L(R, U)$, it easily follows that $f(r) = L(Q_1, Q_2) = s$ and $f(l_3) = L(S, V) = m_3$. Since $P_3 = r \cap l_3$ and $Q_3 = s \cap m_3$, one can conclude that $f(P_3) = Q_3$. Finally, the fact that $f(P_4) = Q_4$ follows from the condition imposed on the cross-ratios.

Solution (2). If we consider a line of $\mathbb{P}^2(\mathbb{R})$ as a point of $\mathbb{P}^2(\mathbb{R})^*$, then the lines described in the statement of the exercise correspond to the points

$$L_1 = [0, 0, 1], L_2 = [0, -1, 1], L_3 = [0, -2, 1], L_4 = [\alpha, -\alpha - 4, 1], R = [1, 0, 0]$$

$$M_1 = [0, 1, 0], M_2 = [-1, 1, 0], M_3 = [1, 0, 0], M_4 = [-\gamma, 1, 0], S = [0, 0, 1],$$

respectively. Therefore, we are looking for a projectivity g of $\mathbb{P}^2(\mathbb{R})^*$ such that $g(L_i) = M_i$ for $i = 1, \dots, 4$, and $g(R) = S$. Since M_1, M_2, M_3, M_4 are collinear (as they correspond to lines in a pencil), the points L_1, L_2, L_3, L_4 must be collinear too, and this occurs if and only if $\alpha = 0$. Moreover, a necessary (and sufficient) condition for the existence of a projectivity g such that $g(L_i) = M_i$ for $i = 1, \dots, 4$ is that the cross-ratio $\beta(L_1, L_2, L_3, L_4)$ coincides with the cross-ratio $\beta(M_1, M_2, M_3, M_4)$; it can be checked that this occurs if and only if $\gamma = -2$.

An easy computation (similar to the one carried out in solution (1)) shows that the projectivity g associated e.g. to the matrix $H = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ is such that $g(L_i) = M_i$ for $i = 1, \dots, 4$ and $g(R) = S$.

We now come back to $\mathbb{P}^2(\mathbb{R})$ via duality. Thus, if we consider the points of $\mathbb{P}^2(\mathbb{R})^*$ as lines of $\mathbb{P}^2(\mathbb{R})$, then the matrix ${}^tH^{-1} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ induces a projectivity that satisfies all the required properties.

- Exercise 43.** (a) Let r, s be distinct lines of $\mathbb{P}^2(\mathbb{K})$ passing through the point A . Let B, C, D be pairwise distinct points of $r \setminus \{A\}$, and let B', C', D' be pairwise distinct points of $s \setminus \{A\}$. Prove that the lines $L(B, B')$, $L(C, C')$, $L(D, D')$ all intersect if and only if $\beta(A, B, C, D) = \beta(A, B', C', D')$.
- (b) Let A, B be distinct points of a line r of $\mathbb{P}^2(\mathbb{K})$. Let r_1, r_2, r_3 be pairwise distinct lines passing through A and distinct from r , and let s_1, s_2, s_3 be pairwise distinct lines passing through B and distinct from r . Prove that the points $r_1 \cap s_1, r_2 \cap s_2, r_3 \cap s_3$ are collinear if and only if $\beta(r, r_1, r_2, r_3) = \beta(r, s_1, s_2, s_3)$.
- (c) Let A, B be pairwise distinct points of $\mathbb{P}^2(\mathbb{K})$ and let \mathcal{F}_A (\mathcal{F}_B , respectively) be the pencil of lines centred at A (at B , respectively). Let $f: \mathcal{F}_A \rightarrow \mathcal{F}_B$ be a projective isomorphism such that $f(L(A, B)) = L(A, B)$. Prove that the set

$$\mathcal{Q} = \bigcup_{s \in \mathcal{F}_A} s \cap f(s)$$

is the union of two distinct lines.

Solution. (a) We denote by O the point at which the distinct lines $L(B, B')$ and $L(C, C')$ intersect; it is immediate to check that $O \notin r \cup s$. Let $\varphi_O: r \rightarrow s$ be the perspectivity centred at O , which maps the points A, B, C to the points A, B', C' , respectively. Since the cross-ratio is invariant under projective isomorphisms, we have $\beta(A, B, C, D) = \beta(A, B', C', \varphi_O(D))$. Therefore, $\beta(A, B, C, D) = \beta(A, B', C', D')$ if and only if $\varphi_O(D) = D'$, i.e., if and only if $L(D, D')$ passes through O .

(b) Via duality, the lines of the pencil centred at A (at B , respectively) form a line of the dual projective plane. Using this, it is not difficult to show that the statement of (b) is just the dual statement of (a), so the conclusion follows from the Duality principle (cf. Theorem 1.4.1).

