p-adic Measures for Hermitian Modular Forms and the Rankin–Selberg Method

Thanasis Bouganis

Abstract In this work we construct p-adic measures associated to an ordinary Hermitian modular form using the Rankin–Selberg method.

Keywords Hermitian modular forms · Special L-values · p-adic measures

1 Introduction

p-adic measures are known to play an important role in Iwasawa theory, since they constitute the analytic part of the various Main Conjectures. In this paper we are interested in p-adic measures attached to an ordinary Hermitian modular form $\mathbf{f}$. There has been work on the subject by Harris et al. [20, 21], where the first steps towards the construction of p-adic measures associated to ordinary Hermitian modular forms were made. Actually in their work they construct a p-adic Eisenstein measure (see also the works of Eischen [15, 16] on this), and provide a sketch of the construction of a p-adic measure associated to an ordinary Hermitian modular form. We also mention here our work [4], where we constructed p-adic measures associated to Hermitian modular forms of definite unitary groups of one and two variables. All these works impose the following assumption on the prime number p: if we denote by K the CM field associated to the Hermitian modular form $\mathbf{f}$ and let F be the maximal totally real subfield of K, then all the primes in F above p must be split in K. One of the main motivation of this work is to consider the case where p does not satisfy this condition.

Actually this work differs from the once mentioned above on the method used to obtain the p-adic measures. Indeed the previous works utilize the doubling method in order to construct the p-adic measures, where in this work we will use the

This work is dedicated with gratefulness and admiration to Professor John Coates in occasion of his 70th birthday.

T. Bouganis (✉)
Department of Mathematical Sciences, Durham University, Science Laboratories, South Rd., Durham DH1 3LE, UK
e-mail: athanasios.bouganis@durham.ac.uk

D. Loeffler and S.L. Zerbes (eds.), *Elliptic Curves, Modular Forms and Iwasawa Theory*, Springer Proceedings in Mathematics & Statistics 188, DOI 10.1007/978-3-319-45032-2_2

Rankin–Selberg method. In the Rankin–Selberg method one obtains an integral representation of the L-values as a Petersson inner product of $\mathbf{f}$ with a product of a theta series and a Siegel-type Eisenstein series, where in the doubling method the L-values can be represented as a Petersson inner product of $\mathbf{f}$ with another Hermitian form, which is obtained by pulling back a Siegel-type Eisenstein series of a larger unitary group. Of course one should remark right away that the use of the Rankin–Selberg method puts some serious restrictions on the unitary groups which may be considered. In particular, the archimedean components of the unitary group must be of the form $U(n, n)$, where the doubling method allows situations of the form $U(n, m)$ with $n \neq m$. However, we believe that it reasonable to expect, with the current stage of knowledge at least, to relax the splitting assumption only in the cases of $U(n, n)$. The reason being that in the cases of $U(n, m)$ with $n \neq m$, in order to obtain the special L-values, one needs to evaluate Siegel-type Eisenstein series on CM points, and in the p-adic setting, one needs that this CM points correspond to abelian varieties with complex multiplication, which are ordinary at p, and hence the need for the splitting assumption. For example, even in the "simplest" case of the definite $U(1) = U(1, 0)$, which is nothing else than the case of p-adic measures for Hecke characters of a CM field K considered by Katz in [24], even today, in this full generality, it is not known how to remove the assumption on the primes above p in F being split in K. We need to remark here that in some special cases (for example elliptic curves over $\mathbb{Q}$ with CM by imaginary quadratic fields), there are results which provide some p-adic distributions associated to Hecke characters of CM fields.

In this work we make some assumptions, which will simplify various technicalities, and we postpone to a later work [7] for a full account. In particular, we fix an odd prime p, and write $\mathfrak{P}_i$ for the prime ideals in F above p, which are inert in K. We write $\mathfrak{p}_i$ for the prime ideal of K above $\mathfrak{P}_i$, and denote by S the set of these primes. We will assume that $S \neq \emptyset$. Then our aim is to construct p-measures for the Galois group $\mathrm{Gal}(K(\prod_i \mathfrak{p}_i^{\infty})/K)$, where $K(\prod_i \mathfrak{p}_i^{\infty})$ denotes the maximal abelian extension of K unramified outside the prime ideals $\mathfrak{p}_i$. As we said already our techniques can also handle the situation of primes split in K, and this will be done in [7]. The other simplifying assumptions which we impose in this work, which will be lifted in [7], are

1. we assume that the class number of the CM field K is equal to the class number of the underlying unitary group with repsect to the standard congruence subgroup. This for example happens when the class number of F is taken equal to one,
2. we will investigate the interpolation properties of the p-adic measures only for the special values for which the corresponding Eisenstein series in the Rankin–Selberg method are holomorphic, and not just nearly-holomorphic.

We should also remark that this present work should be seen as the unitary analogue of the work of Panchishkin [27], and Courtieu and Panchishkin [12] in the Siegel modular form case. We should say here that the second assumption above can be lifted by developing the techniques of Courtieu and Panchishkin on the holomorphic projection in the unitary case. Actually the techniques of this present work grew out of the efforts of the author to extend the work of Courtieu and Panchishkin in the following directions, which is also one of the aims of [7],

1. to consider the situation of totally real fields (they consider the case of $\mathbb{Q}$),
2. to obtain the interpolation properties also for Hecke characters which are not totally ramified.
3. to construct the measures also for symplectic groups of odd genus. In their work they consider the case of even genus, and hence no half-integral theta, and Eisenstein series appear in the construction. We remark here that, over $\mathbb{Q}$, the work of Böcherer and Schmidt [2], provides the existence of these p-adic measures, in both odd and even genus. However their techniques seem to be hard to extend to the totally real field situation.

Indeed in this paper we work completely adelically, which allow us to work over any field. Moreover, we use a more precise form of the so-called Adrianov–Kalinin identity, shown by Shimura, which allows us to obtain a better understanding of the bad Euler factors above p. And finally, we work here the interpolation properties for characters that may be unramified at some of the primes of the set S. Note that only at these primes one sees the needed modification of the Euler factors above p at the interpolation properties.

Notation: Since our main references for this work are the two books of Shimura [29, 30] our notation is the one used by Shimura in his books.

2 Hermitian Modular Forms

In this section, which is similar to the corresponding section in [6], we introduce the notion of a Hermitian modular form, both classically and adelically. We follow closely the books of Shimura [29, 30], and we remark that we adopt the convention done in the second book with respect to the weight of Hermitian modular forms (see the discussion on p. 32, Sect. 5.4 in [30]).

Let K be an algebra equipped with an involution ρ. For a positive integer $n \in \mathbb{N}$ we define the matrix $\eta := \eta_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in GL_{2n}(K)$, and the group $G := U(n,n) := \{\alpha \in GL_{2n}(K) | \alpha^* \eta \alpha = \eta\}$, where $\alpha^* := {}^t\alpha^{\rho}$. Moreover we define $\hat{\alpha} := (\alpha^*)^{-1}$ and $S := S^n := \{s \in M_n(K) | s^* = s\}$ for the set of Hermitian matrices with entries in K. If we take $K = \mathbb{C}$ and let ρ to denote the complex conjugation then the group $G(\mathbb{R}) = \{\alpha \in GL_{2n}(\mathbb{C}) | \alpha^* \eta \alpha = \eta\}$ acts on the symmetric space (Hermitian upper half space)

$$\mathbb{H}_n := \{z \in M_n(\mathbb{C}) | i(z^* - z) > 0\},$$

by linear fractional transformations,

$$\alpha \cdot z := (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1} \in \mathbb{H}_n, \quad \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G(\mathbb{R}), \quad z \in \mathbb{H}_n,$$

where the $a_\alpha, b_\alpha, c_\alpha, d_\alpha$ are taken in $M_n(\mathbb{C})$.

Let now K be a CM field of degree $2d := [K : \mathbb{Q}]$ and we write F for its maximal totally real subfield. Moreover we write $\mathfrak{r}$ for the ring of integers of K, $\mathfrak{g}$ for that of F, D_F and D_K for their discriminants and $\mathfrak{d}$ for the different ideal of F. We write $\mathbf{a}$ for the set of archimedean places of F. We now pick a CM type $(K, \{\tau_v\}_{v\in\mathbf{a}})$ of K, where $\tau_v \in Hom(K, \mathbb{C})$. For an element $a \in K$ we set $a_v := \tau_v(a) \in \mathbb{C}$. We will also regard $\mathbf{a}$ as the archimedean places of K corresponding to the embeddings τ_v of the selected CM type. Finally we let $\mathbf{b}$ be the set of all complex embeddings of K, and we note that $\mathbf{b} = \{\tau_v, \tau_v\rho | v \in \mathbf{a}\}$, where ρ denotes complex conjugation acting on the CM field K. By abusing the notation we may also write $\mathbf{b} = \mathbf{a} \coprod \mathbf{a}\rho$.

We write $G_{\mathbb{A}}$ for the adelic group of G, and $G_{\mathbf{h}} = \prod_v' G_v$ (restricted product) for its finite part, and $G_{\mathbf{a}} = \prod_{v\in\mathbf{a}} G_v$ for its archimedean part. Note that we understand G as an algebraic group over F, and hence the finite places v above are finite places of F, which will be denoted by $\mathbf{h}$. For a description of G_v at a finite place we refer to [29, Chap. 2]. Given two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{g}$, we define following Shimura the subgroup of $G_{\mathbb{A}}$,

$$D[\mathfrak{a}, \mathfrak{b}] := \left\{ \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_{\mathbb{A}} | a_x \prec \mathfrak{g}_v, b_x \prec \mathfrak{a}_v, c_x \prec \mathfrak{b}_v, d_x \prec \mathfrak{g}_v, \quad \forall v \in \mathbf{h} \right\},$$

where we use the notation $\prec$ in [30, p. 11], where $x \prec \mathfrak{b}_v$ means that the v-component of the matrix x has are all its entries in $\mathfrak{b}_v$. Again we take a_x, b_x, c_x, d_x to be n by n matrices. For a finite adele $q \in G_{\mathbf{h}}$ we define $\Gamma^q = \Gamma^q(\mathfrak{b}, \mathfrak{c}) := G \cap q D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]q^{-1}$, a congruence subgroup of G. Given a finite order Hecke character ψ of K of conductor dividing $\mathfrak{c}$ we define a character on $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ by $\psi(x) = \prod_{v|\mathfrak{c}} \psi_v(det(a_x)_v)^{-1}$, where ψ_v denotes the local component of ψ at the finite place v, and a character ψ_q on Γ^q by $\psi_q(\gamma) = \psi(q^{-1}\gamma q)$.

We write $\mathbb{Z}^{\mathbf{a}} := \prod_{v\in\mathbf{a}} \mathbb{Z}$, $\mathbb{Z}^{\mathbf{b}} := \prod_{v\in\mathbf{b}} \mathbb{Z}$ and $\mathcal{H} := \prod_{v\in\mathbf{a}} \mathbb{H}_n$. We embed $\mathbb{Z} \hookrightarrow \mathbb{Z}^{\mathbf{a}}$ diagonally and for an $m \in \mathbb{Z}$ we write $m\mathbf{a} \in \mathbb{Z}^{\mathbf{a}}$ for its image. We will simply write $\mathbf{a}$ for $1\mathbf{a}$. We define an action of $G_{\mathbb{A}}$ on $\mathcal{H}$ by $g \cdot z := g_{\mathbf{a}} \cdot z := (g_v \cdot z_v)_{v\in\mathbf{a}}$, with $g \in G_{\mathbb{A}}$ and $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$. For a function $f : \mathcal{H} \to \mathbb{C}$ and an element $k \in \mathbb{Z}^{\mathbf{b}}$ we define

$$(f|_k\alpha)(z) := j_\alpha(z)^{-k} f(\alpha \cdot z), \ \ \alpha \in G_{\mathbb{A}}, \ \ z \in \mathcal{H},$$

where,

$$j_\alpha(z)^{-k} := \prod_{v\in\mathbf{a}} det(c_{\alpha_v} z_v + d_{\alpha_v})^{-k_v} det(c_{\alpha_v}^{\rho}\, {}^t z_v + d_{\alpha_v}^{\rho})^{-k_{v\rho}}, \quad z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}.$$

For fixed $\mathfrak{b}$ and $\mathfrak{c}$ as above, and $q \in G_{\mathbf{h}}$ and a Hecke character ψ of K, we define,

Definition 2.1 [30, p. 31] A function $f : \mathcal{H} \to \mathbb{C}$ is called a Hermitian modular form for the congruence subgroup Γ^q of weight $k \in \mathbb{Z}^{\mathbf{b}}$ and nebentype ψ_q if:

1. f is holomorphic,
2. $f|_k\gamma = \psi_q(\gamma) f$ for all $\gamma \in \Gamma^q$,
3. f is holomorphic at cusps (see [30, p. 31] for this notion).

The space of Hermitian modular forms of weight k for the congruences group Γ^q and nebentype ψ_q will be denoted by $\mathcal{M}_k(\Gamma^q, \psi_q)$. For any $\gamma \in G$ we have a Fourier expansion of the form (see [30, p. 33])

$$(f|_k\gamma)(z) = \sum_{s\in\mathfrak{S}} c(s,\gamma; f)e_{\mathbf{a}}(sz), \;\; c(s,\gamma; f) \in \mathbb{C}, \tag{1}$$

where $\mathfrak{S}$ a lattice in $S_+ := \{s \in S| \; s_v \geq 0, \;\; \forall v \in \mathbf{a}\}$, and

$$e_{\mathbf{a}}(x) := exp(2\pi i \sum_v tr(x_v)).$$

An f is called a cusp form if $c(s,\gamma; f) = 0$ for any $\gamma \in G$ and s with $det(s) = 0$. The space of cusp forms we will be denoted by $\mathcal{S}_k(\Gamma^q, \psi_q)$. When we do not wish to determine the nebentype we will be writing $f \in \mathcal{M}_k(\Gamma^q)$, and this should be understood that there exists some ψ_q as above such that $f \in \mathcal{M}_k(\Gamma^q, \psi_q)$.

We now turn to the adelic Hermitian modular forms. If we write D for a group of the form $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$, and ψ a Hecke character of finite order then we define,

Definition 2.2 [30, p. 166] A function $\mathbf{f} : G_{\mathbb{A}} \to \mathbb{C}$ is called an adelic Hermitian modular form if

1. $\mathbf{f}(\alpha x w) = \psi(w) j_w^k(\mathbf{i})\mathbf{f}(x)$ for $\alpha \in G$, $w \in D$ with $w_{\mathbf{a}}(\mathbf{i}) = \mathbf{i}$,
2. For every $p \in G_{\mathbf{h}}$ there exists $f_p \in \mathcal{M}_k(\Gamma^p, \psi_p)$, where $\Gamma^p := G \cap pCp^{-1}$ such that $\mathbf{f}(py) = (f_p|_k y)(\mathbf{i})$ for every $y \in G_{\mathbf{a}}$.

Here we write $\mathbf{i} := (i1_n, \ldots, i1_n) \in \mathcal{H}$. We denote this space by $\mathcal{M}_k(D, \psi)$, and the space of cusp forms by $\mathcal{S}_k(D, \psi)$. As in the classical case above, we will write just $\mathcal{M}_k(D)$ if we do not wish to determine the nebentype. A simple computation shows, if $\mathbf{f} \in \mathcal{M}_k(D, \psi)$ then the form $\mathbf{f}^*(x) := \mathbf{f}(x\eta_{\mathbf{h}}^{-1})$ belongs to $\mathcal{M}_k(D', \psi^{-c})$ where $D' := D[\mathfrak{bc}, \mathfrak{b}^{-1}]$ and $\psi^{-c}(x) := \psi(x^\rho)^{-1}$.

By [29, Chap. 2] there exists a finite set $\mathcal{B} \subset G_{\mathbf{h}}$ such that $G_{\mathbb{A}} = \coprod_{b\in\mathcal{B}} GbD$ and an isomorphism $\mathcal{M}_k(D, \psi) \cong \oplus_{b\in\mathcal{B}}\mathcal{M}_k(\Gamma^b, \psi_b)$ (see [29, Chap. 2]). We note here that for the congruence subgroups $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$ the cardinality of the set $\mathcal{B}$ does not depend on the ideal $\mathfrak{c}$ and its elements can be selected to be of the form $\begin{pmatrix} \hat{q} & 0 \\ 0 & q \end{pmatrix}$ with $q \in GL_n(K)_{\mathbf{h}}$, and $q_v = 1$ for $v|\mathfrak{c}$, (see for example [6, Lemma 2.6]). For a $q \in GL_n(K)_{\mathbb{A}}$ and an $s \in S_{\mathbb{A}}$ we have

$$\mathbf{f}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \sum_{\tau\in S_+} c_{\mathbf{f}}(\tau, q)e_{\mathbb{A}}(\tau s).$$

For the properties of $c_{\mathbf{f}}(\tau, q)$ we refer to the [30, Proposition 20.2] and for the definition of $e_{\mathbb{A}}$ to [30, p. 127]. We also note that sometimes we may write $c(\tau, q; \mathbf{f})$ for $c_{\mathbf{f}}(\tau, q)$.

For a subfield L of $\mathbb{C}$ we will be writing $\mathcal{M}_k(\Gamma^q, \psi, L)$ for the Hermitian modular forms in $\mathcal{M}_k(\Gamma^q, \psi)$ whose Fourier expansion at infinity, that is γ is the identity in Eq. 1, has coefficients in L. For a fixed set $\mathcal{B}$ as above we will be writing $\mathcal{M}_k(D, \psi, L)$ for the subspace of $\mathcal{M}_k(D, \psi)$ consisting of elements whose image under the above isomorphism lies in $\oplus_{b\in\mathcal{B}}\mathcal{M}_k(\Gamma^b, \psi_b, L)$. Finally we define the adelic cusp forms $\mathcal{S}_k(D, \psi)$ to be the subspace of $\mathcal{M}_k(D, \psi)$, which maps to $\oplus_{b\in\mathcal{B}}\mathcal{S}_k(\Gamma^b, \psi_b)$. As above, when we do not wish to determine the nebentype we simply write $\mathcal{M}_k(\Gamma^q, L)$ and $\mathcal{M}_k(D, L)$.

We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and write F^{cl} for the Galois closure of F over $\mathbb{Q}$. Then by [30, Chap. II, Sect. 10] we have a well-defined action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F^{cl})$ on $\mathcal{M}_k(\Gamma^q, \overline{\mathbb{Q}})$ given by an action on the Fourier-coefficients of the expansion at infinity. This action will be denoted by f^σ for an $f \in \mathcal{M}_k(\Gamma^q, \overline{\mathbb{Q}})$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F^{cl})$. A similar action can be defined on the space $\mathcal{M}_k(D, \overline{\mathbb{Q}})$ (see [30, p. 193, Lemma 23.14]), and will be also denoted by $\mathbf{f}^\sigma$ for an $\mathbf{f} \in \mathcal{M}_k(D, \overline{\mathbb{Q}})$. In both cases (classical and adelic) the action of the absolute Galois group preserves the space of cusp forms.

We close this section with a final remark concerning Hecke characters. Given an (adelic) Hecke character χ of K (or F), we will be abusing the notation and write χ also for the corresponding ideal character.

3 Eisenstein and Theta Series

3.1 Eisenstein Series

In this section we collect some facts concerning Siegel-type Eisenstein series. We closely follow [30, Chap. IV].

We consider a $k \in \mathbb{Z}^{\mathbf{b}}$, an integral ideal $\mathfrak{c}$ in F and a unitary Hecke character χ of K with infinity component of the form $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$, where $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$ and of conductor dividing $\mathfrak{c}$. For a fractional ideal $\mathfrak{b}$ we write C for $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Then for a pair $(x, s) \in G_{\mathbb{A}} \times \mathbb{C}$, we denote by $E_{\mathbb{A}}(x, s)$ or $E_{\mathbb{A}}(x, s; \chi, \mathfrak{c})$ the Siegel type Eisenstein series associated to the character χ and the weight k. We recall here its definition, taken from [30, p. 131],

$$E_{\mathbb{A}}(x, s) = \sum_{\gamma\in P\backslash G} \mu(\gamma x)\epsilon(\gamma x)^{-s}, \quad \Re(s) >> 0,$$

where P is the standard Siegel parabolic subgroup and the function $\mu : G_{\mathbb{A}} \to \mathbb{C}$ is supported on $P_{\mathbb{A}}C \subset G_{\mathbb{A}}$, defined by,

$$\mu(x) = \chi_{\mathbf{h}}(det(d_p))^{-1}\chi_{\mathfrak{c}}(det(d_w))^{-1} j_x(\mathbf{i})^{-k}|j_x(\mathbf{i})|^m,$$

where $x = pw$ with $p \in P_{\mathbb{A}}$ and $w \in C$, and $m = (k_v + k_{v\rho})_v$. Here we define $|j_x(\mathbf{i})|^m := \prod_{v \in \mathbf{a}} |j_{x_v}(i 1_n)|^{m_v}$. The function $\epsilon : G_{\mathbb{A}} \to \mathbb{C}$ is defined as $\epsilon(x) = |det(d_p d_p^*)|_{\mathbb{A}}$ where $x = pw$ with $p \in P_{\mathbb{A}}$ and $w \in D[\mathfrak{b}^{-1}, \mathfrak{b}]$. Here for an adele $x \in F_{\mathbb{A}}^{\times}$ we write $|x|_{\mathbb{A}}$ for the adele norm normalized as in [29, 30]. Moreover we define the normalized Eisenstein series

$$D_{\mathbb{A}}(x, s) = E_{\mathbb{A}}(x, s) \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s - i, \chi_1 \theta^i),$$

where θ is the non-trivial character associated to K/F and χ_1 is the restriction of the Hecke character χ to $F_{\mathbb{A}}^{\times}$. We note that since we consider unitary characters the infinity part of such a character is of the form $(\chi_1)_{\mathbf{a}}(x) = \prod_{v \in \mathbf{a}} \left(\frac{x_v}{|x_v|}\right)^{\ell_v}$, and it will be often denoted by $sgn(x_{\mathbf{a}})^{\ell}$. Moreover for a Hecke character ϕ of F, we write $L_{\mathfrak{c}}(s, \phi)$ for the Dirichlet series associated to ϕ with the Euler factors at the primes dividing $\mathfrak{c}$ removed.

