Chapter 2
General Properties of the Morse–Smale Diffeomorphisms

Abstract In this chapter we consider an important class of discrete structurally stable systems which adequately describe processes with regular dynamics, Morse-Smale diffeomorphisms. We present with proofs the properties of Morse-Smale diffeomorphisms which are necessary for the topological classification. The asymptotic behavior and the embedding into the ambient manifold (the phase space) of the stable and the unstable manifolds of the saddle periodic points plays the key role in understanding of the dynamics of such diffeomorphisms. To describe the topological invariants which reflect these properties we consider the space of wandering orbits which belong to some specially chosen invariant sets of the diffeomorphism. We describe the important (for the subsequent results) construction of the sequence of the "attractor-repeller" pairs suggested by C. Conley. This construction is based on introduction of an order on the set of the periodic orbits which satisfies the Smale partial relation. The proof of existence of a trapping neighborhood of an attractor (a repeller) relies on the local Morse-Lyapunov function constructed in this chapter. All the proofs are presented for the class $MS(M^n)$ of the orientation preserving Morse-Smale diffeomorphisms $f : M^n \to M^n$ on an orientable manifold M^n. The results are partly announced and proved in the surveys [1–3, 9] and the papers [4–8].

2.1 Embedding and Asymptotic Behavior of the Invariant Manifolds of Periodic Points

Definition 2.1 A diffeomorphism $f : M^n \to M^n$ of a smooth closed (compact without boundary) connected orientable n-manifold ($n \geq 1$) M^n is called a *Morse–Smale diffeomorphism* if

1. the non-wandering set Ω_f is finite and hyperbolic;
2. for every two distinct periodic points p, q the manifolds W^s_p, W^u_q intersect transversally.

The class of these diffeomorphisms we denote by $MS(M^n)$.

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It follows from the condition 1) of Definition 2.1 that Ω_f consists of finite number of periodic points: $\Omega_f = \mathrm{Per}_f$. For $q \in \{0, \dots, n\}$ let Ω_q denote the set of the periodic points of Morse index q. For a hyperbolic periodic point p of a diffeomorphism $f \in MS(M^n)$ we use the following denotations:

- m_p is the period of the point p;
- q_p is the Morse index of the point p;
- ν_p is the orientation type of the point p, i.e $\nu_p = +1$ (-1) if the map $f^{m_p}|_{W^u_p}$ preserves (reverses) orientation;
- ℓ^u_p is the connected component of the set $W^u_p \setminus p$ (separatrix);
- $\mathcal{O}_p$ is the orbit of the point p for which we also assume $m_{\mathcal{O}_p} = m_p$, $q_{\mathcal{O}_p} = q_p$, $\nu_{\mathcal{O}_p} = \nu_p$.
- (m_p, q_p, ν_p) is the periodic data of the orbit $\mathcal{O}_p$.

Furthermore for the diffeomorphism $f \in MS(M^n)$ we denote

- by k_f the number of the periodic orbits;
- by m_f the minimal natural number for which $\Omega_{f^{m_f}}$ consists of the fixed points of the diffeomorphism f^{m_f} with orientation type $+1$.

Dynamical properties and topological type of a Morse–Smale diffeomorphism are largely determined by the properties of the embedding and by the mutual disposition of the invariant manifolds of the periodic points. The key role here belongs to the study of asymptotic properties of the invariant manifolds of the saddle periodic points.

The main result of this section is the following theorem:

Theorem 2.1 *Let $f \in MS(M^n)$. Then*

1. $M^n = \bigcup\limits_{p \in \Omega_f} W^u_p$;
2. *W^u_p is a smooth submanifold of the manifold M^n which is diffeomorphic to $\mathbb{R}^{q_p}$ for every periodic point $p \in \Omega_f$;*
3. $\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup\limits_{r \in \Omega_f : \ell^u_p \cap W^s_r \neq \emptyset} W^u_r$ *for every unstable separatrix ℓ^u_p of a periodic point $p \in \Omega_f$.*

The item (1) of Theorem 2.1 is immediate from the spectral decomposition theorem, nevertheless we present its proof here for the sake of fullness. All the propositions formulated for the unstable manifolds hold for the stable manifolds as well. One gets them if one formally changes "u" to "s" because the stable manifolds of the periodic points of a diffeomorphism f are the unstable manifolds of the periodic points of the diffeomorphism f^{-1}.

The items (1), (2), (3) of Theorem 2.1 are proved in parts 2.1.1, 2.1.2, 2.1.5 resp. Now we present some important corollaries of Theorem 2.1.

According to the item (2) of Theorem 2.1 W^u_p is a smooth q_p-submanifold of the manifold M^n for any periodic point p of a diffeomorphism $f \in MS(M^n)$. Then from Statement 10.48 it follows that the map $f|_{W^u_{\mathcal{O}_p}} : W^u_{\mathcal{O}_p} \to W^u_{\mathcal{O}_p}$ is a diffeomorphism. Furthermore, the class of topological conjugacy of the diffeomorphism $f^{m_p}|_{W^u_p}$ is completely determined by the Morse index q_p and the orientation type ν_p of the point p. Namely, the following proposition holds.

Proposition 2.1 *Let $f \in MS(M^n)$. Then for every periodic point $p \in \Omega_f$ the diffeomorphism $f^{m_p}|_{W^u_p} : W^u_p \to W^u_p$ is topologically conjugate to the canonical expansion $a^u_{q_p,\nu_p} : \mathbb{R}^{q_p} \to \mathbb{R}^{q_p}$.*

If a periodic point of a diffeomorphism $f \in MS(M^n)$ is a saddle then the embedding of its f-invariant neighborhood is also of important. We begin with the linear case.

For $q \in \{1, \dots, n-1\}$, $t \in (0, 1]$ let $\mathcal{N}^t_q = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ (x_1^2 + \dots + x_q^2)(x_{q+1}^2 + \dots + x_n^2) < t\}$ and $\mathcal{N}^1_q = \mathcal{N}_q$. Notice that the set $\mathcal{N}^t_q$ is invariant with respect to the canonical diffeomorphism $a_{q,\nu}$ which has the only fixed saddle point at the coordinate origin O, its unstable manifold being $W^u_O = Ox_1 \dots x_q$ and its stable manifold being $W^s_O = Ox_{q+1} \dots x_n$.

Definition 2.2 Let $f \in MS(M^n)$. We call a neighborhood N_σ of a saddle point $\sigma \in \Omega_f$ *linearizing* if there is a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}_{q_\sigma}$ which conjugates the diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ to the canonical diffeomorphism $a_{q_\sigma,\nu_\sigma}|_{\mathcal{N}_{q_\sigma}}$.

The neighborhood $N_{\mathcal{O}_\sigma} = \bigcup\limits_{k=0}^{m_\sigma - 1} f^k(N_\sigma)$ equipped with the map $\mu_{\mathcal{O}_\sigma}$ made up of the homeomorphisms $\mu_\sigma f^{-k} : f^k(N_\sigma) \to \mathcal{N}_n, \ k = 0, \dots, m_\sigma - 1$ is called the *linearizing neighborhood of the orbit $\mathcal{O}_\sigma$*.

Theorem 2.2 *Every saddle point (orbit) of the diffeomorphism $f \in MS(M^n)$ has a linearizing neighborhood.*

According to the item (2) of Theorem 2.1 invariant manifolds of the periodic points of a diffeomorphism $f \in MS(M^n)$ are submanifolds of the manifold M^n. Nevertheless, the closure of an invariant manifold of a saddle point can have a complicate topological structure. The nature of this phenomenon can be dynamic as well as topological. The first refers to the case when a separatrix of the saddle point has heteroclinic intersections.

Definition 2.3 If σ_1, σ_2 are distinct periodic saddle points of a diffeomorphism $f \in MS(M^n)$ for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is said to be *heteroclinic*.

- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$ then a connected component of the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic manifold* and if $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 1$ then it is called a *heteroclinic curve*.
- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 0$ then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is countable, each point of this set is called a *heteroclinic point* and the orbit of a heteroclinic point is called the *heteroclinic orbit*.

Figure 2.1 shows the phase portraits of two Morse–Smale diffeomorphisms on $\mathbb{S}^3$. The wandering sets of these diffeomorphisms contain heteroclinic curves and the

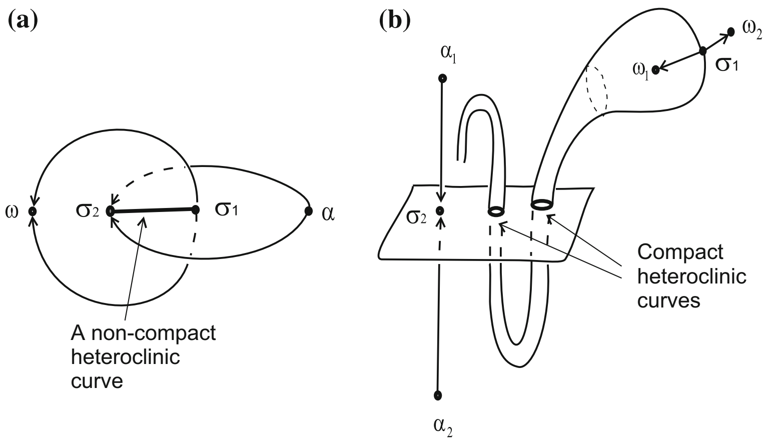


Fig. 2.1 Heteroclinic curves two Morse–Smale diffeomorphisms on the 3-sphere

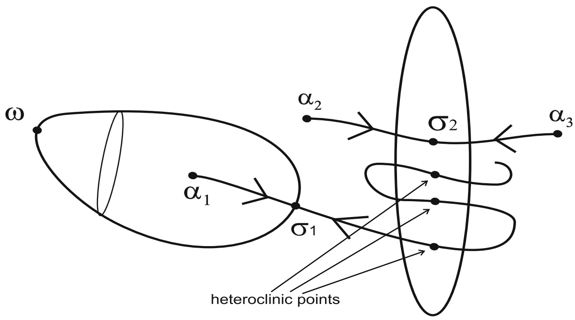
non-wandering sets consist of the fixed points: $\alpha, \omega, \sigma_1, \sigma_2$ and $\alpha_1, \alpha_2, \omega_1, \omega_2, \sigma_1, \sigma_2$, respectively. Here $\alpha, \alpha_1, \alpha_2$ are sources, $\omega, \omega_1, \omega_2$ are sinks, and σ_1, σ_2 are saddles. In Figure 2.1 (a) the intersection $W^u_{\sigma_2} \cap W^s_{\sigma_1}$ consists of one non-compact heteroclinic curve. On Figure 2.1 (b) the intersection $W^u_{\sigma_2} \cap W^s_{\sigma_1}$ consists of the countable set of compact heteroclinic curves.

Figure 2.2 shows the phase portrait of a Morse–Smale diffeomorphism on $\mathbb{S}^3$. The wandering set of this diffeomorphism contains the heteroclinic points and the non-wandering set consists of the fixed points: the three sources $\alpha_1, \alpha_2, \alpha_3$, the one sink ω and the two saddles σ_1, σ_2.

Definition 2.4 A diffeomorphism $f \in MS(M^n)$ is said to be *gradient-like* if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $\dim\ W^u_{\sigma_1} < \dim\ W^u_{\sigma_2}$.

The following proposition gives a geometrical interpretation to the property of a homeomorphism to be gradient-like.

Fig. 2.2 Heteroclinic points of a diffeomorphism on the 3-sphere



Proposition 2.2 *A diffeomorphism $f \in MS(M^n)$ is gradient-like if and only if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for distinct $\sigma_1, \sigma_2 \in \Omega_f$ it follows that* $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$.

Thus a Morse–Smale diffeomorphism is gradient-like if and only if it has no heteroclinic points. The diffeomorphisms with the phase portraits shown in Figure 2.1 are gradient-like but the diffeomorphism with the phase portrait shown in Figure 2.2 is not.

According to the item (3) of Theorem 2.1 the closure of a separatrix of a saddle point which has heteroclinic intersections is not a topological manifold in general, but the closure of a separatrix of a saddle with no heteroclinic intersections is a topologically embedded manifold. The following proposition holds.

Proposition 2.3 *Let $f \in MS(M^n)$ and let σ be a saddle point of f such that the unstable separatrix ℓ^u_σ has no heteroclinic intersections. Then*

$$\mathrm{cl}(\ell^u_\sigma) \setminus (\ell^u_\sigma \cup \sigma) = \{\omega\},$$

where ω is a sink point. If $q_\sigma = 1$ then $\mathrm{cl}(\ell^u_\sigma)$ is an arc topologically embedded into M^n and if $q_\sigma \geq 2$ then $\mathrm{cl}(\ell^u_\sigma)$ is the sphere $\mathbb{S}^{q_\sigma}$ topologically embedded into M^n.

Thus if σ is a saddle of f such that ℓ^u_σ has no heteroclinic intersections then $\mathrm{cl}(\ell^u_\sigma)$ is a topologically embedded manifold. According to the item (2) of Theorem 2.1, $\ell^u_\sigma \cup \sigma$ is a smooth submanifold of the manifold M^n. But the manifold $\mathrm{cl}(\ell^u_\sigma)$ can be wild at the point ω.

Definition 2.5 A separatrix ℓ^u_σ of a saddle point σ which has no heteroclinic intersections is called *tame* or *tamely embedded* into M^n if the closure $\mathrm{cl}(\ell^u_\sigma)$ is a submanifold of the manifold M^n. Otherwise the separatrix ℓ^u_σ is called *wild* or *wildly embedded* into M^n.

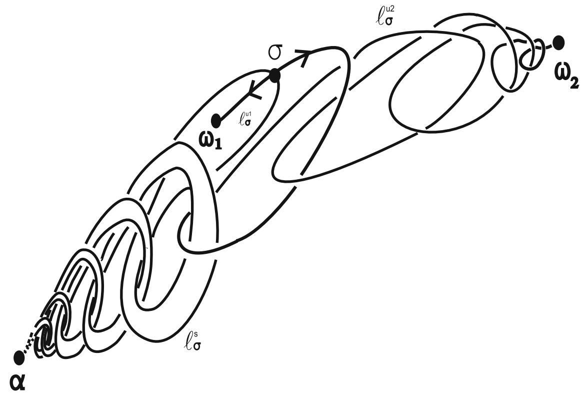
For $n = 2$ according to Statement 10.72 any separatrix with no heteroclinic points is tamely embedded into M^2. In the section 10.4.1 we present an example of a wild compact arc in $\mathbb{S}^3$ (that has nothing to do with the dynamics) with one point of wildness. The first example of a Morse–Smale diffeomorphism of the manifold $\mathbb{S}^3$ with wildly embedded 1-dimensional and 2-dimensional separatrices was constructed by D. Pixton in 1977 (see Figure 2.3). We present the proof of the wildness of the 2-dimensional separatrix ℓ^s_σ and the 1-dimensional separatrix ℓ^{u2}_σ of this example in the section 4.2.

