Chapter 2
Gaussian Optics

2.1 Gaussian Approximation for a Single Surface

In this chapter, we present the Gaussian analysis of an optical system $\mathbb{S}$. In particular, we introduce all of the optical characteristics of $\mathbb{S}$ (pupils, focal points, nodal points, principal planes, etc.). All these data can be obtained using the notebook ***TotalAberrations***.

Among the many potential ways of presenting this topic, we use Fermat's principle in order to introduce an approach that will be extended to third-order aberration theory.[1]

Let $\mathbb{S}$ be a single surface S with a symmetry of revolution about an axis a. N and N' denote the (constant) refractive indices of the media before and after S. Finally, let $Oxyz$ be a Cartesian frame of reference with $Oz \equiv a$ and the origin O located at the vertex of S (see Figure 2.1).

Note that Oyz is the meridional plane whereas Oxz is the sagittal plane (see Section 1.2).

Suppose that the object lies in a plane and denote by P a point of π. Finally, let π' be a plane in the image space. Due to the symmetry of S, it is always possible to choose P to be on the axis Oy, so it will have coordinates $(0, y, z)$. Consider the bundle Γ of polygonal lines γ that, starting from $(0, y, z)$, intersect S at the point (X, Y, Z) and the plane π' at $(0, My, z')$ (see Figure 2.2). In particular, the ray γ_0, which originates at $(0, y, z)$ and intersects S at its vertex, and the plane π' at $(0, My, z')$, belongs to Γ. We wish to know the conditions under which Γ is formed from *rays* or, equivalently, the points $(0, y, z)$ and $(0, My, z')$ are stigmatic (see Section 1.1).

Due to Fermat's principle and the remark 3 in Section 1.1, this happens if and only if the optical paths along the polygonal lines of Γ have the same value or, equivalently, if the following optical path difference vanishes:

[1] Another elementary way of introducing the paraxial optics is shown in Section 2.10 of this chapter.

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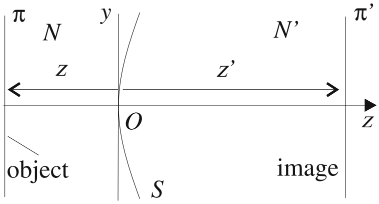


Fig. 2.1 Notations for a single surface

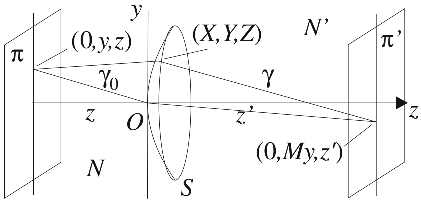


Fig. 2.2 γ and γ_0 rays

$$\begin{aligned}\Phi \equiv\ & N\sqrt{X^2 + (Y - y)^2 + (Z - z)^2} \\ & + N'\sqrt{X^2 + (My - Y)^2 + (z' - Z)^2} \\ & - N\sqrt{y^2 + z^2} - N'\sqrt{M^2y^2 + z'^2} = 0\end{aligned} \tag{2.1}$$

for any choice of $(X, Y, Z) \in S$. It is very simple to verify that the quantities

$$\xi = y^2, \quad \eta = X^2 + Y^2, \quad \zeta = yY, \tag{2.2}$$

are rotational invariants, i.e., they are invariant under an arbitrary rotation of the plane Oxy about the Oz-axis.[2] Introducing (2.2) into (2.1) we obtain

$$\begin{aligned}\Phi \equiv\ & N\sqrt{z^2 + \eta - 2\zeta + \xi - 2zZ + Z^2} \\ & + N'\sqrt{z'^2 + \eta - 2M\zeta + M^2\xi - 2z'Z + Z^2} \\ & - N\sqrt{z^2 + \xi} - N'\sqrt{z'^2 + M^2\xi} = 0.\end{aligned} \tag{2.3}$$

This condition is analyzed here using the ***Gaussian or paraxial approximation*** which is defined as follows:

the optical path difference Φ ***among all the optical paths*** $\gamma \in \Gamma$ ***and*** γ_0 ***is constant to within second-order terms in the variables*** ξ, η ***and*** ζ ***or, equivalently, to within fourth-order terms in the variables*** y, X, Y.

In other words, we are supposing that the angles, that all of the rays that intersect with the surface S form with the optical axis a, as well as the distances of the object points from a, comply with the following condition: that the coordinates y, X, Y of

[2] In fact, ξ, η are the lengths of the vectors $\mathbf{u} = (0, y)$ and $\mathbf{v} = (X, Y)$ belonging to the plane Oxy and $\zeta = \mathbf{u} \cdot \mathbf{v}$.

all the points at which the rays intersect with the base-planes π, π' and the surface S are so small that, to a first approximation, it is possible to neglect the fourth-order powers of y, X, and Y in the optical paths.

In order to determine the consequences of (2.3), we start by noting that, due to the symmetry of revolution of S, the equation $Z = F(X, Y)$ of S necessarily takes the form

$$Z = F(X^2 + Y^2).$$

Consequently, in the Gaussian approximation, it can be written as follows (see Section 1.5):

$$Z = \frac{1}{2R}(X^2 + Y^2) = \frac{1}{2R}\eta, \tag{2.4}$$

where R denotes the curvature radius of S at its vertex O.

In view of Taylor's expansion

$$\sqrt{a^2 + x} = \pm a \pm \frac{x}{2a} + O(2), \tag{2.5}$$

where we take the sign $+$ when $a > 0$ and the sign minus in the opposite case, from (2.3) we obtain

$$\Phi = \frac{1}{2}\left(\frac{N}{R} - \frac{N}{z} - \frac{N'}{R} + \frac{N'}{z'}\right)\eta + \left(\frac{N}{z} - M\frac{N'}{z'}\right)\zeta + O(2) = 0. \tag{2.6}$$

It is evident that the first-order terms vanish for any value of ξ, η, and ζ, if and only if

$$Q \equiv \frac{N}{R} - \frac{N}{z} = \frac{N'}{R} - \frac{N'}{z'}, \tag{2.7}$$

$$M \equiv \frac{y'}{y} = \frac{Nz'}{N'z}. \tag{2.8}$$

The following conclusions can be derived from the above equations:

1. The image of an object point along the axis Oy is located along the axis $O'y'$, for any (X, Y). Consequently, in this approximation, we only need to consider rays in this plane in order to obtain the image of a given object.
2. Relation (2.7) supplies the position of the image plane π' with respect to the surface S when S and the position of the object plane π are given. The quantity Q is called ***Abbe's invariant***. The planes π and π' are said to be ***conjugate*** when z and z' satisfy (2.7).
3. The ***magnification factor*** (2.8) is independent of y, meaning that the image is proportionally reduced if $|M| < 1$ and enlarged if $|M| > 1$. When $M < 0$, the image is inverted with respect to the object.

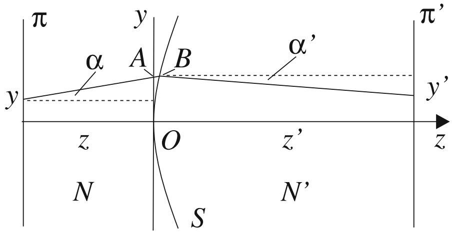


Fig. 2.3 Principal and marginal rays

We conclude this section by showing that, using the above results, it is possible to deduce the correspondence that exists between the angles α and α', which are the angles that a ray originating from an object point on y forms with the optical axis, before and after the refraction, respectively. The angles are supposed to be positive if the oriented optical axis coincides with the oriented ray after a counterclockwise rotation, negative in the opposite case[3].