For a $q \in GL_n(K)_{\mathbf{h}}$ we define $D_q(z, s; k, \chi, \mathfrak{c})$, a function on $(z, s) \in \mathcal{H} \times \mathbb{C}$, associated to $D_{\mathbb{A}}(x, s)$ by the rule (see [30, p. 146]),

$$D_q(x \cdot \mathbf{i}, s; k, \chi, \mathfrak{c}) = j_x^k(\mathbf{i}) D_{\mathbb{A}}(diag[q, \hat{q}]x, s).$$

We now introduce yet another Eisenstein series for which we have explicit information about their Fourier expansion. In particular we define the $E_{\mathbb{A}}^*(x, s) := E_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, s)$ and $D_{\mathbb{A}}^*(x, s) := D_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, s)$, and as before we write $D_q^*(z, s; k, \chi, \mathfrak{c})$ for the series associated to $D_{\mathbb{A}}^*(x, s)$. We now write the Fourier expansion of $E_{\mathbb{A}}^*(x, s)$ as,

$$E_{\mathbb{A}}^* \left(\begin{pmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{pmatrix}, s \right) = \sum_{h \in S} c(h, q, s) \mathbf{e}_{\mathbb{A}}(h\sigma), \tag{2}$$

where $q \in GL_n(K)_{\mathbb{A}}$ and $\sigma \in S_{\mathbb{A}}$. We now state a result of Shimura on the coefficients $c(h, q, s)$. We first define an $\mathfrak{r}$-lattice in $S := S^n$, by

$$T := T^n := \{x \in S | tr(xy) \subset \mathfrak{g}, \quad \forall y \in S(\mathfrak{r})\},$$

where $S(\mathfrak{r}) := S \cap M_n(\mathfrak{r})$. T is usually called the dual lattice to $S(\mathfrak{r})$. For a finite place v of F we write T_v for $T \otimes_{\mathfrak{r}} \mathfrak{r}_v$.

Proposition 3.1 (Shimura, Proposition 18.14 and Proposition 19.2 in [29]). *Suppose that $\mathfrak{c} \neq \mathfrak{g}$. Then $c(h, q, s) \neq 0$ only if $({}^t qhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v T_v^n$ for every $v \in \mathbf{h}$. In this case*

$$c(h, q, s) = C(S) \chi(det(-q))^{-1} |det(qq^*)|_{\mathbf{h}}^{n-s} |det(qq^*)|_{\mathbf{a}}^{s} N(\mathfrak{b}\mathfrak{c})^{-n^2} \times$$

$$\alpha_{\mathfrak{c}}(\omega \cdot {}^t qhq, 2s, \chi_1) \prod_{v \in \mathbf{a}} \xi(q_v q_v^*, h_v; s + (k_v + k_{v\rho})/2, s - (k_v + k_{v\rho})/2)),$$

where $N(\cdot)$ *denotes the norm from* F *to* $\mathbb{Q}$, $|x|_{\mathbf{h}} := \prod_{v \in \mathbf{h}} |x_v|_v$ *with* $|\cdot|_v$ *the normalized absolute value at the finite place* v, ω *is a finite idele such that* $\omega \mathfrak{r} = \mathfrak{b}\mathfrak{d}$, *and*

$$C(S) := 2^{n(n-1)d} |D_F|^{-n/2} |D_K|^{-n(n-1)/4}.$$

For the function $\xi(g_v, h_v, s, s')$ *with* $0 < g_v \in S_v$, $h_v \in S_v$, $s, s' \in \mathbb{C}$, $v \in \mathbf{a}$ *we refer to [30, p. 134].*

Moreover if we write r *for the rank of* h *and let* $g \in GL_n(F)$ *such that* $g^{-1}hg = diag[h', 0]$ *with* $h' \in S^r$. *Then*

$$\alpha_{\mathfrak{c}}(\omega \cdot {}^t q h q, 2s, \chi_1) = \Lambda_{\mathfrak{c}}(s)^{-1} \Lambda_h(s) \prod_{v \in \mathbf{c}} f_{h,q,v}\left(\chi(\pi_v)|\pi_v|^{2s}\right),$$

where

$$\Lambda_{\mathfrak{c}}(s) = \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s - i, \chi_1 \theta^i), \quad \Lambda_h(s) = \prod_{i=0}^{n-r+1} L_{\mathfrak{c}}(2s - n - i, \chi_1 \theta^{n+i-1}).$$

Here $f_{h,q,v}$ *are polynomials with constant term 1 and coefficients in* $\mathbb{Z}$*; they are independent of* χ. *The set* $\mathbf{c}$ *is determined as follows:* $\mathbf{c} = \emptyset$ *if* $r = 0$. *If* $r > 0$, *then take* $g_v \in GL_n(\mathfrak{r}_v)$ *for each* $v \nmid \mathfrak{c}$ *so that* $(\omega q^* h q)_v = g_v^* diag[\xi_v, 0] g_v$ *with* $\xi_v \in T_v^r$. *Then* $\mathbf{c}$ *consists of all the* v *prime to* $\mathfrak{c}$ *of the following two types: (i)* v *is ramified in* K *and (ii)* v *is unramified in* K *and* $det(\xi_v) \notin \mathfrak{g}_v^{\times}$.

For a number field W, a $k \in \mathbb{Z}^{\mathbf{b}}$ and $r \in \mathbb{Z}^{\mathbf{a}}$ we follow [30] and write $\mathcal{N}_k^r(W)$ for the space of W-rational nearly holomorphic modular forms of weight k (see [30, p. 103 and p. 110] for the definition). Regarding the near holomorphicity of the Eisenstein series $D_q(z, s; \chi, \mathfrak{c})$ we have the following theorem of Shimura,

Theorem 3.2 (Shimura, Theorem 17.12 in [30]) *We set* $m := (k_v + k_{v\rho})_{v \in \mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$. *Let* K' *be the reflex field of* K *with respect to the selected CM type and* K_χ *the field generated over* K' *by the values of* χ. *Let* Φ *be the Galois closure of* K *over* $\mathbb{Q}$ *and* $\mu \in \mathbb{Z}$ *with* $2n - m_v \leqslant \mu \leqslant m_v$ *and* $m_v - \mu \in 2\mathbb{Z}$ *for every* $v \in \mathbf{a}$. *Then* $D_q(z, \mu/2; k, \chi, \mathfrak{c})$ *belongs to* $\pi^\beta \mathcal{N}_k^r(\Phi K_\chi \mathbb{Q}_{ab})$, *except when* $0 \leqslant \mu < n$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi_1 = \theta^\mu$, *where* $\beta = (n/2) \sum_{v \in \mathbf{a}} (m_v + \mu) - dn(n-1)/2$. *Moreover* $r = n(m - \mu + 2)/2$ *if* $\mu = n + 1$, $F = \mathbb{Q}$ *and* $\chi_1 = \theta^{n+1}$. *In all other cases we have* $r = (n/2)(m - |\mu - n|\mathbf{a} - n\mathbf{a})$.

We now work out the positivity of the Fourier expansion of some holomoprhic Eisenstein series. In particular we assume that $m = \mu\mathbf{a}$ and we consider the series $D^*_{\mathbb{A}}(x, s)$ for $s = \frac{\mu}{2}$ and for $s = n - \frac{\mu}{2}$. For an $h \in S$, and $c(h, q, s)$ as in Eq. 2, we define $c(h, s) := \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s - i, \chi_1 \theta^i) c(h, q, s)$, that is the hth Fourier coefficient of $D^*_{\mathbb{A}}(x, s)$. Then we have the following,

Proposition 3.3 (Shimura, Proposition 17.6 in [30]) *Exclude the case where* $\mu = n + 1$, $F = \mathbb{Q}$ *and* $\chi = \theta^{n+1}$. *Then we have that* $c(h, \frac{\mu}{2}) \neq 0$ *only in the following situations*

1. *$h = 0$, and $\mu = n$,*
2. *$h \neq 0$, $\mu > n$ and $h_v > 0$ for all $v \in \mathbf{a}$,*
3. *$h \neq 0$, $\mu = n$ and $h_v \geqslant 0$ for all $v \in \mathbf{a}$.*

Proof This follows directly from [30, Proposition 17.6], where the positivity of $c(h, q, \frac{\mu}{2})$ is considered, after observing that $\Lambda_c(\mu/2) = \prod_{i=0}^{n-1} L_{\mathfrak{c}}(\mu - i, \chi_1\theta^i) \neq 0$ for $\mu > n$. For $\mu = n$ we need to observe that $L(s, \chi_1\theta^{n-1})$ does not have a pole at $s = 1$, since $\chi_1\theta^{n-1}$ is not the trivial character, since $(\chi_1)_{\mathbf{a}}(x) = sgn(x_{\mathbf{a}})^{n\mathbf{a}}$, and hence $(\chi_1\theta^{n-1})_{\mathbf{a}}(x) = sgn(x_{\mathbf{a}})$. hence not trivial. □

The other holomorphic Eisenstein series, i.e. $s = n - \frac{\mu}{2}$, has a completely different behaviour. Namely, independently of μ, it may have non-trivial Fourier coefficients even for $h \geqslant 0$ not of full rank, that is with $det(h) = 0$. Let us explain this. By Proposition 3.1 we observe that $c(h, s)$ is equal to a finite non-vanishing factor times

$$f(s)\Lambda_h(s)\prod_{v\in\mathbf{a}}\xi\left(y_v, h_v; s+\mu/2, s-\mu/2\right), \quad y_v := q_v q_v^*,$$

where $f(s) := \prod_{v\in\mathbf{c}} f_{h,q,v}\left(\chi(\pi_v)|\pi_v|^{2s}\right)$, and for the function ξ we have (see [30, p. 140]) that

$$\xi(y_v, h_v; a, b) = i^{nb-na}2^{\tau}\pi^{\epsilon}\frac{\Gamma_t(a+b-n)}{\Gamma_{n-q}(a)\Gamma_{n-p}(b)}det(y_v)^{n-a-b}\times$$

$$\delta_+(h_v y_v)^{a-n+q/2}\delta_-(h_v y_v)^{b-n+p/2}\omega(2\pi y_v, h_v; a, b),$$

where p (resp. q) is the number of positive (resp. negative) eigenvalues of h_v and $t = n - p - q$; $\delta_+(x)$ is the product of all positive eigenvalues of x and $\delta_-(x) = \delta_+(-x)$, and

$$\Gamma_n(s) := \pi^{n(n-1)/2}\prod_{\nu=0}^{n-1}\Gamma(s-\nu).$$

For the quantities τ, ϵ and the function $\omega(\cdot)$ we refer to [30, p. 140], since they do not play any role in the argument below. We are interested in the values

$$f(n-\mu/2)\Lambda_h(n-\mu/2)\prod_{v\in\mathbf{a}}\xi\left(y_v, h_v; n, n-\mu\right),$$

with $\mu \geqslant n$.

Let us write r for the rank of h, then $\Lambda_h(s) = \prod_{i=0}^{n-1-r} L_{\mathfrak{c}}(2s - n - i, \chi_1\theta^{n+i-1})$ and hence $\Lambda_h(n - \mu/2) = \prod_{i=0}^{n-1-r} L_{\mathfrak{c}}(n - \mu - i, \chi_1\theta^{n+i-1})$. We now note that $(\chi_1)_{\mathbf{a}}(x) = sgn(x_{\mathbf{a}})^{\mu\mathbf{a}}$ and hence after setting $\psi_i := \chi_1\theta^{n+i-1}$ we obtain $(\psi_i)_{\mathbf{a}}(x) = sgn(x_{\mathbf{a}})^{(\mu+n+i-1)\mathbf{a}}$. We now conclude that the quantity $\Lambda_h(n - \mu/2)$ may not be zero since by [30, Lemma 17.5] we have that $L(n - \mu - i, \psi_i) = 0$ if $n - \mu - i \equiv \mu + n + i - 1 \mod 2$ (the so-called trivial zeros), which never holds. For the gamma factors we have for $h = 0$,

$$\prod_{v\in\mathbf{a}}\frac{\Gamma_n(n-\mu)}{\Gamma_n(n)\Gamma_n(n-\mu)}=\prod_{v\in\mathbf{a}}\frac{1}{\Gamma_n(n)}\neq 0.$$

Suppose that $h\neq 0$ and let $r=rank(h)$. Then

$$\prod_{v\in\mathbf{a}}\frac{\Gamma_{n-r}(n-\mu)}{\Gamma_n(n)\Gamma_{n-r}(n-\mu)}=\prod_{v\in\mathbf{a}}\frac{1}{\Gamma_n(n)}\neq 0.$$

In particular we conclude that in the case of $s=n-\frac{\mu}{2}$ we may have non-trivial Fourier coefficients even if the matrix h is not positive definite.

3.2 Theta Series

We start by recalling some results of Shimura in (the appendices of) [29, 30] regarding Hermitian theta series. We set $V := M_n(K)$ and we let $\mathcal{S}(V_{\mathbf{h}})$ to denote the space of Schwartz–Bruhat functions on $V_{\mathbf{h}} := \prod'_{v\in\mathbf{h}} V_v$. We consider an element $\lambda\in\mathcal{S}(V_{\mathbf{h}})$ and an $\mu\in\mathbb{Z}^{\mathbf{b}}$ such that $\mu_v\mu_{v\rho}=0$ for all $v\in\mathbf{a}$ and $\mu_v\geqslant 0$ for all $v\in\mathbf{b}$. For a $\tau\in S_+\cap GL_n(K)$ we then consider the theta series defined in [30, p. 277]),

$$\theta(z,\lambda):=\sum_{\xi\in V}\lambda(\xi)det(\xi)^{\mu\rho}\mathbf{e}^n_{\mathbf{a}}(\xi^*\tau\xi),\quad z\in\mathcal{H},$$

where $det(\xi)^{\mu\rho} := \left(\prod_{v\in\mathbf{b}} det(\xi_v)^{\mu_v}\right)^{\rho}$. We fix a Hecke character ϕ of K with infinity type $\phi_{\mathbf{a}}(y)=y^{-\mathbf{a}}|y|^{\mathbf{a}}$ and such that $\phi_1=\theta$, where we recall that we write θ for the non-trivial character of K/F. Such a character ϕ always exists, [30, Lemma A5.1], but may not be unique. We now let ω be a Hecke character of K and we write $\mathfrak{f}$ for its conductor and define $\mathfrak{h}=\mathfrak{f}\cap\mathfrak{g}$. Following Shimura we introduce the notation,

$$R^*=\{w\in M_n(K)_{\mathbb{A}}|w_v\prec\mathfrak{r}_v,\forall v\in\mathbf{h}\},$$

and we fix an element $r\in GL_n(K)_{\mathbf{h}}$. Then we define the function $\lambda\in\mathcal{S}(V_{\mathbf{h}})$ by

$$\lambda(x):=\omega(det(r)^{-1})\prod_{v|\mathfrak{h}}\omega_v(det(r_vx_v^{-1})),$$

if $r^{-1}x\in R^*$ and $r_v^{-1}x_v\in GL_n(\mathfrak{r}_v)$ for all $v|\mathfrak{h}$, and we set $\lambda(x)=0$ otherwise.
As it is explained in Shimura [30, Theorem A5.4] there is an action of $G_{\mathbb{A}}$ on $\mathcal{S}(V_{\mathbf{h}})$, which will be denoted by ${}^x\ell$ for $x\in G_{\mathbb{A}}$ and $\ell\in\mathcal{S}(V_{\mathbf{h}})$. Then we define the adelic theta function $\theta_{\mathbb{A}}$ on $G_{\mathbb{A}}$ by

$$\theta_{\mathbb{A}}(x,\omega):=\theta_{\mathbb{A}}(x,\lambda):=j_x^l(\mathbf{i})\theta(x\cdot\mathbf{i},{}^x\lambda),\quad x\in G_{\mathbb{A}},$$

where $l = \mu + n\mathbf{a} \in \mathbb{Z}^{\mathbf{b}}$. Then Shimura shows that

$$\theta_{\mathbb{A}}(\alpha x w, \lambda) = j_w^l(\mathbf{i})^{-1}\theta_{\mathbb{A}}(x, {}^w\lambda), \quad \alpha \in G, w \in G_{\mathbb{A}}, \text{and } w \cdot \mathbf{i} = \mathbf{i}. \tag{3}$$

and,

Theorem 3.4 (Shimura, Sect. A5.5 in [30] and Proposition A7.16 in [29]) $\theta_{\mathbb{A}}(x, \omega)$ *is an element in* $M_l(C, \omega')$ *with* $C = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$ *and* $\omega' = \omega\phi^{-n}$, *and* $l = \mu + n\mathbf{a}$. *Moreover* $\theta_{\mathbb{A}}(x, \omega)$ *is a cusp form if* $\mu \neq 0$. *The ideals* $\mathfrak{b}$ *and* $\mathfrak{c}$ *are given as follows. We define a fractional ideals* $\mathfrak{y}$ *and* $\mathfrak{t}$ *in* F *such that* $g^*\tau g \in \mathfrak{y}$ *and* $h^*\tau^{-1}h \in \mathfrak{t}^{-1}$ *for all* $g \in r\mathfrak{g}^n$ *and* $h \in \mathfrak{r}^n$. *Then we can take*

$$(\mathfrak{b}, \mathfrak{bc}) = (\mathfrak{d}\mathfrak{y}, \mathfrak{d}(\mathfrak{tef}^{\rho}\mathfrak{f} \cap \mathfrak{ye} \cap \mathfrak{yf})),$$

where $\mathfrak{e}$ *is the relative discriminant of* K *over* F. *For an element* $q \in GL_n(K)_{\mathbf{h}}$ *we have that the qth component of the theta series is given by*

$$\theta_{q,\omega}(z) = \omega'(det(q)^{-1})|det(q)|_K^{n/2} \times$$

$$\sum_{\xi \in V \cap rR^*q^{-1}} \omega_{\mathbf{a}}(det(\xi))\omega(det(r^{-1}\xi q)\mathfrak{r})det(\xi)^{\mu\rho}\mathbf{e}_{\mathbf{a}}(\xi^*\tau\xi z).$$

where $\xi \in V \cap rR^*q^{-1}$ *such that* $\xi^*\tau\xi = \sigma$.

For our later applications we now work out the functional equation with respect to the action of the element $\eta = \eta_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. In particular we are interested in the theta series $\theta^*_{\mathbb{A}}(x, \omega) := \theta_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, \omega)$. We note that by Eq. 3 we have that

$$\theta^*_{\mathbb{A}}(x, \omega) = \theta_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, \lambda) =$$

$$\theta_{\mathbb{A}}((-1)_{\mathbf{h}} x\eta_{\mathbf{h}}, \lambda) = \theta_{\mathbb{A}}((-1)_{\mathbf{h}} x, {}^{\eta}\lambda) = \omega'_{\mathfrak{c}}(-1)\theta_{\mathbb{A}}(x, {}^{\eta}\lambda),$$

and by [30, Theorem A5.4 (6)] we have that

$${}^{\eta}\lambda(x) = i^p|N_{F/\mathbb{Q}}(det(2\tau^{-1}))|^n \int_{V_{\mathbf{h}}} \lambda(y)\mathbf{e}_{\mathbf{h}}(-2^{-1}Tr_{K/F}(tr(y^*\tau x)))dy,$$

where $p = n^2[F : \mathbb{Q}]$ and dy is the Haar measure on $V_{\mathbf{h}}$ such that the volume of $M_n(\mathfrak{r})_{\mathbf{h}}$ is $|D_K|^{-n^2/2}$. We now compute the integral

$$I(x) := \int_{V_{\mathbf{h}}} \lambda(y)\mathbf{e}_{\mathbf{h}}(-2^{-1}Tr_{K/F}(tr(y^*\tau x)))dy.$$

We have

$$I(x)=\omega(det(r))^{-1}\left(\prod_{v\nmid\mathfrak{h}}\int_{r_vM_n(\mathfrak{r}_v)}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*\tau_vx_v)))d_vy\right)\times$$

$$\left(\prod_{v|\mathfrak{h}}\int_{r_vGL_n(\mathfrak{r}_v)}\omega(det(r_v^{-1}y_v))^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*\tau_vx_v)))d_vy\right).$$

We compute the local integrals separately. For a prime $v\nmid\mathfrak{h}$ we have

$$\int_{r_vM_n(\mathfrak{r}_v)}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*\tau_vx_v)))d_vy=$$

$$|det(r)|_v\int_{M_n(\mathfrak{r}_v)}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*r_v^*\tau_vx_v)))d_vy=$$

$$|det(r)|_v\int_{M_n(\mathfrak{r}_v)}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(x_v^*\tau_v^*r_vy_v)))d_vy=$$

$$\begin{cases}0, & \text{if } x_v^*\tau_v^*r_v\notin T;\\ |det(r)|_v|D_{K_v}|_v^{n^2/2}, & \text{otherwise.}\end{cases},$$

where $T:=\{x\in M_n(K_v)|tr(xy)\in\mathfrak{d}_v^{-1},\ \forall y\in M_n(\mathfrak{r}_v))\}$ and D_{K_v} is the discriminant of K_v. For the other finite places, we obtain generalized Gauss sums. We have

$$\int_{r_vGL_n(\mathfrak{r}_v)}\omega(det(r_v^{-1}y_v))^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*\tau_vx_v)))d_vy=$$

$$|det(r)|_v\int_{GL_n(\mathfrak{r}_v)}\omega(det(y_v))^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v^*r_v^*\tau_vx_v)))d_vy=$$

$$|det(r)|_v\int_{GL_n(\mathfrak{r}_v)}\omega(det(y_v))^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(x_v^*\tau_v^*r_vy_v)))d_vy.$$

By a standard argument (see for example [22, pp. 259–260]), this integral is zero, if $x_v^*\tau_v^*r_v\mathfrak{r}_v\neq(\mathfrak{f}\mathfrak{d}_K)^{-1}T_v^{\times}$, where $T_v^{\times}:=T_v\cap GL_n(\mathfrak{r}_v)$. If $x_v^*\tau_v^*r_v\mathfrak{r}_v=(\mathfrak{f}\mathfrak{d}_K)^{-1}T_v^{\times}$, then after the change of variable $y_v\mapsto(x_v^*\tau_v^*r_v)^{-1}y_v$we have that the integral is equal to

$$|det(r)|_v|det(x_v^*\tau_v^*r_v)|^{-1}\times$$

$$\int_{\mathfrak{f}^{-1}\mathfrak{d}_K^{-1}GL_n(\mathfrak{r}_v)}\omega(det((x_v^*\tau_v^*r_v)^{-1}y_v))^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v)))d_vy=$$

$$|det(x_v^*\tau_v^*)|^{-1}\omega(det(\tau_v^* r_v x_v^*))\times$$

$$\int_{\mathfrak{f}^{-1}\mathfrak{d}_K^{-1}GL_n(\mathfrak{r}_v)} \omega(y_v)^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v)))d_v y$$

We then have that $|det(x_v^*\tau_v^*)|^{-1} = |det(r_v)|_v N(\mathfrak{f}\mathfrak{d})^{-n}$ and hence we can rewrite the above expression as

$$|det(r_v)|_v N(\mathfrak{f}\mathfrak{d})^{-n}\omega(det((fd)^n\tau_v^* r_v x_v^*))\omega(fd)^{-n}\times$$

$$\int_{\mathfrak{f}^{-1}\mathfrak{d}_K^{-1}GL_n(\mathfrak{r}_v)} \omega(y_v)^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v)))d_v y$$

for some elements f, d such that $(f) = \mathfrak{f}_v$ and $(d) = \mathfrak{d}_v$. By a standard argument (see for example [22, p. 259]), we obtain

$$N(\mathfrak{d})^{-n}\omega(fd)^{-n}\int_{\mathfrak{f}^{-1}\mathfrak{d}_K^{-1}GL_n(\mathfrak{r}_v)} \omega(y_v)^{-1}\mathbf{e}_v(-2^{-1}Tr_{K_v/F_v}(tr(y_v)))d_v y =$$

$$\sum_{y\in(M_n(\mathfrak{r}_v)/M_n(\mathfrak{d}\mathfrak{f}_v)} \omega_v(det(y))^{-1}\mathbf{e}_v(-tr(y)).$$

We set $\tau_n(\omega^{-1}) := \sum_{y\in(M_n(\mathfrak{r}_v)/M_n(\mathfrak{d}\mathfrak{f}_v)} \omega_v(det(y))^{-1}\mathbf{e}_v(-tr(y))$, and we note that in the case that ω is primitive we have that the last integral can be related to one-dimensional standard Gauss sums (see for example [2, p. 1410]). In particular in such a case we have $\tau_n(\omega^{-1}) = N(\mathfrak{d})^{\frac{n(n-1)}{2}}\tau(\omega^{-1})^n$ where $\tau(\omega^{-1})$ the standard one dimensional Gauss sum, associated to the character ω^{-1}. We summarize the above calculations in the following Proposition.