2.1.1 Representation of the Ambient Manifold as the Unit of the Invariant Manifolds of the Periodic Points

Proof of the item (1) of Theorem 2.1

Now we prove that $M^n = \bigcup\limits_{p \in \Omega_f} W^u_p$ for any diffeomorphism $f \in MS(M^n)$.

Fig. 2.3 Pixton's example



Proof Without loss of generality we assume that the non-wandering set of the diffeomorphism f is fixed (otherwise the same reasoning applies for a suitable iteration of the diffeomorphism f). Then Ω_f is the union of the finite number of the fixed points $\Omega_f = p_1 \cup \cdots \cup p_r$.

Let $x \in M^n$. According to Statement 1.1 and Exercise 1.3 the set $\alpha(x)$ is not empty and it is the subset of Ω_f. We now show that $\alpha(x)$ consists of exactly one fixed point which depends on x. Assume the contrary, i.e, there are distinct fixed points $p_v,\ p_w \in \alpha(x)$. Since Ω_f is finite there is $\rho > 0$ such that $d(p_i, p_j) > \rho$ whenever $i \neq j$. Denote $V_i = \{y \in M^n \ :\ d(y, p_i) < \frac{\rho}{3}\}$. Since all the points $p_i,\ i = \overline{1, r}$ are fixed there is a neighborhood U_i such that $\mathrm{cl}(U_i) \subset V_i$ and $f^{-1}(\mathrm{cl}(U_i)) \cap V_j = \emptyset$ for every $j \neq i$. By the assumption there is an increasing sequence q_ℓ of the iterations of f^{-1} such that $f^{-q_{2m}}(x) \in U_v, f^{-q_{2m+1}}(x) \in U_w$ and $q_{2m+1} - q_{2m} \geq 2$. We pick the sequence n_m so that n_m is the maximal natural number belonging to the interval (q_{2m}, q_{2m+1}) for each $f^{-(n_m-1)}(x) \in \mathrm{cl}(U_v)$. Then $f^{-n_m}(x) \notin \mathrm{cl}(U_v)$. On the other hand $f^{-n_m}(x) = f^{-1}(f^{-(n_m-1)}(x)) \notin V_j$ for $j \neq v$ and hence $f^{-n_m}(x) \in (M^n \setminus \bigcup\limits_{i=1}^{r} U_i)$. But then $\alpha(x)$ is not a subset of Ω_f and we have a contradiction.

Thus for each point $x \in M^n$ there is the unique point $p_v(x) \in \Omega_f$ such that $\alpha(x) = p_v(x)$, i.e., there is a sequence $k_n \to +\infty$ such that $\lim\limits_{k_n \to +\infty} d(f^{-k_n}(x), p_v(x)) = 0$. It follows from the dynamic properties of the diffeomorphism f in the neighborhood of the point $p_v(x)$ (see Theorem 1.4) that $f^{-k_n}(x) \in W^u_{p_v(x)}$ for all n greater then some n_0. Then $x \in W^u_{p_v(x)}$ because the unstable manifold is invariant. $\square$

2.1.2 *Embedding of Invariant Manifolds of Periodic Points into the Ambient Manifold*

Lemma 2.1 *Let σ be a hyperbolic saddle fixed point of a diffeomorphism $f : M^n \to M^n$, let $T_\sigma \subset W^s_\sigma$ be a compact neighborhood of the point σ and $\xi \in T_\sigma$. Then* *for every sequence of points* $\{\xi_m\} \subset (M^n \setminus T_\sigma)$ *converging to the point* ξ *there are a subsequence* $\{\xi_{m_j}\}$, *a sequence of natural numbers* $k_{m_j} \to +\infty$ *and a point* $\eta \in (W^u_\sigma \setminus \sigma)$ *such that the sequence of points* $\{f^{k_{m_j}}(\xi_{m_j})\}$ *converges to the point* η.

Proof Without loss of generality we assume $\nu_\sigma = +1$ (otherwise the same reasoning applies for the diffeomorphism f^2). In accordance with Theorem 1.4 there are neighborhoods $V_\sigma \subset M^n$, $V_O \subset \mathbb{R}^n$ of the points $\sigma, O \in \mathbb{R}^n$, respectively, and there is a homeomorphism $\psi : V_\sigma \to V_O$ such that $\psi(f(x)) = a_{q_\sigma,+1}(\psi(x))$ for every point $x \in (V_\sigma \cap f(V_\sigma))$, where $a_{q_\sigma,+1}$ is the canonical diffeomorphism. Without loss of generality one assumes $(V_\sigma \cap W^s_\sigma) \subset T_\sigma$, $\xi \in (V_\sigma \cap f(V_\sigma))$ and $\{\xi_m\} \subset (V_\sigma \cap f(V_\sigma))$. We pick a number $r > 0$ so that the ball $B_r(O) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1^2 + \cdots + x_n^2) \leq r\}$ would be a subset of the set $V_O \cap a_{q_\sigma,+1}(V(O))$.

Let $\psi(\xi_m) = \bar{\xi}_m = (\bar{\xi}_{1,m}, \ldots, \bar{\xi}_{q_\sigma,m}, \bar{\xi}_{q_\sigma+1,m}, \ldots, \bar{\xi}_{n,m})$. The set $K^u = \{(x_1, \ldots, x_{q_\sigma}) \in Ox_1 \ldots x_{q_\sigma} : \frac{r^2}{4} \leq x_1^2 + \cdots + x_{q_\sigma}^2 \leq r^2\}$ is the fundamental domain of the restriction of the diffeomorphism $a_{q_\sigma,+1}$ to $Ox_1 \ldots, x_{q_\sigma} \setminus O$. Then for every $m \in \mathbb{N}$ there is the unique integer k_m such that $\frac{r^2}{4} \leq 4^{k_m}\left((\bar{\xi}_{1,m})^2 + \cdots + (\bar{\xi}_{q_\sigma,m})^2\right) < r^2$. Let $\bar{\eta}_m = a^{k_m}_{q_\sigma,+1}(\bar{\xi}_m)$. Since $\lim\limits_{m\to\infty} \bar{\xi}_m = \psi(\xi) \in (Ox_{q_\sigma+1} \ldots x_n \setminus O)$ for every $i \in \{1, \ldots, q_\sigma\}$ the limit $\lim\limits_{m\to\infty} \bar{\xi}_{i,m}$ equals to 0 and hence $\lim\limits_{m\to\infty} k_m = +\infty$. Furthermore, the sequence $\{\bar{\xi}_{i,m}\}$ is bounded for every $i \in \{q_\sigma + 1, \ldots, n\}$ and hence $\bar{\eta}_{i,m} = \left(\frac{1}{2}\right)^{k_m} \bar{\xi}_{i,m} \to 0$ for $m \to +\infty$ and $i \in \{q_\sigma + 1, \ldots, n\}$.

Therefore the coordinates of the points $\bar{\eta}_m = (\bar{\eta}_{1,m}, \ldots, \bar{\eta}_{n,m})$ satisfy $\frac{r^2}{4} \leq (\bar{\eta}_{1,m})^2 + \cdots + (\bar{\eta}_{q_\sigma,m})^2 < r^2$ for $i \in \{1, \ldots, q_\sigma\}$ and $\bar{\eta}_{i,m} \to 0$ as $m \to \infty$ for $i \in \{q_\sigma + 1, \ldots, n\}$, i.e., the points η_m are inside some compact subset of $\mathbb{R}^n$. Since any sequence of points of a compact set has a converging subsequence (see item 4 of Statement 10.21) there is a subsequence $\{k_{m_j}\}$ of the sequence $\{k_m\}$ and there is a point $\bar{\eta} \in (W^u_O \setminus O)$ such that $\lim\limits_{j\to\infty} \bar{\eta}_{m_j} = \bar{\eta}$. Then $\xi_{m_j} = \psi^{-1}(a^{-k_{m_j}}_{q_\sigma,+1}(\bar{\eta}_{m_j}))$ is the desired subsequence. □

Proof of the item (2) of Theorem 2.1

We now prove that for every periodic point $p \in \Omega_f$ of the diffeomorphism $f \in MS(M^n)$ W^u_p is a smooth submanifold of the manifold M^n which is diffeomorphic to $\mathbb{R}^{q_p}$.

Proof Let $x \in W^u_p$ and let $T_p(x) \subset W^u_p$ be a compact neighborhood of the point p containing the point x. It follows from Theorem 1.1 that $W^u_p = J^u_p(\mathbb{R}^{q_p})$, where $J^u_p : \mathbb{R}^{q_p} \to M^n$ is an injective immersion. Since by Statement 10.46 an injective immersion of a compact space is an embedding, there is a chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M^n such that $\psi_x(U_x \cap T_p(x)) = \mathbb{R}^{q_p}$. If $q_p = n$ or $q_p = 0$ then $\psi_x(U_x \cap T_p(x)) = \psi_x(U_x \cap W^u_p)$. Therefore the unstable manifold of every node point is a smooth submanifold.

Now we show that W^u_p is a smooth submanifold of M^n diffeomorphic to $\mathbb{R}^{q_p}$ for every saddle point $p \in \Omega_f$ as well. Suppose the contrary: W^u_p is not a smooth submanifold of M^n. Without loss of generality we assume the saddle p to be fixed. It follows from the assumption that there is a point $x \in W^u_p$ such that $(U_x \setminus T_p(x)) \cap W^u_p \neq \emptyset$ for every chart $\psi_x : U_x \to \mathbb{R}^n$ of the manifold M^n for which $\psi_x(U_x \cap T_p(x)) = \mathbb{R}^{q_p}$. Hence there is a sequence $\{x_m\} \subset (W^u_p \setminus T_p(x))$ such that $d(x_m, x) \to 0$ for $m \to +\infty$.

Lemma 2.1 gives us that there is a subsequence x_{m_j} and there is a sequence k_j such that the sequence $y_j = f^{-k_j}(x_{m_j}) \subset W^u_p$ converges to a point $y \in (W^s_p \setminus p)$. According to the item (1) of Theorem 2.1 there is a point $r \in \Omega_f$ such that $y \in W^u_r$. Since a Morse–Smale diffeomorphism has no homoclinic points (see Statement 1.6), $p \neq r$. Applying similar arguments to the sequence y_j we get a subsequence $z_i \subset W^u_p$ converging to a point $z \in (W^s_r \setminus r)$, and a point $v \in \Omega_f$ such that $z \in W^u_v$. Due to the λ-lemma and due to the absence of homoclinic points the point v is distinct from the points p and r. Repeating the arguments we get an infinite sequence of different periodic points and that is impossible because the non-wandering set of the diffeomorphism f is finite. □

Corollary 2.1 *Let $f \in MS(M^n)$. Then Ω_f contains at least one source periodic point.*

Proof Assume the contrary that the non-wandering set Ω_f contains no source periodic points. Then from the items (1) and (2) of Theorem 2.1 it follows that the manifold M^n is the union of the finite number of the submanifolds $\bigcup\limits_{i=1}^{k} W^u_{p_i}$, $\bigcup\limits_{i=1}^{k} p_i = \Omega_f$ each of which is of dimension less then n. Let x_1 be an arbitrary point of the manifold $W^u_{p_1}$. Then there is a chart $\psi_1 : U_1 \to \mathbb{R}^n$ of the manifold M^n such that $\psi_1(U_1 \cap W^u_{p_1}) = \mathbb{R}^{q_{p_1}}$. Since $q_{p_1} < n$ the set $V_1 = U_1 \setminus (U_1 \cap W^u_{p_1})$ is open and it is disjoint from $W^u_{p_1}$. Repeating the arguments for a point $x_2 \in V_1$ belonging to $W^u_{p_2}$ we get that there is an open subset V_2 of the manifold M^n disjoint from $W^u_{p_1} \cup W^u_{p_2}$. We continue the process and construct a nonempty set $V_k \subset M^n$ disjoint from the unstable manifolds of the periodic points of the diffeomorphism f and that contradicts the item (1) of Theorem 2.1. □

The proof of Proposition 2.1

Let $f \in MS(M^n)$. We now prove that for every hyperbolic periodic point $p \in \Omega_f$ the diffeomorphism $f^{m_p}|_{W^u_p} : W^u_p \to W^u_p$ is topologically conjugate to the canonical expansion $a^u_{q_p,\nu_p} : \mathbb{R}^{q_p} \to \mathbb{R}^{q_p}$.

Proof Without loss of generality assume that p is a fixed point of the diffeomorphism f (otherwise all the following arguments should be for the diffeomorphism f^{m_p}). By Theorem 1.4 the diffeomorphism $f|_{W^u_p}$ in a neighborhood $U_p \subset W^u_p$ of the point p is topologically conjugate by a homeomorphism $H^u_p : U_p \to \mathbb{R}^{q_p}$ to the map $a^u_{q_p,\nu_p}$. According to the item (2) of Theorem 2.1 W^u_p is a smooth q_p-submanifold of the manifold M^n. Then the inner and the ambient topology of W^u_p coincide and from Statement 10.48 it follows that the map $f|_{W^u_{\mathcal{O}_p}} : W^u_{\mathcal{O}_p} \to W^u_{\mathcal{O}_p}$ is a diffeomorphism.