In the Gaussian approximation, the points A and B in Figure 2.3 coincide and we have the relation

$$y - z\alpha = y' - z'\alpha', \tag{2.9}$$

which, in view of (2.8), can also be written as follows

$$\alpha' = \frac{1}{N'}\left(\frac{N}{z} - \frac{N'}{z'}\right)y + \frac{z}{z'}\alpha. \tag{2.10}$$

Taking into account (2.7), we finally obtain

$$\alpha' = -\frac{P}{N'}y + \frac{z}{z'}\alpha, \tag{2.11}$$

where

$$P = \frac{N' - N}{R}, \tag{2.12}$$

is the ***power*** of the surface S.

Equations (2.8) and (2.11) lead to the following system

$$y' = \frac{Nz'}{N'z}y,$$
$$\alpha' = -\frac{P}{N'}y + \frac{z}{z'}\alpha,$$

that, in view of (2.8), becomes

[3]Note that in the Figure 2.3, $z < 0$, $\alpha > 0$, $z' > 0$, $\alpha' < 0$.

$$
\begin{aligned}
y' &= My, \\
\alpha' &= -\frac{P}{N'}y + \frac{N}{N'}\frac{1}{M}\alpha.
\end{aligned} \tag{2.13}
$$

Remark 2.1 If θ and θ' denote the angles that the ray starting from (y, z) and reaching the vertex O of S forms with the optical axis before and after the refraction, the following relations hold:

$$
y = \theta z, \quad y' = \theta' z',
$$

and the equations (2.13) can equivalently be written as follows

$$
\begin{aligned}
y' &= Mz\theta, \\
\alpha' &= -\frac{P}{N'}z\theta + \frac{N}{N'}\frac{1}{M}\alpha.
\end{aligned} \tag{2.14}
$$

When the object is at infinity, (2.13) become meaningless and they can be replaced by (2.14). The corresponding value f' of z' is called the ***posterior focal length*** of the surface. From (2.7) we can derive

$$
f' = \frac{N'}{P}, \tag{2.15}
$$

and $(2.14)_1$ gives

$$
y' = \frac{N}{P}\theta. \tag{2.16}
$$

In particular, if S is a mirror, $N = -N' = 1$, $P = -2/R$, and we have

$$
f' = \frac{R}{2}, \quad y' = -\frac{R}{2}\theta. \tag{2.17}
$$

2.2 Compound Systems

Let $\mathbb{S}$ be an optical system containing n surfaces of revolution $S_1 \ldots S_n$ with the same optical axis a. The distance and the refractive index between the surface S_i and the next surface S_{i+1} will be denoted by t_i and N'_i, respectively. Finally, z_1 and z'_n represent the distances of the object plane π from S_1 and the distance of the image plane π' from S_n (see Figure 2.4).

By applying (2.7) and (2.13) to each surface S_i, we get

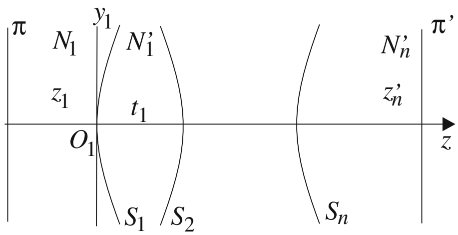


Fig. 2.4 Notations for a compound system

$$\frac{N_i}{R_i} - \frac{N_i}{z_i} = \frac{N_i'}{R_i} - \frac{N_i'}{z_i'}, \tag{2.18}$$

$$y_i' = M_i y_i, \tag{2.19}$$

$$\alpha_i' = -\frac{P_i}{N_i'} y_i + \frac{N_i}{N_i'} \frac{1}{M_i} \alpha_i, \tag{2.20}$$

and using the evident equations

$$y_{i+1} = y_i', \tag{2.21}$$

$$z_{i+1} = z_i' - t_i, \tag{2.22}$$

$$\alpha_{i+1} = \alpha_i', \tag{2.23}$$

$$N_{i+1} = N_i', \tag{2.24}$$

it is possible to evaluate the path of any ray through the whole system $\mathbb{S}$.

Although the above formulae allow us to determine the image plane and the size of the image of any object in the Gaussian approximation, it is interesting to rewrite them in a form which gives prominence to some fundamental properties of a compound system $\mathbb{S}$. First, we write relations (2.19), (2.20) in the matrix form

$$\begin{pmatrix} y_i' \\ \alpha_i' \end{pmatrix} = \mathbb{T}_i \begin{pmatrix} y_i \\ \alpha_i \end{pmatrix}, \tag{2.25}$$

where

$$\mathbb{T}_i = \begin{pmatrix} M_i & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{N_i}{N_i'}\dfrac{1}{M_i} \end{pmatrix} = \begin{pmatrix} \dfrac{N_i z_i'}{N_i' z_i} & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{z_i}{z_i'} \end{pmatrix}. \tag{2.26}$$

In (2.26) we have taken into account (2.8). In conclusion, the linear map between the object plane π_1 and the image plane π_n' is obtained composing the n linear mappings (2.25)

$$\begin{pmatrix} y_n' \\ \alpha_n' \end{pmatrix} = \mathbb{T} \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix}, \tag{2.27}$$

where the matrix $\mathbb{T}$

$$\mathbb{T} = \mathbb{T}_n \cdots \mathbb{T}_1 \tag{2.28}$$

has the following form

$$\mathbb{T} = \begin{pmatrix} M & 0 \\ A & \dfrac{N_1}{N'_n}\dfrac{1}{M} \end{pmatrix}. \tag{2.29}$$

Further,

$$M = M_1 \cdots M_n$$

is the total magnification factor of $\mathbb{S}$ and A is a convenient combination of the powers of the single surfaces S_i. For the sake of simplicity, we limit ourselves to prove (2.29) when $n = 2$. In this case, remembering the condition $N'_1 = N_2$, we have that

$$\mathbb{T} = \begin{pmatrix} M_2 & 0 \\ -\dfrac{P_2}{N'_2} & \dfrac{N_2}{N'_2}\dfrac{1}{M_2} \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ -\dfrac{P_1}{N_2} & \dfrac{N_1}{N_2}\dfrac{1}{M_1} \end{pmatrix}.$$

The product of the matrices on the left-hand side of the above equality gives

$$\mathbb{T} = \begin{pmatrix} M_1 M_2 & 0 \\ -\dfrac{1}{N'_2}\left(\dfrac{P_1}{M_2} + M_1 P_2\right) & \dfrac{N_1}{N'_2}\dfrac{1}{M_1 M_2} \end{pmatrix}$$

and (2.29) is proven.

Finally, the linear transformation (2.27) assume the final form

$$\begin{aligned} y'_n &= M y_1, \\ \alpha'_n &= A y_1 + \frac{N_1}{N'_n}\frac{1}{M}\alpha_1. \end{aligned} \tag{2.30}$$

2.3 Useful Decompositions of Matrix $\mathbb{T}$

In this section, we propose two important decompositions of the matrix $\mathbb{T}$. To this end, we first decompose the matrix $\mathbb{T}_i$ into the product of three matrices. In the Exercise 1 of this chapter the following identity is proven:

$$\mathbb{T}_i = \mathbb{D}_i^l \mathbb{R}_i \mathbb{D}_i^r, \tag{2.31}$$

where

$$\mathbb{D}_i^l = \begin{pmatrix} 1 & z_i' \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{D}_i^r = \begin{pmatrix} 1 & -z_i \\ 0 & 1 \end{pmatrix}, \tag{2.32}$$

and

$$\mathbb{R}_i = \begin{pmatrix} 1 & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{N_i}{N_i'} \end{pmatrix}. \tag{2.33}$$

It is evident that the matrix $\mathbb{D}_i^r$ represents the linear transformation

$$y_i^* = y_i - z_i \alpha_i, \quad \alpha_i^* = \alpha_i, \tag{2.34}$$

determined by the rectilinear rays between the object plane π_i of S_i and the tangent plane π_i^* to S_i at its vertex. A similar meaning must be attributed to the matrix $\mathbb{D}_i^l$ which refers to the corresponding transformation between π_i^* and the image plane π_i' of S_i. In other words, in the decomposition (2.31), the linear transformation between π_i and π_i' is regarded as the combination of the following linear transformations:

1.
$$y_i^* = y_i - z_i \alpha_i,$$
$$\alpha_i^* = \alpha_i,$$

between π_i and π_i^*,

2.
$$y_i^{*\prime} = y_i^*,$$
$$\alpha_i^* = -\frac{P_i}{N_i'} y_i^* + \frac{N_i}{N_i'} \alpha_i^*,$$

between π_i^* and π_i^* itself, which takes into account the refraction that occurs at the surface S_i, and

3.
$$y_i' = y_i^{*\prime} - z_i' \alpha_i^{*\prime},$$
$$\alpha_i' = \alpha_i^{*\prime},$$

between π_i^* and π_i'.