Proposition 3.5 *Let ω be a primitive character of conductor $\mathfrak{f}$. For the theta series $\theta_{\mathbb{A}}^*(x,\omega) \in \mathcal{M}_l(C', \omega'^{-c})$ with $C' := D[\mathfrak{b}\mathfrak{c}, \mathfrak{b}^{-1}]$ we have*

$$\theta_{\mathbb{A}}^*(x,\omega) = i^{n^2[F:\mathbb{Q}]}|N(2det(\tau)^{-1})|^n\omega'_{\mathfrak{c}}(-1)|det(r)|_{\mathbf{h}}N(\mathfrak{f})^{-n}N(\mathfrak{d})^{(\frac{-n}{2})}\times$$

$$\prod_{v|\mathfrak{f}} N(\mathfrak{d}_v)^{\frac{n^2}{2}}\tau(\omega^{-1})^n\theta_{\mathbb{A}}(x,\lambda^*),$$

where $\lambda^*(x) = \omega_{\mathfrak{f}}((fd)^n det(\tau_v r_v x_v^*))$ for $x \in T$ and $x_v^*\tau_v^* r_v \in \mathfrak{f}\mathfrak{d}^{-1}T_v^\times$ for all $v|\mathfrak{f}$, and zero otherwise.

We close this section by making a remark on the support of the q-expansion of θ^*. We first set,

$$C(\omega) := i^{n^2[F:\mathbb{Q}]}|N(2det(\tau)^{-1})|^n\omega'_{\mathfrak{c}}(-1)|det(r)|_{\mathbf{h}}N(\mathfrak{f})^{-n}N(\mathfrak{d})^{\frac{-n}{2}}\prod_{v|\mathfrak{f}}N(\mathfrak{d}_v)^{\frac{n^2}{2}}. \tag{4}$$

and take some $q \in GL_n(K)_{\mathbf{h}}$. Then, the qth component of θ^* is given by

$$\theta^*_q(z) = i^{n^2d}|det(q)|_{\mathbf{h}}^{n/2}\phi(det(q))^n\sum_{\xi\in V}I(\xi q)det(\xi)^{\mu\rho}e_{\mathbf{a}}(\xi^*\tau\xi z)$$

If $det(\xi) \neq 0$, then $I(\xi q) \neq 0$ only when $(\tau^* r q^* \xi^*)_v \in (\mathfrak{f}\mathfrak{d})^{-1}T_v^{\times}$ for all $v|\mathfrak{f}\mathfrak{d}$. That is,

$$\theta^*_q(z) = C(\omega)\tau(\omega^{-1})^n \sum_{\xi\in\widehat{\mathfrak{f}\mathfrak{d}}R^{\times}_{\mathfrak{f}\mathfrak{d}}\tau^{-1}\hat{r}q^{-1}\cap V}\omega_{\mathfrak{f}}((fd)^n\tau^* r q^*\xi^*)det(\xi)^{\mu\rho}e_{\mathbf{a}}(\xi^*\tau\xi z).$$

In particular we have that $(\xi^*\tau\xi)_v \in (\mathfrak{f}\mathfrak{d})^{-1}\hat{q}r^{-1}\hat{\tau}T_v^{\times}\tau T_v^{\times}\tau^{-1}\hat{r}q^{-1}\widehat{\mathfrak{f}\mathfrak{d}}$ for all $v|\mathfrak{f}$.

4 The L-function Attached to a Hermitian Modular Form

4.1 The Standard L-function

We fix a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of F. We set $C = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. For the fixed group C and for an integral ideal $\mathfrak{a}$ of K we write $T(\mathfrak{a})$ for the Hecke operator associated to it as it is defined for example in [30, p. 162].

We consider a non-zero adelic Hermitian modular form $\mathbf{f} \in \mathcal{M}_k(C, \psi)$ and assume that we have $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a}$. If χ denotes a Hecke character of K of conductor $\mathfrak{f}$, for $s \in \mathbb{C}$ with $\Re(s) >> 0$ we consider the Dirichlet series

$$Z(s, \mathbf{f}, \chi) := \left(\prod_{i=1}^{2n} L_{\mathfrak{c}}(2s - i + 1, \chi_1\theta^{i-1})\right) \times \sum_{\mathfrak{a}}\lambda(\mathfrak{a})\chi(\mathfrak{a})N(\mathfrak{a})^{-s}, \tag{5}$$

where the sum runs over all integral ideals of K. It is shown in [30, p. 171] that this series has an Euler product representation, which we write as $Z(s, \mathbf{f}, \chi) = \prod_{\mathfrak{q}} Z_{\mathfrak{q}}\left(\chi(\mathfrak{q})N(\mathfrak{q})^{-s}\right)$, where the product is over all prime ideals of K. Here we remind the reader (see introduction) that we abuse the notation and write χ also for the ideal character associated to the Hecke character χ. For the description of the Euler factors $Z_{\mathfrak{q}}$ at the prime ideal $\mathfrak{q}$ of K we have (see [30, p. 171]),

1. $Z_{\mathfrak{q}}(X) = \prod_{i=1}^n\left((1 - N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1 - N(\mathfrak{q})^n t_{\mathfrak{q},i}^{-1}X)\right)^{-1}$, if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q} \nmid \mathfrak{c}$,
2. $$Z_{\mathfrak{q}_1}(X_1)Z_{\mathfrak{q}_2}(X_2) =$$

$$\prod_{i=1}^{2n}\left((1-N(\mathfrak{q}_1)^{2n}t^{-1}_{\mathfrak{q}_1\mathfrak{q}_2,i}X_1)(1-N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_2)\right)^{-1},$$

if $\mathfrak{q}_1\neq\mathfrak{q}_2$, $\mathfrak{q}_1^\rho=\mathfrak{q}_2$ and $\mathfrak{q}_i\nmid\mathfrak{c}$ for $i=1,2$,

3. $Z_\mathfrak{q}(X)=\prod_{i=1}^n\left((1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)\right)^{-1}$, if $\mathfrak{q}^\rho=\mathfrak{q}$ and $\mathfrak{q}|\mathfrak{c}$,
4.

$$Z_{\mathfrak{q}_1}(X_1)Z_{\mathfrak{q}_2}(X_2)=$$

$$\prod_{i=1}^{2n}\left((1-N(\mathfrak{q}_1)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_1)(1-N(\mathfrak{q}_2)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,n+i}X_2)\right)^{-1},$$

if $\mathfrak{q}_1\neq\mathfrak{q}_2$, $\mathfrak{q}_1^\rho=\mathfrak{q}_2$ and $\mathfrak{q}_i|\mathfrak{c}$ for $i=1,2$,

where the $t_{?,i}$ above for $?=\mathfrak{q},\mathfrak{q}_1\mathfrak{q}_2$ are the Satake parameters associated to the eigenform $\mathbf{f}$. We also introduce the L-function,

$$L(s,\mathbf{f},\chi):=\prod_{\mathfrak{q}}Z_\mathfrak{q}\left(\chi(\mathfrak{q})(\psi/\psi_\mathfrak{c})(\pi_\mathfrak{q})N(\mathfrak{q})^{-s}\right),\quad \Re(s)>>0 \tag{6}$$

where $\pi_\mathfrak{q}$ a uniformizer of $K_\mathfrak{q}$. We note here that we may obtain the Dirichelt series in Eq. 5 from the one in Eq. 6, up to a finite number of Euler factors, by setting $\chi\psi^{-1}$ for χ. Moreover if ψ is trivial then the two series coincide.

4.2 The Rankin–Selberg Integral Representation

We recall that in Sect. 3.2 we have fixed a Hecke character ϕ of K of infinity part $\phi_\mathbf{a}(y)=y_\mathbf{a}^{-\mathbf{a}}|y_\mathbf{a}|^\mathbf{a}$ and the restriction of ϕ to $F_\mathbb{A}^\times$ is the non-trivial Hecke character θ corresponding to the extension K/F. Keeping the notations from above we define $t\in\mathbb{Z}^\mathbf{a}$ to be the infinity type of χ, that is $\chi_\mathbf{a}(x)=x_\mathbf{a}^{-t}|x_\mathbf{a}|^t$. We then define $\mu\in\mathbb{Z}^\mathbf{b}$ by

$$\mu_v:=t_v-k_{v\rho}+k_v,\ \text{ and }\ \mu_{v\rho}:=0\ \text{ if }\ t_v\geqslant k_{v\rho}-k_v,$$

and

$$\mu_v:=0,\ \text{ and }\ \mu_{v\rho}:=k_{v\rho}-k_v-t_v\ \text{ if }\ t_v<k_{v\rho}-k_v.$$

We moreover set $l:=\mu+n\mathbf{a}$, $\psi':=\chi^{-1}\phi^{-n}$ and $h:=1/2(k_v+k_{v\rho}+l_v+l_{v\rho})_{v\in\mathbf{a}}$. Given μ,ϕ,τ and χ as above we write $\theta_\chi(x):=\theta_\mathbb{A}(x,\lambda)\in\mathcal{M}_l(C',\psi')$ for the theta series that we can associate to $(\mu,\phi,\tau,\chi^{-1})$ by taking $\omega:=\chi^{-1}$ in Theorem 3.4. We write $\mathfrak{c}'$ for the integral ideal defined by $C'=D[\mathfrak{b}'^{-1},\mathfrak{b}'\mathfrak{c}']$.

We now fix a decomposition $GL_n(K)_\mathbb{A}=\coprod_{q\in Q}GL_n(K)qEGL_n(K)_\mathbf{a}$, where $E=\prod_{v\in\mathbf{h}}GL_n(\mathfrak{r}_v)$. In particular the size of the set Q is nothing else than the class number of K. Given an element $f \in \mathcal{S}_k(\Gamma^q, \psi_q)$, and a function g on $\mathcal{H}$ such that $g|_k\gamma = \psi_q(\gamma)f$ for all $\gamma \in \Gamma^q$ we define the Petersson inner product

$$< f, g > := < f, g >_{\Gamma^q} := \int_{\Gamma^q \backslash \mathcal{H}} f(z)\overline{g(z)}\delta(z)^m dz,$$

where $\delta(z) := det(\frac{i}{2}(z^* - z))$ and dz a measure on $\Gamma^q \setminus \mathcal{H}$ defined as in [30, Lemma 3.4] and $m = (m_v)_{v\in\mathbf{a}}$ with $m_v = k_v + k_{v\rho}$.

The following theorem (see also [25, Theorem 7.8]) is obtained by combining results of Shimura [30] and Klosin [25]. For details we refer to [6, Sect. 4].

Theorem 4.1 (Shimura, Klosin) *Let* $0 \neq \mathbf{f} \in \mathcal{M}_k(C, \psi))$ *such that* $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *for every* $\mathfrak{a}$*, and assume that* $k_v + k_{v\rho} \geqslant n$ *for some* $v \in \mathbf{a}$*, then there exists* $\tau \in S_+ \cap GL_n(K)$ *and* $r \in GL_n(K)_{\mathbf{h}}$ *such that*

$$\Gamma((s))\psi_{\mathfrak{c}}(det(r))c_{\mathbf{f}}(\tau, r)L(s + 3n/2, \mathbf{f}, \chi) =$$

$$\Lambda_{\mathfrak{c}}(s + 3n/2, \theta(\psi\chi)_1) \cdot \left(\prod_{v\in\mathbf{b}} g_v(\chi(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-2s-3n})\right) det(\tau)^{s\mathbf{a}+h}|det(r)|_K^{-s-n/2} \times$$

$$C_0 \sum_{q\in Q} |det(qq^*)|_F^{-n} < f_q(z), \theta_{q,\chi}(z)E_q(z, \bar{s} + n; k - l, (\psi'/\psi)^c, \mathfrak{c}'') >_{\Gamma^q(\mathfrak{c}'')},$$

where

$$\Gamma((s)) := \prod_{v\in\mathbf{a}} (4\pi)^{-n(s+h_v)}\Gamma_n(s + h_v), \quad and \quad C_0 := \frac{[\Gamma_0(\mathfrak{c}'') : \Gamma]A}{\sharp X}.$$

where $\mathfrak{c}''$ *any non-trivial integral ideal of* F *such that* $\mathfrak{c}\mathfrak{c}'|\mathfrak{c}''$*,* $\Gamma^q(\mathfrak{c}'') := G \cap qD[\mathfrak{e}, \mathfrak{e}\mathfrak{h}]q^{-1}$*, with* $\mathfrak{e} = \mathfrak{b} + \mathfrak{b}'$ *and* $\mathfrak{h} = \mathfrak{e}^{-1}(\mathfrak{b}\mathfrak{c}'' \cap \mathfrak{b}'\mathfrak{c}'')$*. Moreover* $g_v(\cdot)$ *are Siegel-series related to the polynomials* $f_{\tau,r,v}(x)$ *mentioned in Proposition 3.1 above, and we refer to [30, Theorem 20.4] for the precise definition. Finally* X *denotes the set of Hecke characters of infinity type* t *and conductor dividing* $\mathfrak{f}_\chi$*,* Γ *is a congruence subgroup of* $SU(n, n)$ *which appears in the [30, p. 179], and* A *some fixed rational number times some powers of* π*, and is independent of* χ*.*

We will make the following assumption (see also the introduction):

Assumption. We assume that the class number of K is equal to the class number of $U(n, n)/F$ with respect to the full congruence subgroup $D[\mathfrak{b}^{-1}, \mathfrak{b}]$. For example this holds when the class number of F is taken equal to one [29, p. 66].
From the above assumption it follows that

$$\sum_{q\in Q} |det(qq^*)|_F^{-n} < f_q(z), \theta_{q,\chi}(z)E_q(z, \bar{s} + n; k - l, (\psi'/\psi)^c, \mathfrak{c}'') >_{\Gamma^q(\mathfrak{c}'')} =$$

$$< \mathbf{f}(x), \theta_{\mathbb{A},\chi}(x)\widetilde{E}_{\mathbb{A}}(x, \bar{s}+n; k-l, (\psi'/\psi)^c, \mathfrak{c}'' >_{\mathfrak{c}''},$$

where $\widetilde{E}_{\mathbb{A}}(x, \bar{s}+n; k-l, (\psi'/\psi)^c, \mathfrak{c}'')$ is the adelic Eisenstein series with q-component $|det(qq^*)|_F^{-n} E_q(z, \bar{s}+n; k-l, (\psi'/\psi)^c, \mathfrak{c}'')$, and $< \cdot, \cdot >_{\mathfrak{c}''}$ is the adelic Petersson inner product associated to the group $D[\mathfrak{e}, \mathfrak{e}\mathfrak{h}]$ as defined for example in [29, Eq. 10.9.6], but not normalized, and hence depends on the level. Moreover we define,

$$\widetilde{D}_{\mathbb{A}}(x, \bar{s}+n; k-l, \Psi, \mathfrak{c}'') := \overline{\Lambda_{\mathfrak{c}}(s+3n/2, \theta(\psi\chi)_1)}\widetilde{E}_{\mathbb{A}}(x, \bar{s}+n; k-l, \Psi, \mathfrak{c}''), \tag{7}$$

where $\Psi := (\psi'/\psi)^c$.

5 Algebraicity of Special L-Values

In this section we present some algebraicity results on the special values of the L-function introduced above, which were obtained in [6]. Results of this kind have been obtained by Shimura [30], but over the algebraic closure of $\mathbb{Q}$, and in [6] we worked out the precise field of definition, as well as, the reciprocity properties. There is also work by Harris [18, 19] and we refer to [6] for a discussion of how the results there compare with the ones presented here.

We consider a cuspidal Hecke eigenform $0 \neq \mathbf{f} \in \mathcal{S}_k(C, \psi; \overline{\mathbb{Q}})$ with $C := D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ for some fractional ideal $\mathfrak{b}$ and integral ideal $\mathfrak{c}$ of F. We start by introducing some periods associated to $\mathbf{f}$. These periods are the analogue in the unitary case of periods introduced by Sturm in [31], and generalized in [3, 5], in the symplectic case (i.e. Siegel modular forms). In the following theorem we write $< \cdot, \cdot >$ for the adelic inner product associated to the group C.

Theorem 5.1 *Let $\mathbf{f} \in \mathcal{S}_k(D, \psi, \overline{\mathbb{Q}})$ be an eigenform, and define $m_v := k_v + k_{\rho v}$ for all $v \in \mathbf{a}$. Let Φ be the Galois closure of K over $\mathbb{Q}$ and write W for the extension of Φ generated by the Fourier coefficients of $\mathbf{f}$ and their complex conjugation. Assume $m_0 := min_v(m_v) > 3n+2$. Then there exists a period $\Omega_{\mathbf{f}} \in \mathbb{C}^{\times}$ and a finite extension Ψ of Φ such that for any $\mathbf{g} \in \mathcal{S}_k(\overline{\mathbb{Q}})$ we have*

$$\left(\frac{< \mathbf{f}, \mathbf{g} >}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \frac{< \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'} >}{\Omega_{\mathbf{f}^{\sigma}}},$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\Psi)$, with $\sigma' := \rho\sigma\rho$. Here $\Omega_{\mathbf{f}^{\sigma}}$ is the period attached to the eigenform $\mathbf{f}^{\sigma}$. Moreover $\Omega_{\mathbf{f}}$ depends only on the eigenvalues of $\mathbf{f}$ and we have $\frac{<\mathbf{f},\mathbf{f}>}{\Omega_{\mathbf{f}}} \in (W\Psi)^{\times}$. In particular we have $\frac{<\mathbf{f},\mathbf{g}>}{<\mathbf{f},\mathbf{f}>} \in (W\Psi)(\mathbf{g}, \mathbf{g}^{\rho})$, where $(W\Psi)(\mathbf{g}, \mathbf{g}^{\rho})$ denotes the extension of $W\Psi$ obtained by adjoining the values of the Fourier coefficients of $\mathbf{g}$ and $\mathbf{g}^{\rho}$.

We note that the extension Ψ does not depend on $\mathbf{f}$, but only on K and n. We refer to [6] for more details on this. The following two theorems were obtained in [6].

Theorem 5.2 *Let $\mathbf{f} \in \mathcal{S}_k(C, \psi; \overline{\mathbb{Q}})$ be an eigenform for all Hecke operators, and assume that $m_0 \geqslant 3n + 2$. Let χ be a character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ with $t \in \mathbb{Z}^{\mathbf{a}}$, and define $\mu \in \mathbb{Z}^{\mathbf{b}}$ by $\mu_v := -t_v - k_{v\rho} + k_v$ and $\mu_{v\rho} = 0$ if $k_{v\rho} - k_v + t_v \leqslant 0$, and $\mu_v = 0$ and $\mu_{v\rho} = k_{v\rho} - k_v + t_v$, if $k_{v\rho} - k_v + t_v > 0$. Assume moreover that either*

1. *there exists $v, v' \in \mathbf{a}$ such that $m_v \neq m_{v'}$, or*
2. *$m_v = m_0$ for all v and $m_0 > 4n - 2$, or*
3. *$\mu \neq 0$.*

Then let $\sigma_0 \in \frac{1}{2}\mathbb{Z}$ such that

$$4n - m_v + |k_v - k_{v\rho} - t_v| \leqslant 2\sigma_0 \leqslant m_v - |k_v - k_{v\rho} - t_v|,$$

and,

$$2\sigma_0 - t_v \in 2\mathbb{Z}, \quad \forall v \in \mathbf{a}.$$

We exclude the following cases: For $n \leqslant 2\sigma_0 < 2n$, if we write $\mathfrak{f}'$ for the conductor of the character χ_1, then there is no choice of the integral ideal $\mathfrak{c}''$ as in Theorem 4.1 such that for any prime ideal $\mathfrak{q}$ of F, $\mathfrak{q}|\mathfrak{c}''\mathfrak{c}^{-1}$ implies either $\mathfrak{q}|\mathfrak{f}'$ or $\mathfrak{q}$ ramifies in K.

We let W be a number field such that $\mathbf{f}, \mathbf{f}^\rho \in \mathcal{S}_k(W)$ and $\Psi\Phi \subset W$, where Φ is the Galois closure of K in $\overline{\mathbb{Q}}$, and Ψ as in the Theorem 5.1 then

$$\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta \tau(\chi_1^n \psi_1^n \theta^{n^2})^\rho i^{n \sum_{v \in \mathbf{a}} p_v} < \mathbf{f}, \mathbf{f} >} \in \mathcal{W} := W(\chi),$$

where $\beta = n(\sum_v m_v) + d(2n\sigma_0 - 2n^2 + n)$, $W(\chi)$ obtained from W by adjoining the values of χ on finite adeles, and $p \in \mathbb{Z}^{\mathbf{a}}$ is defined for $v \in \mathbf{a}$ as $p_v = \frac{m_v - |k_v - k_{v\rho} - t_v| - 2\sigma_0}{2}$ if $\sigma_0 \geqslant n$, and $p_v = \frac{m_v - |k_v - k_{v\rho} - t_v| - 4n + 2\sigma_0}{2}$ if $\sigma_0 < n$.

Theorem 5.3 *Let $\mathbf{f} \in \mathcal{S}_k(C, \psi; \overline{\mathbb{Q}})$ be an eigenform for all Hecke operators. With notation as before we take $m_0 > 3n + 2$. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ with $t \in \mathbb{Z}^{\mathbf{a}}$. Define $\mu \in \mathbb{Z}^{\mathbf{b}}$ as in the previous theorem. With the same assumptions as in the previous theorem and with $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ as defined in Theorem 5.1 we have for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\Psi_Q)$ that*

$$\left(\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta \tau(\chi_1^n \psi_1^n \theta^{n^2})^\rho i^{n \sum_{v \in \mathbf{a}} p_v} \Omega_{\mathbf{f}}} \right)^\sigma = \frac{L(\sigma_0, \mathbf{f}^\sigma, \chi^\sigma)}{\pi^\beta \tau((\chi_1^n \psi_1^n \theta^{n^2})^\sigma)^\rho i^{n \sum_{v \in \mathbf{a}} p_v} \Omega_{\mathbf{f}^\sigma}},$$

where $\Psi_Q = \Psi$ if $\sigma_0 \in \mathbb{Z}$ and it is the algebraic extension of Ψ obtained by adjoining $|det(qq^*)|_{\mathbf{h}}^{1/2}$ for all $q \in Q$, if $\sigma_0 \in \frac{1}{2}\mathbb{Z}$, where the set Q is defined in Sect. 4.