(c) By hypothesis $f(L(A, B)) = L(A, B)$, so $L(A, B) \subseteq \mathcal{Q}$. Let r_1, r_2 be distinct lines passing through A and distinct from $L(A, B)$. Observe that $r_i \neq f(r_i)$ for $i = 1, 2$, and $r_1 \cap f(r_1), r_2 \cap f(r_2)$ are distinct points. If t is the line joining the points $r_1 \cap f(r_1)$ and $r_2 \cap f(r_2)$, then point (b) implies that, for every $r \in \mathcal{F}_A \setminus \{L(A, B)\}$, the point $r \cap f(r)$ belongs to t , and so $\mathcal{Q} \subseteq L(A, B) \cup t$. On the other hand, for every $P \in t$, if $l = L(A, P)$ then $f(l)$ is a line passing through B and such that $f(l) \cap l \in t$, so $f(l) \cap l = P$. This proves that every point of t belongs to \mathcal{Q} , thus $\mathcal{Q} = L(A, B) \cup t$.

Note. It is possible to prove point (c) via an argument based on the Duality principle. In fact, the statement of Exercise 40 implies that, if $f: \mathcal{F}_A \rightarrow \mathcal{F}_B$ is a projective isomorphism such that $f(L(A, B)) = L(A, B)$, then there exists a line $l \subseteq \mathbb{P}^2(\mathbb{C})$ such that $A \notin l, B \notin l$, and $f(s) = L(B, l \cap s)$ for every $s \in \mathcal{F}_A$. This proves that, if $s \in \mathcal{F}_A$ is distinct from $r = L(A, B)$, then $s \cap f(s) = l \cap s$, so in particular $s \cap f(s) \subset l$. It is now easy to show (as described above) that indeed $\mathcal{Q} = l \cup r$.

Exercise 44. (*Invariant sets of projectivities of the projective plane*) Let f be a projectivity of $\mathbb{P}^2(\mathbb{K})$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Find the possible configurations of fixed points, invariant lines, axes (an axis is a pointwise fixed line) and centres (a centre is a fixed point such that every line passing through it is invariant) of f .

Solution. If $\mathbb{K} = \mathbb{C}$, in a suitable system of homogeneous coordinates the projectivity f is represented by one of the following Jordan matrices (see also Sect. 1.5.3):

$$(a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{with } \lambda, \mu \in \mathbb{K} \setminus \{0, 1\}, \lambda \neq \mu;$$

$$(b) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{with } \lambda \in \mathbb{K} \setminus \{0, 1\};$$

$$(c) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$(d) \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{with } \lambda \in \mathbb{K} \setminus \{0, 1\};$$

$$(e) \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$(f) \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let now $\mathbb{K} = \mathbb{R}$. If a matrix (hence, every matrix) representing f has only real eigenvalues, then in a suitable system of homogeneous coordinates of $\mathbb{P}^2(\mathbb{R})$ the projectivity f is represented by one of the matrices listed above. Otherwise, every matrix representing f has one real and two conjugate non-real eigenvalues. In this case, there exists a system of homogeneous coordinates of $\mathbb{P}^2(\mathbb{R})$ with respect to which f is represented by the matrix

$$(g) \quad A = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } a \in \mathbb{R}, b \in \mathbb{R}^*.$$

Let us now analyze the cases listed above, focusing on the fixed-point set and on the existence of invariant lines, axes and centres.

Recall that $P = [X]$ is a fixed point of f if and only if X is an eigenvector of A , and a line r of equation ${}^tCX = 0$ is invariant under f if and only if C is an eigenvector of the matrix tA (cf. Sect. 1.4.5).

We set $P_0 = [1, 0, 0]$, $P_1 = [0, 1, 0]$, $P_2 = [0, 0, 1]$ and we denote by r_i the line of equation $x_i = 0$ for $i = 0, 1, 2$.

Case (a). The eigenspaces of the matrix A are the lines generated by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, respectively, so f has three fixed points, namely the points P_0, P_1, P_2 . Moreover, f admits three invariant lines, namely the lines r_0, r_1, r_2 . We observe that $f|_{r_i}$ is a hyperbolic projectivity of r_i for $i = 0, 1, 2$.

Case (b). The eigenvectors of the matrix A are of the form $(0, 0, c)$ or $(a, b, 0)$, so P_2 is a fixed point and the line r_2 consists of fixed points, i.e., it is an axis. Any line passing through P_2 has an equation of the form $ax_0 + bx_1 = 0$; since $(a, b, 0)$ is an eigenvector of ${}^tA = A$, every line passing through P_2 is invariant, so P_2 is a center. Summarizing, we have a centre P_2 and an axis r_2 such that $P_2 \notin r_2$.

Case (c). In this case f is the identity, so every point is a centre and every line is an axis.

Case (d). The same argument as above shows that f has two fixed points, namely the points P_0 and P_2 , and two invariant lines, namely the lines r_1 and r_2 . Therefore, there are no centres and no axes. We observe that $f|_{r_1}$ is a hyperbolic projectivity and $f|_{r_2}$ is a parabolic projectivity.