Define a map $h^u_p : W^u_p \to \mathbb{R}^{q_p}$ as follows: for $x \in (W^u_p \setminus U_p)$ let $h^u_p(x) = (a^u_{q_p,\nu_p})^{-i}(H^u_p(f^i(x)))$, where $i \in \mathbb{Z}$ is such that $f^i(x) \in U_p$ and $h^u_p(x) = H^u_p(x)$ for $x \in U_p$. It follows from the construction that the map h^u_p is a homeomorphism between W^u_p and $\mathbb{R}^{q_p}$ which conjugates $f|_{W^u_p}$ to $a^u_{q_p,\nu_p}$. □

2.1.3 Topological Invariants Related to the Embedding of the Invariant Manifolds of the Periodic Points into the Ambient Manifold

Many important properties of the mutual position and the embedding of the separatrices of distinct periodic points (into the ambient manifold) reveal themselves when one studies the orbits space of separatrices of these points. So we now study the orbits space $(\mathbb{R}^q \setminus O)/a^u_{q,\nu}$ of the action of the canonical expansion $a^u_{q,\nu}$ of $\mathbb{R}^q \setminus O$ for $q \in \{1, \ldots, n\},\ \nu \in \{+1, -1\}$.

Proposition 2.4 *Group $A^u_{q,\nu} = \{(a^u_{q,\nu})^k, k \in \mathbb{Z}\}$ acts freely and discontinuously on $\mathbb{R}^q \setminus O$.*

Proof The set $\mathbb{R}^q \setminus O$ contains no fixed points of the map $a^u_{q,\nu}$ and thus the action of the group $A^u_{q,\nu}$ is free. Let K be a compact subset of the set $\mathbb{R}^q \setminus O$. Then due to Statement 10.18 it is bounded and therefore there is an annulus $K_N = \{(x_1, \ldots, x_q) \in \mathbb{R}^q\ :\ 4^{-N} \le x_1^2 + \cdots + x_q^2 \le 4^N\}$ such that $K \subset K_N$. Thus $(a^u_{q,\nu})^k(K) \cap K = \emptyset$ for $|k| > 2N$ and hence the action of the group $A^u_{q,\nu}$ is discontinuous. □

Let $\hat{\mathscr{W}}^u_{q,\nu} = (\mathbb{R}^q \setminus O)/a^u_{q,\nu}$. We say $\hat{\mathscr{W}}^u_{q,\nu}$ to be the *orbits space of the canonical expansion*. Let $p_{\hat{\mathscr{W}}^u_{q,\nu}} : \mathbb{R}^q \setminus O \to \hat{\mathscr{W}}^u_{q,\nu}$ denote the natural projection. From Proposition 2.4 and Statement 10.30 it follows that the projection $p_{\hat{\mathscr{W}}^u_{q,\nu}}$ is a covering map and it induces a structure of smooth q-manifold to the orbits space $\hat{\mathscr{W}}^u_{q,\nu}$. It also induces the map $\eta_{\hat{\mathscr{W}}^u_{q,\nu}}$ from the union of the fundamental groups of the connected components of the manifold $\hat{\mathscr{W}}^u_{q,\nu}$ into the group $\mathbb{Z}$ in the following way. Due to the Monodromy theorem (Statement 2) for every closed curve $\hat{c} \subset \hat{\mathscr{W}}^u_{q,\nu}$ there is the lifting $c \subset (\mathbb{R}^q \setminus O)$ which is the arc with x being its one boundary point and $(a^u_{q,\nu})^k(x)$ being the other boundary point. Then $\eta_{\hat{\mathscr{W}}^u_{q,\nu}}([\hat{c}]) = k$. By Statement 10.30 the restriction of the map $\hat{\mathscr{W}}^u_{q,\nu}$ to the fundamental group of each connected component of the submanifold $\hat{\mathscr{W}}^u_{q,\nu}$ is a nontrivial homomorphism into the group $\mathbb{Z}$. The following proposition describes the topological structure of the space $\hat{\mathscr{W}}^u_{q,\nu}$.

Proposition 2.5 *The orbits space $\hat{\mathscr{W}}^u_{q,\nu}$ of the canonical expansion and the map $\eta_{\hat{\mathscr{W}}^u_{q,\nu}}$ have the following properties:*

1. *for $q = 1,\ \nu = -1$ the space $\hat{\mathscr{W}}^u_{1,-1}$ is homeomorphic to the circle and the map $\eta_{\hat{\mathscr{W}}^u_{1,-1}} : \pi_1(\hat{\mathscr{W}}^u_{1,-1})) \to 2\mathbb{Z}$ is an epimorphism;*
2. *for $q = 1,\ \nu = +1$ the space $\hat{\mathscr{W}}^u_{1,+1}$ is homeomorphic to the pair of the circles $\hat{\mathscr{W}}^{u1}_{1,+1}, \hat{\mathscr{W}}^{u2}_{1,+1}$ and the map $\eta_{\hat{W}^u_{1,+1}}$ consists of epimorphisms $\eta_{\hat{W}^{ui}_{1,+1}} : \pi_1(\hat{W}^{ui}_{1,+1}) \to \mathbb{Z},\ i = 1, 2$;*
3. *for $q = 2,\ \nu = -1$ the space $\hat{\mathscr{W}}^u_{2,-1}$ is homeomorphic to the Klein bottle and the map $\eta_{\hat{\mathscr{W}}^u_{2,-1}} : \pi_1(\hat{\mathscr{W}}^u_{2,-1}) \to \mathbb{Z}$ is an epimorphism;*

4. *for* $q = 2$, $\nu = +1$ *the space* $\hat{\mathscr{W}}^u_{2,+1}$ *is homeomorphic to the 2-torus and the map* $\eta_{\hat{\mathscr{W}}^u_{2,+1}} : \pi_1(\hat{\mathscr{W}}^u_{2,+1}) \to \mathbb{Z}$ *is an epimorphism;*
5. *for* $q \geq 3$, $\nu = -1$ *the space* $\hat{\mathscr{W}}^u_{q,-1}$ *is homeomorphic to the generalized Klein bottle and the map* $\eta_{\hat{\mathscr{W}}^u_{q,-1}} : \pi_1(\hat{\mathscr{W}}^u_{q,-1}) \to \mathbb{Z}$ *is an epimorphism;*
6. *for* $q \geq 3$, $\nu = +1$ *the space* $\hat{\mathscr{W}}^u_{q,+1}$ *is homeomorphic to* $\mathbb{S}^{q-1} \times \mathbb{S}^1$ *and the map* $\eta_{\hat{\mathscr{W}}^u_{3,+1}} : \pi_1(\hat{\mathscr{W}}^u_{3,+1}) \to \mathbb{Z}$ *is an epimorphism.*

Proof Due to Statement 10.30 the restriction of the map $\eta_{\hat{\mathscr{W}}^u_{q,\nu}}$ to the fundamental group of each connected component of the manifold $\hat{\mathscr{W}}^u_{q,\nu}$ is a nontrivial homomorphism into the group $\mathbb{Z}$. Due to Statement 10.31 to prove this proposition it is sufficient to study the geometry of the gluing of the points of the boundary of the fundamental domain of the canonical expansion $a^u_{q,\nu}$ on $\mathbb{R}^q \setminus O$. Figure 2.4 shows the gluing.

1) If $q = 1$, $\nu = -1$ then the fundamental domain of the action of the diffeomorphism $a_{1,-1}$ on $\mathbb{R} \setminus O$ is the segment $a = \{x_1 \in \mathbb{R} \ : \ 1 \leq x_1 \leq 4\}$. Having glued its boundaries by the map $a^u_{1,-1}$ we get the circle $\hat{a}$ which coincides with the space $\hat{\mathscr{W}}^u_{1,-1}$ and $\hat{a}$ is the generator of the fundamental group of $\hat{\mathscr{W}}^u_{1,-1}$. Since a is the lifting of the curve $\hat{a}$ which joins the point 1 and the point $4 = (a^u_{1,-1})^2(1)$ it follows that $\eta_{\hat{\mathscr{W}}^u_{1,-1}}(\pi_1(\hat{\mathscr{W}}^u_{1,-1})) = 2\mathbb{Z}$.

2) If $q = 1$, $\nu = +1$ then the fundamental domain of the action of the diffeomorphism $a^u_{1,+1}$ on $\mathbb{R} \setminus O$ is the two segments $\{x_1 \in \mathbb{R} \ : \ 1 \leq x_1^2 \leq 4\}$. Having glued their boundaries by the map $a^u_{1,+1}$ we get the two connected components $\hat{a}^1$, $\hat{a}^2$ coinciding with the connected components $\hat{\mathscr{W}}^{u1}_{1,+1}$, $\hat{\mathscr{W}}^{u2}_{1,+1}$ of the space $\hat{\mathscr{W}}^u_{1,+1}$. The components $\hat{a}^1$, $\hat{a}^2$ are also the generators of their respective fundamental groups. Since the segment $[1, 2]$ ($[-2, -1]$) of the fundamental domain is the lifting of the curve $\hat{a}^1$ ($\hat{a}^2$) joining the point 1 (-1) and the point $a^u_{1,+1}$ ($a^u_{1,+1}(-1)$) it follows that $\eta_{\hat{\mathscr{W}}^{ui}_{1,+1}}(\pi_1(\hat{\mathscr{W}}^{ui}_{1,+1}) = \mathbb{Z}$ for $i = 1, 2$.

3), 4) If $q = 2$, $\nu = \pm 1$ then the fundamental domain of the action of the map $a^u_{2,\pm 1}$ on $\mathbb{R}^2 \setminus O$ is the 2-annulus $\{(x_1, x_2) \in \mathbb{R}^2 \ : \ 1 \leq x_1^2 + x_2^2 \leq 4\}$. Having glued their boundaries by the diffeomorphism $a^u_{2,+1}$ ($a^u_{2,-1}$) we get the 2-torus (the Klein bottle) which coincides with the space $\hat{\mathscr{W}}^u_{2,\pm 1}$. The curves $a = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 = 0, \ 1 \leq x_2 \leq 2\}$ and $b = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1\}$ are the liftings of the generators $\hat{a}$ and $\hat{b}$ of the fundamental group of the torus (the Klein bottle). Since the curve a joins the point $(0, 1)$ to the point $(0, 2) = a^u_{2,\pm 1}(0, 1)$ and the curve b joins point $(1, 0)$ to the point $(1, 0) = (a^u_{2,\pm 1})^0(1, 0)$ we have $\eta_{\hat{\mathscr{W}}^u_{2,+1}}(\pi_1(\hat{\mathscr{W}}^u_{2,+1})) = \mathbb{Z}$.

5), 6) If $q \geq 3$, $\nu = \pm 1$ then the fundamental domain of the action of the map $a^u_{q,\nu}$ on $\mathbb{R}^q \setminus O$ is the q-annulus $\{(x_1, \ldots, x_q) \in \mathbb{R}^q \ : \ 1 \leq x_1^2 + \cdots + x_q^2 \leq 4\}$. Having glued its boundaries by the diffeomorphism $a^u_{q,+1}$ ($a^u_{q,-1}$) we get the manifold $\mathbb{S}^{q-1} \times \mathbb{S}^1$ (the generalized Klein bottle) which coincides with the space $\hat{\mathscr{W}}^u_{q,\nu}$. The curve $a = \{(x_1, \ldots, x_q) \in \mathbb{R}^q \ : \ x_1 = \cdots = x_{q-1} = 0, 1 \leq x_q \leq 2\}$ is the lifting

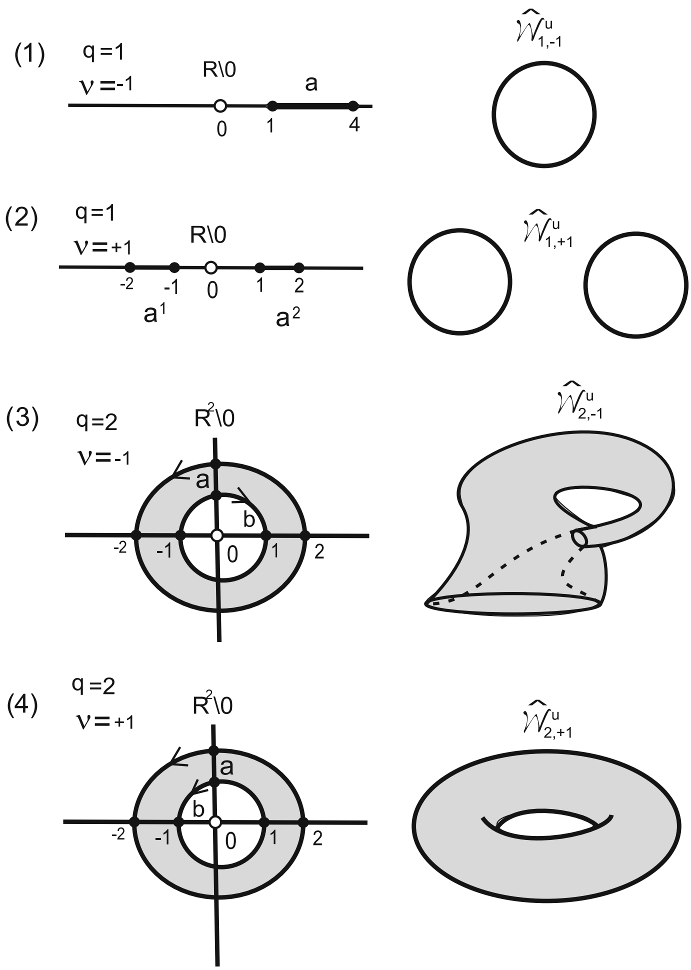


Fig. 2.4 Orbits spaces of the action of the canonical expansion $a^u_{q,\nu}$ on $\mathbb{R}^q \setminus O$

of the generator $\hat{a}$ of the fundamental group $\mathbb{S}^{q-1} \times \mathbb{S}^1$ and the generalized Klein bottle. Since the curve b joins the point $(0, \dots, 0, 1)$ and the point $(0, \dots, 0, 2) = a^u_{q,\nu}(0, \dots, 0, 1)$ we have $\eta_{\hat{\mathcal{W}}^u_{q,\nu}}(\pi_1(\hat{\mathcal{W}}^u_{q,\nu})) = \mathbb{Z}$. $\square$

We call the generators of the fundamental groups introduced in the proof of Proposition 2.5 the *canonical generators of the orbits space of the canonical expansion*.

