Chapter 2
Cosmic Strings

Abstract The velocity-dependent one-scale model of cosmic string network evolution is summarized. Treating the average string velocity as well as the characteristic lengthscale as dynamical variables, one can obtain a fully quantitative model, describing the complete evolution of a string network, including the prediction of previously unknown transient scaling regimes. We also discuss extensions to open, anisotropic and contracting universes, and the effect of radiation backreaction. Finally we discuss the calibration of the model parameters by comparing it to both Abelian-Higgs and Goto–Nambu simulations, in both a static and expanding backgrounds, and highlight the non-trivial fractal properties of cosmic strings.

2.1 Cosmic String Dynamics

We start by summarizing the original derivation of the model discussed in [1, 2]. A string sweeps out a two-dimensional surface (the worldsheet) which can be described by spacetime coordinates x^μ and worldsheet coordinates σ^a, $x^\mu = x^\mu(\sigma^a)$; the line element is then

$$ds^2 = g_{\mu\nu}\, x^\mu_{,a}\, x^\nu_{,b}\, d\sigma^a\, d\sigma^b = \gamma_{ab}\, d\sigma^a\, d\sigma^b, \tag{2.1}$$

where $g_{\mu\nu}$ and γ_{ab} are respectively the 4D spacetime and 2D string worldsheet metrics. For the case of a gauge (global) string, one can then derive the Nambu (Kalb-Ramond) action from the Abelian-Higgs (Goldstone) model on the assumption that the scale of perturbations along the string is much larger than its width δ. (In the global case, one also makes use of the equivalence between a real massless scalar field and a two-index antisymmetric tensor field.) One finds

$$S = \begin{cases} \mu_o \int \sqrt{-\gamma}\, d\sigma^2 & \text{Gauge} \\ \mu_o \int \sqrt{-\gamma}\, d\sigma^2 + \frac{1}{6}\int \sqrt{-g} H^2 d^4x + 2\pi\eta \int B_{\mu\nu} d\sigma^{\mu\nu} & \text{Global} \end{cases} \tag{2.2}$$

where $B_{\mu\nu}$ is the antisymmetric tensor field, $H_{\mu\nu\lambda}$ is its field strength and $d\sigma^{\mu\nu}$ is the worldsheet area element. Hence the Nambu action is proportional to the area swept out by the string. Varying this action one obtains the equations of motion

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$$x^{\nu}{}_{,a}{}^{;a} + \Gamma^{\nu}_{\tau\lambda}\,\gamma^{ab}\,x^{\tau}{}_{,a}\,x^{\lambda}_{,b} = \begin{cases} 0 & \text{Gauge} \\ \frac{2\pi\eta}{\mu_o} H^{\nu}_{\tau\lambda}\,\varepsilon^{ab}\,x^{\tau}{}_{,a}\,x^{\lambda}_{,b} & \text{Global} \end{cases} \tag{2.3}$$

Since strings move through a background radiation fluid, their motion is retarded by particle scattering.This effect can be described by a frictional force per unit length [3]

$$\mathbf{F}_{\mathrm{f}} = -\frac{\mu}{\ell_{\mathrm{f}}}\frac{\mathbf{v}}{\sqrt{1-v^2}}, \tag{2.4}$$

where $\mathbf{v}$ is the string velocity and ℓ_{f} will be called the 'friction lengthscale'; its explicit value depends on the type of symmetry involved. For a gauge string, the main contribution comes from Aharonov–Bohm scattering [4], while in the global case it comes from Everett scattering [5]. Then we have

$$\ell_{\mathrm{f}} = \begin{cases} \frac{\mu}{\beta T^3} & \text{Gauge} \\ \frac{\mu}{\beta T^3}\ln^2(T\delta) & \text{Global} \end{cases} \tag{2.5}$$

where T is the background temperature, δ is the string thickness and β is a numerical factor related to the number of particle species interacting with the string.This force can be included in the equations of motion (2.3) by adding the term

$$\left(U^{\nu} - x^{\nu}_{,a}x^{\sigma,a}U_{\sigma}\right)\frac{1}{\ell_{\mathrm{f}}}, \tag{2.6}$$

(U^{ν} being the four-velocity of the background fluid) on its right-hand side.

Now consider string motion in an FRW universe with the line element,

$$ds^2 = a^2(\tau)\,\left(d\tau^2 - \mathbf{dx}^2\right)\,; \tag{2.7}$$

then $U^{\nu} = \left(a^{-1}, \mathbf{0}\right)$ and choosing the gauge conditions $\sigma^0 = \tau$ (identifying conformal and worldsheet times) and $\dot{\mathbf{x}}\cdot\mathbf{x}' = 0$ (imposing that the string velocity be orthogonal to the string direction) the string equations of motion can be expressed as [3, 6]

$$\ddot{\mathbf{x}} + \left(2\frac{\dot{a}}{a} + \frac{a}{\ell_{\mathrm{f}}}\right)\left(1-\dot{\mathbf{x}}^2\right)\dot{\mathbf{x}} = \frac{1}{\varepsilon}\left(\frac{\mathbf{x}'}{\varepsilon}\right)', \tag{2.8}$$

$$\dot{\varepsilon} + \left(2\frac{\dot{a}}{a} + \frac{a}{\ell_{\mathrm{f}}}\right)\dot{\mathbf{x}}^2\varepsilon = 0, \tag{2.9}$$

where the 'coordinate energy per unit length' ε is defined by

$$\varepsilon^2 = \frac{\mathbf{x}'^2}{1-\dot{\mathbf{x}}^2}, \tag{2.10}$$

and dots and primes respectively denote derivatives with respect to τ and σ.