Case (e). The eigenvectors of A are of the form $(a, 0, b)$, so the line r_1 is an axis. Moreover, the eigenvectors of tA are of the form $(0, a, b)$, so any line of equation $ax_1 + bx_2 = 0$ is invariant; these lines are exactly the elements of the pencil centred at P_0 , so P_0 is a center. In this case, the centre P_0 belongs to the axis r_1 .

Case (f). It is readily seen that there exists a unique fixed point P_0 and, by looking at tA , a unique invariant line r_2 . We observe that the fixed point P_0 belongs to the invariant line r_2 , and $f|_{r_2}$ is a parabolic projectivity.

Case (g). Recall that this case occurs only when $\mathbb{K} = \mathbb{R}$ and A has one real and two conjugate non-real eigenvalues. Therefore, P_2 is the unique fixed point and r_2 is the unique invariant line; moreover, the invariant line does not contain the fixed point, and the restriction $f|_{r_2}$ is an elliptic projectivity.

Note. The solution of Exercise 44 shows that projectivities of $\mathbb{P}^2(\mathbb{K})$ can be partitioned into 6 (when $\mathbb{K} = \mathbb{C}$) or 7 (when $\mathbb{K} = \mathbb{R}$) disjoint families, which can be characterized according to the number of fixed points, invariant lines, centres and axes, and to the incidence relations that hold among these objects. These families are classified by the Jordan form (when $\mathbb{K} = \mathbb{C}$) or by the “real Jordan form” (when $\mathbb{K} = \mathbb{R}$) of the associated matrices.

Exercise 45. Let $P = [1, 1, 1]$ and $l = \{x_0 + x_1 - 2x_2 = 0\}$ be a point and a line of $\mathbb{P}^2(\mathbb{R})$, respectively. Find an explicit formula for a projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ such that $f(l) = l$ and that the fixed-point set of f coincides with $\{P\}$.

Solution. If g is the projectivity of $\mathbb{P}^2(\mathbb{R})$ induced by the matrix $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, then $[1, 0, 0]$ is the unique fixed point of g , and it is contained in the line $\{x_2 = 0\}$,

which is invariant under g (cf. Exercise 44). Therefore, if $h: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ is a projectivity such that $h([1, 0, 0]) = P$, $h(\{x_2 = 0\}) = l$, then the map $h \circ g \circ h^{-1}$ gives the required projectivity.

Since $l = \mathbb{P}(W)$, where $W \subseteq \mathbb{R}^3$ is the linear subspace generated by $(1, 1, 1)$ and $(2, 0, 1)$, such a projectivity h is induced for example by the invertible matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. As $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, we have $ABA^{-1} = \begin{pmatrix} 3 & 1 & -3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, so we can set $f([x_0, x_1, x_2]) = [3x_0 + x_1 - 3x_2, x_2, x_0]$.

Exercise 46. In $\mathbb{P}^2(\mathbb{R})$ consider the points

$$P_1 = [1, 0, 0], \quad P_2 = [0, 1, 0], \quad P_3 = [0, 0, 1], \quad P_4 = [1, 1, 1],$$

$$Q_1 = [1, -1, -1], \quad Q_2 = [1, 3, 1], \quad Q_3 = [1, 1, -1], \quad Q_4 = [1, 1, 1].$$

- (a) Provide an explicit formula for a projectivity f of $\mathbb{P}^2(\mathbb{R})$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3, 4$. Is such a projectivity unique?
 (b) Find all the lines of $\mathbb{P}^2(\mathbb{R})$ that are invariant under f .

Solution. (a) The points P_1, P_2, P_3, P_4 are in general position, so they define a projective frame with associated normalized basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. In a similar way, an easy computation shows that Q_1, Q_2, Q_3, Q_4 define a projective frame with associated normalized basis $\{(1, -1, -1), (1, 3, 1), (-1, -1, 1)\}$. By the Fundamental theorem of projective transformations, there exists a unique projectivity f satisfying the required properties, and such an f is induced by the linear map defined

by the matrix $B = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

(b) An easy computation shows that the characteristic polynomial of B is $(2 - t)^2(1 - t)$. The eigenspace of B relative to the eigenvalue 2 is 2-dimensional and it is described by the equation $x_0 - x_1 + x_2 = 0$, where (x_0, x_1, x_2) are the standard coordinates of \mathbb{R}^3 . The eigenspace of B relative to the eigenvalue 1 is 1-dimensional, and it is generated by the vector $v = (1, 1, 1)$. Now it follows from Exercise 44 that, if r is the line of $\mathbb{P}^2(\mathbb{R})$ of equation $x_0 - x_1 + x_2 = 0$ and $P = [1, 1, 1] = [v] \in \mathbb{P}^2(\mathbb{R})$, then r is pointwise fixed by f , and a line of $\mathbb{P}^2(\mathbb{R})$ distinct from r is f -invariant if and only if it passes through P .