6 The Euler Factors Above p and the Trace Operator

We now fix an odd prime p and write S for the set of prime ideals in K above p such that they are inert with respect to the totally real subfield F. We assume of course that $S \neq \emptyset$. A typical element in this set will be denoted by $\mathfrak{p}$.

For a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of F, which are taken prime to the ideals in the set S, we define $C := D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. We consider a non-zero $\mathbf{f} \in \mathcal{S}_k(C, \psi)$, which we take to be an eigenform for all Hecke operators with respect to C. Furthermore we let χ be a Hecke character of K of conductor $\mathfrak{f}_\chi$ (or simply $\mathfrak{f}$ if there is no danger of confusion), supported in the set S. As we mentioned in the introduction our aim is to obtain measures that interpolate special values of $L(s, \mathbf{f}, \chi)$ such that the Eisenstein series involved in the Theorem 4.1 are holomorphic. In particular if we write $t \in \mathbb{Z}^{\mathbf{a}}$ for the infinite type of the character χ and define $\mu \in \mathbb{Z}^{\mathbf{b}}$ as in Sect. 4, then we will assume that

$$(k_v - \mu_v - n) + (k_{v\rho} - \mu_{v\rho}) = r, \quad \forall v \in \mathbf{a},$$

for some $r \geqslant n$, where we exclude the case of $r = n + 1$, $F = \mathbb{Q}$ and $\chi_1 = \theta$. For a fixed character χ we define

1. $\Theta_\chi := \Theta := \theta_{\mathbb{A}}(x, \chi^{-1})$, where we put some special condition on the element $r \in GL_n(K)_{\mathbf{h}}$ in the definition of the theta series. Namely we pick the element $r \in GL_n(K)_{\mathbf{h}}$ such that $r_v = \pi_v r'_v$ with $r'_v \in GL_n(\mathfrak{r}_v)$ for v not dividing the conductor and $v \in S$, and $r_v \in GL_n(\mathfrak{r}_v)$ for $v \in S$ and dividing the conductor. For τ we assume that $\tau_v \in GL_n(\mathfrak{r}_v)$.
2. $\Theta^*_\chi := \Theta^* := \theta^*_{\mathbb{A}}(x, \chi^{-1})$, with similar conditions on r and τ as above.
3. $\mathbf{E}_{\chi,+} := \mathbf{E}_+ := \widetilde{D}_{\mathbb{A}}(x, \frac{r}{2}; k - l, \Psi, \mathfrak{c}'')$,
4. $\mathbf{E}^*_{\chi,+} := \mathbf{E}^*_+ := \widetilde{D}^*_{\mathbb{A}}(x, \frac{r}{2}; k - l, \Psi, \mathfrak{c}'')$,
5. $\mathbf{E}_{\chi,-} := \mathbf{E}_- := \widetilde{D}_{\mathbb{A}}(x, n - \frac{r}{2}; k - l, \Psi, \mathfrak{c}'')$,
6. $\mathbf{E}^*_{\chi,-} := \mathbf{E}^*_- := \widetilde{D}^*_{\mathbb{A}}(x, n - \frac{r}{2}; k - l, \Psi, \mathfrak{c}'')$,

where $\Psi := (\chi^{-1}\phi^{-n}\psi^{-1})^c$, $\mathfrak{c}''$ is as in Theorem 4.1 and the Eisenstein series $\widetilde{D}_{\mathbb{A}}$ was introduced in Eq. 7.

We now recall some facts about Hecke operators taken from [29, 30]. The action of the Hecke operator $T_C(\xi) := T(\xi) := C\xi C$ for some $\xi \in G_{\mathbf{h}}$, such that $C\xi C = \bigsqcup_{y \in Y} Cy$ for a finite set Y, is defined by,

$$(\mathbf{f}|C\xi C)(x) := \sum_{y \in Y} \psi_{\mathfrak{c}}(det(a_y))^{-1}\mathbf{f}(xy^{-1}).$$

Following Shimura, we introduce the notation $E := \prod_{v \in \mathbf{h}} GL_n(\mathfrak{r}_v)$ and $B := \{x \in GL_n(K)_{\mathbf{h}} | x \prec \mathfrak{r}\}$. We have,

Lemma 6.1 (Shimura, Lemma 19.2 in [30]) *Let $\sigma = diag[\hat{q}, q] \in G_v$ with $q \in B_v$ and $v|\mathfrak{c}$. Then*

$$C_v \sigma C_v = \bigsqcup_{d,b} C_v \begin{pmatrix} \hat{d} & \widehat{db} \\ 0 & d \end{pmatrix},$$

with $d \in E_v \setminus E_v q E_v$ and $b \in S(\mathfrak{b}^{-1})_v / d^* S(\mathfrak{b}^{-1})_v d$, where $S(\mathfrak{b}^{-1}) := S \cap M_n(\mathfrak{b}^{-1})$.

We now introduce the following notation. Let $v \in \mathbf{h}$ be a finite place of F which correspond to a prime ideal of F, that is inert in K. We write $\mathfrak{p}$ for the ideal in K corresponding to the place in K above v, and π_v (or π when there is no fear of confusion) for a uniformizer of $\mathfrak{p}$. Since the choice of v determines uniquely a place of K (since we deal with the inert situation) we will often abuse the notation and write v also for this place of K.

For an integral ideal $\mathfrak{c}$ such that $v|\mathfrak{c}$ we write $U(\pi_i)$, for an $i = 1, \dots, n$, for the operator $C\xi C$ defined by taking $\xi_{v'} = 1_{2n}$ for v' not equal to v and $\xi_v = diag[\hat{q}, q]$ with $q = diag[\pi, \dots, \pi, 1, \dots, 1]$ where there are i-many π's. Sometimes, we will also write $U(\pi)$ or $U(\mathfrak{p})$ for $U(\pi_n)$.

6.1 The Unramified Part of the Character

We now describe how we can choose the elements d in Lemma 6.1 for the operators $U(\pi_i)$. We have,

Lemma 6.2 *Let $q = diag[\pi, \pi, \dots \pi, 1, \dots, 1]$ with m many π's. Then we have that in the decomposition*

$$E_v q E_v = \bigsqcup_d E_v d,$$

the representatives $d = (d_{ij})_{i,j}$'s are all the lower triangular matrices such that,

1. *there exist $n - m$ many 1 on the diagonal and the rest elements of the diagonal are equal to π. Write S for the subset of $\{1, \dots, n\}$ such that $i \in S$ if and only if $d_{ii} = \pi$.*
2. *For any $i > j$, we have*

$$d_{ij} = \begin{cases} 0 & \text{if } j \notin S \text{ and } i \in S \\ 0 & \text{if } j \in S \text{ and } i \in S \\ \alpha & \text{if } j \in S \text{ and } i \notin S \end{cases},$$

where $\alpha \in \mathfrak{r}_v$ runs over some fixed representatives of $\mathfrak{r}_v/\mathfrak{p}_v$, where $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{r}_v$.

Proof See [8, pp. 55–56] □

We now let λ_i be the eigenvalues of $\mathbf{f}$ with respect to the operators $U(\pi_i)$. For the fixed prime ideal $\mathfrak{p}$ as above we write t_i for the Satake parameters $t_{\mathfrak{p},i}$ associated to $\mathbf{f}$ as introduced in Sect. 4.

Lemma 6.3 *We have the identity*

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+ni+\frac{n(n-1)}{2}}\lambda_i X^i\right)=$$

$$(-1)^n N(\mathfrak{p})^{n(2n-1)}X^n\prod_{i=1}^{n}(1-t_i^{-1}N(\mathfrak{p})^{1-n}X^{-1}).$$

Proof We first note that,

$$\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+ni}\lambda_i X^i=\prod_{i=1}^{n}(1-N(\mathfrak{p})^{n-1}t_i X). \tag{8}$$

This follows from [30, Lemma 19.13] and the fact that (see [30, p. 163])

$$\sum_{d\in E_v\backslash B_v}\omega_0(E_v d)|det(d)|_v^{-n}X^{v_{\mathfrak{p}}(det(d))}=\prod_{i=1}^{n}(1-N(\mathfrak{p})^{n-1}t_i X)^{-1},$$

where $v_{\mathfrak{p}}(\cdot)$ is the discrete valuation corresponding to the prime $\mathfrak{p}$, $|\cdot|_v$ the absolute value at v normalized as $|\pi|_v=N(\mathfrak{p})^{-1}$. For the definition of $\omega_0(E_v d)$, we first find an upper triangular matrix g so that $E_v d=E_v g$ and then we define $\omega_0(E_v d):=\prod_{i=1}^{n}\left(N(\mathfrak{p})^{-2i}t_i\right)^{e_i}$, where the $e_i\in\mathbb{Z}$ are so that $g_{ii}=\pi^{e_i}$ for $g=(g_{ij})$.

We can rewrite the right hand side of Eq. 8 as

$$\prod_{i=1}^{n}(1-N(\mathfrak{p})^{n-1}t_i X)=$$

$$N(\mathfrak{p})^{n(n-1)}(-1)^n(t_1t_2\dots t_n)X^n\prod_{i=1}^{n}(1-t_i^{-1}N(\mathfrak{p})^{1-n}X^{-1}).$$

Moreover we have by Eq. 8 that $\lambda_n=N(\mathfrak{p})^{-\frac{n(n+1)}{2}}t_1t_2\dots t_n$. So we conclude that

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+ni}\lambda_i X^i\right)=$$

$$(-1)^n N(\mathfrak{p})^{n(n-1)+\frac{n(n+1)}{2}}X^n\prod_{i=1}^{n}(1-t_i^{-1}N(\mathfrak{p})^{1-n}X^{-1}),$$

or,

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+ni+\frac{n(n-1)}{2}}\lambda_i X^i\right)=$$

$$(-1)^n N(\mathfrak{p})^{n(2n-1)} X^n \prod_{i=1}^{n}(1-t_i^{-1}N(\mathfrak{p})^{1-n}X^{-1}).$$

□

In particular if χ is a Hecke character of K which is taken unramified at $\mathfrak{p}$ and we set $X := \chi(\mathfrak{p})N(\mathfrak{p})^{s_+}$ with $s_+ := -\frac{n+r}{2}$ for some $r \in \mathbb{Z}$ we obtain,

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{(n-i)(n-i-1)}{2}-\frac{r-3n+2}{2}}\lambda_i \chi(\mathfrak{p})^i\right)=$$

$$(-1)^n N(\mathfrak{p})^{n(2n-1)-n(\frac{r+n}{2})}\chi(\mathfrak{p})^n \prod_{i=1}^{n}(1-\chi(\mathfrak{p})^{-1}t_i^{-1}N(\mathfrak{p})^{\frac{r-n+2}{2}}),$$

or

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{(n-i)(n-i-1)}{2}-\frac{r-3n+2}{2}}\lambda_i \chi(\mathfrak{p})^{i-n}\right)= \tag{9}$$

$$(-1)^n N(\mathfrak{p})^{n(2n-1)-n(\frac{r+n}{2})} \prod_{i=1}^{n}(1-\chi(\mathfrak{p})^{-1}t_i^{-1}N(\mathfrak{p})^{\frac{r-n+2}{2}}),$$

and if we set $X := \chi(\mathfrak{p})N(\mathfrak{p})^{s_-}$ with $s_- := -\frac{3n-r}{2}$, we obtain,

$$\lambda_n^{-1}\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{(n-i)(n-i-1)}{2}-\frac{-n-r+2}{2}}\lambda_i \chi(\mathfrak{p})^{i-n}\right)= \tag{10}$$

$$(-1)^n N(\mathfrak{p})^{n(2n-1)-n(\frac{3n-r}{2})} \prod_{i=1}^{n}(1-\chi(\mathfrak{p})^{-1}t_i^{-1}N(\mathfrak{p})^{\frac{n-r+2}{2}}).$$

We also make a general remark about the adjoint operator of the Hecke operators introduced in Lemma 6.1. First we note that,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

In particular we have

$$\eta_{\mathbf{h}}^{-1} D[\mathfrak{b}^{-1}, \mathfrak{bc}]\eta_{\mathbf{h}} = D[\mathfrak{bc}, \mathfrak{b}^{-1}].$$

Now if we write W for the operator $(\mathbf{f}|W)(x) := \mathbf{f}(x\eta_{\mathbf{h}}^{-1})$ we have,

Lemma 6.4 *For* $\mathbf{f}, \mathbf{g} \in \mathcal{M}_k(C, \psi)$ *we have*

$$< \mathbf{f}|C\sigma C, \mathbf{g} >_{\mathfrak{c}} = < \mathbf{f}, \mathbf{g}|W\widetilde{C}\tilde{\sigma}\widetilde{C}W^{-1} >_{\mathfrak{c}}$$

where $\widetilde{C} := D[\mathfrak{bc}, \mathfrak{b}^{-1}]$*, and* $\tilde{\sigma} := diag[\widehat{q^*}, q^*]$ *if* $\sigma = diag[\hat{q}, q]$.

Proof By Proposition 11.7 in [29] we have that $< \mathbf{f}|C\sigma C, \mathbf{g} > = < \mathbf{f}, \mathbf{g}|C\sigma^{-1}C >$. Of course we have $\sigma^{-1} = diag[q^*, q^{-1}]$. Moreover we have that

$$C\sigma^{-1}C = WW^{-1}CWW^{-1}\sigma^{-1}WW^{-1}CWW^{-1}$$

and we have that $W\sigma^{-1}W^{-1} = diag[q^{-1}, q^*] = diag[\widehat{q^*}, q^*]$. Moreover the group $W^{-1}CW = D[\mathfrak{bc}, \mathfrak{b}^{-1}]$ if $C = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$. Moreover we note that we may write $D[\mathfrak{bc}, \mathfrak{b}^{-1}] = D[\tilde{\mathfrak{b}}^{-1}, \tilde{\mathfrak{b}}\mathfrak{c}]$ by taking $\tilde{\mathfrak{b}} = \mathfrak{b}^{-1}\mathfrak{c}^{-1}$. □

For the fixed ideal $\mathfrak{p}$, and an $s \in \mathbb{C}$, we define the operator $J(\mathfrak{p}, s)$ on $\mathcal{M}_k(C, \psi)$ as

$$J(\mathfrak{p}, s) := \sum_{i=0}^{n} (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2} + i(n+s) + \frac{n(n-1)}{2}} (\chi)(\mathfrak{p})^{i-n} U(\pi_i).$$

We now note by Lemma 6.3 we have that for the eigenform $\mathbf{f}$

$$\mathbf{f}|J(\mathfrak{p}, s) = \lambda_n (-1)^n N(\mathfrak{p})^{n(2n-1)} N(\mathfrak{p})^{ns} \prod_{i=1}^{n} (1 - N(\mathfrak{p})^{1-n} \chi(\mathfrak{p})^{-1} t_i^{-1} N(\mathfrak{p})^{-s})\mathbf{f}$$

We will need to consider the adjoint operator of $J(\mathfrak{p}, s)$ with respect to the Petersson inner product. In particular if we write

$$< \mathbf{f}|J(\mathfrak{p}, s), \mathbf{g} > = < \mathbf{f}, \mathbf{g}|W\widetilde{J(\mathfrak{p}, s)}W^{-1} >,$$

then by Lemma 6.4 we have that

$$\widetilde{J(\mathfrak{p}, s)} = \sum_{i=0}^{n} (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2} + i(n+\bar{s}) + \frac{n(n-1)}{2}} \chi(\mathfrak{p})^{n-i} U(\pi_i),$$

where we keep writing $U(\pi_i)$ for the Hecke operator

$$D[\mathfrak{bc}, \mathfrak{b}^{-1}]diag[\pi, \pi, \ldots, \pi, 1 \ldots, 1]D[\mathfrak{bc}, \mathfrak{b}^{-1}].$$

We note here that

$$D[\mathfrak{bc},\mathfrak{b}^{-1}]diag[\pi,\pi,\dots,\pi,1\dots,1]D[\mathfrak{bc},\mathfrak{b}^{-1}]=$$

$$D[\mathfrak{bc},\mathfrak{b}^{-1}]diag[\pi^{\rho},\pi^{\rho},\dots,\pi^{\rho},1\dots,1]D[\mathfrak{bc},\mathfrak{b}^{-1}]$$

Of particular interest for us are the operators $\widetilde{J(\mathfrak{p},s_{\pm})}$ where we recall we have defined $s_{+}:=-\frac{r+n}{2}$ and $s_{-}:=\frac{3n-r}{2}$. We set $\mathfrak{m}_0:=\mathfrak{cpp}^{\rho}$. We note that $\Theta^{*}\mathbf{E}^{*}_{\pm}\in D[\mathfrak{bam}_0,\mathfrak{b}^{-1}]$ for some ideals $\mathfrak{a},\mathfrak{b}$ prime to $\mathfrak{q}$. This is clear for the Eisenstein series by its definition, and for the theta series we need to observe that since we are taking an $r\in GL_n(K)_{\mathbf{h}}$ of the form $\pi\pi^{\rho}r'$ for some $r'\in GL_n(K)_{\mathbf{h}}$ with $r_v\in GL_n(\mathfrak{r}_v)$ we have that the ideals $\mathfrak{t}$ and $\mathfrak{y}$ are equal to $\mathfrak{qq}^{\rho}$. Hence we have that $\theta\in\mathcal{M}_l(D[(\mathfrak{dqq}^{\rho})^{-1},\mathfrak{dqq}^{\rho}\mathfrak{eff}^{\rho}])$. Hecne $\theta^{*}\in\mathcal{M}_l(D[\mathfrak{dqq}^{\rho}\mathfrak{eff}^{\rho},(\mathfrak{dqq}^{\rho})^{-1}])\subset\mathcal{M}_l(D[\mathfrak{dqq}^{\rho}\mathfrak{eff}^{\rho},\mathfrak{d}^{-1}])\subset\mathcal{M}_l(D[\mathfrak{dcqq}^{\rho}\mathfrak{eff}^{\rho},\mathfrak{d}^{-1}])$. We then take $\mathfrak{b}=\mathfrak{d}^{-1}$ and $\mathfrak{a}=\mathfrak{eff}^{\rho}$.
Before we go further, we collect some facts which will be needed in the proof of the following Theorem. We start by recalling the so-called generalized Möbius function as for example defined by Shimura in [30, pp. 163–164]. We restrict ourselves to the local version of it, since this will be enough for our purposes. We have fixed a finite place v of the filed K (recall here our abusing of notation explained above), and write K_v for the completion at v and $\mathfrak{r}_v$ for its ring of integers. We continue writing $\mathfrak{p}$ for the prime ideal of $\mathfrak{r}$ corresponding to the finite place v, and $\mathfrak{p}_v$ for the maximal ideal of $\mathfrak{r}_v$. Finally we write π for a fixed uniformizer of $\mathfrak{r}_v$.
The generalized Möbius function will be denoted by μ, and it is defined on the set of $\mathfrak{r}_v$-submodules of a torsion $\mathfrak{r}_v$-module. In particular we cite the following lemma [30, Lemma 19.10].

Lemma 6.5 *To every finitely generated torsion $\mathfrak{r}_v$-module A we can uniquely assign an integer $\mu(A)$ so that*

$$\sum_{B\subset A}\mu(B)=\begin{cases}1 & \text{if } A=\{0\}\\ 0 & \text{if } A\neq\{0\}\end{cases}.$$

We also recall two properties (see [30] for a proof) of this generalized Möbius function, which will play an important role later. We have

1. $\mu((\mathfrak{r}_v/\mathfrak{p}_v)^r)=(-1)^r N(\mathfrak{p})^{r(r-1)/2}$ if $0\leqslant r\in\mathbb{Z}$.
2. $\mu(A)\neq 0$ if and only if A is annihilated by a square free integral ideal of K_v.

Let us now denote by $\mathcal{L}:=\mathcal{L}_{\ell}$ the set of $\mathfrak{r}_v$-lattices in K_v^{ℓ}. Given an $y\in GL_{\ell}(K_v)$ and an $L\in\mathcal{L}$ we define a new lattice by $yL:=\{yx|x\in L\}\in\mathcal{L}$. Conversely it is clear that given two lattices $M,L\in\mathcal{L}$ there exists a $y\in GL_{\ell}(K_v)$ such that $M=yL$. We also note that if $L,M\in\mathcal{L}$ and $M\subset L$ then we can write $\mu(L/M)$. Let us now take $L:=\mathfrak{r}_v^{\ell}\subset K_v^{\ell}$. Then by [30, Lemma 19.13] we have

$$\sum_{L \supset M \in \mathcal{L}} \mu(L/M) X^{v_{\mathfrak{p}}(det(y))} = \prod_{i=1}^{\ell} (1 - N(\mathfrak{p})^{i-1} X), \tag{11}$$

where the sum runs over all lattices $M \in \mathcal{L}$ contained in L, and y is defined so that $M = yL$. Here we write $v_{\mathfrak{p}}(\cdot)$ for the normalized discrete valuation of K_v.
We will now use the above equality to obtain a relation between the number of left cosets in the decomposition of Lemma 6.2. We set $E_\ell := GL_\ell(\mathfrak{r}_v)$ and for an $m \leqslant \ell$ we set $\pi_m^{(\ell)} := diag[\pi, \pi, \ldots, \pi, 1, \ldots, 1] \in GL_\ell(K_v)$ with m-many π's. As we have seen in Lemma 6.2 we have a decomposition

$$E_\ell \pi_m^{(\ell)} E_\ell = \bigsqcup_{d_m^{(\ell)}} E_\ell d_m^{(\ell)},$$

for some $d_m^{(\ell)} \in GL_\ell(K_v) \cap M_\ell(\mathfrak{r}_v)$. We write $\mu_m^{(\ell)}$ for the number of the cosets in the above decomposition. Then we have,

Lemma 6.6 *With notation as above,*

$$\sum_{i=0}^{\ell} (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}} \mu_i^{(\ell)} = 0.$$

Proof We first note that by taking the transpose of the decomposition above we may also work with right cosets, that is $E_\ell \pi_m^{(\ell)} E_\ell = \bigsqcup_{d_m^{(\ell)}} {}^t d_m^{(\ell)} E_\ell$. We now let $L := \mathfrak{r}_v^\ell$, and we see that to every coset ${}^t d_m^{(\ell)} E_\ell$ for $0 \leqslant m \leqslant \ell$ we can associate a lattice $M \in \mathcal{L}$ by $M := {}^t d_m^{(\ell)} L$. Since ${}^t d_m^{(\ell)}$ are integral we have $M \subset L$. Moreover in the sum $\sum_{L \supset M \in \mathcal{L}} \mu(L/M) X^{v_{\mathfrak{p}}(det(y))}$, because of property (ii) of the Möbius function, we have that the $y's$ have square free elementary divisors. Indeed it is enough to notice (see for example [9, Theorem 1.4.1]) that for the lattice $M = yL$ we have that L/M is isomorphic to $\oplus_{0 \leqslant i \leqslant r} (\mathfrak{r}_v/\mathfrak{p}_v)^{e_i}$ where e_i are the (powers) of the elementary divisors of y, and r its rank. In particular we can conclude that each y in the sum $\sum_{L \supset M \in \mathcal{L}} \mu(L/M) X^{v_{\mathfrak{p}}(det(y))}$ belongs to some ${}^t d_m^{(\ell)} E_\ell$ for m equal to $v_{\mathfrak{p}}(det(y))$. That is, we may write

$$\sum_{L \supset M \in \mathcal{L}} \mu(L/M) X^{v_{\mathfrak{p}}(det(y))} = \sum_{i=0}^{\ell} (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}} \mu_i^{(\ell)} X^i,$$

where we have used property (i) of the Möbius function. We now set $X = 1$ and use Eq. 11 to conclude the lemma. □

We are now ready to prove the following theorem.