2.1.1 Lengthscale Evolution

We can average the string equations of motion to describe the large-scale evolution of the string network. Define the total string energy and the average RMS string velocity to be

$$E = \mu a(\tau) \int \varepsilon d\sigma, \tag{2.11}$$

$$v^2 \equiv \langle \dot{\mathbf{x}}^2 \rangle = \frac{\int \dot{\mathbf{x}}^2 \varepsilon d\sigma}{\int \varepsilon d\sigma}. \tag{2.12}$$

Differentiating (2.11) and using (2.9) and (2.12), we see that the total string energy density $\rho \propto E/a^3$ will obey (in terms of physical time t)

$$\frac{d\rho}{dt} + \left[2H\left(1+v^2\right) + \frac{v^2}{\ell_{\rm f}}\right]\rho = 0\,. \tag{2.13}$$

Equation (2.13) incorporates both long strings and small, short-lived loops which usually have a low probability of interacting with other strings before their demise. We shall study the evolution of the long-string network on the assumption that it can be characterized by a single lengthscale L; this can be interpreted as the inter-string distance or the 'correlation length'. Strings larger than L will be called long or 'infinite'; otherwise they will be called loops. For Brownian long strings, we can define the 'correlation length' L in terms of the network density ρ_∞ as

$$\rho_\infty \equiv \frac{\mu}{L^2}\,. \tag{2.14}$$

Following Kibble [7], the rate of loop production from long-string collisions can be written as

$$\left(\frac{d\rho_\infty}{dt}\right)_{\text{to loops}} = \rho_\infty \frac{v_\infty}{L} \int w\left(\frac{\ell}{L}\right) \frac{\ell}{L} \frac{d\ell}{L} \equiv \tilde{c} v_\infty \frac{\rho_\infty}{L}, \tag{2.15}$$

where the loop 'chopping' efficiency $\tilde{c}$ is assumed to be constant. Finally, by subtracting the loop energy losses (2.15) from (2.13) and then using (2.14), we obtain the overall evolution equation for the characteristic lengthscale L,

$$2\frac{dL}{dt} = 2HL(1+v_\infty^2) + \frac{Lv_\infty^2}{\ell_{\rm f}} + \tilde{c} v_\infty\,. \tag{2.16}$$

Note that with the exception of the expansion term, all terms on the right-hand side are velocity-dependent.

2.1.2 Loop Evolution

Define $n_\ell(\ell, t)d\ell$ to be the number density of loops with length in the range $(\ell, \ell + d\ell)$ at time t; the corresponding loop energy density distribution is

$$\rho_\ell(\ell, t)d\ell = \mu\ell n_\ell(\ell, t)d\ell\,. \tag{2.17}$$

Note that the total loop energy density is

$$\rho_o \equiv \int \rho_\ell(\ell, t)d\ell\,; \tag{2.18}$$

the subscript 'o' referring to properties of the entire loop population, while 'ℓ' refers to the loops with length in the range $(\ell, \ell + d\ell)$. From our assumptions on the loop production rate (2.15) we get

$$\frac{d\rho_\ell}{dt} + \left[2H\left(1 + v_\ell^2\right) + \frac{v_\ell^2}{\ell_\mathrm{f}}\right]\rho_\ell = g\mu\frac{v_\infty \ell}{L^5}w\left(\frac{\ell}{L}\right), \tag{2.19}$$

where g is a Lorentz factor accounting for the initial non-zero center-of-mass kinetic energy of the loops (lost through velocity redshift). Note that this equation is 'static': it does not include loop decay mechanisms.

The physical size of a loop is simply given by

$$\ell = a(\tau)\int_{loop} \varepsilon d\sigma\,; \tag{2.20}$$

its time derivative can be easily calculated using (2.9). However one must still subtract energy (hence length) losses due to radiative processes. For a gauge string, this can be roughly estimated from the quadrupole formula

$$\left(\frac{dE}{dt}\right)_{rad} \sim G\left(\frac{d^3D}{dt^3}\right)^2 \sim G\mu^2v^6, \tag{2.21}$$

($D \sim \mu\ell^3$ being the loop's quadrupole moment). Then we define

$$\left(\frac{d\ell}{dt}\right)_{rad} \equiv -\Gamma' G\mu v^6, \tag{2.22}$$

where according to numerical estimates $\Gamma' \sim 8 \times 65$. Then the evolution equation for the physical loop size is

$$\frac{d\ell}{dt} = (1 - 2v_\ell^2)H\ell - \frac{\ell v_\ell^2}{\ell_f} - \Gamma' G\mu v_\ell^6\,. \tag{2.23}$$