Exercise 47. In $\mathbb{P}^2(\mathbb{R})$ consider the points

$$P_1 = [1, 0, 0], \quad P_2 = [0, -1, 1], \quad P_3 = [0, 0, -1], \quad P_4 = [1, -1, 2],$$

$$Q_1 = [3, 1, -1], \quad Q_2 = [-1, -3, 3], \quad Q_3 = [-1, 1, 3], \quad Q_4 = [1, -1, 5].$$

- (a) Construct a projectivity f of $\mathbb{P}^2(\mathbb{R})$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3, 4$. Is such a projectivity unique?

- (b) Prove that there exist a point P and a line r such that $P \notin r$, $f(P) = P$ and r is pointwise fixed by f .
- (c) Let s be a line passing through P and set $Q = s \cap r$. Prove that the cross-ratio $\beta(P, Q, R, f(R))$ is independent of s and of R , when s varies in the pencil of lines centred at P and R varies in $s \setminus \{P, Q\}$.

Solution. It is easy to check that the points P_1, P_2, P_3, P_4 define a projective frame of $\mathbb{P}^2(\mathbb{R})$ with associated normalized basis $v_1 = (1, 0, 0)$, $v_2 = (0, -1, 1)$, $v_3 = (0, 0, 1)$. Moreover, the points Q_1, Q_2, Q_3, Q_4 define a projective frame of $\mathbb{P}^2(\mathbb{R})$ with associated normalized basis $w_1 = (3, 1, -1)$, $w_2 = (-1, -3, 3)$, $w_3 = (-1, 1, 3)$. Therefore, by the Fundamental theorem of projective transformations, there exists a unique projectivity f of $\mathbb{P}^2(\mathbb{R})$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3, 4$, and this projectivity is induced e.g. by the linear map $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\varphi(v_i) = w_i$ for $i = 1, 2, 3$. Since $(0, 1, 0) = -v_2 + v_3$ and $-w_2 + w_3 = (0, 4, 0)$, the map φ is represented, with respect to the canonical basis of \mathbb{R}^3 , by the matrix $A = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 4 & 1 \\ -1 & 0 & 3 \end{pmatrix}$.

The characteristic polynomial of A is equal to $(4 - t)^2(2 - t)$. It is immediate to check that $\dim \ker(A - 4I) = 2$, so φ admits a 2-dimensional eigenspace V_4 relative to the eigenvalue 4, and a 1-dimensional eigenspace V_2 relative to the eigenvalue 2. It follows that $P = \mathbb{P}(V_2)$ and $r = \mathbb{P}(V_4)$ are the point and the line requested in (b). An easy computation shows that $P = [1, -1, 1]$ and that r has equation $x_0 + x_2 = 0$.

Let us now take $Q \in r$. We have seen in Sect. 1.5.4 that, since $P = [v]$, where v is an eigenvector of φ relative to the eigenvalue 2, and $Q = [w]$, where w is an eigenvector of φ relative to the eigenvalue 4, for every $R \in L(P, Q) \setminus \{P, Q\}$ we have $\beta(P, Q, R, f(R)) = 4/2 = 2$.

Exercise 48. Let r, s be distinct lines of $\mathbb{P}^2(\mathbb{R})$ and set $R = r \cap s$. Let A, B, C be pairwise distinct points of $r \setminus \{R\}$, and let $g: r \rightarrow r$ be the unique projectivity such that $g(A) = A$, $g(R) = R$ and $g(B) = C$. For every $P \in \mathbb{P}^2(\mathbb{R}) \setminus r$, set $t(P) = L(B, P) \cap s$, and $h(P) = L(C, t(P)) \cap L(A, P)$.

- (a) Prove that there exists a unique projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ such that $f|_{\mathbb{P}^2(\mathbb{R}) \setminus r} = h$ and $f|_r = g$.
- (b) Find the fixed points of f .
- (c) Show that f is an involution if and only if $\beta(A, R, B, C) = -1$ (Fig. 2.13).

Solution. If M, N are distinct points of $s \setminus \{R\}$, then the points A, B, M, N define a projective frame of $\mathbb{P}^2(\mathbb{R})$. Let us fix the system of homogeneous coordinates induced by this frame. Then $A = [1, 0, 0]$, $B = [0, 1, 0]$, $r = \{x_2 = 0\}$, $s = \{x_0 = x_1\}$, $R = [1, 1, 0]$, and $C = [1, \beta, 0]$ with $\beta \neq 0, 1$.