Theorem 6.7 *Let $\mathfrak{p} \in S$ and write v for the finite place of F corresponding to $\mathfrak{p}$ as above. Consider a Hecke character χ of K unramified at the prime $\mathfrak{p}$. Let $\mathbf{F}_\pm := \Theta^* \mathbf{E}_\pm^*$ and write*

$$\mathbf{g}_{\pm} := \mathbf{F}_{\pm}|\widetilde{J(\mathfrak{p}, s_{\pm})}.$$

Then, for $q \in GL(K)_{\mathbf{h}}$*, with* $q_v \in GL_n(\mathfrak{r}_v)$*, we have*

$$\mathbf{g}_{\pm}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = C(\mathfrak{p}, s_{\pm}) \sum_{\tau \in S_+} c(\tau, q; \mathbf{g}_{\pm})\mathbf{e}^n_{\mathbb{A}}(\tau s)$$

with $C(\mathfrak{p}, s_{\pm}) := (-1)^n N(\mathfrak{p})^{n(n-1)+n(n+s_{\pm})}\psi(\mathfrak{p})^{-n}$ *and,*

$$c(\tau, q; \mathbf{g}_{\pm}) = \sum_{\tau_1+\tau_2=\tau} c(\tau_1, q\pi, \Theta^*)c(\tau_2, q\pi, \mathbf{E}^*_{\pm}),$$

where $(\tau_1)_v \in (\pi_v\pi_v^{\rho})^{-1}T_v^{\times}$*, where* $T_v^{\times} = T_v \cap GL_n(\mathfrak{r}_v)$ *and we recall that*

$$T = \{x \in S | tr(S(\mathfrak{r})x) \subset \mathfrak{g}\},$$

where $S(\mathfrak{r}) = S \cap M_n(\mathfrak{r})$*, and* $T_v := T \otimes_{\mathfrak{r}} \mathfrak{r}_v$*.*

Proof We will show the Theorem when $\mathbf{F} := \mathbf{F}_+ = \Theta^*\mathbf{E}^*_+$, and a similar proof shows also the case of $\mathbf{F}_- = \Theta^*\mathbf{E}^*_-$. We set $\mathbf{g} := \mathbf{g}_+$, and we note that the Nebentype of $\Theta^*\mathbf{E}^*_+$ is ψ^{-c}. We then have,

$$\mathbf{g}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) =$$

$$N(\mathfrak{p})^{\frac{n(n-1)}{2}} \sum_{i=0}^{n} B_i \sum_{d_i} \psi_v(det(d_i))^{-1} \sum_{b_i} \mathbf{F}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\begin{pmatrix} \hat{d}_i^{-1} & -b_i d_i^{-1} \\ 0 & d_i^{-1} \end{pmatrix}\right),$$

where here we write d_i and b_i for the d's and b's corresponding to the Hecke operator $U(\pi_i)$ as described in Lemma 6.1, and in order to make the formulas a bit shorter we have introduced the notation $B_i := (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+i(n+s_+)}\chi(\mathfrak{p})^{i-n}$. In particular we have that

$$N(\mathfrak{p})^{-\frac{n(n-1)}{2}} \mathbf{g}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) =$$

$$\sum_{i=0}^{n} B_i \sum_{d_i} \psi_v(det(d_i))^{-1} \sum_{b_i} \mathbf{F}\left(\begin{pmatrix} q\hat{d}_i^{-1} & -qb_i d_i^{-1} + s\hat{q}d_i^{-1} \\ 0 & \hat{q}d_i^{-1} \end{pmatrix}\right) =$$

$$\sum_{i=0}^{n} B_i \sum_{d_i} \psi_v(det(d_i))^{-1} \sum_{b_i} \mathbf{F}\left(\begin{pmatrix} qd_i^* & (-qb_i q^* + s)\widehat{qd_i^*} \\ 0 & \widehat{qd_i^*} \end{pmatrix}\right) =$$

$$\sum_{i=0}^{n} B_i \sum_{d_i} \psi_v(det(d_i))^{-1} \sum_{b_i} \sum_{\tau \in S_+} c(\tau, qd_i^*; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau(-qb_iq^* + s)) =$$

$$\sum_{i=0}^{n} B_i \sum_{d_i} \psi_v(det(d_i))^{-1} \sum_{\tau \in S_+} c(\tau, qd_i^*; \mathbf{F}) \left(\sum_{b_i} \mathbf{e}_{\mathbf{h}}^n(-\tau q b_i q^*) \right) \mathbf{e}_{\mathbb{A}}^n(\tau s)$$

Since $b_i \in S(\mathfrak{bc})_v$, we have by [30, Lemma 19.6] that

$$\sum_{b_i} \mathbf{e}_{\mathbf{h}}^n(-\tau q b_i q^*) = |det(d_i)|_{\mathbb{A}}^{-n},$$

if $(q^* \tau q)_v \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1} T_v$ for all $v \in \mathbf{h}$ and zero otherwise. We now write

$$c(\tau, qd_i^*; \mathbf{F}) = \sum_{\tau_1 + \tau_2 = \tau} c(\tau_1, qd_i^*; \Theta^*) c(\tau_2, qd_i^*; \mathbf{E}_+^*),$$

and from above we have that $(q^* \tau q)_v \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1} T_v$ for all $v \in \mathbf{h}$. Moreover we have that $c(\tau_1, qd_i^*; \Theta^*) \neq 0$ only if $(q^* \tau_1 q)_v \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1} d_i^{-1} T_v \widehat{d_i}$ for all $v \in \mathbf{h}$ and $c(\tau_2, qd_i^*; \mathbf{E}_+^*) \neq 0$, only if $(q^* \tau_2 q)_v \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1} d_i^{-1} T_v \widehat{d_i}$ for all $v \in \mathbf{h}$. In the above sum we run over all possible pairs of positive semi-definite hermitian matrices τ_1, τ_2 with $\tau_1 + \tau_2 = \tau$, and set $c(\tau_1, qd_i^*; \Theta^*) = c(\tau_2, qd_i^*; \mathbf{E}_+^*) = 0$ if τ_1, τ_2 are not in the set described above.
From now on we will be writing v for the finite place of F corresponding to the prime ideal $\mathfrak{p}$. We introduce the notation

$$S_i := \{s \in S : q^* s q \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1} d_i^{-1} T \widehat{d_i}, ord_{\mathfrak{p}}(\mathfrak{dbc}\nu(s)) = 2i\},$$

where $\nu(s)$ is the so-called denominator ideal associated to a matrix s, as for example defined in [29, Chap. I, Sect. 3]. That is, the valuation at $\mathfrak{p}$ of the denominator-ideal of the symmetric matrix $q^* s q$ is exactly i, after clearing powers of $\mathfrak{p}$ coming from $\mathfrak{dcb}$. We note that since $\tau \in S_0$ we have that $\tau_1 \in S_i$ if and only if $\tau_2 \in S_i$ if $\tau_1 + \tau_2 = \tau$. We now rewrite the Fourier expansion of $\mathbf{g}$ as

$$\sum_{\tau \in S_+} N(\mathfrak{p})^{\frac{n(n-1)}{2}} \chi(\mathfrak{p})^{-n} \sum_{i=0}^{n} (-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2} + i(n+s_+)} \chi(\mathfrak{p}^i) \sum_{d_i} \psi_v(det(d_i))^{-1} \times$$

$$\sum_{\tau_1 + \tau_2 = \tau} c(\tau_1, qd_i^*; \Theta^*) c(\tau_2, qd_i^*; \mathbf{E}_+^*) |det(d_i)|_v^{-n} \mathbf{e}_{\mathbf{a}}^n(\mathbf{i}\hat{A}'^t q \tau q) \mathbf{e}_{\mathbb{A}}^n(\tau s),$$

where we have used the fact that $|det(d_i)|_{\mathbb{A}} = |det(d_i)|_v$ since $(d_i)_{v'} = 1_n$ for any finite place v' not equal to v. We now work the inner sum for any fixed τ. That is,

$$\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}+i(n+s_+)}\chi(\mathfrak{p}^i)\sum_{d_i}\psi_v(det(d_i))^{-1}\times$$

$$\sum_{\tau_1+\tau_2=\tau} c(\tau_1, qd_i^*; \Theta^*)c(\tau_2, qd_i^*; \mathbf{E}_+^*)|det(d_i)|_v^{-n}, \tag{12}$$

or

$$\sum_{\tau_1+\tau_2=\tau}\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}} N(\mathfrak{p})^{i(n+s_+)}\chi(\mathfrak{p}^i)\times$$

$$\sum_{d_i}\psi_v(det(d_i))^{-1}c(\tau_1, qd_i^*; \Theta^*)c(\tau_2, qd_i^*; \mathbf{E}_+^*)|det(d_i)|_v^{-n} \tag{13}$$

We claim that this sum is equal to

$$N(\mathfrak{p})^{\frac{n(n-1)}{2}+n(n+s_+)}(-1)^n\chi(\mathfrak{p})^n\psi(\mathfrak{p})^{-n}\sum_{\tau_1+\tau_2=\tau} c(\tau_1, q\pi, \Theta^*)c(\tau_2, q\pi, \mathbf{E}_+^*), \tag{14}$$

where $(q^*\tau_1 q)_v, (q^*\tau_2 q)_v \in \pi^{-2}\mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1}T_v^\times = S_n$. Note that this is enough in order to establish the claim of the Theorem.
To show this, we consider the nth term of the Eq. 12, that is the summand with $i = n$ and we recall that the d_n's run over the single element πI_n. That is, the nth term is equal to

$$N(\mathfrak{p})^{\frac{n(n-1)}{2}+n(n+s_+)}(-1)^n\chi(\mathfrak{p})^n\psi(\mathfrak{p})^{-n}\sum_{\tau_1+\tau_2=\tau} c(\tau_1, q\pi, \Theta^*)c(\tau_2, q\pi, \mathbf{E}_+^*), \tag{15}$$

where $(q^*\tau_1 q)_v, (q^*\tau_2 q)_v \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1}d_n^{-1}T_v\widehat{d_n} = \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1}\pi^{-2}T_v$.

Note that the difference of the expression in Eq. 15, and the claimed sum in Eq. 14 is the difference of the support of the Fourier coefficients. Indeed note that in Eq. 15 (or better say in the line right after) we write T_v where in Eq. 14 we write $T_v^\times$, and of course $T_v^\times \subset T_v$. Hence our aim is to prove that for every pair (τ_1, τ_2) with $\tau_1 + \tau_2 = \tau$ and $\tau_i \in S_j$ with $j < n$ that contributes a non-trivial term in Eq. 15, its contribution will be cancelled out by the lower terms (i.e. $i < n$) that appear in Eq. 13. So the only terms that "survive" the cancellation will be the ones with $\tau_1, \tau_2 \in S_n$. Moreover all lower terms will be cancelled out.

We note that if we consider a $\tau_1 \in S_j$ (hence $\tau_2 \in S_j$) with $j < n$, then we observe that given such a τ_1 and τ_2, we have that $c(\tau_1, qd_m^*; \Theta^*)c(\tau_2, qd_m^*; \mathbf{E}_+^*) \neq 0$ implies that $m \geqslant j$. Indeed since $\tau_1, \tau_2 \in S_j$ we have for any $m < j$ that $(q^*\tau_1 q)_v, (q^*\tau_2 q)_v \notin \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1}d_i^{-1}T_v\widehat{d_i}$.

So in what follows we fix a pair τ_1 and τ_2 in S_j for some $j \geqslant 0$ with $j < n$. By [29, Lemma 13.3] since we are interested in the question whether τ_1, τ_2 belong to a particular lattice, we may assume without loss of generality that our τ_1, τ_2, locally at v, are of the form $diag[s_1, \dots, s_n]$ for $s_i \in K_v$. After reordering the s_i's we may assume that $s_{j+1}, \dots, s_n$ are integral, while the rest have non-trivial denominators. That means, that the d_m's for $j \leqslant m \leqslant n$ with $c(\tau_1, qd_m^*; \Theta^*)c(\tau_2, qd_m^*; \mathbf{E}_+^*) \neq 0$ can be taken of a very particular form, namely we can take them to be lower triangular matrices (by Lemma 6.2) with the diagonal of the form $diag[\pi, \dots, \pi, \pi^{e_{j+1}}, \dots, \pi^{e_n}]$, where $e_{j+1}, \dots, e_n \in \{0, 1\}$ and $e_{j+1} + \dots + e_n = m - j$. Indeed the first j many π's on the diagonal are imposed to us in order $d_j \tau 1 d_j^*, d_j \tau_2 d_j^*$ to have integral coefficients along the diagonal. Given such a pair of indices m and j, with $m \geqslant j$ we will write $\lambda_m^{(j)}$ for the number of left cosets $E_v d_m$ with diagonal of d_m as just described. From now on when we write a d_m or d_j it will be always one of this particular form (i.e. lower diagonal and with the above mentioned description of the diagonal).
We now claim that we may write

$$c(\tau_1, qd_n^*, \Theta^*) = \alpha_{n,j} c(\tau_1, qd_j^*, \Theta^*),$$

and

$$c(\tau_2, qd_n^*, \mathbf{E}_+^*) = \beta_{n,j} c(\tau_2, qd_j^*, \mathbf{E}_+^*),$$

for some $\alpha_{n,j}$ and $\beta_{n,j}$, and any d_j. The terms $c(\tau_1, qd_j^*, \Theta^*)$, $c(\tau_2, qd_j^*, \mathbf{E}_+^*)$ are not trivially zero since $(d_j q^* \tau_i d_j^*)_v \in \mathfrak{b}\mathfrak{d}\mathfrak{c}^{-1} T_v$. Actually for any m with $n \geqslant m \geqslant j$, and for any d_m and d_j of the form mentioned in the previous paragraph regarding their diagonal we may write

$$c(\tau_1, qd_m^*, \Theta^*) = \alpha_{m,j} c(\tau_1, qd_j^*, \Theta^*),$$

and

$$c(\tau_2, qd_m^*, \mathbf{E}_+^*) = \beta_{m,j} c(\tau_2, qd_j^*, \mathbf{E}_+^*),$$

for $\tau_1, \tau_2 \in S_j$. We now compute the $\alpha_{m,j}, \beta_{m,j}$. We have by the explicit description of the Fourier coefficients in Proposition 3.1 that,

$$c(\tau_2, qd_m^*, \mathbf{E}_+^*) = (\psi\chi)(det(d_m d_j^{-1}))\phi(det(d_m d_j^{-1}))^n \times$$

$$|det(d_m d_m^{\rho}) d_j^{-1} d_j^{-\rho}|_v^{n-r/2} c(\tau_2, qd_j^*, \mathbf{E}_+^*).$$

Now we consider the theta series. We first notice that in order to compute the coefficients $c(\tau_1, qd_i^*; \Theta_\chi^*)$ for any i with $0 \leqslant i \leqslant n$ it is enough to compute the Fourier coefficients of $\theta_\chi(xw)$ with $w = diag[d_i^*, d_i^{-1}]_{\mathbf{h}}$. We now note that by [30, Eq. (A5.7)] we have that

$$\theta_\chi(xw) = |det(d_i^{\rho})|_v^{n/2} \phi_{\mathbf{h}}(det(d_i^{\rho})^n \chi_{\mathfrak{f}_\chi}(det(d_i))\theta_\chi(x),$$

where we have used [30, Theorem A5.4] and the definition of the theta series. In particular we conclude that

$$c(\tau_1, qd_m^*, \Theta^*) = |det(d_m^{\rho})det(d_j^{-\rho})|_v^{n/2}\phi_{\mathbf{h}}(det(d_m^{\rho})det(d_j^{-\rho}))^n c(\tau_1, qd_j^*, \Theta^*).$$

where we have used the fact that the character χ is unramified at $\mathfrak{p}$, and hence $\chi_{\mathfrak{f}_\chi}$ can be ignored.

We now note that $det(d_m) = \pi^m$ and $det(d_j) = \pi^j$. In particular we have

$$\beta_{m,j} = (\chi\psi)(\pi^{m-j})\phi(\pi^{m-j})^n|\pi^{m-j}|_v^{n-r/2},$$

and

$$\alpha_{m,j} = |\pi^{m-j}|_v^{n/2}\phi(\pi^{(m-j)\rho})^n$$

In particular we observe that the $\alpha_{m,j}$ and $\beta_{m,j}$ do not depend on the specific class of Ed_m and Ed_j.

Now we remark that the coefficients $c(\tau_1, qd_j^*, \Theta^*)$ and $c(\tau_2, qd_j^*, \mathbf{E}_+^*)$ depend only on the determinant of d_j, and not on the particular choice of the d_j, as it follows from the explicit description of the Fourier coefficients of the Eisenstein series in Propositions 3.1 and of the theta series in 3.5. Especially for the theta series we remark that it is important here that the character χ is unramified at $\mathfrak{p}$. Hence going back to the Eq. 13, we observe that we can factor the term $c(\tau_1, qd_j^*, \Theta^*)c(\tau_2, qd_j^*, \mathbf{E}_+^*)$ since it does not depend on a particular choice of d_j. Here we remind the reader the convention done above, that the d_j's are taken of a particular form, i.e. lower diagonal and a condition on the diagonal are described above. So for the fixed choice of the pair τ_1 and τ_2, we see that in order to establish the cancellation of the contribution of the fixed pair (τ_1, τ_2) in the sum, we need to show, that

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2}}\chi(\mathfrak{p})^{n-m}\alpha_{m,j}\beta_{m,j}|det(d_m)|_v^{-n}\lambda_m^{(j)} = 0.$$

(We remark one more time here that the outer summation runs from j to n, since for the fixed choice of τ_1 and τ_2 we have that $c(\tau_1, qd_i^*; \Theta^*)c(\tau_2, qd_i^*; \mathbf{E}_+^*) = 0$ for $i < j$.)

Using the fact that $\phi((\pi\pi^{\rho})^{m-j})^n$ is equal to $\phi_1(\pi\pi^{\rho})^{(m-j)n}$ and the restriction $\phi_1 = \theta$, a quadratic character, we obtain $\phi((\pi\pi^{\rho})^{m-j})^n = 1$. Hence we may rewrite the above sum as

$$(\chi^{n-j}\psi^{-j})(\mathfrak{p})\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2}}|\pi^{m-j}|_v^{n-r/2}|\pi^{m-j}|_v^{n/2}\times$$

$$|det(d_m)|_v^{-n}\lambda_m^{(j)} = 0.$$

Of course $|\pi|_v = N(\mathfrak{p})^{-1}$ and hence we have

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2}+(j-m)(n-r/2)+(j-m)n/2+mn}\lambda_m^{(j)} =$$

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2}+(j-m)(n-r/2)+(j-m)n/2+mn}\lambda_m^{(j)} =$$

$$N(\mathfrak{p})^{j(n-r/2+n/2)}\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{2(\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2})+mr-mn}{2}}\lambda_m^{(j)}.$$

That is, we need to establish that

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{2(\frac{m(m-1)}{2}+m(n+s_+)+\frac{n(n-1)}{2})+mr-mn}{2}}\lambda_m^{(j)} = 0,$$

which is equivalent to

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1+n+2s_++r)}{2}}\lambda_m^{(j)} = 0,$$

and since $s_+ = -\frac{r+n}{2}$ we get that we need to show that,

$$\sum_{m=j}^{n}(-1)^m N(\mathfrak{p})^{\frac{m(m-1)}{2}}\lambda_m^{(j)} = 0. \tag{16}$$

We now recall that we are considering d_m's of very particular form, namely lower diagonal matrices where the diagonal is of the form $diag[\pi, \ldots, \pi, \pi^{e_{j+1}}, \ldots, \pi^{e_n}]$, where $e_{j+1}, \ldots, e_n \in \{0, 1\}$ and $e_{j+1} + \ldots + e_n = m - j$. We wrote $\lambda_m^{(j)}$ for the number of them. Recalling now the notation introduced in Lemma 6.6, we claim that

$$\lambda_m^{(j)} = \mu_{m-j}^{(n-j)} \times N(\mathfrak{p})^{(n-m)j}. \tag{17}$$

We first recall that by Lemma 6.2 we may pick the d_m's in the decomposition $E_v\pi_m E_v = \bigsqcup_{dm} E_v d_m$ such that, if we write $d_m = (a_{ik})$ we have that $a_{ik} = 0$ for $i < k$ (i.e. lower triangular), and for $i > k$ we have that a_{ik} could be any representative in $\mathfrak{r}_v$ of $\mathfrak{r}_v/\mathfrak{p}_v$ for $k \in S$ and $i \notin S$ and zero otherwise, where S is the subset of $\{1, \ldots, n\}$ of cardinality m indicating the indices of the π's in the diagonal of d_m.
Since we consider d_m's with π in the first j entries of the diagonal we have that $a_{ik} = 0$ for $1 \leqslant k < i \leqslant j$. Moreover the number of choices for the lower right $n - j \times n - j$ part of d_m is equal to $\mu_{m-j}^{(n-j)}$ since we are putting $m - j$ many π on a

diagonal of length $n-j$. We can conclude the claimed equality after observing that we are free to pick for the entry a_{ik} with $i>k$ and $j+1\leqslant i\leqslant n$, and $1\leqslant k\leqslant j$ (i.e. the lower left $(n-j)\times j$ part) any representative of $\mathfrak{r}_v/\mathfrak{p}_v$ as long as $a_{ii}=1$. That is we have $N(\mathfrak{p})^{(n-m)j}$ many choices, since we place $n-m$ many ones in the $n-j$ many lower entries of the diagonal of d_m.