The above considerations allow us to conclude that the total transformation between π_1 and π_n' generated by the whole system $\mathbb{S}$ is described by the linear transformation

$$\begin{pmatrix} y_n' \\ \alpha_n' \end{pmatrix} = \begin{pmatrix} 1 & z_n' \\ 0 & 1 \end{pmatrix} \mathbb{M} \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix}, \tag{2.35}$$

where

$$\mathbb{M} = \mathbb{R}_n \mathbb{D}_n^r \mathbb{D}_{n-1}^l \cdots \mathbb{D}_2^r \mathbb{D}_1^l \mathbb{R}_1. \tag{2.36}$$

The expression of $\mathbb{M}$ can be simplified by noting that, using (2.22) and (2.32), we have

$$\mathbb{D}_i \equiv \mathbb{D}_{i+1}^r \mathbb{D}_i^l = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix}. \tag{2.37}$$

This matrix describes the linear transformation between the tangent planes π_i^* and π_{i+1}^* at the vertices of the surfaces S_i and S_{i+1}. In view of this result, we write the matrix $\mathbb{M}$ in the form

$$\mathbb{M} = \mathbb{R}_n \mathbb{D}_{n-1} \cdots \mathbb{D}_1 \mathbb{R}_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{2.38}$$

so that $\mathbb{M}$ is the composition of the all transformations (refractions at π_i^* and translations from π_i^* to π_{i+1}^*) inside the system $\mathbb{S}$ and, consequently, $\mathbb{M}$ ***depends only on the characteristics of*** $\mathbb{S}$. From (2.33), (2.37), and (2.38) it follows that

$$\det \mathbb{M} = \frac{N_1}{N_n'}. \tag{2.39}$$

We conclude this section writing (2.35) in another useful form. Consider the matrices

$$\mathbb{D}_1^r = \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{D}_n^l = \begin{pmatrix} 1 & z_n' \\ 0 & 1 \end{pmatrix}. \tag{2.40}$$

The first matrix corresponds to the translation from the object plane π_1 to the first surface S_1 of the optical system $\mathbb{S}$. The second matrix corresponds to the translation from the last surface S_n of $\mathbb{S}$ to the image plane π_n'. Consider a plane $\overline{\pi}_1$ at a distance $\overline{z}_1$ from the first surface S_1 of S and a plane $\overline{\pi}_n$ at a distance $\overline{z}_n$ from the last surface S_n of $\mathbb{S}$ (see Figure 2.5). Translation $\mathbb{D}_1^r$ can be obtained as the composition of the translation from π_1 to $\overline{\pi}_1$ and the translation from $\overline{\pi}_1$ to π_1^*, which is tangent to S_1 at its vertex, that is

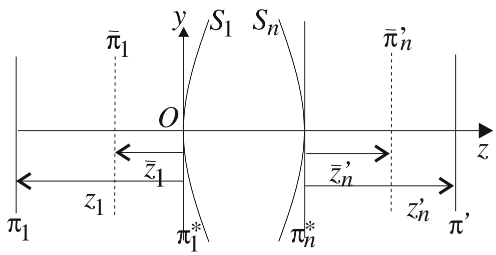


Fig. 2.5 A change of reference planes

$$\mathbb{D}_1^r = \begin{pmatrix} 1 & -\overline{z}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z_1 + \overline{z}_1 \\ 0 & 1 \end{pmatrix} \equiv \overline{\mathbb{D}}_1 \mathbb{L}_1. \tag{2.41}$$

Similarly, we can write

$$\mathbb{D}_n^l = \begin{pmatrix} 1 & -z_n' + \overline{z}_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\overline{z}_n \\ 0 & 1 \end{pmatrix} \equiv \mathbb{L}_n \overline{\mathbb{D}}_n. \tag{2.42}$$

Introducing (2.41) and (2.42) in (2.35), we finally get

$$\begin{pmatrix} \overline{y}_n' \\ \alpha_n' \end{pmatrix} = \begin{pmatrix} 1 & \overline{z}_n' \\ 0 & 1 \end{pmatrix} \mathbb{M} \begin{pmatrix} 1 & -\overline{z}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{y}_1 \\ \alpha_1 \end{pmatrix}, \tag{2.43}$$

for any pair of planes $\overline{\pi}_1$ *and* $\overline{\pi}_n'$.

2.4 Principal Planes and Focal Lengths

In this section and the next, the matrix formulation of the linear transformation between the object and image planes is shown to be very useful for defining some global properties of the optical system $\mathbb{S}$.

First, omitting the overline, the matrix equation (2.43) explicitly gives

$$\begin{aligned} y_n' &= (M_{11} + M_{21} z_n') y_1 \\ &\quad + (-M_{11} z_1 + M_{12} - M_{21} z_1 z_n' + M_{22} z_n') \alpha_1, \qquad (2.44) \\ \alpha_n' &= M_{21} y_1 + (M_{22} - M_{21} z_1) \alpha_1. \qquad (2.45) \end{aligned}$$

The plane π_1 (whose distance from S_1 is z_1) and the plane π_n' (whose distance from S_n is z_n') are conjugate if y_n' is independent of α_1; in other words, if the following condition of stigmatism is valid

$$M_{12} - M_{11} z_1 - M_{21} z_1 z_n' + M_{22} z_n' = 0. \tag{2.46}$$

When the optical system $\mathbb{S}$ (i.e., the matrix $\mathbb{M}$) and the distance z_1 are given, it supplies the value of z_n' corresponding to the conjugate plane π_n'

$$z_n' = -\frac{M_{11} z_1 - M_{12}}{M_{21} z_1 - M_{22}}. \tag{2.47}$$

Moreover, the quantity

$$M = M_{11} + M_{21} z'_n = \frac{1}{M_{22} - M_{21} z_1}, \tag{2.48}$$

gives the total magnification of $\mathbb{S}$ with respect to the conjugate planes π and π'_n. For the last equality, we used (2.39) and (2.47).

In particular, from (2.47) we have

$$\lim_{z_1 \to \infty} z'_n = -\frac{M_{11}}{M_{21}} \equiv f'_b, \tag{2.49}$$

and the quantity f_b is called ***back focal*** of $\mathbb{S}$. The ***frontal focal*** f_f can be defined in a similar way.

Two conjugate planes π_p and π'_p are called ***anterior principal plane*** and ***posterior principal plane***, if the magnification related to them is equal to 1

$$M_{11} + M_{21} z'_n = 1. \tag{2.50}$$

From the conditions (2.47), (2.50), and (2.39), we obtain the following expressions for the distances z_p and z'_p of π_p and π'_p from the surfaces S_1 and S_n, respectively:

$$z_p = \frac{1}{M_{21}} \left(M_{22} - \frac{N_1}{N'_n} \right), \quad z'_p = \frac{1 - M_{11}}{M_{21}}, \tag{2.51}$$

Further, (2.44) and (2.45) become

$$y'_p = y_p, \tag{2.52}$$

$$\alpha'_p = M_{21} y_p + \frac{N_1}{N'_n} \alpha_p. \tag{2.53}$$

Moreover, the quantities (see (2.51) and (2.49))

$$f = f_f - z_p, \quad f' = f'_b - z'_p = -\frac{1}{M_{21}}, \tag{2.54}$$

are the ***anterior focal length*** and the ***posterior focal length***.