(a) Take $P \in \mathbb{P}^2(\mathbb{R}) \setminus r$. Then $P = [a, b, c]$ with $c \neq 0$. The line $L(B, P)$ has equation $cx_0 - ax_2 = 0$, so $t(P) = L(B, P) \cap s = [a, a, c]$. Therefore, the line $L(C, t(P))$ has equation $-c\beta x_0 + cx_1 + a(\beta - 1)x_2 = 0$. Since $L(A, P)$ has equation $cx_1 - bx_2 = 0$, we thus get $h(P) = [a(\beta - 1) + b, \beta b, \beta c]$. It follows that, on the set $\mathbb{P}^2(\mathbb{R}) \setminus r$, the map h coincides with the projectivity $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^2(\mathbb{R})$ represented

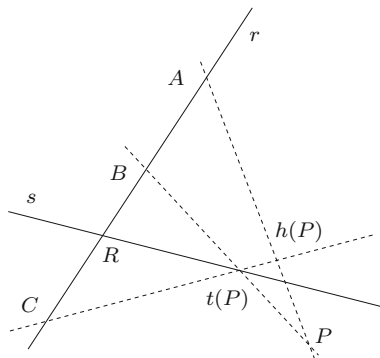


Fig. 2.13 The construction described in Exercise 48

(with respect to the chosen coordinates) by the invertible matrix $\begin{pmatrix} \beta - 1 & 1 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}$.

Therefore, in order to prove (a) we are left to show that $f|_r = g$. However, since $f([x_0, x_1, x_2]) = [(\beta - 1)x_0 + x_1, \beta x_1, \beta x_2]$ for every $[x_0, x_1, x_2] \in \mathbb{P}^2(\mathbb{R})$, we have $f(A) = f([1, 0, 0]) = [1, 0, 0] = A$, $f(R) = f([1, 1, 0]) = [1, 1, 0] = R$ and $f(B) = f([0, 1, 0]) = [1, \beta, 0] = C$, so $f|_r$ and g coincide on three pairwise distinct points of r , so that $f|_r = g$.

(b) If $P \in s \setminus \{R\}$, then $t(P) = P$, and $f(P) = h(P) = P$. Moreover, we have proved in point (a) that $f(R) = R$ and $f(A) = A$. Therefore, every point of $s \cup \{A\}$ is fixed by f . Now, if $f(M) = M$ for some $M \notin s \cup \{A\}$, then f is the identity (cf. Sect. 1.2.5), and this contradicts the fact that $f(B) \neq B$. We have thus shown that the fixed-point set of f is equal to $s \cup \{A\}$.

(c) Since $f(A) = A$, $f(R) = R$ and $f(B) = C$, Exercise 23 implies that $f^2|_r = (f|_r)^2 = \text{Id}_r$ if and only if $\beta(A, R, B, C) = -1$. Therefore, if f is an involution, then $\beta(A, R, B, C) = -1$. Conversely, let us suppose that $\beta(A, R, B, C) = -1$. Then $f^2|_r = \text{Id}_r$. But $f|_s = \text{Id}_s$, so $f^2|_{r \cup s} = \text{Id}_{r \cup s}$. Since the fixed-point set of a projectivity is given by the union of pairwise skew projective subspaces (cf. Sect. 1.2.5), we deduce that f^2 coincides with the identity on $L(r, s) = \mathbb{P}^2(\mathbb{R})$, and this concludes the proof.

Exercise 49. Let $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ be a non-trivial involution.

- Show that the fixed-point set of f is given by $l \cup \{P\}$, where l, P are a line and a point of $\mathbb{P}^2(\mathbb{K})$, respectively, and $P \notin l$.
- Show that there exists an affine chart $h: \mathbb{P}^2(\mathbb{K}) \setminus l \rightarrow \mathbb{K}^2$ such that $h(f(h^{-1}(v))) = -v$ for every $v \in \mathbb{K}^2$.

Solution. (a) We first observe that the minimal polynomial of a linear endomorphism g of \mathbb{K}^3 cannot be irreducible of degree 2. In fact, if this were the case, then g would not have any eigenvalue. However, the characteristic polynomial of g would be divisible by the minimal polynomial of g . Therefore, having degree 3, the characteristic


polynomial of g would be divisible by a linear factor, and this would imply that g admits an eigenvalue, a contradiction.

Let now $\varphi: \mathbb{K}^3 \rightarrow \mathbb{K}^3$ be a linear map inducing f , and let m be the minimal polynomial of φ . As $f \neq \text{Id}$, we have that $\deg m \geq 2$. Let us show that m is the product of two non-proportional linear factors. By hypothesis there exists $\lambda \in \mathbb{K}^*$ such that $\varphi^2 = \lambda \text{Id}_{\mathbb{K}^3}$. If λ is not a square in \mathbb{K} then m , being a factor of $t^2 - \lambda$, is irreducible of degree 2, and this contradicts what we have proved above. So let $\alpha \in \mathbb{K}$ be a square root of λ . Since m divides $t^2 - \lambda = (t - \alpha)(t + \alpha)$ and $\deg m \geq 2$, we have $m = (t - \alpha)(t + \alpha)$. Also observe that, since $\alpha \neq 0$ and $\mathbb{K} \subseteq \mathbb{C}$, we have $\alpha \neq -\alpha$, so m is indeed the product of two non-proportional linear factors.