Now, we will assume that loop production is 'monochromatic', i.e. that loops formed at a time t_p have an initial length

$$\ell(t_p) = \alpha(t_p)\, L(t_p)\,. \tag{2.24}$$

Notice that we are implicitly saying that the loop size at formation depends both on the large-scale properties of the network (through the correlation length) and on the small-scale structure it contains (through the parameter α). With this ansatz the scale-invariant loop production function w becomes

$$w\left(\frac{\ell}{L}\right) = \frac{\tilde{c}}{\alpha}\delta\left(\frac{\ell}{L} - \alpha\right), \tag{2.25}$$

and the rate of energy loss into loops becomes

$$\left(\frac{d\rho_\infty}{dt}\right)_{\text{to loops}} = g\mu\tilde{c}\frac{v_\infty}{L^3}\,. \tag{2.26}$$

Hence the energy density converted into loops from time t to $t + dt$ is

$$d\rho_o(t) = g\mu\tilde{c}\frac{v_\infty}{L^3}dt\,; \tag{2.27}$$

this corresponds to a fraction

$$\frac{d\rho_o(t)}{\rho_\infty(t)} = g\tilde{c}\frac{v_\infty}{L}dt \tag{2.28}$$

of the energy density in the form of long strings at time t. Then using Eq. (2.25), the number of loops produced in a volume V is

$$dN(t) = g\frac{\tilde{c}}{\alpha}\frac{v_\infty}{L^4}V(t)dt\,; \tag{2.29}$$

hence the ratio of the energy densities in 'dynamic' loops and long strings is

$$\varrho(t)_{dyn} \equiv \frac{\rho_o(t)_{dyn}}{\rho_\infty(t)} = gL^2(t)\int_{t_c}^{t}\frac{dN(t')\ell(t,t')}{V} = g\tilde{c}L^2(t)\int_{t_c}^{t}\frac{a^3(t')}{a^3(t)}\frac{v_\infty(t')}{L^4(t')}\frac{\ell(t,t')}{\alpha(t')}dt', \tag{2.30}$$

where t_c is the moment of the network formation and $\ell(t, t')$ is the length at time t of loops produced at time t'.

We can also find the ratio of the energy densities in 'primordial' loops and long strings with a modification of our counting strategy: instead of integrating over time, we integrate over the possible loop lengths in the initial distribution

$$\varrho(t)_{pri} \equiv \frac{\rho_o(t)_{pri}}{\rho_\infty(t)} = L^2(t)\frac{a^3(t_c)}{a^3(t)}\int_{L_c}^{L_{cut}} n_\ell(\ell', t_c)\ell(\ell', t_c)d\ell', \tag{2.31}$$

where L_c is the value of the 'correlation length' at time t_c, $L_{cut} \gg L_c$ is a cutoff length, $\ell(\ell', t_c)$ is the length at time t of a (primordial) loop with length ℓ' at t_c and the loop number density n_ℓ has the well-know Vachaspati–Vilenkin form [8]. We can therefore numerically (and, in some simple limit cases, analytically) determine the loop density at all times.

2.1.3 Velocity Evolution

We must now consider the evolution of the average string velocity v. A non-relativistic equation can be easily obtained: it is just Newton's law,

$$\mu\frac{dv}{dt} = \frac{\mu}{R} - \mu v\left(2H + \frac{1}{\ell_{\mathrm{f}}}\right). \tag{2.32}$$

This merely states that curvature accelerates the strings while damping (both from friction and expansion) slows them down. On dimensional grounds, the force per unit length due to curvature should be μ over the curvature radius R. The form of the damping force can be found similarly.

A relativistic generalization can be obtained more rigorously by differentiating (2.12):

$$\frac{dv}{dt} = \left(1 - v^2\right)\left[\frac{k}{R} - v\left(2H + \frac{1}{\ell_{\mathrm{f}}}\right)\right]. \tag{2.33}$$

This is exact up to second-order terms. To obtain the damping term we have taken $\langle\dot{\mathbf{x}}^4\rangle = \langle\dot{\mathbf{x}}^2\rangle^2$. Writing $\dot{\mathbf{x}}^2 = (1 + \mathbf{p}\cdot\mathbf{q})/2$ ($\mathbf{p}$ and $\mathbf{p}$ being unit left- and right-movers along the string) and defining $\varsigma \equiv -\langle\mathbf{p}\cdot\mathbf{q}\rangle$ the difference between the two is

$$\langle\dot{\mathbf{x}}^4\rangle - \langle\dot{\mathbf{x}}^2\rangle^2 = \frac{1}{4}\left[\langle(\mathbf{p}\cdot\mathbf{q})^2\rangle - \varsigma^2\right], \tag{2.34}$$

and numerical simulations of string evolution indicate that $\varsigma_{rad} \sim 0.14$ and $\varsigma_{mat} \sim 0.26$, so this difference should be small [9]. As for the curvature term, we have introduced R via the definition of the curvature radius vector,