Finally, the ***nodal points*** are two *conjugate* points on the optical axis a $(y = 0)$ such that any ray which intercepts one of them forming an angle α with a, intercepts the other one, forming the same angle with a. From (2.44) and (2.45), the values z_{nd} and z'_{nd}, which define these two points, are given by the system

$$z'_{nd} = -\frac{M_{11} z_{nd} - M_{12}}{M_{21} z_{nd} - M_{22}}, \quad M_{22} - M_{21} z_{nd} = 1. \tag{2.55}$$

Relations (2.52) and (2.53) show that the nodal points coincide with the intersections of the principal planes with a, if $N_1 = N'_n$.

An optical system $\mathbb{S}$ is said to be ***telescopic*** or ***afocal*** if $M_{21} = 0$, i.e., if $f' = \infty$.

2.5 Stops and Pupils

Predicting the Gaussian position and the size of the image produced by an optical system $\mathbb{S}$ is not the only important task of the paraxial analysis. It is equally important to establish the brightness of the image and the size of the field of view. These characteristics of the image are strongly conditioned by the presence of obstacles in the optical system as well as by the existence of lens rims, which control both the angular and spatial extension of the light beams.

Consider an object point P located on the optical axis of the system $\mathbb{S}$ and the collection c of all the rays emitted from P that pass through $\mathbb{S}$. The **aperture stop** AS is the stop that effectively controls the angular extension of c and consequently determines the amount of light arriving at the image. Figures 2.6, 2.7 and 2.8 show different positions of the aperture stop.

Let K^a be the part of the optical system $\mathbb{S}$ that is situated before the aperture stop; in other words, the part through which the light passes before meeting the aperture stop.

The image EnP of AS formed by K^a is called the **entrance pupil** of $\mathbb{S}$. Similarly, let K^b denote the part of $\mathbb{S}$ that light passes through after AS. The image ExP of AS formed by K^b is called the **exit pupil** (see Figures 2.7–2.9 and the exercises at the end of the chapter). A ray from an off-axis object P^* through the center of the entrance pupil is called a **principal ray**. Since AS, EnP and ExP are conjugate,

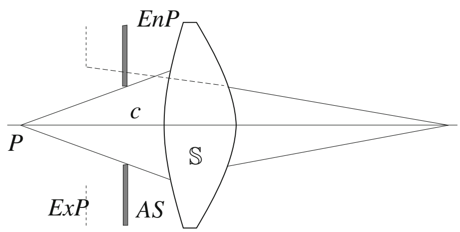


Fig. 2.6 Stop before the system

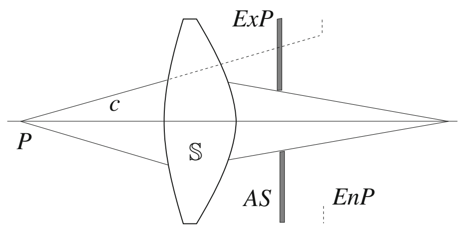


Fig. 2.7 Stop after the system

Fig. 2.8 Stop inside the system

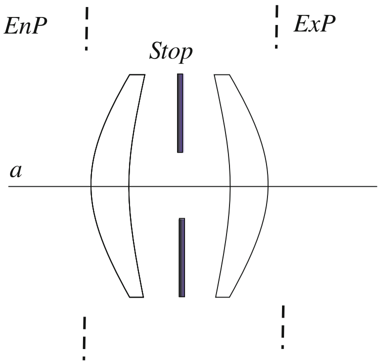


Fig. 2.9 Principal ray

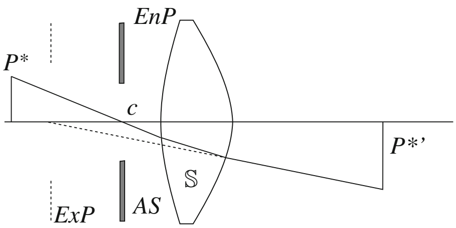
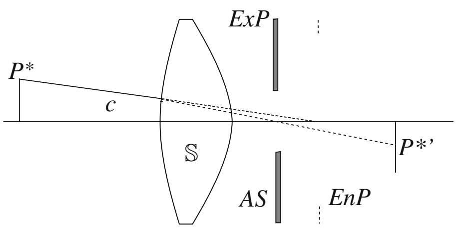


Fig. 2.10 Principal ray



the principal ray will intercept the optical axis at the centers of the pupils and at the aperture stop.

Figures 2.9 and 2.10 show the path of the principal ray for the same cases depicted in Figures 2.6 and 2.7. The presence of the aperture stop usually determines a different distribution of light on the image surface. This phenomenon is called the **vignetting**. Because of it, the intensity of the light decreases from the axis to the periphery of the image. Figure 2.11 shows why the vignetting occurs.

As the off-axis object point P^* is shifted away from the optical axis, the angular extension of the beam c' limited by the aperture stop reduces; moreover, this reduced amount of light is distributed over the same surface S which is not orthogonal to the axis of c'.

Upon increasing the distance of P^* from the axis still further, some of the beam c' is intercepted by the rim of the lens in the optical system, and so the light arriving at the image is further reduced.

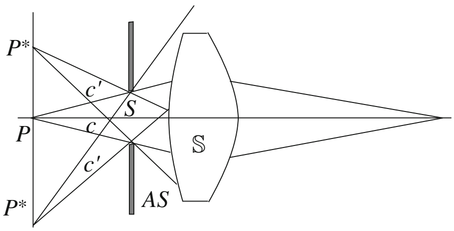


Fig. 2.11 Vignetting

2.6 Some Gaussian Optical Invariants

At the end of Section 2.2 we proved that the transformation between *any* pair of conjugate planes π_1 and π'_n assumes the form

$$y'_n = My_1, \quad \alpha'_n = Ay_1 + \frac{N_1}{N'_n}\frac{1}{M}\alpha_1. \tag{2.56}$$

When these relations refer to the conjugate principal planes, $M = 1$. Moreover, the ray intersecting the anterior principal plane at $y_1 = 0$ has a corresponding ray that crosses the posterior principal plane on the optical axis ($y'_n = 0$) and forms an angle with it of (see (2.56))

$$\alpha'_n = \frac{N_1}{N'_n}\alpha_1.$$

We find again that *if* $N_1 = N'_n$, *then the intersection points of the principal planes with a are also nodal points.*

Some important Gaussian invariants will now be derived. Let π, π' an arbitrary pair of object and image planes of the optical system $\mathbb{S}$ or of an arbitrary part S_p of $\mathbb{S}$. By π_e and π'_e we denote, respectively, the entrance and exit pupils of the considered part of $\mathbb{S}$. Due to (2.56), the marginal ray originating from O ($y = 0$, see Figure 2.12) and forming the angle α with the optical axis Oz will be transformed into the ray intersecting the optical axis at O' ($y' = 0$) and forming the angle α' with Oz, that in view of (2.56) is given by

$$\alpha' = \frac{N}{N'M}\alpha, \tag{2.57}$$

where M is the magnification relative to the conjugate planes π, π' and N, N' are the refractive indices before and after S_p, respectively. We now denote by y the starting point of the principal ray γ_p that intercepts the entrance pupil π_e at the point A and forms the angle α_p with Oz. This ray is transformed into the ray crossing the exit pupil π'_e at A' and the image plane π' at y'. Since it is

$$y' = My, \tag{2.58}$$

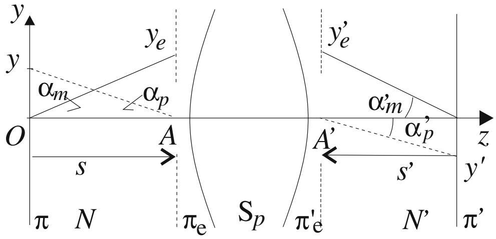


Fig. 2.12 Optical invariants

we obtain from (2.57) the ***Helmholtz-Lagrange invariant***

$$\mathfrak{H} \equiv N'y'\alpha'_m = Ny\alpha_m, \tag{2.59}$$

which, with the notations of Figure 2.12, can also be written

$$\frac{N'_n y'_n y'_e}{s'} = \frac{Nyy_e}{s}, \tag{2.60}$$

since in our approximation it is $\alpha = y_e/s$, $\alpha' = y'_e/s'$.