Let now p be a periodic point of Morse index $q_p \geq 1$ for a diffeomorphism $f \in MS(M^n)$. Consider the orbits space $\hat{W}^u_{\mathcal{O}_p} = (W^u_{\mathcal{O}_p} \setminus \mathcal{O}_p)/f$ of the action of the group $F = \{f^k,\ k \in \mathbb{Z}\}$ on $W^u_{\mathcal{O}_p} \setminus \mathcal{O}_p$. Let $p_{\hat{W}^u_{\mathcal{O}_p}} : W^u_{\mathcal{O}_p} \setminus \mathcal{O}_p \to \hat{W}^u_{\mathcal{O}_p}$ denote the natural projection. The following theorem shows the connection between the orbits space $\hat{W}^u_{\mathcal{O}_p}$ and the linear model.

Theorem 2.3 *Let p be a periodic point of period m_p, of orientation type ν_p and of Morse index $q_p \geq 1$ for a diffeomorphism $f \in MS(M^n)$. Then the projection $p_{\hat{W}^u_{\mathcal{O}_p}}$ is a covering map which induces a structure of the smooth q_p-manifold on the orbits space $\hat{W}^u_{\mathcal{O}_p}$ and it induces such a map $\eta_{\hat{W}^u_{\mathcal{O}_p}}$ from the union of the fundamental groups of the connected components of the manifold $\hat{W}^u_{\mathcal{O}_p}$ to the group $\mathbb{Z}$ that there is a homeomorphism $\hat{h}^u_{\mathcal{O}_p} : \hat{W}^u_{\mathcal{O}_p} \to \hat{\mathscr{W}}^u_{q_p,\nu_p}$ such that $\eta_{\hat{W}^u_{\mathcal{O}_p}}([\hat{c}]) = m_p \eta_{\hat{\mathscr{W}}^u_{q_p,\nu_p}}([\hat{h}^u_{\mathcal{O}_p}(\hat{c})])$ for every closed curve $\hat{c} \subset \hat{W}^u_{\mathcal{O}_p}$.*

Proof Due to Proposition 2.1 there is a homeomorphism $h^u_p : W^u_p \to \mathbb{R}^{q_p}$ conjugating diffeomorphisms $f^{m_p}|_{W^u_p}$ and $a^u_{q_p,\nu_p}$. Let $\hat{W}^u_p = (W^u_p \setminus p)/f^{m_p}$ and let $p_{\hat{W}^u_p} : W^u_p \setminus p \to \hat{W}^u_p$ denote the natural projection; $p_{\hat{W}^u_p}$ is a covering map because the diffeomorphisms $f^{m_p}|_{W^u_p}$ and $a^u_{q_p,\nu_p}$ are conjugate. Then from Statement 10.35 it follows that the map $\hat{h}^u_p = p_{\hat{\mathscr{W}}^u_{q_p,\nu_p}} h^u_p p^{-1}_{\hat{W}^u_p} : \hat{W}^u_p \to \hat{\mathscr{W}}^u_{q_p,\nu_p} : \hat{W}^u_p \to \hat{\mathscr{W}}^u_{q_p,\nu_p}$ is a homeomorphism such that

$$\eta_{\hat{W}^u_p}([\hat{c}]) = \eta_{\hat{\mathscr{W}}^u_{q,\nu}}([\hat{h}^u_p(\hat{c})]) \tag{2.1}$$

for every closed curve $\hat{c} \subset \hat{W}^u_p$. Let $r = p_{\hat{W}^u_p}(p_{\hat{W}^u_{\mathcal{O}_p}}|_{W^u_p \setminus p})^{-1} : \hat{W}^u_{\mathcal{O}_p} \to \hat{W}^u_p$. By construction the map r is a homeomorphism such that

$$\eta_{\hat{W}^u_{\mathcal{O}_p}}([\hat{c}]) = m_p \eta_{\hat{W}^u_p}([r(\hat{c})]) \tag{2.2}$$

for every closed curve $\hat{c} \subset \hat{W}^u_{\mathcal{O}_p}$. Let $\hat{h}^u_{\mathcal{O}_p} = \hat{h}^u_p r : \hat{W}^u_{\mathcal{O}_p} \to \hat{\mathscr{W}}^u_{q_p,\nu_p}$. Then the map $\hat{h}^u_{\mathcal{O}_p}$ is a homeomorphism for which due to (2.1), (2.2) $\eta_{\hat{W}^u_{\mathcal{O}_p}}([\hat{c}]) = m_p \eta_{\hat{\mathscr{W}}^u_{q_p,\nu_p}}([\hat{h}^u_{\mathcal{O}_p}(\hat{c})])$ holds for every closed curve $\hat{c} \subset \hat{W}^u_{\mathcal{O}_p}$. □

Similarly we define the *orbits space* $\hat{\mathscr{W}}^s_{q,\nu} = (\mathbb{R}^{n-q} \setminus O)/a^s_{q,\nu}$ of the *canonical contraction* for $q \in \{0, \dots, n-1\}$, $\nu \in \{+1, -1\}$ and the orbits space $\hat{W}^s_{\mathcal{O}_p} = (W^s_{\mathcal{O}_p} \setminus \mathcal{O}_p)/f$ of the action of the group F on the set of separatrices $W^s_{\mathcal{O}_p} \setminus \mathcal{O}_p$ of a periodic point p with Morse index $q_p \leq (n-1)$.

Figure 2.5 (a) shows a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ the non-wandering set of which consists of eight periodic points with the following periodic data: $\mathcal{O}_{\omega_1}(1, 0, +1)$, $\mathcal{O}_{\omega_2}(1, 0, +1)$, $\mathcal{O}_{\omega_3}(3, 0, +1)$, $\mathcal{O}_{\sigma_1}(1, 2, +1)$, $\mathcal{O}_{\sigma_2}(1, 1, +1)$, $\mathcal{O}_{\sigma_3}(3, 1, +1)$, $\mathcal{O}_{\alpha_1}(1, 3, +1)$, $\mathcal{O}_{\alpha_2}(1, 3, +1)$. Figure 2.5 (b) shows the fundamental domains of the action of the diffeomorphism f on $W^s_{\mathcal{O}_{\omega_i}} \setminus \mathcal{O}_{\omega_i}$, $i = 1, 2, 3$, $W^u_{\mathcal{O}_{\alpha_i}} \setminus \mathcal{O}_{\alpha_i}$, $i = 1, 2$. Each fundamental domain is the 3-annulus from which the orbits spaces $\hat{W}^s_{\mathcal{O}_{\omega_i}}$, $i = 1, 2, 3$, $\hat{W}^u_{\mathcal{O}_{\alpha_i}}$, $i = 1, 2$ are obtained by gluing the boundary

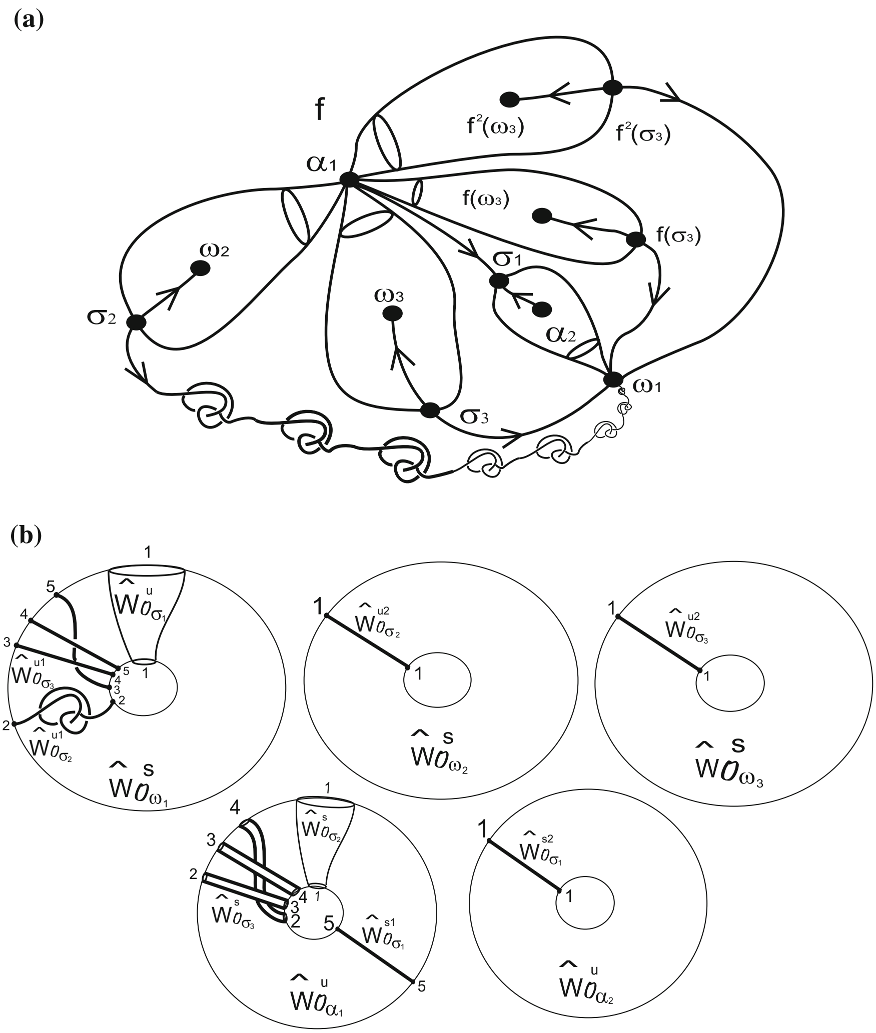


Fig. 2.5 Orbits spaces of the separatrices of the periodic points

spheres by the diffeomorphism $f^{m_{\omega_i}},\ i = 1, 2, 3, f^{m_{\alpha_i}},\ i = 1, 2$, respectively. The orbits spaces $\hat{W}^s_{\mathcal{O}_{\sigma_i}},\ \hat{W}^u_{\mathcal{O}_{\sigma_i}},\ i = 1, 2, 3$ are obtained from the arcs and the cylinders by gluing the points with the same numbers and of the circles with the same numbers.

Figure 2.6 (a) shows a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ the non-wandering set of which consists of five periodic points with the following periodic data: $\mathcal{O}_\omega(1, 0, +1), \mathcal{O}_{\sigma_1}(1, 2, -1), \mathcal{O}_{\sigma_2}(2, 2, +1), \mathcal{O}_{\alpha_1}(2, 3, +1), \mathcal{O}_{\alpha_2}(2, 3, +1)$. Figure 2.6 (b) shows the fundamental domains of the action of the diffeomorphism f on $W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$ and $W^u_{\mathcal{O}_{\alpha_i}} \setminus \mathcal{O}_{\alpha_i},\ i = 1, 2$. Each fundamental domain is the 3-annulus

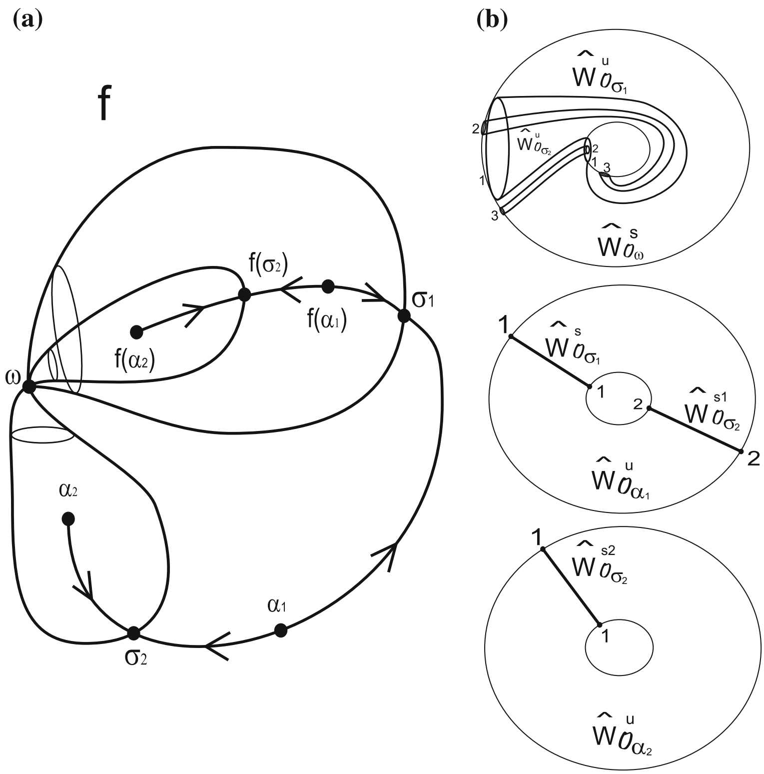


Fig. 2.6 Orbits spaces of the separatrices of periodic points

from which the orbits spaces $\hat{W}^s_{\mathcal{O}_\omega}, \hat{W}^u_{\mathcal{O}_{\alpha_i}},\ i = 1, 2$ are obtained by gluing the boundary spheres of the annulus by the diffeomorphism $f^{m_\omega}, f^{m_{\alpha_i}},\ i = 1, 2$, respectively. The orbits spaces $\hat{W}^s_{\mathcal{O}_{\sigma_i}},\ \hat{W}^u_{\mathcal{O}_{\sigma_i}},\ i = 1, 2$ are obtained from the arcs and the cylinders by gluing the points with the same numbers and the circles with the same numbers.

2.1.4 A Linearizing Neighborhood

Proof of Theorem 2.2

Now we prove that every saddle point (orbit) of the diffeomorphism $f \in MS(M^n)$ has a linearizing neighborhood.

Proof Let σ be a saddle periodic point of the diffeomorphism $f \in MS(M^n)$. It follows from the definition of a linearizing neighborhood that it is sufficient to construct a neighborhood N_σ for the saddle point $\sigma \in \Omega_f$ for which there is a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}_{q_\sigma}$ conjugating the diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ to the canonical diffeomorphism $a_{q_\sigma,\nu_\sigma}|_{\mathcal{N}_{q_\sigma}}$. Without loss of generality we assume that σ is a fixed point (otherwise the same reasoning applies for the diffeomorphism f^{m_σ}).