$$\frac{a(\tau)}{R}\hat{\mathbf{u}} = \frac{d^2\mathbf{x}}{ds^2}, \tag{2.35}$$

where $\hat{\mathbf{u}}$ is a unit vector and s is the physical length along the string (related to the coordinate length σ by $ds = |\mathbf{x}'|d\sigma = \left(1 - \dot{\mathbf{x}}^2\right)^{1/2} \varepsilon d\sigma$). The dimensionless parameter k is defined by

$$\langle (1 - \dot{\mathbf{x}}^2)(\dot{\mathbf{x}} \cdot \hat{\mathbf{u}}) \rangle \equiv kv(1 - v^2) \tag{2.36}$$

and is related to the presence of small-scale structure on strings: on a perfectly smooth string, $\hat{\mathbf{u}}$ and $\dot{\mathbf{x}}$ will be parallel so $k = 1$ (up to a second-order term as above), but this need not be so for a wiggly string. The following phenomenological function is found to provide a good fit to simulations [10]

$$k(v) = \frac{2\sqrt{2}}{\pi}(1 - v^2)(1 + 2\sqrt{2}v^3)\frac{1 - 8v^6}{1 + 8v^6}\,. \tag{2.37}$$

If one is only interested in the relativistic regime then

$$k_{\rm rel}(v) = \frac{2\sqrt{2}}{\pi}\,\frac{1 - 8v^6}{1 + 8v^6}, \tag{2.38}$$

should be sufficiently accurate to provide reliable results. On the other hand, a reliable approximation for small non-relativistic velocities is

$$k_{\rm nr}(v) = \frac{2\sqrt{2}}{\pi}(1 - v^2)\,. \tag{2.39}$$

2.2 Scaling Results

In the early universe the friction lengthscale increases with time, so friction will only be important at early times. Let T_c be the temperature of the string-forming phase transition; the corresponding time of formation is

$$t_c = \frac{1}{f}\frac{m_{Pl}}{T_c^2}, \tag{2.40}$$

where $f = 4\pi\sqrt{\pi\mathcal{N}/45}$ and $\mathcal{N}$ is the number of effectively massless degrees of freedom in the model (e.g., $\mathcal{N} = 106.75$ for a minimal GUT model, but it can be as high as 10^4 for particular extensions of it). Then in the case of a gauge symmetry breaking the friction lengthscale can be written

$$\ell_{\rm f} = \begin{cases} \frac{1}{\theta}\frac{t^{3/2}}{t_c^{1/2}} & \text{Radiation} \\ \left(\frac{3}{4}\right)^{3/2}\frac{1}{\theta}\frac{t^2}{\left(t_c t_{eq}\right)^{1/2}} & \text{Matter} \end{cases} \tag{2.41}$$

and for the case of a global symmetry

$$\ell_{\mathrm{f}} = \begin{cases} \frac{1}{4\theta}\frac{t^{3/2}}{t_c^{1/2}} \ln\left(\frac{L}{\delta}\right)\left[\ln\left(\frac{6}{\lambda}\frac{t_c}{t}\right)\right]^2 & \text{Radiation} \\ \left(\frac{3}{4}\right)^{3/2}\frac{1}{4\theta}\frac{t^2}{(t_c t_{eq})^{1/2}} \ln\left(\frac{L}{\delta}\right)\left[\ln\left(\frac{8}{\lambda}\frac{t_c t_{eq}^{1/3}}{t^{4/3}}\right)\right]^2 & \text{Matter} \end{cases} \tag{2.42}$$

The constant θ is a measure of the importance of the friction term in the evolution equations; its value is

$$\theta = \frac{\beta}{\sqrt{f}}\left(\frac{t_c}{t_{Pl}}\right)^{1/2}. \tag{2.43}$$

The string energy per unit length can be written

$$\mu = \begin{cases} T_c^2 & \text{Gauge} \\ T_c^2 \ln\left(\frac{L}{\delta}\right) & \text{Global} \end{cases} \tag{2.44}$$

Defining t_* as the time at which the two damping terms in (2.8) and (6.4) have equal magnitude we find

$$\frac{t_*}{t_c} = \begin{cases} \theta^2 & \text{Gauge} \\ 16\theta^2\left(\ln\frac{L}{\delta}\right)^{-2}\left[\ln\left(\frac{6}{\lambda}\frac{t_c}{t_\star}\right)\right]^{-4} & \text{Global} \end{cases} \tag{2.45}$$

provided this is still in the radiation era; otherwise, in the matter era we obtain

$$\frac{t_*}{t_c} = \begin{cases} \left(\frac{4}{3}\right)^{1/2}\theta\left(\frac{t_{eq}}{t_c}\right)^{1/2} & \text{Gauge} \\ 4\left(\frac{4}{3}\right)^{1/2}\theta\left(\frac{t_{eq}}{t_c}\right)^{1/2}\left(\ln\frac{L}{\delta}\right)^{-1}\left[\ln\left(\frac{8}{\lambda}\left(\frac{t_{eq}}{t_c}\right)^{1/3}\left(\frac{t_c}{t_\star}\right)^{4/3}\right)\right]^{-2} & \text{Global} \end{cases} \tag{2.46}$$

String dynamics is friction-dominated from t_c until t_*, after which motion becomes relativistic or 'free'. A simple heuristic argument due to Kibble [7] first suggested that in the damped phase the correlation length will scale as $L \propto t^{5/4}$.