We can derive important consequences from (2.59), (2.60).

- First, exchange the role of the pairs of conjugate planes π, π' and π_e, π'_e. In other words, π_e, π'_e become the object and image planes whereas π, π' are the entrance and exit pupils. Consequently, the marginal ray Oy_e now becomes the principal ray and the principal ray yA is now the marginal ray. Equation (2.59) in this new interpretation gives
$$N'y'_e\alpha'_p = Ny_e\alpha_p, \tag{2.61}$$
where α_p denotes the angle that the principal ray forms with the optical axis and α'_p the corresponding angle in the exit pupil.
- Further, in view of (2.60) and (2.61), we obtain another optical invariant that is still named Lagrange invariant
$$\Gamma \equiv N'(y'\alpha'_m - y'_e\alpha'_p) = N(y\alpha_m - y_e\alpha_p), \tag{2.62}$$
- If the conjugate planes π_e and π'_e coincide, respectively, with the entrance and the exit pupils *of the whole system* $\mathbb{S}$, we have $y'_e = M_e y_e$, $y = y_1$, and $y' = y'_n$. From (2.60), we then derive
$$\frac{y'_n}{s'_n} = \frac{N_1}{N'_n}\frac{1}{M_e}\frac{y_1}{s_1}, \tag{2.63}$$
where M_e is the magnification related to the pair π_e and π'_e. Since $y = -s\alpha_p$, $y' = -s'\alpha'_p$, the previous relation assumes the form

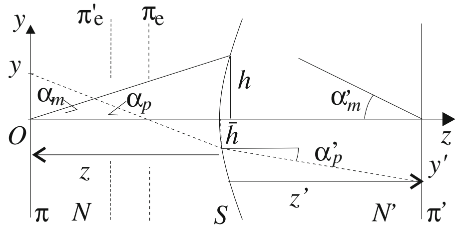


Fig. 2.13 Optical invariants

$$\alpha'_p = \frac{N}{N'}\frac{1}{M_e}\alpha_p. \tag{2.64}$$

- Moreover, if π_e and π'_e coincide, respectively, with the anterior and posterior principal planes π_p and π'_p *of the whole system* $\mathbb{S}$, the magnification related to this pair of planes is equal to unity, and (2.63) implies that

$$\frac{s}{y} = \frac{N}{N'}\frac{s'}{y'}. \tag{2.65}$$

If the object goes to infinity ($s,\ y \to \infty$) s' coincides with the posterior focal length f' of $\mathbb{S}$. Therefore, since $\alpha_p = -y/s$, from the previous relation we have

$$f' = -\frac{N'}{N}\frac{y'}{\alpha_p}. \tag{2.66}$$

- Finally, we apply the Lagrange invariant to a single dioptric S of $\mathbb{S}$ and use the notations of Figure 2.13. Due to the conventions about the sign of the optical quantities that we have introduced in Section 2.1, we have that y, α_m, h_m, z' are positive whereas z, h_p, y', α_p and α'_p are negative. Therefore, we obtain

$$y = \alpha_p z + \overline{h}, \quad y' = \alpha'_p z' + \overline{h}. \tag{2.67}$$

In view of (2.67), the Lagrange invariant (2.59) becomes

$$\mathfrak{H} \equiv N(h\alpha_p - \alpha_m \overline{h}) = N'(h\alpha'_p - \alpha'_m \overline{h}). \tag{2.68}$$

2.7 Gaussian Analysis of Compound Systems

It has already been emphasized that the transformation (2.44)–(2.45) can be used to determine the image formed by an axially symmetric optical system $\mathbb{S}$, provided that y_1, α_1 are first-order quantities. In other words, the results of the above transformation are acceptable only for points next to the optical axis a and for rays which form small angles with a. However, the paraxial approximation is very important since the paraxial behavior of an optical system usually represents its ideal behavior. For this reason, defects or aberrations of $\mathbb{S}$ are defined as deviations from the paraxial response.

In this section, a complete Gaussian description of an optical system $\mathbb{S}$ is generated in detail using all of the considerations discussed in the above sections. The subject is presented in a way that facilitate its implementation in a programming language.

- First, the matrices (2.32) and (2.33)

$$\mathbb{R}_i = \begin{pmatrix} 1 & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{N_i}{N_i'} \end{pmatrix}, \tag{2.69}$$

$$\mathbb{D}_i = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix}, \tag{2.70}$$

  have to be written for each surface of the system. They describe, respectively, the refraction at the surface S_i and the behavior of meridional rays in the space between S_i and S_{i+1}.
- Consequently, the matrices

$$\mathbb{B}_i = \begin{cases} \mathbb{D}_i \mathbb{R}_i, & i = 1, \ldots, n-1, \\ \mathbb{R}_n, & i = n, \end{cases} \tag{2.71}$$

  fully describe the behavior of meridional rays from S_i to S_{i+1}.
- Then, the three matrices

$$\mathbb{M} = \mathbb{B}_n \cdots \mathbb{B}_1, \tag{2.72}$$

$$\mathbb{M}^{(en)} = \mathbb{R}_k \mathbb{B}_{k-1} \cdots \mathbb{B}_1, \quad \mathbb{M}^{(ex)} = \mathbb{B}_n \cdots \mathbb{B}_{k+1}, \tag{2.73}$$

  must be evaluated. In (2.73) the index k, $0 \leq k \leq n$, relates to last surface S_k before the stop. The matrix $\mathbb{M}$ describes the behavior of meridional rays throughout the whole system, whereas $\mathbb{M}^{(en)}$ and $\mathbb{M}^{(ex)}$ represent the action on these rays at the surfaces before and after the stop, respectively. It is worth noting that

$$\begin{aligned} \mathbb{M}^{(en)} &= \mathbb{I}, \qquad \text{if } k = 0, \\ \mathbb{M}^{(ex)} &= \mathbb{I}, \qquad \text{if } k = n. \end{aligned} \tag{2.74}$$

- The distances z_p and z_p' of the principal planes, respectively from the surface S_1 and S_n, are given by the relations (see (2.47) and (2.50))

$$z_p = \frac{M_{22} - 1}{M_{21}}, \quad z'_p = \frac{1 - M_{11}}{M_{21}}. \tag{2.75}$$