Due to Theorem 1.4 the diffeomorphism f in some neighborhood $U_\sigma \subset M^n$ of the point σ is topologically conjugate to the map a_{q_σ,ν_σ} by the topological embedding $g_\sigma : U_\sigma \to \mathbb{R}^n$. Since W^s_σ and W^u_σ are smooth submanifolds of the manifold M^n one can pick a neighborhood U_σ so that the set $\tilde{N}_\sigma = \bigcup\limits_{k\in\mathbb{Z}} f^k(U_\sigma)$ is a smooth submanifold of the manifold M^n as well. Define the map $\tilde{\mu}_\sigma : \tilde{N}_\sigma \to \mathbb{R}^n$ as follows: for $x \in (\tilde{N}_\sigma \setminus U_\sigma)$ let $\tilde{\mu}_\sigma(x) = a^{-i}_{q_\sigma,\nu_\sigma}(g_\sigma(f^i(x)))$, where $i \in \mathbb{Z}$ is such that $f^i(x) \in U_\sigma$. Similarly to Proposition 2.1 one proves that the map $\tilde{\mu}_\sigma$ is the topological embedding conjugating the diffeomorphisms $f|_{\tilde{N}_\sigma}$ and $a_{q_\sigma,\nu_\sigma}|_{\tilde{\mu}_\sigma(\tilde{N}_\sigma)}$.

Pick $t_0 \in (0, 1]$ such that $\mathcal{N}^{t_0}_{q_\sigma} \subset \tilde{\mu}_\sigma(\tilde{N}_\sigma)$. Notice that $a_{q_\sigma,\nu_\sigma}|_{\mathcal{N}^{t_0}_{q_\sigma}}$ is conjugate to the canonical diffeomorphism $a_{q_\sigma,\nu_\sigma}|_{\mathcal{N}_{q_\sigma}}$ by the diffeomorphism $h(x_1, \ldots, x_n) = (\frac{x_1}{\sqrt{t_0}}, \ldots, \frac{x_n}{\sqrt{t_0}})$. Then $N_\sigma = \tilde{\mu}^{-1}_\sigma(\mathcal{N}^{t_0}_{q_\sigma})$ is the desired neighborhood and $\mu_\sigma = h\tilde{\mu}_\sigma : N_\sigma \to \mathcal{N}_{q_\sigma}$ is the conjugating homeomorphism. □

Note that the action of the group $A_{q,\nu} = \{a^k_{q,\nu},\ k \in \mathbb{Z}\}$ on $\mathcal{N}_q$ is not free since it has the fixed point O and it is not discontinuous on $\mathcal{N}_q \setminus O$ since it has the compact subset $K = \{(x_1, \ldots, x_n) \in (\mathcal{N}_q \setminus O) : \frac{1}{16} \le x_1^2 + \cdots + x_n^2 \le \frac{1}{4}\}$ which intersects each of its iterations $f^k(K),\ k \in \mathbb{Z}$. This leads to the fact that the orbits space $\mathcal{N}_q/a_{q,\nu}$ is not Hausdorff (see Statement 10.30) and thus it is not a manifold. Nevertheless on the set $\mathcal{N}^u_q = \mathcal{N}_q \setminus W^s_O$ the action of the group $A_{q,\nu}$ is free and discontinuous.

Exercise 2.1 Find a fundamental domain of the action of the group $A_{q,\nu}$ on the manifold $\mathcal{N}^u_q$ (see Figures 2.7, 2.8) and show that this action is free and discontinuous.

By Exercise 2.1 and Statements 10.30, 10.31 the orbits space $\hat{\mathcal{N}}^u_{q,\nu} = (\mathcal{N}^u_q)/a_{q,\nu}$ is a smooth n-manifold with boundary and the natural projection $p_{\hat{\mathcal{N}}^u_{q,\nu}} : \mathcal{N}^u_q \to \hat{\mathcal{N}}^u_{q,\nu}$ is a covering map which induces the map $\eta_{\hat{\mathcal{N}}^u_{q,\nu}}$ from the union of fundamental groups of connected components of the space $\hat{\mathcal{N}}^u_{q,\nu}$ into the group $\mathbb{Z}$.

Since $a_{q,\nu}|_{W^u_O \setminus O} = a^u_{q,\nu}|_{W^u_O \setminus O}$, the space $\hat{\mathcal{N}}^u_{q,\nu}$ is the tubular neighborhood of the orbits space $\hat{\mathcal{W}}^u_{q,\nu}$ of the canonical expansion. The manifold $\hat{\mathcal{W}}^u_{q,+1}$ is homeomorphic to the manifold $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \{0\}$ and its neighborhood $\hat{\mathcal{N}}^u_{q,+1}$ is homeomorphic to the manifold $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \operatorname{int} \mathbb{D}^{n-q}$. The following exercise clarifies the structure of the neighborhood $\hat{\mathcal{N}}^u_{q,-1}$.

Exercise 2.2 Using the facts that $a^2_{q,-1} = a^2_{q,+1}$ and that the maps $a^2_{q,+1}$ and $a_{q,+1}$ are topologically conjugate (according to Theorem 1.4) show that the manifold $\hat{\mathcal{W}}^u_{q,+1}$ is the twofold cover for the manifold $\hat{\mathcal{W}}^u_{q,-1}$ and the manifold $\hat{\mathcal{N}}^u_{q,+1}$ is the twofold cover for the neighborhood $\hat{\mathcal{N}}^u_{q,-1}$.

Similarly one defines the orbits space $\hat{\mathcal{N}}^s_{q,\nu} = \mathcal{N}^s_q/a^s_{q,\nu}$ (where $\mathcal{N}^s_q = \mathcal{N}_q \setminus W^u_O$), the covering map $p_{\hat{\mathcal{N}}^s_{q,\nu}} : \mathcal{N}^s_q \to \hat{\mathcal{N}}^s_{q,\nu}$ and the map $\eta_{\hat{\mathcal{N}}^s_{q,\nu}}$ from the union of the

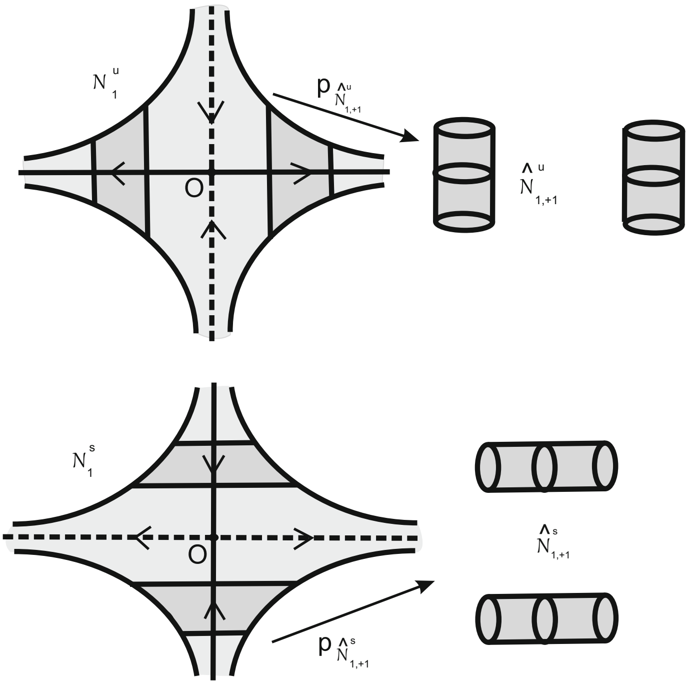


Fig. 2.7 Neighborhoods of the orbits spaces of the canonical contraction and expansion for $n = 2$

fundamental groups of the connected components of the manifold $\hat{\mathscr{N}}^s_{q,\nu}$ into the group $\mathbb{Z}$.

Figures 2.7 and 2.8 show these objects for $n = 2, 3$; $q = 1$; $\nu = +1$. To make the structure of the orbits space $\hat{\mathscr{N}}^s_{q,\nu}$, $\hat{\mathscr{N}}^u_{q,\nu}$ more clear we mark out the fundamental domain of the action of the canonical diffeomorphism $a_{q,\nu}$ on the sets $\mathscr{N}^s_q$, $\mathscr{N}^u_q$.

Now let σ be a saddle periodic point with Morse index q_σ of a diffeomorphism $f \in MS(M^n)$ and let $N_{\mathscr{O}_\sigma}$ be a linearizing neighborhood of the orbit $\mathscr{O}_\sigma$. Denote $N^u_{\mathscr{O}_\sigma} = N_{\mathscr{O}_\sigma} \setminus W^s_{\mathscr{O}_\sigma}$. Consider the orbits space $\hat{N}^u_{\mathscr{O}_\sigma} = N^u_{\mathscr{O}_\sigma}/f$ of the action of the diffeomorphism f on $N^u_{\mathscr{O}_\sigma}$. Denote by $p_{\hat{N}^u_{\mathscr{O}_\sigma}} : N^u_{\mathscr{O}_\sigma} \to \hat{N}^u_{\mathscr{O}_\sigma}$ the natural projection. The following theorem shows the connection between the orbits space $\hat{N}^u_{\mathscr{O}_\sigma}$ and the linear model. It can be proved similarly to Theorem 2.3.

Theorem 2.4 *Let σ be a saddle periodic point of period m_σ with orientation type ν_σ and Morse index q_σ for a diffeomorphism $f \in MS(M^n)$. Then the projection $p_{\hat{N}^u_{\mathscr{O}_\sigma}}$ is the covering map; it induces a structure of a smooth n-manifold on the orbits space $\hat{N}^u_{\mathscr{O}_\sigma}$ and it induces a map $\eta_{\hat{N}^u_{\mathscr{O}_\sigma}}$ from the union of the fundamental groups of the connected components of the manifold $\hat{N}^u_{\mathscr{O}_\sigma}$ into the group $\mathbb{Z}$ such*

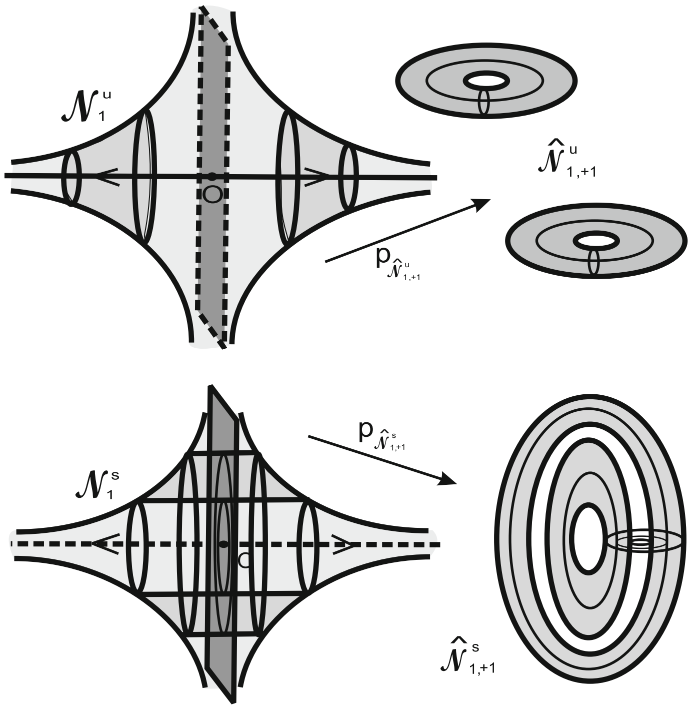


Fig. 2.8 Neighborhoods of the orbits spaces of the canonical expansion and contraction for $n = 3$

that there is a homeomorphism $\hat{\mu}^u_{\mathscr{O}_\sigma} : \hat{N}^u_{\mathscr{O}_\sigma} \to \hat{\mathscr{N}}^u_{q_\sigma,\nu_\sigma}$, *which satisfies* $\eta_{\hat{N}^u_{\mathscr{O}_\sigma}}([\hat{c}]) = m_\sigma \eta_{\hat{\mathscr{N}}^u_{q_\sigma,\nu_\sigma}}([\hat{\mu}^u_{\mathscr{O}_\sigma}(\hat{c})])$ *for any closed curve* $\hat{c} \subset \hat{N}^u_{\mathscr{O}_\sigma}$.

Similarly one defines the orbits space $\hat{N}^s_{\mathscr{O}_\sigma} = N^s_{\mathscr{O}_\sigma}/f$ of the action of the group F on $N^s_{\mathscr{O}_\sigma} = N_{\mathscr{O}_\sigma} \setminus W^u_{\mathscr{O}_\sigma}$, the covering map $p_{\hat{N}^s_{\mathscr{O}_\sigma}} : N^s_{\mathscr{O}_\sigma} \to \hat{N}^s_{\mathscr{O}_\sigma}$ and the map $\eta_{\hat{N}^s_{\mathscr{O}_\sigma}}$ consisting of nontrivial homomorphisms into the group $\mathbb{Z}$ on the fundamental group of each connected component of the manifold $\hat{N}^s_{\mathscr{O}_\sigma}$.

Below for any $t \in (0, 1]$ we denote $N^t_\sigma = (\mu_\sigma)^{-1}(\mathscr{N}^t_{q_\sigma})$, $N^t_{\mathscr{O}_\sigma} = \bigcup\limits_{k=0}^{m_\sigma-1} f^k(N^t_\sigma)$, $\mathscr{N}^{ut}_{q_\sigma} = \mathscr{N}^t_{q_\sigma,\nu_\sigma} \setminus W^s_O$, $N^{ut}_\sigma = (\mu_\sigma)^{-1}(\mathscr{N}^{ut}_{q_\sigma})$, $N^{ut}_{\mathscr{O}_\sigma} = \bigcup\limits_{k=0}^{m_\sigma-1} f^k(N^{ut}_\sigma)$, $\mathscr{N}^{st}_{q_\sigma} = \mathscr{N}^t_{q_\sigma} \setminus W^u_O$, $N^{st}_\sigma = (\mu_\sigma)^{-1}(\mathscr{N}^{st}_{q_\sigma})$, $N^{st}_{\mathscr{O}_\sigma} = \bigcup\limits_{k=0}^{m_\sigma-1} f^k(N^{st}_\sigma)$, $\hat{\mathscr{N}}^{ut}_{q_\sigma,\nu_\sigma} = p_{\hat{\mathscr{N}}^{ut}_{q_\sigma,\nu_\sigma}}(\mathscr{N}^{ut}_{q_\sigma})$, and $\hat{\mathscr{N}}^{st}_{q_\sigma,\nu_\sigma} = p_{\hat{\mathscr{N}}^{st}_{q_\sigma,\nu_\sigma}}(\mathscr{N}^{st}_{q_\sigma})$.