Analysis of the evolution equations (2.16), (2.33) reveals the existence of three types of scaling regimes, which we now describe in detail. We should also note that in scaling regimes the velocity and correlation length are generically related via

$$v \propto \frac{\ell_d}{L}, \tag{2.47}$$

where we have defined an overall damping length.

$$\frac{1}{\ell_d} = 2H + \frac{1}{\ell_f}. \tag{2.48}$$

2.2.1 Scale-Invariant Solutions

Scale-invariant solutions of the form $L \propto t$ or $L \propto H^{-1}$, together with $v_\infty = const.$, only exist when the scale factor is a power law of the form

$$a(t) \propto t^\lambda, \qquad \lambda = const., \qquad 0 < \lambda < 1 \ . \tag{2.49}$$

This condition implies that

$$L \propto t \propto H^{-1} \propto d_H, \tag{2.50}$$

with the proportionality factors dependent on λ

$$\left(\frac{L}{t}\right)^2 = \frac{k(k+\tilde{c})}{4\lambda(1-\lambda)}, \qquad v^2 = \frac{k(1-\lambda)}{\lambda(k+\tilde{c})}, \tag{2.51}$$

where k is the constant value of $k(v)$ given by solving the second (implicit) equation for the velocity. It is easy to verify numerically that this solution is well-behaved and stable for all realistic parameter values.

2.2.2 Friction-Dominated Solutions

During friction-dominated epochs one has two different scaling solutions, which are transient and no longer 'scale-invariant'. In this case the network retains a memory of its initial conditions, and in particular of the epoch of formation. This can be expressed by the parameter θ in Eq. (2.43) which is the ratio of the damping terms due to friction and Hubble damping, measured at the epoch of string formation.

The first solution is a conformal 'stretching' regime,

$$\frac{L}{L_c} = \left(\frac{t}{t_c}\right)^{1/2}, \qquad v = \frac{t}{\theta L_c}, \tag{2.52}$$

which will occur when the initial string density and velocity are sufficiently low—for example, as a result of a slow first-order phase transition. In this case the network starts out with a correlation length significantly larger than the damping length and so is 'frozen', and is conformally stretched. However the damping length is growing as $\ell_f \propto t^{3/2}$, so it quickly catches up with it, ending this regime. Although this is not cosmologically relevant except for extremely light strings, the analogous regime in the matter-dominated case would be $L \propto t^{2/3}$, $v \propto t^{4/3}$.

The attractor solution for a friction-dominated epoch, which follows the stretching regime (if this exists) is the Kibble regime, which in the radiation era is

$$\frac{L}{L_c} = \left[\frac{2k_{nr}(\tilde{c}+k_{nr})}{3\theta}\right]^{1/2} \left(\frac{t}{t_c}\right)^{5/4}, \qquad v = \left[\frac{3k_{nr}}{2\theta(\tilde{c}+k_{nr})}\right]^{1/2} \left(\frac{t}{t_c}\right)^{1/4}, \tag{2.53}$$

where k_{nr} is the value of the momentum parameter in the nonrelativistic limit. In this case the correlation length stays halfway between the damping length and the horizon length. Again there is a matter era analogue, $L \propto t^{3/2}$, $v \propto t^{1/2}$, but this is rarely relevant cosmologically.

2.2.3 *A Cosmological Constant*

We can also use the VOS model in a flat background to discuss the domination at late times by a cosmological constant. In the extreme asymptotic case when the universe is inflating we have $a \propto \exp(Ht)$ with $H = \sqrt{\Lambda/3}$. The network will 'freeze out' and will simply be conformally stretched, that is,

$$L \propto a, \qquad v_\infty \propto a^{-1}, \tag{2.54}$$

where, as soon as the strings become nonrelativistic $k_{\rm nr} = 2\sqrt{2}/\pi$, their product satisfies

$$L v_\infty = \frac{2\sqrt{2}}{\pi} H\,. \tag{2.55}$$

2.3 Some Extensions

We now provide short discussions of some extensions of the VOS model. Some of these, as in the case of open universes, mostly have an historic interest, while others are highly relevant. In either case, the goal is to demonstrate the model's versatility, which will be further addressed in the following chapters.