Therefore, \mathbb{K}^3 decomposes as the direct sum of two eigenspaces W_1, W_2 of φ having dimensions 1 and 2, respectively. If $P = \mathbb{P}(W_1)$, $l = \mathbb{P}(W_2)$, then $P \not\subseteq l$, and the fixed-point set of f coincides with $\{P\} \cup l$.

(b) After replacing φ by $\alpha^{-1}\varphi$ or by $-\alpha^{-1}\varphi$, we may suppose that $\varphi|_{W_1} = \text{Id}_{W_1}$ and $\varphi|_{W_2} = -\text{Id}_{W_2}$. Now, if $v_1 \in W_1 \setminus \{0\}$ and $\{v_2, v_3\}$ is a basis of W_2 , then $\{v_1, v_2, v_3\}$ is a basis of \mathbb{K}^3 . It is not difficult to show that, with respect to the homogeneous coordinates of $\mathbb{P}^2(\mathbb{K})$ induced by $\{v_1, v_2, v_3\}$, the map f is described by the formula $f([x_0, x_1, x_2]) = [x_0, -x_1, -x_2]$. Moreover, with respect to these coordinates we have $l = \mathbb{P}(\{x_0 = 0\})$. After setting $h: \mathbb{P}^2(\mathbb{K}) \setminus l \rightarrow \mathbb{K}^2$, $h[1, x, y] = (x, y)$, we finally have $h(f(h^{-1}(v))) = -v$ for every $v \in \mathbb{K}^2$.

Note. If $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, Exercise 49 admits an easier solution that makes use of the Jordan form (or of the “real Jordan form”) (cf. Exercise 44).

 **Exercise 50.** Let $f: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{P}^2(\mathbb{Q})$ be a projectivity such that $f^4 = \text{Id}$, $f^2 \neq \text{Id}$. Compute the number of fixed points of f .

Solution (1). Let $\varphi: \mathbb{Q}^3 \rightarrow \mathbb{Q}^3$ be a linear map inducing f , and let $m, p \in \mathbb{Q}[t]$ be the minimal and the characteristic polynomial of φ , respectively. Since the fixed-point set of f coincides with the projection on $\mathbb{P}^2(\mathbb{Q})$ of the set of eigenvectors of φ , we analyze the number and the dimensions of the eigenspaces of φ . To this aim, we study the factorization of m and p .

Since $f^4 = \text{Id}_{\mathbb{P}^2(\mathbb{Q})}$, there exists $\lambda \in \mathbb{Q}^*$ such that $\varphi^4 = \lambda \text{Id}_{\mathbb{Q}^3}$, so m divides $t^4 - \lambda$ in $\mathbb{Q}[t]$. Let us prove that λ is positive. If λ were negative, then $t^4 - \lambda$ would not have any rational root, so φ would not have any eigenvalue. As a consequence, p would not have any rational root so it would be irreducible, being of degree 3. By Hamilton-Cayley Theorem, we would have $m = p$, so m would have degree 3. Being divided by m , the polynomial $t^4 - \lambda$ would be divided by a linear factor, thus admitting a rational root. So λ would be positive, against our hypothesis. We have thus shown that λ is positive.

Let us prove that p is the product of a linear factor and an irreducible factor of degree 2. This implies that φ has exactly one eigenspace, and that this eigenspace is one-dimensional, so that f necessarily has exactly one fixed point. So let $\alpha \in \mathbb{R}^*$ be the positive fourth root of λ . If p were irreducible over \mathbb{Q} , then Hamilton-Cayley Theorem would imply that $m = p$, so the polynomial $t^4 - \lambda = (t - \alpha)(t + \alpha)(t - i\alpha)(t + i\alpha)$ would be divided by a factor of degree 3 and irreducible over

\mathbb{Q} . By analyzing the cases $\alpha \in \mathbb{Q}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, it can be easily shown that this is impossible, so p is reducible, and it is divided by a linear factor. As a consequence, φ admits an eigenvalue, so α is rational. We now observe that $t^2 + \alpha^2$ must divide m , because otherwise m would divide $t^2 - \alpha^2$, and we would have $f^2 = \text{Id}$. Therefore, we may conclude that either $p = m = (t - \alpha)(t^2 + \alpha^2)$ or $p = m = (t + \alpha)(t^2 + \alpha^2)$, as desired.

Solution (2). Set $g = f^2$. We have seen in Exercise 49 that the fixed-point set of g is given by $\{P\} \cup l$, where P is a point and l is a line not containing P . Now, if a point Q is fixed by g , then $g(f(Q)) = f^3(Q) = f(g(Q)) = f(Q)$, and $f(Q)$ is also fixed by g . It follows that $f(\{P\} \cup l) \subseteq \{P\} \cup l$, and $f(P) = P, f(l) = l$, since f is a projectivity. So P is a fixed point of f .