2.1.5 The Asymptotic Behavior of the Invariant Manifolds of the Periodic Points

Proof of Proposition 2.2

Recall that a diffeomorphism $f \in MS(M^n)$ is gradient-like if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for any two different points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that dim $W^u_{\sigma_1} <$ dim $W^u_{\sigma_2}$. Now we prove that a diffeomorphism $f \in MS(M^n)$ is gradient-like if and only if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for every two different points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$.

Proof Since the invariant manifolds of the periodic points intersect transversally, if $z \in (W^s_{\sigma_1} \cap W^u_{\sigma_2})$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ then $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = \dim\ W^s_{\sigma_1} + \dim\ W^u_{\sigma_2} - n$ (see Statement (10.56)). On the other hand since the periodic points are hyperbolic we have $\dim\ W^s_{\sigma_1} + \dim\ W^u_{\sigma_1} = n$. Thus $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = \dim\ W^u_{\sigma_2} - \dim\ W^u_{\sigma_1}$. Then the inequalities $\dim\ W^u_{\sigma_1} < \dim\ W^u_{\sigma_2}$ and $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$ are equivalent. □

Proof of the item (3) of Theorem 2.1

Now we prove that $\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup\limits_{r \in \Omega_f : \ell^u_p \cap W^s_r \neq \emptyset} W^u_r$ for every periodic point $p \in \Omega_f$ of the diffeomorphism $f \in MS(M^n)$.

Proof It is sufficient to prove the following:

(i) if $x \in (\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p))$ then $x \in W^u_r$ for some point $r \in \Omega_f$ such that $\ell^u_p \cap W^s_r \neq \emptyset$;

(ii) if $\ell^u_p \cap W^s_r \neq \emptyset$ for a point $r \in \Omega_f$ then $W^u_r \subset (\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p))$.

Without loss of generality we assume that the non-wandering set of the diffeomorphism f is fixed (otherwise the same reasoning applies for the diffeomorphism f^n for the appropriate n).

Now we prove (i). Let $x \in (\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p))$. Then there is a sequence $\{x_m\} \subset \ell^u_p$ such that $d(x_m, x) \to 0$ for $m \to +\infty$. By the item (1) of Theorem 2.1 $x \in W^u_r$ for some point $r \in \Omega_f$. We show that r cannot be a source. Assume the contrary, then $x_m \in W^u_r$ for all m large enough. But then $r = p$ and $\ell^u_p \cup p = W^u_r$ and $x \notin W^u_r$. We come to contradiction.

Thus we have two possibilities: (a) r is a sink, (b) r is a saddle.

(a) If r is a sink then $W^u_r = r$, $x = r$ and $x_m \in W^s_r$ for all m large enough. Then $\ell^u_p \cap W^s_r \neq \emptyset$ and (i) is true.

(b) If r is a saddle then by Lemma 2.1 there is a subsequence x_{m_j} and there is a sequence k_j such that the sequence $y_j = f^{-k_j}(x_{m_j})$ converges to a point $y \in (W^s_r \setminus r)$. By the item (1) of Theorem 2.1 there is a point $v \in \Omega_f$ such that $y \in \ell^u_v$. Arguing as above we get that the point v cannot be a source. The point v is evidently not a sink because for a sink $\ell^u_v = \emptyset$. Thus due to the absence of homoclinic points (see Statement 1.6) the point v is a saddle different from r. If $\ell^u_v = \ell^u_p$ then the proposition is proved. Otherwise repeating the arguments from the λ-lemma and the fact that the non-wandering set is finite we prove the proposition in a finite number of steps.

Now we prove (*ii*). There are two possibilities: (a) r is a sink, (b) r is a saddle.

(a) Let $y \in (\ell_p^u \cap W_r^s)$. Then $d(y_k, r) \to 0$ for $y_k = f^k(y)$ and $k \to +\infty$. But then $r = W_r^u \subset (\mathrm{cl}(\ell_p^u) \setminus (\ell_p^u \cup p))$ and (*ii*) holds.

(b) It is immediate from the λ-lemma and the fact that the intersections $\ell_p^u \cap W_r^s$ are transversal. □

Proof of Proposition 2.3

Let $f \in MS(M^n)$ and let σ be a saddle point of f such that ℓ_σ^u has no heteroclinic intersections. We now prove that in this case

$$\mathrm{cl}(\ell_\sigma^u) \setminus (\ell_\sigma^u \cup \sigma) = \{\omega\},$$

where ω is a sink periodic point. If $q_\sigma = 1$ then $\mathrm{cl}(\ell_\sigma^u)$ is the arc topologically embedded into M^n and if $q_\sigma \geq 2$ then $\mathrm{cl}(\ell_\sigma^u)$ is the sphere $\mathbb{S}^{q_\sigma}$ topologically embedded into M^n.

Proof Let ℓ_σ^u have no heteroclinic intersections for some saddle point $\sigma \in \Omega_f$. Then $\mathrm{cl}(\ell_\sigma^u) \setminus (\ell_\sigma^u \cup \sigma) = \bigcup\limits_{p \in \Omega_f : \ell_\sigma^u \cap W_p^s \neq \emptyset} W_p^u$. The point p cannot be saddle because ℓ_σ^u has no heteroclinic intersections; it cannot be a source as well for in this case $W_p^s = p$. Thus $\ell_\sigma^u \subset \bigcup\limits_{\omega \in \Omega_0} W_\omega^s$. Since the separatrix ℓ_σ^u is connected there is the unique sink ω such that $\ell_\sigma^u \subset W_\omega^s$ and $\mathrm{cl}(\ell_\sigma^u) = \ell_\sigma^u \cup \{\sigma, \omega\}$.

We now show that for $q_\sigma = 1$ the set $\mathrm{cl}(\ell_\sigma^u)$ is an arc topologically embedded into M^n. By Proposition 2.1 there is a homeomorphism $H : W_\sigma^u \to \mathbb{R}$ such that $H(\sigma) = O$. Then there is a homeomorphism $\tilde{H} : \ell_\sigma^u \cup \sigma \to [0, 1)$. The homeomorphism $\tilde{H}$ can obviously be extended to a homeomorphism $\tilde{H} : \mathrm{cl}(\ell_\sigma^u) \to [0, 1]$. Then $\mathrm{cl}(\ell_\sigma^u)$ is an arc topologically embedded into M^n.

Now we show that for $q_\sigma \geq 2$ the set $\mathrm{cl}(\ell_\sigma^u)$ is the q_σ-sphere topologically embedded into M^n.

By Proposition 2.1 there is a homeomorphism $H : W_\sigma^u \to \mathbb{R}^{q_\sigma}$. Let $\psi = H^{-1}\vartheta_- : \mathbb{S}^{q_\sigma} \setminus N \to \mathrm{cl}(\ell_\sigma^u) \setminus \omega$, where $\vartheta_- : \mathbb{S}^{q_\sigma} \setminus \{N\} \to \mathbb{R}^{q_\sigma}$ is the stereographic projection and N is the north pole (see formula 10.4). The homeomorphism ψ obviously extends to the homeomorphism $\psi : \mathbb{S}^{q_\sigma} \to \mathrm{cl}(\ell_\sigma^u)$ if we set $\psi(N) = \omega$. Then $\mathrm{cl}(\ell_\sigma^u)$ is the q_σ-sphere topologically embedded into M^n. □

2.2 Morse–Lyapunov Functions. Attractors and Repellers

In this section we give general approaches to the dynamics of the Morse–Smale diffeomorphisms which often make it possible to give the topological classification of these diffeomorphisms.

Definition 2.6 A diffeomorphism $f \in MS(M^n)$ is called a "source-sink" diffeomorphism or a "north pole-south pole" diffeomorphism if its non-wandering set consists of a single sink and a single source.

Theorem 2.5 *If a diffeomorphism* $f \in MS(M^n)$, $n > 1$ *has no saddle points then*

1. *f is a "source-sink" diffeomorphism;*
2. *the space of the wandering orbits of the diffeomorphism f is homeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{S}^1$*;*
3. *all the "source-sink" diffeomorphisms are topologically conjugate to one another (for n fixed) and the manifold M^n is homeomorphic to the n-sphere* $\mathbb{S}^n$.

Theorem 2.5 shows that the "source-sink" diffeomorphisms have trivial dynamics: all the non-fixed points are wandering and under the action of the diffeomorphism they move from the source to the sink (see Figure 2.9 (a)). Topological conjugacy of all these diffeomorphisms follows from the fact that the spaces of their wandering orbits are homeomorphic. When one studies a more complicated Morse–Smale diffeomorphism the dynamics looks similar but "the source" and "the sink" then stand for the closed invariant sets of as simple topological structure as possible, one of them A being the attracting set and the other R being the repelling set (see Figure 2.9 (b)).

If the orbit $\hat{V} = V/f$, where $V = M^n \setminus (A \cup R)$ can be described for some class of diffeomorphisms then it gives rise to topological classification for these diffeomorphisms. This approach is used in the Chapter 3 and in the Chapter 5.

More explicitly, since a diffeomorphism $f \in MS(M^n)$ is structurally stable its basic sets coincide with its periodic orbits and from the results of Chapter 1 it follows that there is an order relation on the set of periodic orbits which is compatible with the partial order relation $\prec$:

$$\mathscr{O}_p \prec \mathscr{O}_r \iff W^s_{\mathscr{O}_p} \cap W^u_{\mathscr{O}_r} \neq \emptyset.$$

Definition 2.7 We say that a numbering of the periodic orbits $\mathscr{O}_1, \dots, \mathscr{O}_{k_f}$ of the diffeomorphism $f \in MS(M^n)$ is *dynamical* if it satisfies the following:

1) if $\mathscr{O}_i \prec \mathscr{O}_j$ then $i \leq j$;

2) if $q_{\mathscr{O}_i} < q_{\mathscr{O}_j}$ then $i < j$.

Proposition 2.6 *For any diffeomorphism $f \in MS(M^n)$ there is a dynamical numbering of its periodic orbits.*

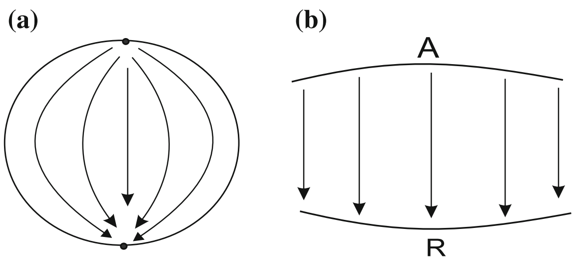


Fig. 2.9 A "source-sink" diffeomorphism (a) and its generalization (b).

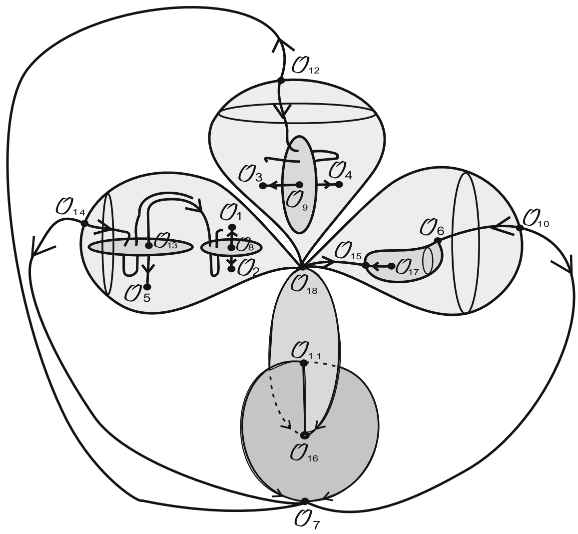


Fig. 2.10 A phase portrait of a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with the dynamically numbered set of the periodic orbits

Figure 2.10 shows the phase portrait of a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with a dynamical numbering of the periodic orbits for the case when the non-wandering set Ω_f consists of fixed points only.

Notice that there are numberings of the periodic orbits of a diffeomorphism $f \in MS(M^n)$ distinct from dynamical but which preserve the partial order relation $\prec$. Everywhere below we assume that the orbits of the diffeomorphism $f \in MS(M^n)$ are dynamically ordered. For each periodic orbit $\mathcal{O}_i$ denotes $m_i = m_{\mathcal{O}_i}$, $q_i = q_{\mathcal{O}_i}$, $\nu_i = \nu_{\mathcal{O}_i}$, $W^s_i = W^s_{\mathcal{O}_i}$, and $W^u_i = W^u_{\mathcal{O}_i}$.

For $i = 1, \dots, k_f - 1$ denote

$$A_i = \bigcup_{j=1}^{i} W^u_j,\ R_i = \bigcup_{j=i+1}^{k_f} W^s_j,\ V_i = M^n \setminus (A_i \cup R_i).$$

Exercise 2.3 Using the item (1) of Theorem 2.1 show that $V_i = \bigcup\limits_{j=1}^{i} W^s_j \setminus \bigcup\limits_{j=1}^{i} W^u_j = \bigcup\limits_{j=i+1}^{k_f} W^u_j \setminus \bigcup\limits_{j=i+1}^{k_f} W^s_j$.

Let $\hat{V}_i = V_i/f$ and let $p_{_i} : V_i \to \hat{V}_i$ denote the natural projection. We say the manifold V_i to be the *characteristic manifold* and we say its orbits space $\hat{V}_i$ to be the *characteristic space*. Notice that the characteristic space $\hat{V}_i$ generally is not connected. Denote by $\hat{V}_i^1, \dots, \hat{V}_i^{r_i}$ the connected components of the space $\hat{V}_i$.

Theorem 2.6 *Let $f \in MS(M^n)$. Then*

1. *the set A_i (R_i) is an attractor (repeller) of the diffeomorphism f and it has a trapping neighborhood $M_i \subset \bigcup\limits_{j=1}^{i} W_j^s$ ($M_i \subset \bigcup\limits_{j=i+1}^{k_f} W_j^u$) such that $M_i \setminus \operatorname{int} f(M_i)$ ($M_i \setminus \operatorname{int} f^{-1}(M_i)$) is the fundamental domain of the restriction of the diffeomorphism f to V_i;*
2. *the projection $p_{_i} : V_i \to \hat{V}_i$ is a covering map which induces a structure of the smooth closed n-manifold on the orbits space $\hat{V}_i$ and it induces the map η_i composed of the nontrivial homomorphisms $\eta_{_{\hat{V}_i^j}} : \pi_1(\hat{V}_i^j) \to \mathbb{Z}$, $j = 1, \dots, r_i$;*
3. *if $\dim\ A_i \leq (n-2)$ ($\dim\ R_i \leq (n-2)$) then the repeller R_i (the attractor A_i) is connected and if $\dim\ (A_i \cup R_i) \leq (n-2)$ then the manifolds V_i, $\hat{V}_i$ are connected and the map $\eta_{_i} : \pi_1(\hat{V}_i) \to \mathbb{Z}$ is an epimorphism.*

Thus, for the given numbering of the periodic orbits of the diffeomorphism $f \in MS(M^n)$ we have $k_f - 1$ distinct representations of the diffeomorphism f as a "source-sink" diffeomorphism.