2.3.1 *Radiation Back-Reaction*

Even though radiation backreaction is closely related to small-scale structure (which the VOS model as described thus far does not explicitly model), its effect on the long-string network can be included in the evolution equation for the correlation length

[10]. For gravitational radiation the following term can be added to the right-hand side of (2.16)

$$2\left(\frac{dL}{dt}\right)_{\rm gr} \equiv 8\Sigma_{\rm gr} v_\infty^6 = 8\tilde{\Gamma} G\mu v_\infty^6 \,. \tag{2.56}$$

Here, $\tilde{\Gamma}$ is a constant which is a long-string analogue of the $\Gamma \approx 65$ found for the radiative decay of strings. For global string radiation into Goldstone bosons or axions, the corresponding radiative decay term at a time t will be

$$2\left(\frac{dL}{dt}\right)_{\rm ax} \equiv 8\Sigma_{\rm ax} v_\infty^6 = \frac{8\tilde{\Gamma} v_\infty^6}{2\pi \ln(t/\delta)}, \tag{2.57}$$

where the logarithmic term arises because of the long-range fields of the global string and δ is the string width. For GUT-scale strings, the backreaction term for local strings is $\Gamma G\mu \sim 10^{-4}$ whereas for global strings it is of order 10^{-1}.

Remarkably, the inclusion of the back-reaction term does not affect the existence of a scale-invariant attractor solution. However, it does influence the quantitative values of the scaling parameters and the timescale necessary for this solution to be reached: the inclusion of back-reaction can make the approach to scaling much faster.

In this case one can distinguish two asymptotic scenarios. Firstly, if Σ is small (of order unity at most) then the effect of back-reaction on the scaling solution is also small. This will be the case, for example, for most local or global string networks in a cosmological context. We can express this as

$$\gamma^2 \approx \gamma_0^2 (1 + \Delta) \,, \qquad v^2 \approx v_0^2 (1 - \Delta) \,, \tag{2.58}$$

where γ_0 and v_0 are the "unperturbed" scaling values, given by Eq. (2.51), and the back-reaction correction has the form

$$\Delta = 8\beta v_0^5 \Sigma = 8\beta \left[\frac{k(1-\beta)}{\beta(k+\tilde{c})}\right]^{5/2} \Sigma \,. \tag{2.59}$$

Second, for larger values of Σ the back-reaction term will dominate the evolution equation for L, and the attractor scale-invariant solution has a different form altogether. It is not possible to write this solution in closed form, even expressing k implicitly as above. However, it is possible to write it as a series. The dominant term and the first correction take the form

$$\gamma = \frac{k}{2\beta}\left[\frac{8\beta\Sigma}{k(1-\beta)}\right]^{1/7} (1 + \Delta_1 + \ldots) \,, \qquad v = \left[\frac{k(1-\beta)}{8\beta\Sigma}\right]^{1/7} (1 - \Delta_1 + \ldots) \,, \tag{2.60}$$

with

$$\Delta_1 = \frac{1}{2^{6/7} 7} (k + \tilde{c}) \left[\frac{\beta}{k(1-\beta)}\right]^{5/7} \Sigma^{-2/7} \,. \tag{2.61}$$

There has been work on numerical simulations of global string networks [11] which explores this strong backreaction regime. These authors report a surprisingly low string density relative to the gauged case. For their expanding universe simulations in the realistic case with periodic boundary conditions, they find the following radiation and matter era densities respectively,

$$\zeta_{\mathrm{rad}} = 0.9 \pm 0.1\,, \qquad \zeta_{\mathrm{mat}} = 0.5 \pm 0.1\,. \tag{2.62}$$

These results are perfectly consistent (within the estimated error bars) with our extended VOS model if we adopt a back-reaction parameter

$$\Sigma_{\text{ax-sim}} \approx 3\,. \tag{2.63}$$

Indeed, this corresponds to the approximate average value for Σ_{ax} that one would estimate for simulations of this resolution. Present limitations on numerical dynamic range give the upper bound $\ln(t/\delta) < 4$ (at the end of the simulation), implying $\Sigma_{\text{ax-sim}} > 2$ throughout.

2.3.2 Open Universes

In this section we discuss the behavior of our model in open universes, for which one must also include an additional correction due to the curvature [12–14]. This is essentially the curvature radius of the strings, which as previously said we assume to coincide with L, divided by the radius of spatial curvature of the universe,

$$\mathscr{R} = \frac{H^{-1}}{\mid 1 - \Omega \mid^{1/2}}\,. \tag{2.64}$$

After a certain amount of algebra, one finds correction terms of the form

$$w = 1 - (1 - \Omega)(HL)^2\,. \tag{2.65}$$

Note that Ω denotes the total density of the universe. For a flat universe, $\Omega = 1$, and we have $w = 1$. The evolution equation for the correlation length L now takes the form

$$2\frac{dL}{dt} = 2HL + \frac{L}{\ell_d}\frac{v_\infty^2}{w^2} + \tilde{c}v_\infty\,, \tag{2.66}$$

while the velocity equation becomes

$$\frac{dv_\infty}{dt} = \left(1 - \frac{v_\infty^2}{w^2}\right)\left(w^2\frac{k}{L} - \frac{v_\infty}{\ell_d}\right)\,. \tag{2.67}$$