Suppose now that $Q \neq P$ is another fixed point of f , and set $s = L(P, Q)$. Of course we have $f(s) = s$ and, if s is endowed with homogeneous coordinates such that $P = [1, 0]$, $Q = [0, 1]$, then the map $f|_s$ can be represented by a matrix of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{Q}^*$. From $f^4 = \text{Id}$ it follows that $\alpha^4 = \beta^4$, so $\alpha^2 = \beta^2$, and $g|_s = f^2|_s = \text{Id}_s$, which contradicts the fact that the fixed-point set of g is contained in $\{P\} \cup l$, while the line s passes through P , so it contains points in $\mathbb{P}^2(\mathbb{K}) \setminus (\{P\} \cup l)$. This proves that P is the unique fixed point of f .

Note. Let \mathcal{H} be the set of projectivities that satisfy the conditions described in the statement. The solutions above show that any $f \in \mathcal{H}$ has exactly one fixed point, but they do not exclude the possibility that $\mathcal{H} = \emptyset$ (and, if this were the case, then any answer to the question “How many fixed points has f ?” would be correct!). However, it is easy to check that the map $f: \mathbb{P}^2(\mathbb{Q}) \rightarrow \mathbb{P}^2(\mathbb{Q})$ defined by $f([x_0, x_1, x_2]) = [x_1, -x_0, x_2]$ belongs to the set \mathcal{H} .

Note. It is easy to check that the solutions above may be adapted to deal with the case when the field \mathbb{Q} is replaced by the field \mathbb{R} . Therefore, the statement of Exercise 50 is still true if we replace $\mathbb{P}^2(\mathbb{Q})$ by $\mathbb{P}^2(\mathbb{R})$.

On the contrary, every projectivity f of $\mathbb{P}^2(\mathbb{C})$ such that $f^4 = \text{Id}$ is induced by a diagonalizable linear map, so it admits at least three fixed points. Moreover, in this case the conditions $f^4 = \text{Id}, f^2 \neq \text{Id}$ are not sufficient to determine the number of fixed points of f : if $g, h: \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ are the projectivities induced by the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}$, respectively, then $g^4 = h^4 = \text{Id}, g^2 \neq \text{Id}, h^2 \neq \text{Id}$, but g and h do not have the same number of fixed points.

Exercise 51. Let P_1, P_2, P_3 be points of $\mathbb{P}^2(\mathbb{K})$ in general position; let r be a line such that $P_i \notin r$ for $i = 1, 2, 3$.

(a) Prove that there exists a unique projectivity f of $\mathbb{P}^2(\mathbb{K})$ such that

$$f(P_1) = P_1, \quad f(P_2) = P_3, \quad f(P_3) = P_2, \quad f(r) = r.$$

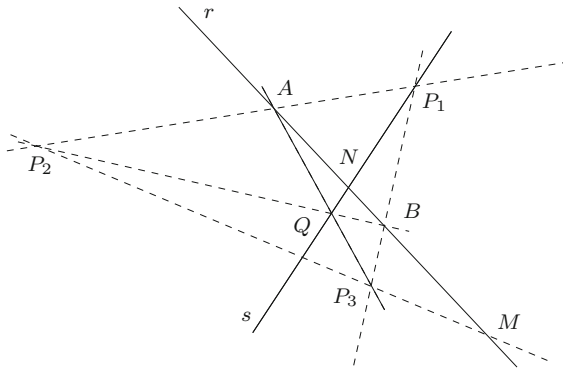


Fig. 2.14 The construction described in the solution of Exercise 51

- (b) Prove that the fixed-point set of f is the union of a point $M \in r$ and a line $s \subseteq \mathbb{P}^2(\mathbb{K})$ such that $M \notin s$.

Solution. (a) Set $A = L(P_1, P_2) \cap r$, $B = L(P_1, P_3) \cap r$ (cf. Fig. 2.14). It is immediate to check that A, B, P_2, P_3 define a projective frame of $\mathbb{P}^2(\mathbb{K})$.

If f is a projectivity satisfying the required properties, then $f(L(P_1, P_2)) = L(f(P_1), f(P_2)) = L(P_1, P_3)$, so $f(A) = f(r \cap L(P_1, P_2)) = r \cap L(P_1, P_3) = B$. In a similar way one shows that $f(B) = A$ so, as by hypothesis $f(P_2) = P_3$ and $f(P_3) = P_2$, the Fundamental theorem of projective transformations implies that the required projectivity is unique, if it exists.

So let $f: \mathbb{P}^2(\mathbb{K}) \rightarrow \mathbb{P}^2(\mathbb{K})$ be the unique projectivity such that $f(P_2) = P_3$, $f(P_3) = P_2$, $f(A) = B$, $f(B) = A$. We have

$$f(P_1) = f(L(A, P_2) \cap L(B, P_3)) = L(B, P_3) \cap L(A, P_2) = P_1$$

and

$$f(r) = f(L(A, B)) = L(f(A), f(B)) = L(B, A) = r,$$

so f satisfies the properties described in the statement.