- Let t_s be the distance of the stop from the surface S_k. Since the entrance pupil π_e is the image of the stop formed by the surfaces which precede it, we evaluate the distance w_e of the entrance pupil from the surface S_1 using the formula of conjugate points (2.47)
$$w_e = \frac{M_{22}^{(en)} t_s + M_{12}^{(en)}}{M_{21}^{(en)} t_s + M_{11}^{(en)}}. \tag{2.76}$$
If $k = 0$ (the whole system $\mathbb{S}$ follows the stop), note that, due to (2.66), $\mathbb{M}^{(en)}$ is the identity matrix and the above formula gives $w_e = t_s$. Moreover, if the stop is situated after the system $\mathbb{S}$, then $\mathbb{M}^{(en)} = \mathbb{M}$.
- Similarly, the distance t_{ex} of the stop from the surface S_{k+1} is given by
$$t_{ex} = \begin{cases} -t_s \quad \text{if} \quad k = 0, \\ -(t_k - t_s) \quad \text{if} \quad 0 < k. \end{cases}$$
Then, the distance w'_e of the exit pupil from the last surface S_n results
$$w'_e = -\frac{M_{11}^{(ex)} t_{ex} - M_{12}^{(ex)}}{M_{21}^{(ex)} t_{ex} - M_{22}^{(ex)}}. \tag{2.77}$$
- If r_s denotes the radius of the aperture stop, the radii r_{en} and r_{ex} of the entrance and exit pupils are given by
$$r_{en} = \frac{r_s}{M_{11}^{(en)} + M_{21}^{(en)} t_s},$$
$$r_{ex} = r_s (M_{11}^{(ex)} + M_{21}^{(ex)} w'),$$
- The distance z'_n of the image from the last surface S_n of the whole system $\mathbb{S}$ is given by the usual formula of conjugate points
$$z'_n = -\frac{M_{11} z_1 - M_{12}}{M_{21} z_1 - M_{22}}, \tag{2.78}$$
where z_1 denotes the distance of the object from S_1. In particular, when $z_1 \to \infty$, we obtain the back focal length f'_b
$$f'_b = -\frac{M_{11}}{M_{21}}, \tag{2.79}$$
so that the posterior focal length f' is given by (see (2.54))

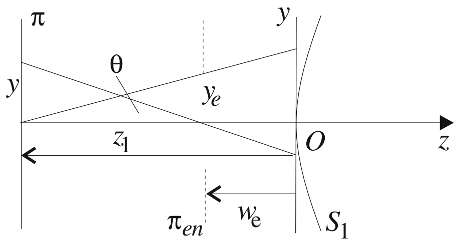


Fig. 2.14 View angle

$$f' = f_b' - z_{np}'. \tag{2.80}$$

- Finally, we need to evaluate the heights of the marginal ray and the principal ray on each surface S_i as well as the angles they form with the optical axis. In the Figure 2.14 are represented the marginal and principal rays. It is easy to verify that the point at which these rays intercept the entrance pupil π_e as well as the angles that they form with the optical axis are given by the relations

$$\text{marginal ray} = \left(y_e, -\frac{y_e}{z_1 - w_e} \right), \tag{2.81}$$

$$\text{principal ray} = \left(0, \frac{y}{z_1 - w_e} \right). \tag{2.82}$$

In particular, if the object is situated at infinity, (2.81), (2.82) become

$$\text{marginal ray} = (y_e, 0)\,, \quad \text{principal ray} = (0, \theta)\,. \tag{2.83}$$

Consequently, the characteristics of these rays across the system are obtained by the relations

$$\begin{pmatrix} y_i' \\ \alpha_i' \end{pmatrix} = \mathbb{B}_i \cdots \mathbb{B}_1 \begin{pmatrix} 1 & -w_e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}, \tag{2.84}$$

where $i = 1, \ldots, n-1$ and the matrix

$$\begin{pmatrix} 1 & -w_e \\ 0 & 1 \end{pmatrix}, \tag{2.85}$$

describes the correspondence between the entrance pupil π_e and the tangent plane π_1^* at the vertex of S_1. In this formula (y, α) is one of the rays (2.81), (2.82) (or (2.83) if the object is at infinity). Finally, the equation

$$\begin{pmatrix} y_n' \\ \alpha_n' \end{pmatrix} = \begin{pmatrix} 1 & z_n' \\ 0 & 1 \end{pmatrix} \mathbb{M} \begin{pmatrix} 1 & -w_e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}, \tag{2.86}$$

where z'_n is the distance of the image plane from the last surface S_n, supplies the characteristic of the marginal and principal rays that cross the image plane.

The notebook ***TotalAberrations*** supplies all the Gaussian characteristic of a compound optical system.

2.8 A Graphical Method

When the characteristic points of $\mathbb{S}$ have been determined, it is possible to obtain graphically the image of an object with the simple procedure presented in this section. For the sake of simplicity, in the sequel we refer to the most common case $N_1 = N'_n$, in which the nodal and principal points coincide.

- Let y_1 be an object point and γ_0 be a ray parallel to a, in other words, a ray for which $\alpha_1 = 0$. Denote the point at which γ_0 (or its extension) meets the first principal plane π_p by y_p (see Figure 2.15)
  In the correspondence between the conjugate principal planes π_p and π'_p, the magnification is equal to 1. Consequently, at the exit of the whole optical system, the ray γ_0, or its extension, will meet the posterior principal plane π'_p at a point $y'_p = y_p$.
- The transformation (2.45) between the object plane and the posterior focal one π'_f, when the condition (2.38) is taken into account, is described by the following matrix:
$$\begin{pmatrix} 1 & -\dfrac{M_{11}}{M_{21}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix},$$
  which reduces to the form
$$\begin{pmatrix} 0 & A \\ M_{21} & B \end{pmatrix},$$
  where A and B have suitable expressions. Consequently, the correspondence between the object plane and the focal one becomes

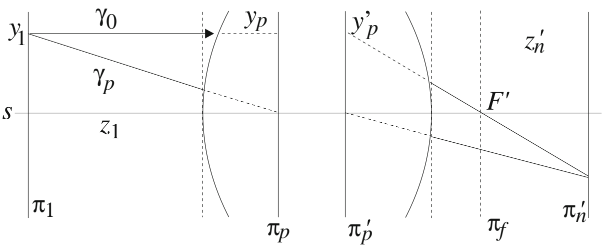


Fig. 2.15 Graphical construction of the image

$$y'_f = A\alpha_1,$$
$$\alpha'_f = M_{21}y_1 + B\alpha_1.$$

In particular, these relations imply that any ray that starts from the object point y_1 and is parallel to the optical axis a ($\alpha_1 = 0$), meets to the focal plane on a.

- The ray γ_p, directed towards the anterior principal point (or node), has a corresponding parallel ray that intersects with the posterior principal point (or node).

Using these properties, the paraxial image can be obtained in the way illustrated in the Figure 2.15.

When the refractive indices N and N' are different, it is sufficient to refer the last conclusion to the nodal points.

2.9 Thick and Thin Lenses

In this section, we consider an optical system $\mathbb{S}$ of a single lens in the air ($N = 1$). Let P_1 and P_2 be the powers, respectively, of the first surface S_1 and the second surface S_2 of $\mathbb{S}$. If t denotes the thickness of the lens and N its refractive index, then the effect of $\mathbb{S}$ on the rays of light is described by the matrix (see Section 2.3)

$$\mathbb{M} = \mathbb{R}_2\mathbb{D}_1\mathbb{R}_1, \tag{2.87}$$

where

$$\mathbb{R}_1 = \begin{pmatrix} 1 & 0 \\ -\dfrac{P_1}{N} & \dfrac{1}{N} \end{pmatrix}, \quad \mathbb{D}_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \tag{2.88}$$

$$\mathbb{R}_2 = \begin{pmatrix} 1 & 0 \\ -P_2 & N \end{pmatrix}. \tag{2.89}$$

Therefore, $\mathbb{M}$ is the matrix

$$\mathbb{M} = \begin{pmatrix} 1 - \dfrac{P_1}{N}t & \dfrac{t}{N} \\ -P_1 - P_2 + \dfrac{P_1P_2t}{N} & 1 - \dfrac{P_2t}{N} \end{pmatrix}. \tag{2.90}$$