We can now re-examine the question of the existence of 'scale invariant' attractor solutions. Scaling solutions of the form $L \propto t$ or $L \propto H^{-1}$ together with $v_\infty = const.$ still only exist provided

$$a(t) \propto t^\lambda, \qquad \lambda = const., \qquad 0 < \lambda < 1, \tag{2.68}$$

but now we also require

$$\Omega = const. \tag{2.69}$$

The simplest example of the second condition is of course a flat universe, but there are examples of cosmological models which have attractors other than $\Omega = 1$ [15]. In any case, note that there can be additional relations between the values of λ and Ω for specific models. The scaling solution is now given in the implicit form

$$\left(\frac{L}{t}\right)^2 = w^2 \frac{k(k+\tilde{c})}{4\lambda(1-\lambda)}, \qquad v^2 = w^2 \frac{k(1-\lambda)}{\lambda(k+\tilde{c})}, \tag{2.70}$$

where k is defined as before, and

$$w = \frac{2(1-\lambda)}{(1-\Omega)\lambda k(k+\tilde{c})} \left[\left(1 + \frac{(1-\Omega)\lambda k(k+\tilde{c})}{(1-\lambda)}\right)^{1/2} - 1 \right]. \tag{2.71}$$

If the two conditions above do not hold, then a scaling solution will not exist.

We should also mention another cosmologically important solution: in an open universe with $\Omega \to 0$, $a \propto t$, the asymptotic solution is

$$L = \left[\frac{k_{nr}\tilde{c}}{2(1-k_{nr})}\right]^{1/2} t\,(\ln t)^{1/2}, \qquad v_\infty = \left[\frac{k_{nr}(1-k_{nr})}{2\tilde{c}}\right]^{1/2} (\ln t)^{-1/2}, \tag{2.72}$$

with k_{nr} given by (2.39). Note that this *is not* a scale-invariant solution, since $H^{-1} = t$ and $d_H = t \ln t$. In other words, by looking at the network one would be able to determine when the curvature-dominated period had started.

2.3.3 Anisotropic Models

Topological defects can be a relic left behind after inflation. The inflationary epoch will dilute the defect density and push the network outside the horizon, freezing it in co-moving coordinates in the process. However, once the inflationary epoch ends the subsequent evolution of the defects is necessarily such as to make them come back inside the horizon [13, 16]. A defect network produced during an anisotropic phase in the very early universe could still be present today.

Let us consider a cosmic string in a flat anisotropic universe of Bianchi type I, with line element:

$$ds^2 = dt^2 - X^2(t)x^2 - Y^2(t)dy^2 - Z^2(t)dz^2 \,. \tag{2.73}$$

Here $X(t)$, $Y(t)$ and $Z(t)$ are the cosmological expansion factors in the x, y and z directions respectively, and t is physical time. We also define $A \equiv \dot{X}/X$, $B \equiv \dot{Y}/Y$ and $C \equiv \dot{Z}/Z$ where the dot represents a derivative with respect to physical time t.

In the limit where the curvature radius of a cosmic string is much larger than its thickness, we can describe it as a one-dimensional object so that its world history can be represented by a 2D world-sheet

$$x^\nu = x^\nu(\zeta^a)\,; \qquad a = 0, 1\,; \quad \nu = 0, 1, 2, 3 \tag{2.74}$$

obeying the usual Goto–Nambu action

$$S = -\mu \int \sqrt{-\gamma}\, d^2\zeta \,, \tag{2.75}$$

where μ is the string mass per unit length, γ_{ab} is the two-dimensional world-sheet metric and $\gamma = \det(\gamma_{ab})$. Let us also define

$$\dot{\mathbf{x}}^2 \equiv g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 1 - X^2\dot{x}^2 - Y^2\dot{y}^2 - Z^2\dot{z}^2 \tag{2.76}$$

$$\mathbf{x}'^2 \equiv g_{\alpha\beta}x'^\alpha x'^\beta = -X^2x'^2 - Y^2y'^2 - Z^2z'^2 \,, \tag{2.77}$$

so that $\gamma = \dot{\mathbf{x}}^2\mathbf{x}'^2$ (using $\dot{\mathbf{x}} \cdot \mathbf{x}' \equiv g_{\alpha\beta}x'^\alpha\dot{x}^\beta = 0$ as a gauge condition).

If we choose $\zeta^0 = t$ and define $\zeta \equiv \zeta^1$ then the string equation of motion is given by [9]

$$\frac{\partial}{\partial t}\left(\frac{\dot{x}^\mu \mathbf{x}'^2}{\sqrt{-\gamma}}\right) + \frac{\partial}{\partial \zeta}\left(\frac{x'^\mu \dot{\mathbf{x}}^2}{\sqrt{-\gamma}}\right) + \frac{1}{\sqrt{-\gamma}}\Gamma^\mu_{\nu\sigma}(\mathbf{x}'^2\dot{x}^\nu\dot{x}^\sigma + \dot{\mathbf{x}}^2x'^\nu x'^\sigma) = 0\,. \tag{2.78}$$