Remark 2.1 The assertion that the set A_i (R_i) is an attractor (repeller) of the diffeomorphism f follows immediately from the existence of the filtration for Morse–Smale diffeomorphisms (see Theorem 1.8). Indeed, from the definition of the filtration $M^n = M_{k_f} \supset M_{k_f-1} \supset \dots \supset M_1 \supset M_0 = \emptyset$ (see Definition 1.28) it immediately follows that the set M_i is the trapping neighborhood for A_i (for the set R_i the same reasoning applies for the diffeomorphism f^{-1}).

For the sake of independence of the proof of Theorem 2.6 we construct the filtration for the diffeomorphism $f \in MS(M^n)$ without reference to Theorem 1.8. The technique of the construction is based on the existence of a local Morse–Lyapunov function stated in the section 2.2.2.

For $q = 0, \dots, n$ denote by k_q the number of all periodic orbits of Morse index less or equal to q. Notice that $k_n = k_f$. For $j = k_0 + 1, \dots, k_{n-1}$ let $\hat{W}^s_{j,i} = p_{_i}(W_j^s \cap V_i)$ and $\hat{W}^u_{j,i} = p_{_i}(W_j^u \cap V_i)$.

Exercise 2.4 Using Exercise 2.3, Proposition 2.5 and Theorems 2.1, 2.3, 2.6 prove that:

1) $\hat{W}^s_{j,i} = \emptyset$ for $j > i$ and $\hat{W}^u_{j,i} = \emptyset$ for $j \leq i$;

2) if $W_j^s \cap V_i \neq \emptyset$ then $\hat{W}^s_{j,i}$ is the smooth $(n - q_j)$-submanifold of the manifold $\hat{V}_i$ for which $\operatorname{cl}(\hat{W}^s_{j,i}) \subset \bigcup\limits_{r=j}^{i} \hat{W}^s_{r,i}$; if $(W_j^s \setminus \mathcal{O}_j) \subset V_i$ then $\hat{W}^s_{j,i}$ is homeomorphic to the orbits space $\hat{\mathscr{W}}^s_{q_j,\nu_j}$ of the canonical contraction;

3) if $W_j^u \cap V_i \neq \emptyset$ then $\hat{W}^u_{j,i}$ is the smooth q_j-submanifold of the manifold $\hat{V}_i$ for which $\operatorname{cl}(\hat{W}^u_{j,i}) \subset \bigcup\limits_{r=i+1}^{j} \hat{W}^u_{r,i}$; if $(W_j^u \setminus \mathcal{O}_j) \subset V_i$ then the manifold $\hat{W}^u_{j,i}$ is homeomorphic to the orbits space $\hat{\mathscr{W}}^u_{q_j,\nu_j}$ of the canonical expansion.

Characteristic spaces play an important role for the topological classification of Morse–Smale diffeomorphisms. Particularly, they are topological invariants in the following sense. Let two Morse–Smale diffeomorphisms $f, f' : M^n \to M^n$ be topologically conjugate by a homeomorphism h. Then the chosen numbering of the periodic orbits of the diffeomorphism f induces by $\mathscr{O}'_i = h(\mathscr{O}_i)$ the numbering of the periodic orbits of the diffeomorphism f'.

Exercise 2.5 Prove that the dynamical numbering of the periodic orbits of the diffeomorphism f' induced by the conjugating homeomorphism h is dynamical and that for every $i = 1, \dots, k_f - 1$ there is a homeomorphism $\hat{h}_i : \hat{V}_i \to \hat{V}'_i$ such that:

1) $\eta_i([c]) = \eta'_i([\hat{h}_i(c)])$ for every closed curve $c \in \hat{V}_i$;

2) $\hat{h}_i(\hat{W}^s_{j,i}) = \hat{W}'^s_{j,i}$ and $\hat{h}_i(\hat{W}^u_{j,i}) = \hat{W}'^u_{j,i}$ for every $j = k_0 + 1, \dots, k_{n-1}$.

2.2.1 "Source-Sink" Diffeomorphisms

Proof of Theorem 2.5

Now we prove that if a diffeomorphism $f \in MS(M^n),\ n > 1$ has no saddle points then

1) f is the "source-sink" diffeomorphism;

2) the space of the wandering orbits of the diffeomorphism f is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$;

3) all "source-sink" diffeomorphisms are topologically conjugate to one another for the fixed n and the manifold M^n is homeomorphic to the n-sphere $\mathbb{S}^n$.

Proof

1) We now prove that the set Ω_0 consists of exactly one sink. Then it would follow that the set Ω_n consists of exactly one source because the set Ω_n of f coincides with the set Ω_0 of f^{-1}.

By Corollary 2.1 the set Ω_0 is not empty. According to the hypothesis of the theorem the diffeomorphism f has no saddle points, therefore the item (1) of Theorem 2.1 implies $M^n = \bigcup\limits_{\omega \in \Omega_0} W^s_\omega \cup \Omega_n$. Then since the set Ω_n is 0-dimensional and $n \geq 2$ from the Dividing sets theorem (Statement 10.37) we get that the set $\bigcup\limits_{\omega \in \Omega_0} W^s_\omega$ is connected. Hence, the set Ω_0 consists of the single sink ω.

2) Since the wandering set of the diffeomorphism f coincides with the separatrix ℓ^u_α of the source α by Proposition 2.5 and Theorem 2.3 we get that the space of the wandering orbits of the diffeomorphism f is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

3) Define the diffeomorphism $g : \mathbb{S}^n \to \mathbb{S}^n$ by $g(x_1, \dots, x_{n+1}) = \left(\frac{4x_1}{5-3x_{n+1}}, \dots, \frac{4x_n}{5-3x_{n+1}}, \frac{5x_{n+1}-3}{5-3x_{n+1}}\right)$, where $x_1^2 + \dots + x_{n+1}^2 = 1$. It is directly checkable that $\vartheta_- g \vartheta_-^{-1} = a^u_{n,+1}$ and $\vartheta_+ g \vartheta_+^{-1} = a^s_{n,+1}$ are the stereographic projections (see formulas 10.3, 10.4), where $\vartheta_- : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n$ and $\vartheta_+ : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$. Then the non-wandering set of the diffeomorphism g consists of exactly two hyperbolic

periodic points: the source $(0, \dots, 0, 1)$ and the sink $(0, \dots, 0, -1)$. By Proposition 2.1 the diffeomorphisms f and g are topologically conjugate in the trapping domains in the basins of the sinks. Then the conjugating homeomorphism can be extended to the source points. Thus the diffeomorphisms f and g are topologically conjugate and M^n is consequently homeomorphic to the n-sphere $\mathbb{S}^n$. □

Corollary 2.2 *If a diffeomorphism $f \in MS(M^n)$ has at least one saddle point then for every sink point ω there is a saddle point σ such that $\omega \in \mathrm{cl}(W^u_\sigma)$.*

Proof Suppose the contrary, then for some sink point ω by the item (3) of Theorem 2.1 we get $\mathrm{cl}(W^s_\omega) = W^s_\omega \cup \bigcup\limits_{i=1}^{k} \alpha_i$, where $\alpha_i,\ i \in \{1, \dots, k\}$ is a sink such that $W^u_{\alpha_i} \cap W^s_\omega \neq \emptyset$. Now we show that $W^u_{\alpha_i} \subset \mathrm{cl}(W^s_\omega)$.

Assume the contrary. Then by the item (1) of Theorem 2.1 there is a point $p \in \Omega_f$ distinct from ω and such that $W^s_p \cap W^u_{\alpha_i} \neq \emptyset$. Let x_ω and x_p be the points of $W^u_{\alpha_i} \cap W^s_\omega$ and $W^u_{\alpha_i} \cap W^s_p$, respectively. Since the manifold $W^u_{\alpha_i} \setminus \alpha_i$ is homeomorphic to $\mathbb{R}^n \setminus O$ (see the item (2) of Theorem 2.1) there is a path $c : [0, 1] \to (W^u_{\alpha_i} \setminus \alpha_i)$ without self intersections which joins the point $x_\omega = c(0)$ with the point $x_p = c(1)$. Then there is a number $\tau \in (0, 1)$ such that $c(\tau) \notin W^s_\omega$ and $c(t) \in W^s_\omega$ for $t < \tau$. Hence, there is a point $r \in \Omega_f$ distinct from ω and such that $c(\tau) \in W^s_r$. Therefore, $c(\tau) \in \mathrm{cl}\, W^s_\omega$ and we get a contradiction.

It follows from the above that the set $\mathrm{cl}(W^s_\omega)$ is open because it contains each point with some open neighborhood. Since the set $\mathrm{cl}(W^s_\omega)$ is closed it coincides with the entire manifold M^n. Then Ω_f contains no saddle points and this contradicts the hypothesis. □

2.2.2 Morse–Lyapunov Functions

Proof of Proposition 2.6 Now we prove that for any diffeomorphism $f \in MS(M^n)$ there is a dynamical numbering of the periodic orbits.

Proof To prove that it is sufficient to show that from $\mathcal{O}_i \prec \mathcal{O}_j$ it follows that $q_{\mathcal{O}_i} \leq q_{\mathcal{O}_j}$. Indeed, the intersection $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j}$ is transversal and from $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset$ it follows that $\dim\ W^s_{\mathcal{O}_i} + \dim\ W^u_{\mathcal{O}_j} - n \geq 0$ (see Statement 10.56). Then $n - q_{\mathcal{O}_i} + q_{\mathcal{O}_j} - n \geq 0$ and hence $q_{\mathcal{O}_i} \leq q_{\mathcal{O}_j}$. □

Definition 2.8 Let $\mathcal{O}_i$ be a periodic orbit of a diffeomorphism $f \in MS(M^n)$ and let U_i be a neighborhood of the orbit $\mathcal{O}_i$. We say a Morse function $\psi_i : U_i \to \mathbb{R}$ to be a local *Morse–Lyapunov function* if it satisfies:

1) $\psi_i(f(x)) < \psi_i(x)$ for every $x \in (f^{-1}(U_i) \setminus \mathcal{O}_i)$ and $\psi_i(f(x)) = \psi_i(x) = 0$ for $x \in \mathcal{O}_i$;

2) the set of the critical points of the function ψ_i coincides with the orbit $\mathcal{O}_i$ and each critical point is of index q_i;

3) $(W^u_r \cap U_i) \subset Ox_1 \dots x_{q_i}$ and $(W^s_r \cap U_i) \subset Ox_{q_i+1} \dots x_n$ for Morse coordinates $x_1, \dots, x_n$ in some neighborhood of the point $r \in \mathcal{O}_i$.

Lemma 2.2 *For every periodic point $\mathcal{O}_i$ of a diffeomorphism $f \in MS(M^n)$ there is a local Morse–Lyapunov function.*

Proof Since $\mathcal{O}_i$ is a hyperbolic set, for each point $r \in \mathcal{O}_i$ the tangent space T_rM^n decomposes into the direct sum of the subspaces $T_rM^n = T_rW^u_r \oplus T_rW^s_r$ such that $D_rf(T_rW^u_r) = T_{f(r)}W^u_{f(r)}$ and $D_rf(T_rW^s_r) = T_{f(r)}W^s_{f(r)}$ (see formula 1.1). Moreover, there is a Lyapunov metric $\|\cdot\|$ on M^n and there is a constant $0 < \lambda < 1$ such that $\left\|Df^{-1}(v^u)\right\| \leq \lambda \|v^u\|$, $\|Df(v^s)\| \leq \lambda \|v^s\|$ for any $v^u \in E^u$ and $v^s \in E^s$, where $E^u = \bigcup\limits_{r\in\mathcal{O}_i} T_rW^u_r$ and $E^s = \bigcup\limits_{r\in\mathcal{O}_i} T_rW^s_r$.

Define the map $\psi : E^u \oplus E^s \to \mathbb{R}$ by $\psi(v^u, v^s) = -\|v^u\|^2 + \|v^s\|^2$. Now we show that $\psi(Df(v^u, v^s)) < \psi(v^u, v^s)$ for every nonzero $v^u \in E^u$ and every nonzero $v^s \in E^s$. Indeed, $\psi(Df(v^u, v^s)) - \psi(v^u, v^s) = -\|Df(v^u)\|^2 + \|Df(v^s)\|^2 + \|v^u\|^2 - \|v^s\|^2 \leq -\frac{1}{\lambda^2}\|v^u\|^2 + \lambda^2\|v^s\|^2 + \|v^u\|^2 - \|v^s\|^2 \leq -(\frac{1}{\lambda^2} - 1)\|v^u\|^2 - (1 - \lambda^2)\|v^s\|^2 < 0$ for every nonzero $v^u \in E^u$ and $v^s \in E^s$.