By Lemma 6.6 we have,

$$\sum_{i=0}^{n-j}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}}\mu_i^{(n-j)}=0,$$

and using Eq. 17 we obtain

$$\sum_{i=0}^{n-j}(-1)^i N(\mathfrak{p})^{\frac{i(i-1)}{2}}N(\mathfrak{p})^{-(n-(i+j))j}\lambda_{i+j}^{(j)}=0,$$

or,

$$\sum_{m=j}^{n}(-1)^{m-j}N(\mathfrak{p})^{\frac{(m-j)(m-j-1)}{2}-(n-m)j}\lambda_m^{(j)}=0,$$

or,

$$\sum_{m=j}^{n}(-1)^{m}N(\mathfrak{p})^{\frac{m(m-1)}{2}}\lambda_m^{(j)}=0,$$

which establishes Equality (16), and hence concludes the proof. □

6.2 The Ramified Part of the Character

We now fix two integral ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of F with $\mathfrak{c}_1|\mathfrak{c}_2$. We write $C_i:=D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}_i]$, for $i=1,2$ and define the trace operator $Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2}:\mathcal{M}_k(C_2,\psi)\rightarrow\mathcal{M}_k(C_1,\psi)$ by

$$\mathbf{f}\mapsto Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2}(\mathbf{f})(x):=\sum_{r\in R}\psi_{\mathfrak{c}_2}(det(a_r))^{-1}\mathbf{f}(xr),$$

where R is a set of left coset representatives of $D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}_2]\setminus D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}_1]$. We note that for a Hermitian cusp form $\mathbf{g}\in\mathcal{S}_k(C_1,\psi)$ we have the well known identity

$$<\mathbf{g},\mathbf{f}>_{\mathfrak{c}_2}=<\mathbf{g},Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2}(\mathbf{f})>_{\mathfrak{c}_1}, \tag{18}$$

where $< \cdot, \cdot >_{\mathfrak{c}_i}$ denotes the adelic inner product with respect to the group $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}_i]$. We now give an explicit description of the trace operator $Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2}$ in the case of $supp(\mathfrak{c}_1) = supp(\mathfrak{c}_2)$, where by $supp(\mathfrak{m})$ of an ideal $\mathfrak{m}$ is defined to be the set of prime ideals $\mathfrak{q}$ of F with $\mathfrak{q}|\mathfrak{m}$. We note that this is similar to the description given in [27, p. 91, p. 136]. We write $\mathfrak{c}_2\mathfrak{c}_1^{-1} = \mathfrak{c}$ for some integral ideal $\mathfrak{c}$ and we fix elements $c, c_1, c_2 \in F_{\mathbb{A}}^{\times}$ such that $c_?\mathfrak{g} = \mathfrak{c}_?$ as well as $b \in F_{\mathbb{A}}^{\times}$ such that $b\mathfrak{g} = \mathfrak{b}$. We first show the following lemma.

Lemma 6.8 *Let $\mathfrak{a}$ be an integral ideal prime to $\mathfrak{c}_2$. Then we have the decomposition*

$$D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{a}\mathfrak{c}_1] = \bigsqcup_{r \in R} D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{a}\mathfrak{c}_2]r,$$

where

$$R = \{\begin{pmatrix} 1 & 0 \\ bac_1u & 1 \end{pmatrix} | u \in S(\mathfrak{g})_{\mathbf{h}} \mod \mathfrak{c}\},$$

with $a \in F_{\mathbb{A}}^{\times}$ such that $a\mathfrak{g} = \mathfrak{a}$.

Proof Clearly without loss of generality we can set $\mathfrak{a} = \mathfrak{g}$. Moreover it is clear that the right hand side of the claimed decomposition is included into the left. To prove the other inclusion we consider an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}_1]$ and show that there exists an $r \in R$ such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} r^{-1} \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}_2]$ or otherwise there exists $u \in S(\mathfrak{g})_{\mathbf{h}} \mod \mathfrak{c}$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ bc_1u & 1 \end{pmatrix} \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}_2].$$

That is, we need to prove that there exists such a u as above so that $C + bc_1Du \prec \mathfrak{b}\mathfrak{c}_2$. Since $C \prec \mathfrak{b}\mathfrak{c}_1\mathfrak{r}$ we can write it as $C = bc_1C_0$ with $C_0 \prec \mathfrak{r}$, and hence we need to show that $bc_1(C_0 + Du) \prec \mathfrak{b}\mathfrak{c}_2\mathfrak{r}$. By our assumption that $supp(\mathfrak{c}_1) = supp(\mathfrak{c}_2)$ we have that $DA^* \equiv 1_n \mod \left(\prod_{\mathfrak{q}|\mathfrak{c}} \mathfrak{q}\right) \mathfrak{r}$. For a prime ideal $\mathfrak{q}$ that divides $\mathfrak{c}$ we write $e_{\mathfrak{q}}$ for the largest power of it that divides $\mathfrak{c}$ and we define $e := max(e_{\mathfrak{q}})$. Then we have that $(DA^* - 1_n)^e \prec \mathfrak{c}\mathfrak{r}$. That means that there exists an element $\tilde{D} \prec \mathfrak{r}$ such that $D\tilde{D} \equiv 1 \mod \mathfrak{c}\mathfrak{r}$ and $\tilde{D}C_0 \in S(\mathfrak{g})_{\mathbf{h}}$. Indeed we have that

$$(DA^* - 1_n)^e = DA^*DA^* \cdots DA^* + ...(-1)^e I_n \prec \mathfrak{c}\mathfrak{r},$$

or equivalently

$$D\left(A^*DA^* \cdots DA^* + \cdots + A^*\right) \equiv (-1)^{e-1} I_n \mod \mathfrak{c}\mathfrak{r}.$$

So we need only to check that the matrix

$$\left(A^*DA^* \cdots DA^* + \cdots + A^*\right) C_0 = A^*DA^* \cdots DA^*C_0 + ... + A^*C_0$$

is hermitian. But we know that A^*C is hermitian and since $bc_1 \in F_{\mathbb{A}}^{\times}$ we have that also A^*C_0 is hermitian. The same reasoning holds for the product DC_0^*. In particular we have

$$(A^*DA^* \cdots DA^*C_0)^* = C_0^*AD^*A \cdots D^*A = A^*C_0D^*A \cdots DA^* =$$

$$A^*DC_0^*A \cdots DA^* =$$

$$\cdots = A^*DA^* \cdots C_0A = A^*DA^* \cdots DA^*C_0.$$

This establishes the claim. Then we can take $u = (-1)^e \tilde{D}C_0$ to conclude the proof. □

Let us now assume that the deal $\mathfrak{c} = \mathfrak{c}_2\mathfrak{c}_1^{-1}$ above is the norm of an integral ideal $\mathfrak{c}_0$ of K, that is $\mathfrak{c} = N_{K/F}(\mathfrak{c}_0)$. We also pick an element $c_0 \in K_{\mathbb{A}}^{\times}$ such that $c_0\mathfrak{r} = \mathfrak{c}_0$. We consider now the Hecke operator $T_C(\mathfrak{c}) := T(\mathfrak{c}) := \prod_{v|\mathfrak{c}} T(\sigma_v)$ for $\sigma_v = diag[\widehat{c_{0v}}1_n, c_{0v}1_n]$, where we take $C = D[\mathfrak{b}\mathfrak{c}_1, \mathfrak{b}^{-1}\mathfrak{c}]$. Note that this group is of the form $D[\tilde{\mathfrak{b}}^{-1}, \tilde{\mathfrak{b}}\tilde{\mathfrak{c}}]$ with $\tilde{\mathfrak{b}} = (\mathfrak{b}\mathfrak{c}_1)^{-1}$ and $\tilde{\mathfrak{c}} = \mathfrak{c}\mathfrak{c}_1 = \mathfrak{c}_2$. By Lemma 6.1 we have that

$$C_v\sigma_vC_v = \coprod_b C_v \begin{pmatrix} \widehat{c_{0v}}1_n & \widehat{c_{0v}}b \\ 0 & c_{0v}1_n \end{pmatrix},$$

where $b \in S(\mathfrak{b}\mathfrak{c}_1)_v/\mathfrak{c}S(\mathfrak{b}\mathfrak{c}_1)_v$. We now observe the identity

$$\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}\begin{pmatrix} c_0^*1_n & -c_0^{-1}b \\ 0 & c_0{}^{-1}1_n \end{pmatrix}\begin{pmatrix} \widehat{c_0}1_n & 0 \\ 0 & c_01_n \end{pmatrix}\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} = \begin{pmatrix} 1_n & 0 \\ b & 1_n \end{pmatrix}.$$

We now write $V(\mathfrak{c}_0) : \mathcal{M}_k(D[\mathfrak{b}\mathfrak{c}_2, \mathfrak{b}^{-1}]) \to \mathcal{M}_k(D[\mathfrak{b}\mathfrak{c}_1, \mathfrak{c}\mathfrak{b}^{-1}])$ for the operator defined by $\mathbf{f}(x) \mapsto \mathbf{f}\left(x\begin{pmatrix} \widehat{c_0}1_n & 0 \\ 0 & c_01_n \end{pmatrix}\right)$. We can conclude from the above calculation that the trace operator can be decomposed as

$$Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2} = W \circ V(\mathfrak{c}_0) \circ T(\mathfrak{c}) \circ W^{-1},$$

where the operators are operating from the right. We note that in general the image of the right hand side is in $\mathcal{M}_k(D[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}\mathfrak{c}_1])$ which contains of course $\mathcal{M}_k(D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}_1])$, where the image of the trace operator lies. We summarize the above calculations to the following lemma.

Lemma 6.9 *With notation as above, and assuming that there exists a* $\mathfrak{c}_0$ *such that* $\mathfrak{c} = N_{K/F}(\mathfrak{c}_0)$ *we have*

$$Tr_{\mathfrak{c}_1}^{\mathfrak{c}_2} = W \circ V(\mathfrak{c}_0) \circ T(\mathfrak{c}) \circ W^{-1}.$$

The effect of $T(\mathfrak{c})$ **on the** q**-expansion.** We now study the effect of the operator $T(\mathfrak{c})$ and of $V(\mathfrak{c}_0)$ on the q-expansion of an automorphic form $\mathbf{F}$ which we take in $\mathcal{M}_k([D[\mathfrak{b}\mathfrak{c}_1, \mathfrak{b}^{-1}\mathfrak{c}]], \psi^{-c})$. We write

$$\mathbf{F}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \sum_{\tau \in S_+} c(\tau, q; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau s).$$

Setting $\mathbf{G} := \mathbf{F}|T(\mathbf{c})$ we have,

$$\mathbf{G}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) =$$

$$\psi(det(c_0))^{-1} \sum_b \mathbf{F}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\begin{pmatrix} {c_0}^* 1_n & -{c_0}^{-1} b \\ 0 & {c_0}^{-1} 1_n \end{pmatrix}\right) =$$

where b runs over the set $S(\mathfrak{b}\mathfrak{c}_1)_v / \mathfrak{c}S(\mathfrak{b}\mathfrak{c}_1)$. In particular

$$\mathbf{G}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) =$$

$$\psi(det(c_0))^{-1} \sum_b \mathbf{F}\left(\begin{pmatrix} qc_0^* & -q{c_0}^{-1} b + s\hat{q}c_0^{-1} \\ 0 & \hat{q}c_0^{-1} \end{pmatrix}\right) =$$

$$\sum_b \mathbf{F}\left(\begin{pmatrix} qc_0^* & (-qbq^* + s)\widehat{qc_0^*} \\ 0 & \widehat{qc_0^*} \end{pmatrix}\right) =$$

$$\psi(det(c_0))^{-1} \sum_b \sum_{\tau \in S_+} c(\tau, qc_0^*; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau(-qbq^* + s)) =$$

$$\psi(det(c_0))^{-1} \sum_{\tau \in S_+} \left(\sum_b \mathbf{e}_{\mathbb{A}}^n(\tau qbq^*)\right) c(\tau, qc_0^*; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau s) =$$

$$\psi(det(c_0))^{-1} \sum_{\tau \in S_+} \left(\sum_b \mathbf{e}_{\mathbf{h}}^n(\tau qbq^*)\right) c(\tau, qc_0^*; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau s).$$

Note that the inner sum is well-defined since by [30, Proposition 20.2] we have that $c(\tau, qc_0^*; \mathbf{f}) = 0$ unless $e_{\mathbf{h}}^n(q^* c_0 \tau q c_0^* s) = 1$ for every $s \in S(\mathfrak{b}\mathfrak{c}_1)_{\mathbf{h}}$. Moreover (see for example [30, Lemma 19.6]) we have that

$$\sum_b \mathbf{e}_{\mathbf{h}}^n(\tau q b q^*) = |c_0|_K^{-n^2},$$

if $\tau \in \Lambda := qT(\mathfrak{b}\mathfrak{c}_1)q^*$, and zero otherwise. Here $T(\mathfrak{b}\mathfrak{c}_1)$ denotes the dual lattice of $S(\mathfrak{b}\mathfrak{c}_1) := S \cap M(\mathfrak{b}\mathfrak{c}_1))$. That is,

$$\mathbf{G}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = |c_0|_K^{-n^2} \psi(det(c_0))^{-1} \sum_{\tau \in \Lambda} c(\tau, qc_0^*; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau s). \tag{19}$$

The effect of $V(\mathfrak{c}_0)$ **on the** q**-expansion.** Now we turn to the operator $V(\mathfrak{c}_0)$. With $\mathbf{F}$ we now consider $\mathbf{G} = \mathbf{F}|V(\mathfrak{c}_0)$. Then for the q-expansion of $\mathbf{G}$ we have,

$$\mathbf{G}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \mathbf{F}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\begin{pmatrix} \widehat{c_0} 1_n & 0 \\ 0 & c_0 1_n \end{pmatrix}\right) = \mathbf{F}\left(\begin{pmatrix} \widehat{c_0} q & s\widehat{q} c_0 \\ 0 & \widehat{q} c_0 \end{pmatrix}\right) =$$

$$\sum_{\tau \in S_+} c(\tau, \widehat{c_0} q; \mathbf{F}) \mathbf{e}_{\mathbb{A}}^n(\tau s).$$

We now take $\mathbf{F}$ of a particular dorm, namely we take $\mathbf{F} = \Theta^* \mathbf{E}*_+$, and assume that the conductor $\mathfrak{f}_\chi$ of the character χ has the property that $\mathfrak{f}_\chi | \mathfrak{c}_0$. We have that

$$c(\tau, q\widehat{c_0}, \Theta^* \mathbf{E}_+^*) = \sum_{\tau_1 + \tau_2 = \tau} c(\tau_1, q\widehat{c_0}, \Theta^*) c(\tau_2, q\widehat{c_0}, \mathbf{E}_+^*)$$

We now note that

$$c(\tau_1, q\widehat{c_0}, \Theta^*) = |c_0^\rho|_K^{-\frac{n}{2}} \phi_{\mathbf{h}}(c_0^\rho)^{n^2} \chi_{\mathfrak{f}_\chi}(c_0)^n c(\tau, q, \Theta^*),$$

and

$$c(\tau_2, q\widehat{c_0}, \mathbf{E}_+^*) = (\psi\chi)(c_0)^{-n} \phi(c_0)^{-n^2} |c_0|^{-n(n-r/2)} c(\tau_2, q, \mathbf{E}_+^*).$$

We then conclude that,

$$c(\tau, q\widehat{c_0}, \Theta^* \mathbf{E}_+^*) = \psi(c_0)^{-n} |c_0|_K^{-\frac{n}{2} - n(n-r/2)} c(\tau, q, \Theta^* \mathbf{E}_+^*). \tag{20}$$

In particular we have that $c(\tau, q\widehat{c_0}, \Theta^* \mathbf{E}_+^*) \neq 0$ only if

$$(c_0^{-\rho} q^* \tau c_0^{-1} q)_v \in (\mathfrak{f}_\chi^{-1} \mathfrak{f}_\chi^{-\rho})_v T_v,$$

for all $v | p$.

Similarly we have for $\Theta^*\mathbf{E}^*_-$ but we need to replace $\frac{r}{2}$ with $n-\frac{r}{2}$ in the above equations. That is

$$c(\tau, q\widehat{c_0}, \Theta^*\mathbf{E}^*_-) = \psi(c_0)^{-n}|c_0|_K^{-\frac{n}{2}-n(r/2)} c(\tau, q, \Theta^*\mathbf{E}^*_-). \tag{21}$$

6.3 Rewriting the Rankin–Selberg Integral

We now use the above identities to rewrite the Rankin–Selberg integrals. We consider a $\mathfrak{p} \in S$ and we let $\mathbf{f}_0 \in \mathcal{S}_k(C, \psi)$ be an eigenform for the Hecke operator $U(\mathfrak{p})$, of eigenvalue $\alpha(\mathfrak{p})$. We take $C := D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{m}_0]$ where $\mathfrak{m}_0 := \mathfrak{c}'\mathfrak{p}\mathfrak{p}^\rho$ for some $\mathfrak{c}'$ prime to $\mathfrak{p}$. We now consider a Hecke character χ of K, of some conductor $\mathfrak{f}_\chi$, and write $\mathfrak{m}_\chi$ for the ideal $\mathfrak{c}'\mathfrak{p}^{n_\mathfrak{p}}\mathfrak{p}^{n_\mathfrak{p}\rho}$ where $\mathfrak{p}^{n_\mathfrak{p}}$ is the smallest power-$\mathfrak{p}$ ideal contained in the conductor $\mathfrak{f}_\chi$. Moreover we take $\mathfrak{c}'$ small enough so that it includes the prime to $\mathfrak{p}$ level of Θ. Note that by Theorem 3.4 the level of Θ supported at $\mathfrak{p}$ is exactly $\mathfrak{p}^{n_\mathfrak{p}}\mathfrak{p}^{n_\mathfrak{p}\rho}$. We then show,

Proposition 6.10 *Consider any $c_\mathfrak{p} \in \mathbb{N}$ with $c_\mathfrak{p} \geqslant n_\mathfrak{p} \geqslant 1$. Then we have*

$$\alpha(\mathfrak{p})^{-n_\mathfrak{p}-1} < \mathbf{f}_0, \Theta\mathbf{E}_\pm >_{\mathfrak{m}_\chi} =$$

$$\alpha(\mathfrak{p})^{-c_\mathfrak{p}-1} < \mathbf{f}_0|W, \Theta^*\mathbf{E}^*_\pm|V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{c_\mathfrak{p}-1} >_{\mathfrak{m}_0} .$$

Proof

$$< \mathbf{f}_0, \Theta\mathbf{E}_\pm >_{\mathfrak{m}_\chi} =$$

$$< \mathbf{f}_0, \Theta\mathbf{E}_\pm|Tr^{\mathfrak{m}_\chi}_{\mathfrak{m}_0} >_{\mathfrak{m}_0} = < \mathbf{f}_0, \Theta\mathbf{E}_\pm|W \circ V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{n_\mathfrak{p}-1} \circ W^{-1} >_{\mathfrak{m}_0} =$$

$$\alpha(\mathfrak{p})^{n_\mathfrak{p}-c_\mathfrak{p}} < \mathbf{f}_0|U(\mathfrak{p})^{c_\mathfrak{p}-n_\mathfrak{p}}, \Theta\mathbf{E}_\pm|W \circ V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{n_\mathfrak{p}-1} \circ W^{-1} >_{\mathfrak{m}_0} =$$

$$\frac{< \mathbf{f}_0, \Theta\mathbf{E}_\pm|W \circ V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{n_\mathfrak{p}-1} \circ W^{-1} \circ W \circ U(\mathfrak{p})^{c_\mathfrak{p}-n_\mathfrak{p}} \circ W^{-1} >_{\mathfrak{m}_0}}{\alpha(\mathfrak{p})^{-n_\mathfrak{p}+c_\mathfrak{p}}} =$$

$$\alpha(\mathfrak{p})^{n_\mathfrak{p}-c_\mathfrak{p}} < \mathbf{f}_0, \Theta\mathbf{E}_\pm|W \circ V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{c_\mathfrak{p}-1} \circ W^{-1} >_{\mathfrak{m}_0} =$$

$$\alpha(\mathfrak{p})^{n_\mathfrak{p}-c_\mathfrak{p}} < \mathbf{f}_0|W, \Theta^*\mathbf{E}^*_\pm|V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{c_\mathfrak{p}-1} >_{\mathfrak{m}_0} .$$

Hence

$$< \mathbf{f}_0, \Theta\mathbf{E}_\pm >_{\mathfrak{m}_\chi} = \alpha(\mathfrak{p})^{n_\mathfrak{p}-c_\mathfrak{p}} < \mathbf{f}_0|W, \Theta^*\mathbf{E}^*_\pm|V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{c_\mathfrak{p}-1} >_{\mathfrak{m}_0},$$

or

$$\alpha(\mathfrak{p})^{-n_\mathfrak{p}} < \mathbf{f}_0, \Theta\mathbf{E}_\pm >_{\mathfrak{m}_\chi} = \alpha(\mathfrak{p})^{-c_\mathfrak{p}} < \mathbf{f}_0|W, \Theta^*\mathbf{E}^*_\pm|V(\mathfrak{p})^{n_\mathfrak{p}-1} \circ U(\mathfrak{p})^{c_\mathfrak{p}-1} >_{\mathfrak{m}_0} .$$

□

7 The p-stabilization

Let us consider $C := D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, where we take the integral ideal $\mathfrak{c}$ prime to the ideals in the fixed set S. We consider a Hermitian cusp form $\mathbf{f}$ in $\mathcal{S}_k(C, \psi)$ which we take to be an eigenform for all the "good" Hecke operators in $\prod_{v \nmid \mathfrak{c}} \mathfrak{R}(C_v, \mathfrak{X}_v)$, where $\mathfrak{R}(C_v, \mathfrak{X}_v)$ is the local Hecke algebra at v defined in [30, Chap. IV]. Our aim in this section is to construct a Hermitian cusp form $\mathbf{f}_0$, of level $\mathfrak{c} \prod_{\mathfrak{p} \in S} \mathfrak{p}\mathfrak{p}^{\rho} =: \mathfrak{c}\mathfrak{m}$ which is an eigenform for all the "good" Hecke operators away from $\mathfrak{c}\mathfrak{m}$ and for the operators $U(\pi_{v,i})$ for all finite places v corresponding to prime ideals in the set S. Our construction is the unitary analogue of the symplectic situation considered in [2, Sect. 9]. It is important to mention here that our construction is adelic, so it can be used to generalize the one in [2] to the totally real field situation. Here, as we mentioned in the introduction, we restrict ourselves to the case where all prime ideals in S are inert, but our arguments generalize also to the split case. We will consider this in [7].