From the time component we can obtain

$$\dot{\varepsilon} + \varepsilon\left[AX^2\left(\dot{x}^2 - \frac{x'^2}{\varepsilon^2}\right) + BY^2\left(\dot{y}^2 - \frac{y'^2}{\varepsilon^2}\right) + CZ^2\left(\dot{z}^2 - \frac{z'^2}{\varepsilon^2}\right)\right] = 0 \tag{2.79}$$

where we have made the further definition

$$\varepsilon \equiv \sqrt{-\mathbf{x}'^2/\dot{\mathbf{x}}^2} = \mathbf{x}'^2/\sqrt{-\gamma} = \sqrt{-\gamma}/\dot{\mathbf{x}}^2 \,. \tag{2.80}$$

On the other hand, the x component gives

$$\ddot{x} + \left(\frac{\dot{\varepsilon}}{\varepsilon} + 2A\right)\dot{x} + \frac{1}{\varepsilon}\left(\frac{x'}{\varepsilon}\right)' = 0\,, \tag{2.81}$$

and analogous equations apply for the y and z components. One can show that in the limit of an isotropic universe these equations reduce to the usual form.

Further analysis shows that the existence of an anisotropic phase through which the network evolved will be imprinted on it much beyond the time when the background becomes isotropic [13, 16]. In fact, it will be imprinted on the network as long as it is frozen outside the horizon. Only when it falls inside the horizon it will start to become relativistic and isotropic. We expect that the evolution towards the relativistic regime will be somewhat slower than in the standard case, which could conceivably have observational implications. Specifically, it is possible to see that a cosmic string network can survive up to about 60 e-foldings of inflation (the exact number being model-dependent), in the sense that any network produced in such a period will still come back inside the horizon in time to have observable consequences by the present day.

2.4 Calibrating the Model with Simulations

The model has been calibrated by detailed comparisons of its predictions to Abelian-Higgs (field theory) [17, 18] and Goto–Nambu numerical simulations [10, 19]. In the field theory case the best fit is provided by

$$\tilde{c} = 0.57 \pm 0.04\,, \tag{2.82}$$

which is is precisely the same value that was found in flat spacetime Goto–Nambu string simulations. On the other hand, for radiation and matter era Goto–Nambu simulations, one finds

$$\tilde{c} = 0.23 \pm 0.04\,. \tag{2.83}$$

This is to be expected given that Goto–Nambu network simulations can probe a much wider range of length scales below the correlation length, thus allowing small-scale wiggles to build up on those scales. We emphasize that no currently available field theory simulation has a spatial resolution or dynamic range sufficiently large to allow for the build-up of small-scale structures on the strings.

The value of $\tilde{c} = 0.57$ can therefore be regarded as a bare loop chopping efficiency, while $\tilde{c} = 0.23$ can be interpreted as a renormalised one. This interpretation is consistent with the fact that Goto- Nambu simulations in the expanding case somewhat surprisingly possess much more small-scale structure than corresponding flat spacetime strings (for example, as quantified by the fractal properties of each network). The approximate factor of two difference between the two loop production rates may be related to the well-known result that the renormalised and bare string mass per unit length differ by about a factor of two in radiation era Nambu simulations [19–21].

The most detailed numerical study of the properties of small-scale structures on cosmic string networks has been carried out in [19]. A first striking feature is that in the expanding universe cosmic string velocities are *anti-correlated* on scales between the correlation length and the horizon. However, such a feature is not present in flat spacetime. This anti-correlation is the result of a 'memory' of the network for recent intercommutings, and its absence in the flat spacetime case highlights the fact that the loop production mechanism is different in the expanding and non-expanding cases. Indeed, such an effect was discussed in [22, 23]. If one defines a 'velocity coherence length', this will be significantly smaller than ξ itself. The network's fractal dimension is unity on small scales and two on very large scales. The interesting question, however, is what happens at intermediate scales. In particular, one should expect (and indeed finds) a range of scales where strings should behave as *self-avoiding random walks*, and these have a fractal dimension $d_s = 5/3$ in three spatial dimensions.

Radiation and matter era networks have similar fractal profiles (if one rescales length scales by the respective correlation lengths). The flat spacetime ones, however, are qualitatively different, which must reflect the differing efficiencies of loop production in flat space and the expanding universe. While the integrated loop production efficiency is much greater in flat spacetime $c = 0.57$, as opposed to the expanding $c = 0.23$, it appears to be relatively less effective around the correlation scale with energy trapped on fairly large scales. This intuitive picture is confirmed by noting that the renormalised mass per unit length μ is smaller than in the expanding case on small scales, but is larger on large scales.

Finally, we note that in all cases the fractal dimension of the network at the scale of the correlation length is well below two: typical values are 1.2 in the expanding case and 1.4 in the flat case. This is to be expected if one interprets ξ as a persistence length. So the intuitive picture that a string network looks Brownian at the scale of the correlation length is clearly incorrect. It's not even true at the scale of the horizon—here the network looks more like a *self-avoiding* random walk, which is an obvious consequence of intercommutings. The Brownian picture is only valid on significantly larger scales.

The two distinguishing characteristics of string evolution in Minkowski spacetime are the absence of velocity anti-correlations on scales around the correlation length, and the apparent existence of a 'preferred' scale (around the correlation length ξ) from which energy does not move to smaller scales. These can have a substantial influence when calculating the network's observational consequences.

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