Formulae (2.75) and (2.90) give us the distance z_p of the first principal point from the vertex of S_1 and the distance z'_p of the second principal point from the vertex of S_2:

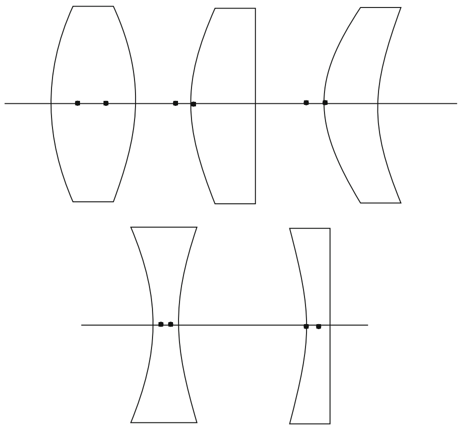


Fig. 2.16 Positions of principal points in thick lenses

$$z_p = \frac{P_2 t}{N(P_1 + P_2) - P_1 P_2 t}, \quad z'_p = -\frac{P_1 t}{N(P_1 + P_2) - P_1 P_2 t}. \tag{2.91}$$

Remembering (2.12), we have

$$P_1 = \frac{N-1}{R_1}, \quad P_2 = -\frac{N-1}{R_2}. \tag{2.92}$$

In view of (2.91) and (2.92), we easily verify that the principal points H_1 and H_2 of different kinds of thick lenses is positioned as it is shown by the black dots in Figure 2.16. Finally, the posterior and anterior focal lengths are equal since $N_1 = N'_2 = 1$. Due to (2.54), the focal length is given by

$$\frac{1}{f} = P_1 + P_2 - \frac{1}{N} P_1 P_2 t. \tag{2.93}$$

When the thickness t can be neglected in the above formulae, we say that S is a ***thin lens***. In this condition, (2.91), (2.93) reduce to

$$z_p = z'_p = 0, \tag{2.94}$$

and

$$\frac{1}{f} = P_1 + P_2 = (N-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{2.95}$$

respectively.

2.10 A Different Approach to Gaussian Optics

In Section 2.1 we derived Gaussian optics starting from a single dioptric S and requiring that all of the stigmatic paths between the object point $(0, y, z)$ and the image point $(0, My, z')$ were rays up to fourth-order terms in the variables y, X, Y, where X and Y are the coordinates of the intersection point of a ray with S. In other words, we derived Gaussian optics by requiring that Fermat's principle were satisfied by all of the broken lines between $(0, y, z)$ and $(0, My, z')$ up to fourth-order terms in the variables y, X, Y. In this section, we present a more traditional approach to Gaussian optics (see, for instance, [5, 13]). According to the conventions we adopted in Section 2.1 about the sign of angles and distances, the quantities of Figure 2.17 have the following signs:

$$\alpha > 0, \;\; \beta < 0, \;\; \alpha' < 0,$$
$$z < 0, \;\; z' > 0, \;\; R > 0,$$
$$\theta < 0, \;\; \theta' < 0, \;\; h > 0.$$

From Figure 2.17 and elementary results of Euclidean geometry, we easily obtain

$$i = |\alpha| + |\beta| = \alpha - \beta, \tag{2.96}$$
$$r = |\beta| - |\alpha'| = -\beta + \alpha'. \tag{2.97}$$

In the paraxial approximation, the refraction law is

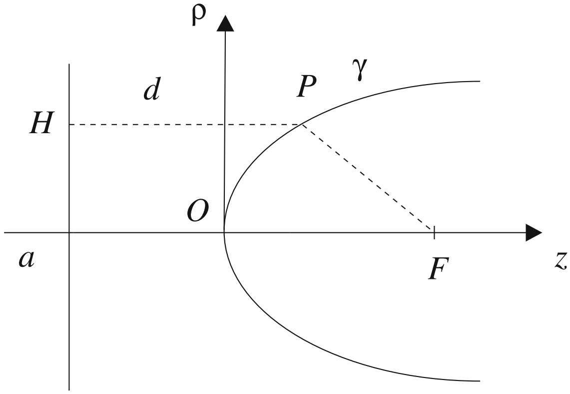


Fig. 2.17 Gaussian approximation

$$Ni = N'r, \tag{2.98}$$

where N and N' are the refractive indices of the media before and after the surface S, respectively. Moreover, denoting by R the radius of S and $c = 1/R$ its curvature, we have

$$\alpha = -\frac{h}{z}, \quad \beta = -\frac{h}{R} = -hc, \quad \alpha' = -\frac{h}{z'}. \tag{2.99}$$

Introducing (2.97) and (2.99) into (2.98), we verify that the refraction law is equivalent to the Abbe invariant

$$\frac{N}{R} - \frac{N}{z} = \frac{N'}{R} - \frac{N'}{z'}. \tag{2.100}$$

Finally, applying the refraction law to the ray γ' that intersects S at the optical axis, we deduce the condition

$$N\theta = N'\theta', \tag{2.101}$$

which can also be written in the form

$$\frac{y'}{y} = \frac{z'N}{zN'}. \tag{2.102}$$

Formula (2.100) of Abbe invariant is often written in terms of curvature c of S, the angles α and α' that the marginal ray r forms with the optical axis and the height h with respect to the optical axis of the intersection point of r with the surface S of dioptric. In view of (2.96), (2.97), we easily verify that (2.100) can also be written as follows:

$$N'\alpha' = N\alpha - (N' - N)ch. \tag{2.103}$$

Further, (2.96) and (2.97) allow us to give the refraction law (2.98) the form

$$a \equiv N(ch + \alpha) = N'(ch + \alpha'), \tag{2.104}$$

where a is said to be the ***refraction invariant***.

We note that applying (2.100) and (2.102) to any dioptric S_i of an optical system $\mathbb{S}$ we localize the image formed by S by resorting to the object and image distances z_i and z_i' and evaluate its dimension by the object and image heights y_i and y_i'.

In contrast, (2.103) localizes the position of the image by the angles α_i and α_i' that the marginal ray r_i forms with the optical axis before and after the refraction on S_i, whereas the dimension of the image is given by (2.101) that contains the angles θ_i, θ_i' that the ray r' forms with the optical axis. It is worthwhile to formulate the Lagrange invariant (2.68) in terms of the new variables we are adopting. It is simple to verify that in view of (2.104) we can write H as follows:

$$H = \overline{a}_i h_i - a_i \overline{h}_i, \tag{2.105}$$

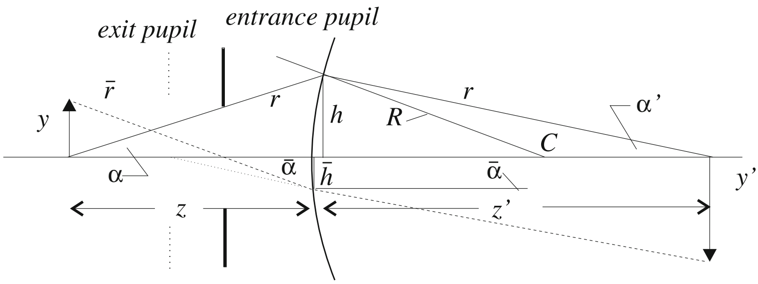


Fig. 2.18 Marginal and principal rays

where

$$\overline{a}_i = N_i(c_i\overline{h}_i + \overline{\alpha}_i) = N'(c_i\overline{h}_i + \overline{\alpha}'_i) \tag{2.106}$$

is the refraction invariant relative to the principal ray (see Figure 2.18).