We write $\mathbf{M}_S$ for the submodule of $\mathcal{S}_k(D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}\mathfrak{m}], \psi)$ generated by $\mathbf{f}$ under the action of the Hecke algebra $\prod_{v \in S} \mathfrak{R}(C'_v, \mathfrak{X}_v)$, where $C' = D(\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}\mathfrak{m})$. We let $\mathbf{f}_0 \in \mathbf{M}_S$ to be a non-trivial eigenform of all the Hecke operators in $\prod_{v \in S} \mathfrak{R}(C'_v, \mathfrak{X}_v)$. In particular $\mathbf{f}_0 \neq 0$. We write the adelic q-expansion of $\mathbf{f}$ as

$$\mathbf{f}\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \sum_{\tau \in S_+} c(\tau, q; \mathbf{f}) \mathbf{e}^n_{\mathbb{A}}(\tau s).$$

and of $\mathbf{f}_0$ as,

$$\mathbf{f}_0\left(\begin{pmatrix} q & s\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \sum_{\tau \in S_+} c(\tau, q; \mathbf{f}_0) \mathbf{e}^n_{\mathbb{A}}(\tau s).$$

We pick a $\tau \in S_+ \cap GL_n(K)$ and $q \in GL_n(K)_{\mathbf{h}}$ such that $c(\tau, q; \mathbf{f}_0) \neq 0$. In particular that means that we have $q^* \tau q \in T$, where as always T denotes the dual lattice to $S(\mathfrak{b}^{-1}) := S \cap M_n(\mathfrak{b}^{-1})$. Then for any finite place v corresponding to a prime ideal $\mathfrak{p} \in S$ we have [30, Eq. (20.15)]

$$Z_v(\mathbf{f}_0, X) c(\tau, q; \mathbf{f}_0) = \sum_d \psi_{\mathfrak{c}}(det(d^*)) |det(d)^*|_v^{-n} X^{v_{\mathfrak{p}}(det(d^*))} c(\tau, qd^*; \mathbf{f}_0), \quad (22)$$

where $d \in E_v \setminus E_v q E_v$, and $Z_v(\mathfrak{f}_0, X)$ denotes the Euler factor $Z_{\mathfrak{p}}(X)$ of Sect. 4.1. Moreover $v_{\mathfrak{p}}(\cdot)$ is the valuation associated to the ideal $\mathfrak{p}$, and $|\cdot|_v$ the normalized norm.

Following Böcherer and Schmidt [2] we now try to describe the right hand side of (22) using the Satake parameters of the form $\mathbf{f}$. As in [loc. cit.] we start with the Andrianov type identity generalized by Shimura [30, Theorem 20.4]. For the selected $\tau \in S_+ \cap GL_n(K)$ and $q \in GL_n(K)_{\mathbf{h}}$ we define (this is the local version at v of the series $D(\tau, q; \mathbf{f})$ considered in [30, p. 169])

$$D_v(\tau, q : \mathbf{f}, X) := \sum_{x \in B_v/E_v} \psi_{\mathfrak{c}}(det(qx))|det(x)|_v^{-n} c(\tau, qx; \mathbf{f}) X^{v_{\mathfrak{p}}(det(x))},$$

where $B_v = GL_n(K_v) \cap M_n(\mathfrak{r}_v)$. We will employ now what may be considered as a local version of the Andrianov–Kalinin equality in the unitary case. Namely we will relate the above series $D_v(\tau, q : \mathbf{f}, X)$ to the Euler factor $Z_v(\mathbf{f}_0, X)$.

We first introduce some notation. We let $\mathcal{L}_\tau$ be the set of $\mathfrak{r}$-lattices L in K^n such that $\ell^* \tau \ell \in \mathfrak{b}\mathfrak{d}^{-1}$ for all $\ell \in L$. Moreover for the chosen ideal $\mathfrak{c}$ above, and for two $\mathfrak{r}$ lattices M, N we write $M < N$ if $M \subset N$ and $M \otimes_{\mathfrak{r}} \mathfrak{r}_v = N \otimes_{\mathfrak{r}} \mathfrak{r}_v$ for every $v \mid \mathfrak{c}$. We now set $L := q\mathfrak{r}^n$. Then we have the following local version of [30, Theorem 20.7],

$$D_v(\tau, q; \mathbf{f}, X) \cdot \mathfrak{L}_{0,v}(X) \cdot g_v(X) =$$

$$Z_v(\mathbf{f}, X) \cdot \sum_{L_v < M_v \in \mathcal{L}_\tau} \mu(M_v/L_v)\psi_{\mathfrak{c}}(det(y)) X^{v_{\mathfrak{p}}(det(q^* \hat{y})} c(\tau, y; \mathbf{f}),$$

where $\mathfrak{L}_{0,v}(X) := \prod_{i=0}^{n-1}(1 - (-1)^{i-1} N(\mathfrak{p})^{n+i} X)^{-1}$, and $g_v(X)$ is a polynomial in X with integers coefficients and constant term equal to 1. In the sum over the M's, we take $y \in GL_n(K_v)$ such that $M_v = y\mathfrak{r}^n$ and $y^{-1}q \in B_v$. Furthermore $\mu(\cdot)$ is the generalized Möbius function introduced in the previous section, and as in the last section we write $v_{\mathfrak{p}}(\cdot)$ for the discrete valuation associated to the prime ideal $\mathfrak{p}$. We now cite the following lemma regarding $g_v(X)$ (see [23, Lemma 5.2.4]).

Lemma 7.1 *Write $(q^* \tau q)_v = diag[1_{n-r}, \pi_v s_1]$ with $s_1 \in S^r(\mathfrak{r}_v)$. Then we have*

$$g_v(X) = \prod_{i=0}^{r-1}(1 - (-1)^{i-1} N(\mathfrak{p})^{n+i} X).$$

In particular we conclude that if $(q^* \tau q)_v$ is divisible by π_v (i.e. $r = n$) then we have that $g_v(X)$ is equal to $\mathfrak{L}_{0,v}^{-1}(X)$.

Our next step is to rewrite the expression

$$\sum_{L_v < M_v \in \mathcal{L}_{\tau,v}} \mu(M_v/L_v)\psi_{\mathfrak{c}}(det(y)) X^{v_{\mathfrak{p}}(det(q^* \hat{y})} c(\tau, y; \mathbf{f}),$$

in terms of the action of the Hecke algebra. By the above lemma if we take $\pi_v q$ instead of q we obtain,

$$D_v(\tau, \pi q; \mathbf{f}, X) = Z_v(\mathbf{f}, X) \times$$

$$\sum_{L_v < M_v \in \mathcal{L}_\tau} \mu(M_v/L_v)\psi_{\mathfrak{c}}(det(y)) X^{v_{\mathfrak{p}}(det(q^* \pi^* \hat{y})} c(\tau, y; \mathbf{f}),$$

where now $L_v = \pi_v \mathfrak{r}_v^n$ and $M = y\mathfrak{r}^n$. Since the y's are supported only at v and we are taking $\mathfrak{p} \not| \mathfrak{c}$ we have $\psi_{\mathfrak{c}}(det(y)) = 1$. That is,

$$D_v(\tau, \pi q; \mathbf{f}, X) = Z_v(\mathbf{f}, X) \cdot \sum_{L_v < M_v \in \mathcal{L}_\tau} \mu(M_v/L_v) X^{v_{\mathfrak{p}}(det(q^* \pi^* \hat{y})} c(\tau, y; \mathbf{f}).$$

Now we rewrite the above expression in terms of the Hecke operators $U(\pi_j)$. In particular we have (see [30, proof of Theorem 19.8]),

$$D_v(\tau, \pi q; \mathbf{f}, X) = Z_v(\mathbf{f}, X) \times$$

$$c\left(\tau, q; \mathbf{f} | \left(\sum_{i=0}^{n} (-1)^n N(\mathfrak{p})^{i(i-1)/2} \psi_v(\pi^{i-n}) N(\mathfrak{p})^{-n(n-i)} U(\pi_{n-i}) X^i\right)\right),$$

where recall that we write the action of the Hecke operators from the right. Using the fact that $\mathbf{f}_0$ is obtained from $\mathbf{f}$ by using the Hecke operators at the prime $\mathfrak{p}$, and the fact that the Hecke algebra is commutative we obtain that the above relation holds also for $\mathbf{f}_0$. That is, we have

$$D_v(\tau, \pi q; \mathbf{f}_0, X) = Z_v(\mathbf{f}, X) \times \tag{23}$$

$$c\left(\tau, q; \mathbf{f}_0 | \left(\sum_{i=0}^{n} (-1)^n N(\mathfrak{p})^{i(i-1)/2} \psi_v(\pi^{i-n}) N(\mathfrak{p})^{-n(n-i)} U(\pi_{n-i}) X^i\right)\right).$$

We first rewrite the left hand side of the above equation. We recall that

$$D_v(\tau, \pi q; \mathbf{f}_0, X) = \sum_{x \in B_v/E_v} \psi_{\mathfrak{c}}(det(qx)) |det(x)|_v^{-n} c(\tau, \pi qx; \mathbf{f}_0) X^{v_{\mathfrak{p}}(det(x))}.$$

Now we use the fact that $\mathbf{f}_0$ is an eigenform for the operators $U(\pi_i)$. We write λ_i for the eigenvalues. Then we have that

$$c(\tau, \pi qx, \mathbf{f}_0) = N(\mathfrak{p})^{-n^2} \psi_v(\pi)^{-n} \lambda_n c(\tau, qx, \mathbf{f}_0).$$

That is we obtain,

$$D_v(\tau, \pi q; \mathbf{f}_0, X) = Z_v(\mathbf{f}_0, X) \lambda_n N(\mathfrak{p})^{-n^2} c(\tau, q, \mathbf{f}_0),$$

and so we can rewrite Eq. 23 as,

$$Z_v(\mathbf{f}_0, X) \lambda_n N(\mathfrak{p})^{-n^2} c(\tau, q, \mathbf{f}_0) =$$

$$Z_v(\mathbf{f}, X)c(\tau, q; \mathbf{f}_0)\left(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{i(i-1)/2}\psi_v(\pi^i)N(\mathfrak{p})^{-n(n-i)}\lambda_{n-i}X^i\right).$$

We note that we have

$$N(\mathfrak{p})^{\frac{i(i-1)}{2}+ni}\lambda_i = N(\mathfrak{p})^{i(n-1)}E_i(t_1, \dots, t_n) \tag{24}$$

and $(t_1 \dots t_n)^{-1}E_{n-i}(t_1, \dots, t_n) = E_i(t_1^{-1}, \dots, t_n^{-1})$ where E_i is the ith symmetric polynomial. Indeed Eq. 24 is the unitary analogue of the formula employed in [2, pp. 1429–1430] of how to obtain the eigenvalues of the Hecke operators $U(\pi_i)$ from the Satake parameters at $\mathfrak{p}$, and it can be shown in the same way. Hence we conclude that after picking τ and q such that $c(\tau, q, \mathbf{f}_0) \neq 0$ we have

$$\lambda_n N(\mathfrak{p})^{-n^2} Z_v(\mathbf{f}_0, X) = Z_v(\mathbf{f}, X)\times$$

$$(\sum_{i=0}^{n}(-1)^i N(\mathfrak{p})^{i(i-1)/2}\psi_v(\pi^i)X^i N(\mathfrak{p})^{-n^2}N(\mathfrak{p})^{-\frac{i(i-1)}{2}+2ni-\frac{n(n+1)}{2}}\times$$

$$(t_1 \dots t_n)E_i(t_1^{-1}, \dots, t_n^{-1})),$$

and using the fact that $N(\mathfrak{p})^{\frac{n(n+1)}{2}}\lambda_n = t_1 \dots t_n$ we get

$$Z_v(\mathbf{f}_0, X) = Z_v(\mathbf{f}, X)\left(\sum_{i=0}^{n}(-1)^i \psi_v(\pi^{n-i})X^i N(\mathfrak{p})^{2ni}E_i(t_1^{-1}, \dots, t_n^{-1})\right) =$$

$$Z_v(\mathbf{f}, X)\left(\sum_{i=0}^{n}(-1)^i \psi_v(\pi^i)N(\mathfrak{p})^{2ni}E_i(t_1^{-1}, \dots, t_n^{-1})X^i\right),$$

and so

$$Z_v(\mathbf{f}_0, X) = Z_v(\mathbf{f}, X)\left(\sum_{i=0}^{n}(-1)^i \psi_v(\pi^i)X^i N(\mathfrak{p})^{2ni}E_i(t_1^{-1}, \dots, t_n^{-1})\right).$$

Equivalently

$$Z_v(\mathbf{f}_0, X) = Z_v(\mathbf{f}, X)\prod_{i=0}^{n}\left(1 - N(\mathfrak{p})^{2n}\psi_v(\pi)^i t_i^{-1}X^i\right),$$

and so we conclude that

$$Z_v(\mathbf{f}_0, X) \prod_{i=0}^{n} \left(1 - N(\mathfrak{p})^{2n} \psi(\pi)^i t_i^{-1} X^i\right)^{-1} = Z_v(\mathbf{f}, X).$$

We now make the following definition

Definition 7.2 Let $\mathbf{f} \in \mathcal{S}_k(C, \psi)$ be a Hecke eigenform for $C = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Let $\mathfrak{p}$ be a prime of K prime to $\mathfrak{c}$, which is inert over F. Then we say that $\mathbf{f}$ is ordinary at $\mathfrak{p}$ if there exists an eigenform $0 \neq \mathbf{f}_0 \in \mathbf{M}_{\{\mathfrak{p}\}} \subset \mathcal{S}_k(D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}\mathfrak{p}\mathfrak{p}^{\rho}], \psi)$ with Satake parameters $t_{\mathfrak{p},i}$ such that

$$\left\| \left(\prod_{i=1}^{n} t_{\mathfrak{p},i}\right) N(\mathfrak{p})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\| \cdot \|_p$ the normalized absolute value at p.

Summarizing the computations of this section we have,

Theorem 7.3 *Let* $\mathbf{f}$ *be an cuspidal Hecke eigenform. Assume that* $\mathbf{f}$ *is ordinary for all primes in* K *above* p *that are inert from* F. *Then we can associate to it a cuspidal Hecke eigenform* $\mathbf{f}_0$ *such that its Euler factors above* p *are related by the equation*

$$Z_{\mathfrak{p}}(\mathbf{f}_0, X) \prod_{i=0}^{n} \left(1 - N(\mathfrak{p})^{2n} \psi_v(\pi)^i t_i^{-1} X^i\right)^{-1} = Z_{\mathfrak{p}}(\mathbf{f}, X),$$

where $Z_{\mathfrak{p}}(\mathbf{f}, X)$ *and* $Z_{\mathfrak{p}}(\mathbf{f}_0, X)$ *are given by* (i) *and* (iii) *respectively of the Euler factors described at the beginning of Sect. 4. Moreover the eigenvalues of* $\mathbf{f}_0$ *with respect to the Hecke operators* $U(\mathfrak{p})$ *are* p*-adic units. For all other primes* $\mathfrak{q}$ *we have* $Z_{\mathfrak{q}}(\mathbf{f}, X) = Z_{\mathfrak{q}}(\mathbf{f}_0, X)$.

8 p-adic Measures for Ordinary Hermitian Modular Forms

We recall that for a fixed odd prime p we write S for the set of all prime ideals above p in K, that are inert from F, and we assume that $S \neq \emptyset$. Moreover we denote by $\mathfrak{v}$ the ideal $\prod_{\mathfrak{p} \in S} \mathfrak{p}$. We denote by $K(S)$ the maximal abelian extension of K unramified outside the set S, and we write G for the Galois group of the extension $K(S)/K$. We consider a Hecke eigenform $\mathbf{f} \in \mathcal{S}_k(C, \psi)$ with $C = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ for some ideals $\mathfrak{b}$ and $\mathfrak{c}$ of F which are prime to p. We assume that $m_0 \geqslant 3n + 2$, where we recall that $m_0 := min_{v \in \mathbf{a}}(m_v)$ with $m_v := k_v + k_{v\rho}$. Moreover we take $\mathbf{f}$ to be ordinary at every prime $\mathfrak{p}$ in the set S in the sense defined in the previous section. By Theorem 7.3 we can associate to it a Hermitian modular form $\mathbf{f}_0$. In particular the eigenvalues of $\mathbf{f}_0$ with respect to the Hecke operators $U(\mathfrak{p})$ for all $\mathfrak{p} \in S$ are p-adic units, where we recall that we write $U(\mathfrak{p})$ for the Hecke operator $U(\pi_n)$ where π is a uniformizer corresponding to the prime ideal $\mathfrak{p}$. In this section we write $\alpha(\mathfrak{p})$ for $U(\mathfrak{p})\mathbf{f}_0 = \alpha(\mathfrak{p})\mathbf{f}_0$. We also write $\{t_{i,\mathfrak{p}}\}$ for the Satake parameters of $\mathbf{f}_0$ at the prime $\mathfrak{p}$.

Given a $k \in \mathbb{Z}^{\mathbf{b}}$ and a $t \in \mathbb{Z}^{\mathbf{a}}$ we define a $\mu \in \mathbb{Z}^{\mathbf{b}}$ as in Sect. 4. Since in this paper we have been working with unitary Hecke characters so far we need to establish a correspondence between Galois characters and unitary Hecke characters. We start by recalling the definition of a Grössencharacter of type A_0 for the CM field K. In the following for an integral ideal $\mathfrak{m}$ of K we write $I(\mathfrak{m})$ for the free abelian group generated by all prime ideals of K prime to $\mathfrak{m}$.

Definition 8.1 A Grössencharacter of type A_0, in the sense of Weil, of conductor dividing a given integral ideal $\mathfrak{m}$ of K, is a homomorphism $\chi : I(\mathfrak{m}) \to \overline{\mathbb{Q}}$ such that there exist integers $\lambda(\tau)$ for each $\tau : K \hookrightarrow \mathbb{C}$, such that for each $\alpha \in K^{\times}$ we have

$$\chi((\alpha)) = \prod_{\tau} \tau(\alpha)^{\lambda(\tau)}, \quad \text{if } \alpha \equiv 1 \mod {}^{\times}\mathfrak{m}.$$

Here the condition $\alpha \equiv 1 \mod {}^{\times}\mathfrak{m}$ means that if we write $\mathfrak{m} = \prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}}$ with $\mathfrak{q}$ distinct prime ideals and $n_{\mathfrak{q}} \in \mathbb{N}$ then $v_{\mathfrak{q}}(\alpha - 1) \geqslant n_{\mathfrak{q}}$, where $v_{\mathfrak{q}}$ the standard discrete valuation associated to the prime ideal $\mathfrak{q}$.

It is well known (see for example [24]) if since we are taking K to be a CM field then the above $\lambda(\tau)$ must satisfy some conditions. In particular if we select a CM type of K, which we identify with the places $\mathbf{a}$ of F, then there exists integers d_v for each $v \in \mathbf{a}$ and an integer k such that

$$\chi((\alpha)) = \prod_{v \in \mathbf{a}} \left(\frac{1}{\alpha_v^k} \left(\frac{\alpha_v^{\rho}}{\alpha_v} \right)^{d_v} \right), \quad \text{if } \alpha \equiv 1 \mod {}^{\times}\mathfrak{m}.$$

We now keep writing χ for the associated, by class field theory, adelic character to χ. As it is explained in [24, p. 286] the infinity type is of the form,

$$\chi_{\mathbf{a}}(x) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+d_v}}{x_v^{\rho d_v}} \right). \tag{25}$$

We now consider the unitary character $\chi^1 := \chi | \cdot |_{\mathbb{A}_K}^{-k/2}$, where $| \cdot |_{\mathbb{A}_K}$ the adelic absolute value with archimedean part $|x|_{\mathbf{a}} = \prod_{v \in \mathbf{a}} |x_v|_v$, where $| \cdot |_v$ is the standard absolute value of $\mathbb{C}$. We then have that

$$\chi_{\mathbf{a}}^1(x) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k/2+d_v}}{\bar{x}_v^{\,k/2+d_v}} \right) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+2d_v}}{(x_v \bar{x}_v)^{k/2+d_v}} \right) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+2d_v}}{|x_v|^{k+2d_v}} \right).$$

In particular to a Grössencharacter χ of type A_0 of infinity type as in Eq. 25 we can associate a unitary character χ^1 of infinity type $\{m_v\}_{v\in\mathbf{a}}$ with $m_v := k + 2d_v$. The relation between the associated L functions is given by

$$L(s,\chi) = L(s + k/2, \chi^1).$$

In particular, in what follows, when we say that we consider a character χ of G of infinite type $t \in \mathbb{Z}^{\mathbf{a}}$ we shall mean that the corresponding unitary character, in the way we explained above, is of infinity type t. And we will keep writing χ, instead of χ^1 for this corresponding unitary character.

Now we return to the general setting introduced at the beginning of this section. Given a character χ of G we write $\mathfrak{f}_\chi = \prod_{\mathfrak{p}\in S}\mathfrak{p}^{n_\mathfrak{p}}$ for its conductor and define the ideal $\mathfrak{m}_\chi := \mathfrak{a}\prod_j(\mathfrak{p}\mathfrak{p}^\rho)^{m_\mathfrak{p}}$ where $m_\mathfrak{p} = n_\mathfrak{p}$ for $n_\mathfrak{p} \neq 0$ and $m_\mathfrak{p} = 1$ for $n_\mathfrak{p} = 0$, and $\mathfrak{a}$ is a small enough ideal so that it is included in $\mathfrak{c}$ and the prime to S level of the theta series Θ_χ, where Θ_χ is defined at the beginning of Sect. 6. Moreover we define $\mathfrak{m}_0 := \mathfrak{a}\prod_{\mathfrak{p}\in S}\mathfrak{p}\mathfrak{p}^\rho$ and

$$A^+(\chi) := C(\chi^{-1})^{-1}C(S)^{-1}N(\mathfrak{f}_\chi)^{n^2-\frac{n}{2}-n(n-\frac{r}{2})}N(\mathfrak{v}),$$

where $C(\chi^{-1})$ was defined in Eq. 4, $C(S)$ in Proposition 3.1, and we recall that $\mathfrak{v} = \prod_{\mathfrak{p}\in S}\mathfrak{p}$. We also define

$$A^-(\chi) := C(\chi^{-1})^{-1}C(S)^{-1}N(\mathfrak{f}_\chi)^{n^2-\frac{n}{2}-\frac{nr}{2}}N(\mathfrak{v}),$$

$$B^+(\chi) := \prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} N(\mathfrak{p})^{n(2n-1)-n(\frac{r}{2}+\frac{3n}{2}-1)-n^2}\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} C(\mathfrak{p}, -\frac{n+r}{2})\right)^{-\rho},$$

$$B^-(\chi) := \prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} N(\mathfrak{p})^{n(2n-1)-n(-\frac{r}{2}+\frac{5n}{2}-1)-n^2}\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} C(\mathfrak{p}, -\frac{3n-r}{2})\right)^{-\rho},$$

where $C(\mathfrak{p}, s)$ was defined in Theorem 6.7. We also write $C_0(\mathfrak{m}_\chi)$ for the quantity appearing in Theorem 4.1 by taking $\mathfrak{c}''$ equal to $\mathfrak{m}_\chi$ there. We then have the following theorems,

Theorem 8.2 *Assume we are given a* $t \in \mathbb{Z}^{\mathbf{a}}$ *such that*

$$(k_v - \mu_v - n) + (k_{v\rho} - \mu_{v\rho}) = r, \quad \forall v \in \mathbf{a}$$

for some $r \geqslant n$. *Moreover assume that* $r > n$ *if* $\psi_1 = 1$ *or* $\mathfrak{c} = \mathfrak{g}$. *Then there exists a measure* $\mu^+_{\mathbf{f},t}$ *of* G *such that for any primitive Hecke character* χ *of conductor* $\mathfrak{f}_\chi = \prod_\mathfrak{p}\mathfrak{p}^{n_\mathfrak{p}}$ *of infinite type* $\chi_\mathbf{a}(x) = x_\mathbf{a}^{-t}|x_\mathbf{a}|^t$ *we have*

$$\int_G \chi d\mu^+_{\mathbf{f},t} = \frac{A^+(\chi)B^+(\chi)}{C_0(\mathfrak{m}_\chi)} \left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right) \tau(\chi)^{-n\rho} \times$$

$$\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \prod_{i=1}^{n} \left(\frac{1-\chi(\mathfrak{p})^{-1}t_{i,\mathfrak{p}}^{-1}N(\mathfrak{p})^{\frac{r-n+2}{2}}}{1-\chi(\mathfrak{p})t_{i,\mathfrak{p}}N(\mathfrak{p})^{\frac{n-r}{2}-1}}\right) \times \frac{L_\mathfrak{v}(\frac{r+n}{2},\mathbf{f},\chi)}{\pi^\beta \Omega_{\mathbf{f}_0}},$$

where β is as in Theorem 5.2, and $\Omega_{\mathbf{f}_0} \in \mathbb{C}^\times$ is the period defined in Theorem 5.1 corresponding to the eigenform $\mathbf{f}_0$. In the case of $r = n+1$ and $F = \mathbb{Q}$ we exclude the characters χ such that $(\chi\psi)_1 = \theta$.