- (b) If $M = L(P_2, P_3) \cap r$ then

$$f(M) = f(L(P_2, P_3)) \cap f(r) = L(P_2, P_3) \cap r = M.$$

Moreover $f(L(A, P_3)) = L(B, P_2)$ and $f(L(B, P_2)) = L(A, P_3)$, so also the point $Q = L(A, P_3) \cap L(B, P_2)$ is fixed by f . As $L(A, P_3) \cap r = A$, $L(B, P_2) \cap r = B$, we also have $Q \notin r$, and $Q \neq P_1$ because otherwise B would lie on $L(P_1, P_2)$ and P_1, P_2, P_3 would be collinear. So if $s = L(Q, P_1)$, then the point $N = s \cap r$ is well defined. Since Q and P_1 are fixed by f , we have $f(s) = s$, so $f(N) = f(s \cap r) = s \cap r = N$. The restriction of f to s fixes the three distinct points P_1, Q, N , so it coincides with the identity of s .

Since P_2, P_3, A, B are in general position, the points $M = L(P_2, P_3) \cap L(A, B)$, $P_1 = L(P_3, B) \cap L(P_2, A)$, $Q = L(A, P_3) \cap L(B, P_2)$ are not collinear (cf. Exercise 6), so $M \notin s = L(P_1, Q)$. Therefore, the fixed-point set of f contains the line s and the point $M \in r$, which does not lie on s . Now, f cannot have other fixed points, because otherwise f would coincide with the identity on a projective frame of $\mathbb{P}^2(\mathbb{K})$, so it would be the identity, and this would contradict the fact that $f(P_2) = P_3 \neq P_2$.

Exercise 52. Let P_1, P_2, P_3, P_4 be points of $\mathbb{P}^1(\mathbb{K})$ such that $\beta(P_1, P_2, P_3, P_4) = -1$ and let $(\mathbb{P}^1(\mathbb{K}) \setminus \{P_4\}, g)$ be any affine chart. Show that $g(P_3)$ is the midpoint of the segment with endpoints $g(P_1), g(P_2)$, i.e., show that, if $\alpha_i = g(P_i)$ for $i = 1, 2, 3$, then $\alpha_3 = \frac{\alpha_1 + \alpha_2}{2}$.

Solution (1). By definition of affine chart (cf. Sect. 1.3.8) we may endow $\mathbb{P}^1(\mathbb{K})$ with a system of homogeneous coordinates such that $P_1 = [1, \alpha_1]$, $P_2 = [1, \alpha_2]$, $P_3 = [1, \alpha_3]$, $P_4 = [0, 1]$. Then it follows from Sect. 1.5.1 that

$$-1 = \frac{(1-0)(\alpha_2 - \alpha_3)}{(\alpha_3 - \alpha_1)(0-1)} = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3},$$

hence $\alpha_3 - \alpha_1 = \alpha_2 - \alpha_3$ and $\alpha_3 = \frac{\alpha_1 + \alpha_2}{2}$, as desired.

Solution (2). Thanks to the symmetries of the cross-ratio (cf. Sect. 1.5.2) we have

$$\beta(P_2, P_1, P_3, P_4) = \beta(P_1, P_2, P_3, P_4)^{-1} = (-1)^{-1} = -1 = \beta(P_1, P_2, P_3, P_4),$$

so there exists a projectivity $f: \mathbb{P}^1(\mathbb{K}) \rightarrow \mathbb{P}^1(\mathbb{K})$ such that $f(P_1) = P_2, f(P_2) = P_1, f(P_3) = P_3, f(P_4) = P_4$. As $f(P_4) = P_4$, the map $h: \mathbb{K} \rightarrow \mathbb{K}$ defined by $h = g \circ f \circ g^{-1}$ is a well-defined affinity. Therefore, there exist $\lambda \in \mathbb{K}^*, \mu \in \mathbb{K}$ such that $h(x) = \lambda x + \mu$ for every $x \in \mathbb{K}$. Moreover, since $f(P_1) = P_2$ and $f(P_2) = P_1$, we have $h(\alpha_1) = \alpha_2, h(\alpha_2) = \alpha_1$, so $\lambda\alpha_1 + \mu = \alpha_2$ and $\lambda\alpha_2 + \mu = \alpha_1$. It follows that $\lambda = -1$ and $\mu = \alpha_1 + \alpha_2$, so $h(\alpha_3) = -\alpha_3 + \alpha_1 + \alpha_2$. Moreover, from $f(P_3) = P_3$ we deduce that $h(\alpha_3) = \alpha_3$, so $-\alpha_3 + \alpha_1 + \alpha_2 = \alpha_3$, as desired.

Projective Geometry

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