It is evident that we can obtain the localization of the image and its size by applying (2.103) and (2.101) to any dioptric S_i of $\mathbb{S}$. If t_i is the distance of the dioptrics S'_i from the dioptric S_i, we have

$$h'_i = h_i + \alpha_i t_i, \quad h_{i+1} = h'_i. \tag{2.107}$$

Therefore, the matrix equation that describes the transfer from S_i to S_{i+1} is

$$\begin{pmatrix} h'_i \\ \alpha'_i \end{pmatrix} = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_i \\ \alpha_i \end{pmatrix}. \tag{2.108}$$

Further, the matrix equation that gives the refraction across S_{i+1} is

$$\begin{pmatrix} h'_i \\ \alpha'_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c_i(1-\mu_i) & \mu_i \end{pmatrix} \begin{pmatrix} h'_i \\ \alpha_i \end{pmatrix}, \tag{2.109}$$

where $\mu = N_i/N'_i$. Finally, by combining (2.108) and (2.109) we obtain the matrix equation that represents a refraction followed by a transfer

$$\begin{pmatrix} h'_i \\ \alpha'_i \end{pmatrix} = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_i(1-\mu_i) & \mu_i \end{pmatrix} \begin{pmatrix} h_i \\ \alpha_i \end{pmatrix}. \tag{2.110}$$

2.11 Exercises

1. Verify that

$$\begin{pmatrix} \dfrac{N_i z_i'}{N_i' z_i} & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{z_i}{z_i'} \end{pmatrix} = \begin{pmatrix} 1 & z_i' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{P_i}{N_i'} & \dfrac{N_i}{N_i'} \end{pmatrix} \begin{pmatrix} 1 & -z_i \\ 0 & 1 \end{pmatrix}. \tag{2.111}$$

The matrix composition on the right-hand side yields

$$\begin{pmatrix} 1 - \dfrac{z_i' P_i}{N_i'} & -z_i \left(1 - \dfrac{z_i' P_i}{N_i'}\right) + \dfrac{z_i' N_i}{N_i'} \\ -\dfrac{P_i}{N_i'} & \dfrac{z_i P_i}{N_i'} + \dfrac{N_i}{N_i'} \end{pmatrix}. \tag{2.112}$$

On the other hand, from (2.7), it is easy to derive

$$\frac{N_i'}{z_i'} - \frac{N_i}{z_i} = P_i,$$

in other words,

$$1 - \frac{P_i z_i'}{N_i'} = \frac{N_i z_i'}{N_i' z_i}.$$

It is now straightforward to show that the matrix (2.112) is equal to the left-hand side of (2.111).

2. Evaluate the Gaussian characteristics of a single spherical mirror S having a curvature radius $R = -1000$ and the aperture stop that lyes in the plane of the center of curvature.
The power of the mirror is

$$P = \frac{1}{500} = 0.002$$

and the refractive matrix is

$$\mathbb{R} = \begin{pmatrix} 1 & 0 \\ 0.002 & -1 \end{pmatrix}$$

whereas the matrices $\mathbb{M}^{en}$ and $\mathbb{M}^{ex}$ (see Section 1.7) are

$$\mathbb{M}^{en} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{M}^{ex} = \begin{pmatrix} 1 & 0 \\ 0.002 & -1 \end{pmatrix},$$

and the entrance and exit pupils are situated in the curvature center. Finally, the matrix M relative for the whole system is

$$\mathbb{M} = \begin{pmatrix} 1 & 0 \\ 0.002 & -1 \end{pmatrix}.$$

Consequently, the distances of the principal planes from S (see (2.36) and (2.44)) are

$$z_{1p} = z_{2p} = 0,$$

and the focal length is (see (2.45))

$$f = -500.$$

3. Evaluate the Gaussian characteristics of a single plane-convex lens for which the anterior curvature radius is 50, the posterior one is infinite, the thickness is equal to 8 and the refractive index in green light is 1.518722. The aperture stop is -6 far from the lens.
   The powers of the two surfaces are respectively,

$$P_1 = 0.0103744, \quad P_2 = 0,$$

   Consequently, the refractive matrices are

$$\mathbb{R}_1 = \begin{pmatrix} 1 & 0 \\ -0.0068 & 0.6584 \end{pmatrix}, \quad \mathbb{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1.518722 \end{pmatrix},$$

   whereas the translation matrix $\mathbb{D}$ is

$$\mathbb{D} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}.$$

   Moreover, the matrices $\mathbb{M}^{en}$, $\mathbb{M}^{ex}$ and $\mathbb{M}$ result

$$\mathbb{M}^{en} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{M}^{ex} = \begin{pmatrix} 0.9453 & 5.2675 \\ -0.0103 & 1 \end{pmatrix},$$

$$\mathbb{M} = \begin{pmatrix} 0.9453 & 5.2675 \\ -0.0103 & 1 \end{pmatrix}.$$

   From the previous matrices, we derive

$$z_{1p} = 0, \quad z_{2p} = -5.267,$$

$$f = 96.39.$$

Evaluate the distance of the image z_2' when $z_1 = -200$.

4. Evaluate the Gaussian characteristics of the following optical system: the radii are $50, \infty, \infty, -50$; the distances are $8, 12, 8$; the refractive indices are $1, 1.518722, 1, 1.58722$; the aperture stop is situated between the second and third surface and its distance from the second surface is 6.
   We have
$$P_1 = 0.0104, \quad P_2 = P_3 = 0, \quad P_4 = 0.0104;$$
$$\mathbb{R}_1 = \begin{pmatrix} 1 & 0 \\ -0.0068 & 0.6584 \end{pmatrix}, \quad \mathbb{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1.5187 \end{pmatrix},$$
$$\mathbb{R}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0.6584 \end{pmatrix}, \quad \mathbb{R}_4 = \begin{pmatrix} 1 & 0 \\ -0.0103 & 1.5187 \end{pmatrix},$$
$$\mathbb{D}_1 = \mathbb{D}_3 \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{D}_2 = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}.$$
   Consequently,
$$\mathbb{M}^{en} = \begin{pmatrix} 0.945 & 5.267 \\ -0.0103 & 1 \end{pmatrix}, \quad \mathbb{M}^{ex} = \begin{pmatrix} 1 & 5.267 \\ -0.0103 & 0.9453 \end{pmatrix},$$
$$\mathbb{M} = \begin{pmatrix} 0.762 & 22.53 \\ -0.018 & 0.766 \end{pmatrix};$$
   $f = 54.57$ and the distances w_e of the entrance pupil from the first surface and w_e' of the exit pupil from the last surface are
$$w_e = 12.76, \quad w_e' = 0.727.$$

5. Let $\mathbb{S}$ the system in Figure 2.19. The radii of the eight surfaces (from left to right) are
$$15.9, 30.2, 18.5, 12, -12, -18.5, -30.2, -15.9.$$
   The thicknesses and refractive indices of lenses are
$$4, 0.5, 1, 15.2, 1, 0.5, 4;$$
$$1, 1.609937, 1, 1.624082, 1, 1.624082, 1, 1.609937, 1,$$

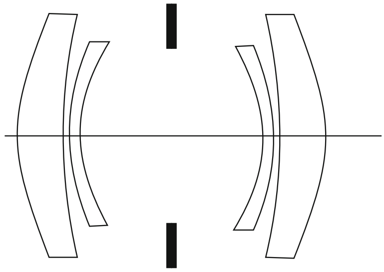


Fig. 2.19 Projector of unit magnification

respectively. Finally the stop is 7.6 far from the fourth surface. Evaluate all the Gaussian characteristics of $\mathbb{S}$. In particular prove that, when the distance of the object plane from the first surface is 212.2, the magnification of S is -1.

6. Using (2.48), (2.54), and (2.51), prove that (2.44) and (2.45) can also be written as follows:

$$y'_n = M y_1, \tag{2.113}$$

$$\alpha'_n = -\frac{1}{f'} y_1 + \frac{1}{M} \alpha_1. \tag{2.114}$$

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