We identify a small neighborhood U_i of the orbit $\mathcal{O}_i$ to a neighborhood of the zero section $E^u \oplus E^s$ by the exponential map such that it sends the stable (unstable) manifold into E^u (E^s). Then for every $v = (v^u, v^s) \in U_i$ we have $f(v^u, v^s) = Df(v^u, v^s) + o(v)$. Hence, $\psi(f(v^u, v^s)) < \psi(v^u, v^s)$ for each nonzero $(v^u, v^s) \in U_i$ if the neighborhood U_i is small enough. Thus $\psi_i = \psi$ is the desired function. □

2.2.3 Attractors and Repellers

Proof of Theorem 2.6

We now prove that for every diffeomorphism $f \in MS(M^n)$:

1) the set A_i (R_i) is an attractor (repeller) of the diffeomorphism f and it has a trapping neighborhood $M_i \subset \bigcup\limits_{j=1}^{i} W^s_j$ ($M_i \subset \bigcup\limits_{j=i+1}^{k_f} W^u_j$) such that $M_i \setminus \operatorname{int} f(M_i)$ ($M_i \setminus \operatorname{int} f^{-1}(M_i)$) is the fundamental domain of the restriction of the diffeomorphism f to V_i;

2) the projection $p_i : V_i \to \hat{V}_i$ is a covering map which induces a structure of the smooth closed n-manifold on the orbits space $\hat{V}_i$ and it induces the map η_i composed of the nontrivial homomorphisms $\eta_{\hat{V}_i^j} : \pi_1(\hat{V}_i^j) \to \mathbb{Z}$, $j = 1, \ldots, r_i$;

3) if dim $A_i \leq (n-2)$ (dim $R_i \leq (n-2)$) then the repeller R_i (attractor A_i) is connected and if dim $(A_i \cup R_i) \leq (n-2)$ then the manifolds V_i, $\hat{V}_i$ are connected and the map $\eta_i : \pi_1(\hat{V}_i) \to \mathbb{Z}$ is an epimorphism.

Proof We prove the theorem for the case of the attractor because for the repeller the same arguments apply for the diffeomorphism f^{-1}. We prove the theorem by induction on $i = 1, \ldots, k_f - 1$.

Let $i = 1$. The definition of the dynamical order and Corollary 2.1 give us that $\mathcal{O}_1$ is a sink periodic orbit, hence $A_1 = W^u_1 = \mathcal{O}_1$. Now we prove the item (1) of the theorem.

1) By Lemma 2.2 there is a neighborhood $U_1 \subset W_1^s$ of the orbit $\mathcal{O}_1$ and there is a local Morse–Lyapunov function $\psi_1 : U_1 \to \mathbb{R}$ such that in the local coordinates $x_1, \dots, x_n$ in the neighborhood of the point $\omega \in \mathcal{O}_1$ it can be expressed by $\psi_1(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$. Then there is a number $\varepsilon_1 > 0$ such that the set $M_1 = \psi_1^{-1}([0, \varepsilon_1])$ is the union of m_1 n-balls for which $f(M_1) \subset \operatorname{int} M_1$ due to the item 1) of Definition 2.8. Since $A_1 \subset M_1$ and $f^k(A_1) = A_1$ for $k \in \mathbb{Z}$ we have $A_1 \subset \bigcap\limits_{k \geq 0} f^k(M_1)$. Let $\tilde{A}_1 = \bigcap\limits_{k \geq 0} f^k(M_1)$. We now show that $\tilde{A}_1 = A_1$. Assume the contrary, i.e., there is a point $x \in (\tilde{A}_1 \setminus A_1)$. From the item (1) of Theorem 2.1 it follows that there is a point $p \in (\Omega_f \setminus \mathcal{O}_1)$ such that $x \in W_p^u$. Since set $\tilde{A}_1$ is closed and invariant we have $\operatorname{cl}(\mathcal{O}_x) \subset \tilde{A}_1$ and hence $p \in \tilde{A}_1$, so we come to contradiction with $\tilde{A}_1 \subset W_1^s$. Thus, A_1 is an attractor and M_1 is its trapping neighborhood. We now show that the set $K_1 = M_1 \setminus \operatorname{int} f(M_1)$ is the fundamental domain of the restriction of the diffeomorphism f to V_1.

It suffices to show that $\bigcup\limits_{k \in \mathbb{Z}} f^k(K_1) = V_1$. Since $K_1 \subset V_1$ we have $\bigcup\limits_{k \in \mathbb{Z}} f^k(K_1) \subset V_1$. Suppose the reverse inclusion not to be true, i.e., there is a point $x \in V_1$ such that $x \notin \bigcup\limits_{k \in \mathbb{Z}} f^k(K_1)$. Since M_1 is a neighborhood of $A_1 = \mathcal{O}_1$ the set M_1 contains a fundamental domain of the restriction of the diffeomorphism f to W_1^s and hence $\bigcup\limits_{k \in \mathbb{Z}} f^k(M_1) \supset W_1^s$. Since $M_1 \subset W_1^s$ we have $\bigcup\limits_{k \in \mathbb{Z}} f^k(M_1) \subset W_1^s$. Then $\bigcup\limits_{k \in \mathbb{Z}} f^k(M_1) = W_1^s$ and, therefore, $\bigcup\limits_{k \in \mathbb{Z}} f^k(M_1) \setminus A_1 = V_1$ and $x \in f^{k_*}(M_1)$ for some $k_* \in \mathbb{Z}$. Since $x \notin A_1$ and $A_1 = \bigcap\limits_{k \geq 0} f^k(M_1)$ there is $k^* > k_*$ such that $x \in f^{k^*}(M_1)$ and $x \notin f^{k^*+1}(M_1)$. Hence $x \in f^{k^*}(K_1)$ and we get a contradiction.

2) By Exercise 2.3 $V_1 = W_1^s \setminus \mathcal{O}_1$. Then from Proposition 2.5 and Theorem 2.3 it follows that the orbits space $\hat{V}_1 = V_1/f$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the projection $p_{_1} : V_1 \to \hat{V}_1$ is the covering map which induces the structure of the smooth n-manifold on $\hat{V}_1$ and it induces the epimorphism $\eta_{_{\hat{V}_1}} : \pi_1(\hat{V}_1) \to m_1\mathbb{Z}$.

Induction step. Suppose we have constructed the smooth n-submanifold M_{i-1} which is a trapping neighborhood of A_{i-1} such that $M_{i-1} \setminus \operatorname{int} f(M_{i-1})$ is the fundamental domain of the restriction of the diffeomorphism f to V_{i-1} and suppose we have proved that the projection $p_{_{i-1}} : V_{i-1} \to \hat{V}_{i-1}$ is the covering map which induces a structure of the smooth closed n-manifold on the orbits space $\hat{V}_{i-1}$ and induces the nontrivial homomorphism $\eta_{_{\hat{v}_{i-1}}} : \pi_1(\hat{v}_{i-1}) \to \mathbb{Z}$ on each connected component $\hat{v}_{i-1}$ of the manifold $\hat{V}_{i-1}$. We now prove the theorem for i. Consider three cases: a) $q_i = 0$, b) $q_i = n$, c) $0 < q_i < n$.

In the case a), $M_i = M_{i-1} \cup \tilde{M}_i$, where $\tilde{M}_i$ is the neighborhood of the orbit $\mathcal{O}_i$ constructed similarly to the case $i = 1$.

In the case b), $M_i = M_{i-1} \cup \tilde{M}_i$, where $\tilde{M}_i = W_i^u$.

We now prove 1) and 2) for the case c).

1) Due to Statement 10.55 without loss of generality we assume that $S_{i-1} = \partial M_{i-1}$ intersect W_i^u transversally. Let $\hat{S}_{i-1} = p_{_{i-1}}(S_{i-1})$ and $\hat{W}^u_{i,i-1} = p_{_{i-1}}(W_i^u \cap V_{i-1})$. By the construction $(W_i^u \setminus \mathcal{O}_i) \subset V_{i-1}$ and by Theorem 2.3 $\hat{W}^u_{i,i-1}$ is a closed smooth

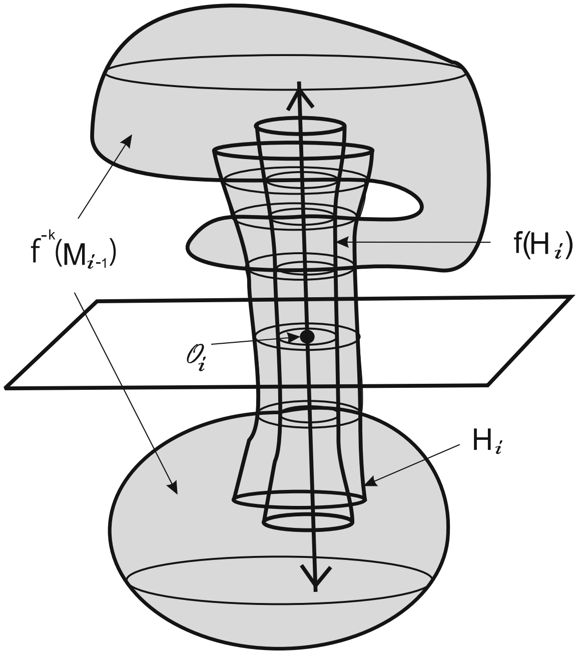


Fig. 2.11 The construction of the filtration

q_i-submanifold of the manifold $\hat{V}_{i-1}$. Then $\hat{S}_{i-1}$ and $\hat{W}^u_{i,i-1}$ are smooth compact submanifolds of the manifold $\hat{V}_{i-1}$. Since they intersect transversally the intersection consists of finite number n_i of connected components. From Section 10.3.6 and Exercise 2.2 it follows that there is a neighborhood $\hat{N}^u_i$ of the manifold $\hat{W}^u_{i,i-1}$ in $\hat{V}_{i-1}$ such that $\hat{N}^u_i \cap \hat{S}_{i-1}$ consists of n_i connected components. Let $N^u_i = p^{-1}_{i-1}(\hat{N}^u_i)$.

By Lemma 2.2 there is a neighborhood $U_i \subset (N^u_i \cup W^s_i)$ of the set $\mathcal{O}_i$ and there is a function $\psi_i : U_i \to \mathbb{R}$ which in the local coordinates $x_1, \dots, x_n$ in a neighborhood of the point $p \in \mathcal{O}_i$ can be expressed by $\psi_i(x_1, \dots, x_n) = -x_1^2 - \cdots - x_{q_i}^2 + x_{q_i+1}^2 + \cdots + x_n^2$. Then by the λ-lemma there is some $\varepsilon_i > 0$ and there is a natural number k such that each set $G_i = \psi_i^{-1}(\varepsilon_i)$ and $f(G_i)$ intersect $f^{-k}(S_{i-1})$ at n_i connected components.

Let $H_i = \psi_i^{-1}((-\infty, \varepsilon_i])$ and $M_i = f^{-k}(M_{i-1}) \cup H_i$ (see Figure 2.11). Then $f(H_i) \setminus \operatorname{int} f^{-k}(M_{i-1}) \subset \operatorname{int} M_i$. We now show that $f(M_i) \subset \operatorname{int} M_i$. Indeed it is true for any point $x \in f^{-k}(M_{i-1})$ because $f(M_{i-1}) \subset \operatorname{int} M_{i-1}$ by the inductive hypothesis. It is also true for every point $x \in (H_i \setminus f^{-k}(M_{i-1}))$ because $f(H_i) \setminus \operatorname{int} f^{-k}(M_{i-1}) \subset \operatorname{int} M_i$ due to the condition 1) of the definition of the Morse–Lyapunov function.

Similarly to the case $i = 1$ one shows that M_i is the desired trapping neighborhood after its corners have been smoothed.

2) We now show that the group F acts freely and discontinuously on V_i.

By construction all the non-wandering points of the diffeomorphism f belong to $A_i \cup R_i$. Therefore the manifold V_i consists of wandering points only and hence the group F acts freely on V_i. If K is a compact subset of the set V_i then due to Statement 10.18 it is bounded. Since the set $K_i = M_i \setminus \operatorname{int} f(M_i)$ is the fundamental domain of the action of the group F on V_i there is a number $N \in \mathbb{N}$ such that $K \subset \bigcup\limits_{|k|\leq N} f^k(K_i)$. Hence $f^k(K) \cap K = \emptyset$ for $|k| > 2N$ and therefore the action of the group F is discontinuous.

From Statement 10.30 it follows that the natural projection $p_{_i} : V_i \to \hat{V}_i$ is a covering map and it induces the structure of Hausdorff space to the orbits space $\hat{V}_i$, it induces the map η_i consisting of nontrivial homomorphisms $\eta_{_{\hat{V}_i^j}} : \pi_1(\hat{V}_i^j) \to \mathbb{Z}$, $j = 1, \dots, r_i$. Then the covering map $p_{_i}$ induces the structure of a smooth n-manifold to $\hat{V}_i$ which according to Statement 10.31 is homeomorphic to the manifold derived from K_i by gluing its boundaries by the diffeomorphism f. Then the manifold $\hat{V}_i$ is closed.

Thus, we have proved the items 1) and 2) of the theorem. Now we prove the item 3), that is if the dimension of the repeller R_i is less or equal to $(n-2)$ then the attractor A_i is connected (the proof that the repeller R_i is connected if the dimension of the attractor $A_i \leq (n-2)$ is analogous because R_i is the attractor for the diffeomorphism f^{-1}).

First we prove that the trapping neighborhood M_i is connected if $\dim R_i \leq (n-2)$. Assume the contrary, let the manifold M_i be disconnected. Then by Statement 10.22 it can be represented as the union of closed disjoint nonempty subsets E_1 and E_2. Without loss of generality assume that $f(E_j) \subset \operatorname{int} E_j$, $j = 1, 2$ (otherwise the same reasoning applies for the diffeomorphism f^2). Let $\tilde{E}_j = \bigcup\limits_{k\geq 0} f^{-k}(\operatorname{int} E_j)$. By the construction $\tilde{E}_1$, $\tilde{E}_2$ are nonempty open disjoint sets such that $\tilde{E}_1 \cup \tilde{E}_2 = M^n \setminus R_i$. Thus the set $M^n \setminus R_i$ is disconnected and that contradicts the Dividing sets theorem (Statement 10.37) because M^n is the connected manifold and R_i is of dimension $\leq (n-2)$.

Then A_i is connected because it is the intersection of the connected compact nested sets $M_i \supset f(M_i) \supset \cdots \supset f^k(M_i) \supset \dots$ (see Statement 10.1).

If the dimensions of both the attractor A_i and the repeller R_i are less or equal to $(n-2)$ then by the Dividing sets theorem (Statement 10.37) the set $A_i \cup R_i$ does not divide the manifold M^n and hence the manifold V_i is connected. The connectedness of the manifold $\hat{V}_i$ in this case follows from the continuity of the covering map $p_{_i}$ and Statement 10.23. The fact that the map $\eta_{_{\hat{V}_i}} : \pi_1(\hat{V}_i) \to \mathbb{Z}$ induced by the covering map $p_{_i} : V_i \to \hat{V}_i$ is the epimorphism follows immediately from Statement 10.33. $\square$

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