We remark here that on the left hand side, χ denotes a Galois character to which by class field theory we can associate a Hecke character of A_0 type, and by the process described above we can further associate to it a unitary character χ^1. Then as it was indicated above it is our convention that in the right hand side of the above theorem we write χ for this χ^1. Moreover we recall that we declared the infinite type of χ to be the infinite type of χ^1.

Furthermore we remark that the archimedean periods we use for our interpolation properties are the ones related to $\mathbf{f}_0$. However it is not hard to see by the definition of these periods in [6] that they are related to $\Omega_\mathbf{f}$ by some algebraic factor, which can be made very precise. For the cases excluded in the above theorem we have the following theorem.

Theorem 8.3 *We let $\mathfrak{q}$ be a prime ideal of F, prime to p. Assume that $r = n$ and further that $\psi_1 = 1$ or $\mathfrak{c} = \mathfrak{g}$ there exists a measure $\mu^+_{\mathbf{f},\mathfrak{q},t}$ such that for all characters χ of G of infinite type t we have*

$$\int_G \chi d\mu^+_{\mathbf{f},\mathfrak{q},t} = \frac{A^+(\chi)B^+(\chi)}{C_0(\mathfrak{m}_\chi)} \left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right) \tau(\chi)^{-n\rho} \times$$

$$\prod_{\substack{i=0 \\ i+n\equiv 1 \,mod\, 2}}^{n-1} (1-(\chi\psi)_1(\mathfrak{q})N(\mathfrak{q})^{i+1}) \prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \prod_{i=1}^{n} \left(\frac{1-\chi(\mathfrak{p})^{-1}t_{i,\mathfrak{p}}^{-1}N(\mathfrak{p})}{1-\chi(\mathfrak{p})t_{i,\mathfrak{p}}N(\mathfrak{p})^{-1}}\right) \frac{L_\mathfrak{v}(n,\mathbf{f},\chi)}{\pi^\beta \Omega_{\mathbf{f}_0}},$$

where $A^+(\chi)$ and $B^+(\chi)$ are defined by taking $r = n$ there.

For the other critical value, which does not involve nearly-holomorphic Eisenstein series we have the following theorem.

Theorem 8.4 *Assume that $\psi_1 \neq 1$, $\mathfrak{c} \neq \mathfrak{g}$ and $r \geqslant n$. Then there exists a measure $\mu^-_{\mathbf{f},t}$ on G such that for all characters χ of G of infinite type t we have,*

$$\int_G \chi d\mu^-_{\mathbf{f},t} = \frac{A^-(\chi)B^-(\chi)}{C_0(\mathfrak{m}_\chi)} \left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right) \tau(\chi)^{-n\rho}$$

$$\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}\prod_{i=1}^{n}\left(\frac{1-\chi(\mathfrak{p})t_{i,\mathfrak{p}}^{-1}N(\mathfrak{p})^{\frac{n-r+2}{2}}}{1-\chi(\mathfrak{p})t_{i,\mathfrak{p}}N(\mathfrak{p})^{\frac{r-n}{2}-1}}\right)\frac{L_{\mathfrak{v}}(\frac{3n-r}{2},\mathbf{f},\chi)}{\pi^{\beta}\Omega_{\mathbf{f}_0}},$$

And finally,

Theorem 8.5 *Assume that $\psi_1 = 1$ or $\mathfrak{c} = \mathfrak{g}$, and moreover $r \geqslant n$. Let $\mathfrak{q}$ be an ideal prime to p. Then there exist a measure $\mu^{-}_{\mathbf{f},\mathfrak{q},t}$ such that*

$$\int_G \chi\, d\mu^{-}_{\mathbf{f},\mathfrak{q}} = \frac{A^{-}(\chi)B^{-}(\chi)}{C_0(\mathfrak{m}_\chi)}\left(\prod_{\mathfrak{p}\mid\mathfrak{f}_\chi}\alpha(\mathfrak{p})^{-n_{\mathfrak{p}}}\right)\tau(\chi)^{-n\rho}\times$$

$$\prod_{\substack{i=0\\ n+i\equiv 1\, mod\, 2}}^{n-1}(1-(\chi\psi)_1(\mathfrak{q})N(\mathfrak{q})^{r+i+1-n})\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}\prod_{i=1}^{n}\left(\frac{1-\chi(\mathfrak{p})^{-1}t_{i,\mathfrak{p}}^{-1}N(\mathfrak{p})^{\frac{n-r+2}{2}}}{1-\chi(\mathfrak{p})t_{i,\mathfrak{p}}N(\mathfrak{p})^{\frac{r-n}{2}-1}}\right)\times$$

$$\frac{L_{\mathfrak{v}}(\frac{3n-r}{2},\mathbf{f},\chi)}{\pi^{\beta}\Omega_{\mathbf{f}_0}}.$$

Remark 8.6 We remark that in the interpolation properties above, at the modified Euler factors above p, we use the Satake parameters of the Hermitian form $\mathbf{f}_0$, and not of $\mathbf{f}$. However Theorem 7.3 provides a relation between them.

The rest of this section is devoted to proving the above theorems. We will establish in details the proof of Theorem 8.2 and then comment on the needed modifications to establish the rest.

We define,

$$\mathcal{F}_\chi^{+} := \Theta^{*}\mathbf{E}_{+}^{*}|\left(\prod_{\mathfrak{p}\mid\mathfrak{f}_\chi}V(\pi_{\mathfrak{p}})^{n_{\mathfrak{p}}-1}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}C(\mathfrak{p},s_{+})^{-1}\widetilde{J(\mathfrak{p},s_{+})}\right),$$

and

$$\mathcal{F}_\chi^{-} := \Theta^{*}\mathbf{E}_{-}^{*}|\left(\prod_{\mathfrak{p}\mid\mathfrak{f}_\chi}V(\pi_{\mathfrak{p}})^{n_{\mathfrak{p}}-1}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}C(\mathfrak{p},s_{-})^{-1}\widetilde{J(\mathfrak{p},s_{-})}\right),$$

where Θ^* and $\mathbf{E}^*_{\pm}$ are the series defined at the beginning of Sect. 6, associated to the character χ, and $C(\mathfrak{p}, s_{\pm})$ is defined in Theorem 6.7. We now define the following distribution on G, which later we will show it is actually a measure. For the definition of the distribution it is enough to give the values at each character χ of infinite type t.

$$\int_G \chi d\mu'_{\mathbf{f},+,t} := \frac{1}{\pi^\beta \Omega_{\mathbf{f}_0}} A^+(\chi) \left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right) \left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-2}\right) \tau(\chi)^{-n\rho} \times$$

$$< \mathbf{f}_0 | W, \mathcal{F}^+_\chi | \prod_{\mathfrak{p}|\mathfrak{f}_\chi} U(\mathfrak{p})^{n_\mathfrak{p}-1} >_{\mathfrak{m}_0},$$

We now show that $\mu'_{\mathbf{f},+,t}$ is actually a measure. We start by recalling the classical Kummer congruences (see [24]). Let Y be a profinite topological space, and R a p-adic ring.

Proposition 8.7 (abstract Kummer congruences) *Suppose R is flat over $\mathbb{Z}_p$, and let $\{f_i\}_{i\in I}$ be a collection of elements of $Cont(Y, R)$, whose $R[1/p]$-span is uniformly dens in $Cont(Y, R[1/p])$. Let $\{a_i\}_{i\in I}$ be a family elements of R with the same indexing set I. Then there exists an R-valued p-adic measure μ on Y such that*

$$\int_Y f_i d\mu = a_i, \quad \forall i \in I$$

if and only if the a_i's satisfy the following "Kummer congruences":

for every collection $\{b_i\}_{i\in I}$ of elements in $R[1/p]$ which are zero for all but finitely many i, and every integer n such that

$$\sum_i b_i f_i(y) \in p^n R, \quad \forall y \in Y,$$

we have

$$\sum_i b_i a_i \in p^n R.$$

Proof [24] □

Proposition 8.8 *The distribution $\mu'_{\mathbf{f},+,t}$ is a measure.*

Proof We establish the Kummer congruences. We first start with a remark. For a character χ of conductor $\mathfrak{f}_\chi = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{n_p}$ we consider any vector $c = (c_\mathfrak{p})_{\mathfrak{p}\in S}$ with $c_\mathfrak{p} \in \mathbb{Z}$, and $c_\mathfrak{p} \geqslant max(n_\mathfrak{p}, 1)$ for all $\mathfrak{p} \in S$. Then, by the same considerations as in the proof of Proposition 6.10, we have that

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right) \left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-2}\right) < \mathbf{f}_0 | W, \mathcal{F}^+_\chi | \prod_{\mathfrak{p}|\mathfrak{f}_\chi} U(\mathfrak{p})^{n_\mathfrak{p}-1} >_{\mathfrak{m}_0}$$

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_\mathfrak{p}-1}\right) \left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-2}\right) < \mathbf{f}_0 | W, \mathcal{F}^+_\chi | \prod_{\mathfrak{p}|\mathfrak{f}_\chi} U(\mathfrak{p})^{n_\mathfrak{p}} >_{\mathfrak{m}_0} =$$

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-c_\mathfrak{p}-1}\right)\left(\prod_{\mathfrak{p}\not|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-c_\mathfrak{p}-1}\right) \times$$

$$< \mathbf{f}_0|W, \mathcal{F}_\chi^+|\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} U(\mathfrak{p})^{c_\mathfrak{p}}\right)\left(\prod_{\mathfrak{p}\not|\mathfrak{f}_\chi} U(\mathfrak{p})^{c_\mathfrak{p}-1}\right) >_{\mathfrak{m}_0} .$$

We now consider a finite set of characters χ_i with $i = 1, \dots \ell$ of conductors $\mathfrak{f}_{\chi_i} = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{n_{\mathfrak{p},i}}$. We define $c = (c_\mathfrak{p})_{\mathfrak{p}\in S}$ with $c_\mathfrak{p} := max(max_i(n_{\mathfrak{p},i}), 1)$. We now let $\mathcal{O}$ be a large enough p-adic ring and take elements $a_i \in \mathcal{O}[1/p]$ such that

$$\sum_{i=0}^{\ell} a_i \chi_i \in p^m\mathcal{O}$$

for some $m \in \mathbb{N}$. We then establish the congruences

$$\sum_{i=0}^{\ell} a_i A^+(\chi_i)\tau(\chi)^{-n\rho}\mathcal{F}_{\chi_i}^+|\left(\prod_{\mathfrak{p}|\mathfrak{f}_{\chi_i}} U(\mathfrak{p})^{c_\mathfrak{p}}\right)\left(\prod_{\mathfrak{p}\not|\mathfrak{f}_\chi} U(\mathfrak{p})^{c_\mathfrak{p}-1}\right) \in p^m\mathcal{O}[[q]].$$

The above statement should be understood that the q-expansion of the Hermitian modular form on the left has coefficients in $p^m\mathcal{O}$.

The first observation here is that by Theorem 6.7 and by the discussion right after Proposition 3.5, the Fourier expansion for all

$$\mathcal{G}_i := \mathcal{F}_{\chi_i}^+|\left(\prod_{\mathfrak{p}|\mathfrak{f}_{\chi_i}} U(\mathfrak{p})^{c_\mathfrak{p}}\right)\left(\prod_{\mathfrak{p}\not|\mathfrak{f}_\chi} U(\mathfrak{p})^{c_\mathfrak{p}-1}\right),$$

is supported at the same Hermitian matrices. That is, the sets $Supp_i := \{(\tau, q) : c(\tau, q; \mathcal{G}_i) \neq 0\}$ for $i = 1, \dots, \ell$, are the same.
We note here that we need to apply one power less of the Hecke operators $U(\mathfrak{p})$ at the primes $\mathfrak{p}$ which divide $\mathfrak{f}_\chi$, since for the rest we have already applied $U(\mathfrak{p})$ as the n'th term of the operator $J(\mathfrak{p}, s_+)$.

It now follows from the explicit description of the Fourier coefficients given in Propositions 3.1 and 3.5 and by Eq. 20 that the coefficients of $A^+(\chi_i)\tau(\chi_i)^{-n\rho}\mathcal{F}_{\chi_i}^+$ are all p-integral and that we have the congruences

$$\sum_{i=0}^{\ell} a_i A^+(\chi_i)\tau(\chi)^{-\rho}\mathcal{F}_{\chi_i}|\left(\prod_{\mathfrak{p}|\mathfrak{f}_{\chi_i}} U(\mathfrak{p})^{c_\mathfrak{p}}\right)\left(\prod_{\mathfrak{p}\in S} U(\mathfrak{p})^{c_{\mathfrak{p}-1}}\right) \in p^m\mathcal{O}[[q]].$$

Indeed, let us write R for the “polynomial” ring $\mathcal{O}[\mathfrak{q}|\mathfrak{q} \in \mathcal{P}_S]$, in the variables $\mathfrak{q} \in \mathcal{P}_S$. where $\mathcal{P}_S$ is the set of prime ideals of K not in the set S. A character χ of G, induces then a ring homomorphism $\chi_R : R \to \overline{\mathbb{Q}}_p$, where we have extended $\mathcal{O}$- linear the multiplicative map $\chi : \mathcal{P}_S \to \overline{\mathbb{Q}}_p^{\times}$. Given an element $P \in R$ we write $P(\chi)$ for $\chi_R(P) \in \overline{\mathbb{Q}}_p$. Then by Propositions 3.5 and 3.1 we have that the Fourier coefficients of $A^+(\chi_i)\tau(\chi_i)^{-n\rho}\mathcal{F}^+_{\chi_i}$ at any given Hermitian matrix τ are of the form $P_{\tau_1}(\chi_i)P_{\tau_2}(\chi_i) = P_\tau(\chi_i)$ for some P_{τ_i}, $P_\tau \in R$, with $P_\tau = P_{\tau_1}P_{\tau_2}$. In particular if we have $\sum_i a_i\chi_i \in p^m\mathcal{O}$ then $\sum_i a_i P_\tau(\chi_i) \in p^m\mathcal{O}$. We also remark here that we need to use also Proposition 3.3, which guarantees that the coefficients of the Eisenstein series are supported only at full rank Hermitian matrices, and hence no L-values of Dirichlet series appear in the Fourier coefficients (and so the polynomial description above is enough). Moreover we also use the fact that the operator $U(\mathfrak{p})$ is p-integral as it was shown using the q-expansion in Eq. 19, where in the notation there $U(\mathfrak{p})^m = T(\mathfrak{p}^m)$ for any $m \in \mathbb{N}$ and $\mathfrak{p} \in S$.

It is now a standard argument using the finite dimension of the space of cusp forms of a particular level (see for example [2, Lemma 9.7] or [11, p. 134]) to show that by taking projection to $\mathbf{f}_0|W$ we obtain a measure. For this of course we use also by Theorem 5.1, $\Omega(\mathbf{f}_0)$ is up to algebraic factor equal to $< \mathbf{f}_0, \mathbf{f}_0 >$. Hence we conclude that $\mu'_{\mathbf{f},t,+}$ is indeed a measure. $\square$

We now define the measure $\mu_{\mathbf{g}}$ on G by

$$\int_G \chi d\mu_{\mathbf{g}} := \prod_{v\in\mathbf{b}} g_v(\chi(\pi_v)|\pi_v|^{r+n}),$$

where $g_v(X)$ are the polynomials appearing in Theorem 4.1. Note that $g_v \in \mathbb{Z}[X]$ with $g_v(0) = 1$, and hence since we evaluate then at places prime to p, we have that $\mu_{\mathbf{g}}$ is indeed a measure. We now define are measure $\mu^+_{\mathbf{f},2}$ as the convolution of $\mu'_{\mathbf{f},+}$ with $\mu_{\mathbf{g}}$. In particular we now obtain after evaluating at a character χ that,

$$\int_G \chi d\mu^+_{\mathbf{f},2} = \left(\int_G \chi d\mu'_{\mathbf{f},+}\right)\left(\int_G \chi d\mu_{\mathbf{g}}\right) =$$

$$\frac{1}{\pi^\beta\Omega_{\mathbf{f}_0}} A^+(\chi)\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi}\alpha(\mathfrak{p})^{-n_\mathfrak{p}}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}\alpha(\mathfrak{p})^{-2}\right)\tau(\chi)^{-n\rho}\times$$

$$< \mathbf{f}_0|W, \mathcal{F}^+_\chi|\prod_{\mathfrak{p}|\mathfrak{f}_\chi}U(\mathfrak{p})^{n_\mathfrak{p}-1} >_{\mathfrak{m}_0} \prod_{v\in\mathbf{b}} g_v(\chi(\pi_v)|\pi_v|^{r+n}).$$

However we have by Proposition 6.10, by taking there $n_{\mathfrak{p}} = c_{\mathfrak{p}}$ for all $\mathfrak{p}|\mathfrak{f}_\chi$ that

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_{\mathfrak{p}}}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-2}\right) < \mathbf{f}_0|W, \mathcal{F}_\chi^+ >_{\mathfrak{m}_0} =$$

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_{\mathfrak{p}}}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-2}\right) < \mathbf{f}_0|\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} J(\mathfrak{p}, s_+), \Theta_\chi \mathbf{E}_{+,\chi} >_{\mathfrak{m}_\chi},$$

and using Lemma 6.3, and in particular Eq. 9, we get that the above is equal to

$$\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_{\mathfrak{p}}}\right)\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-1}\right)\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}(-1)^n N(\mathfrak{p})^{n(2n-1)-n(\frac{n+r}{2})} \times$$

$$\prod_{i=1}^{n}(1-\chi(\mathfrak{p})^{-1}t_i^{-1}N(\mathfrak{p})^{\frac{r-n+2}{2}}) < \mathbf{f}_0, \Theta_\chi \mathbf{E}_{\chi,+} >_{\mathfrak{m}_\chi} .$$

We now use Theorem 4.1, where we pick an invertible τ such that $c(\tau, r, \mathbf{f}_0) \neq 0$, which is of course always possible since $\mathbf{f}_0$ is a cusp form. Moreover after using the fact that $c(\tau, \pi r, \mathbf{f}_0) = N(\mathfrak{p})^{-n^2}\alpha(\mathfrak{p})c(\tau, r, \mathbf{f}_0)$ we have that

$$\int_G \chi d\mu^+_{\mathbf{f},2} = B \times C_0^{-1}\left(\prod_{\mathfrak{p}|\mathfrak{f}_\chi} \alpha(\mathfrak{p})^{-n_{\mathfrak{p}}}\right) A^+(\chi)\tau(\chi)^{-n\rho} \times$$

$$\left(\prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi} N(\mathfrak{p})^{n(2n-1)-n(\frac{n+r}{2})-n^2}\prod_{i=1}^{n}(1-\chi(\mathfrak{p})^{-1}t_i^{-1}N(\mathfrak{p})^{\frac{r-n+2}{2}})\right)\frac{L(\frac{r+n}{2}, \mathbf{f}_0, \chi)}{\pi^\beta \Omega_{\mathbf{f}_0}},$$

where B is some non-zero algebraic constant independent of χ. We then define the measure $\mu_{\mathbf{f},t,+} := B^{-1}\mu_{\mathbf{f},2}$. Using the fact that $\mathbf{f}$ and $\mathbf{f}_0$ have the same Satake parameters away from p, we obtain the claimed interpolation properties of Theorem 8.2.

The proofs of Theorems 8.3, 8.4 and 8.5 are similar, we just need to take some extra care for the fact that in the Fourier coefficients of the Eisenstein series involve values of various Dirichlet series. In order to establish the congruences we use the Barsky, Cassou-Noguès, Deligne–Ribet p-adic L-function [1, 10, 14]. Let us write $F(p^\infty)$ for the maximal abelian extension of F unramified outside p and infinity. Then it is known that if we pick an ideal $\mathfrak{q}$ of F prime to p, then there exists a measure $\mu_{F,\mathfrak{q}}$ of the Galois group $G' := \mathrm{Gal}(F(p^\infty)/F)$, such that for any $k \geqslant 1$ we have,

$$\int_{G'} \chi N^k d\mu_{F,\mathfrak{q}} = \left(1-\chi(\mathfrak{q})N(\mathfrak{q})^k\right) L_{(p)}(1-k, \chi),$$

where $\mathcal{N}$ denotes the cyclotomic character. Moreover if we select some primitive character ψ, of some non-trivial conductor prime to p, then we can define a twisted measure $\mu_{F,\psi}$ on G' such that for any $k \geqslant 1$ we have,

$$\int_{G'} \chi \mathcal{N}^k d\mu_{F,\psi} = L_{(p)}(1-k, \chi\psi),$$

where in both equations $L_{(p)}(1-k, ?)$ means that we remove the Euler factors above p. Now we are ready to deal with the proof of the theorems. We explain it for Theorem 8.3, and similarly we argue for the rest. The main difference is the fact that the τ'th Fourier expansion of $\Theta^*\mathbf{E}^*$ is of the form $P_{\tau_1}(\chi)P_{\tau_2}(\chi)$ (with notation as before) multiplied by the L-values $\prod_{i=0}^{n-1-r_2} L_{\mathfrak{c}}(-i, \chi_1\theta^{n+i-1})$, where r_2 is the rank of the matrix τ_2. That is we need to establish congruences of the form

$$\sum_i a_i P_{\tau_1}(\chi_i)P_{\tau_2}(\chi_i) \prod_{\substack{i=0 \\ i+n\equiv 1 \bmod 2}}^{n-1} (1 - \chi_{i,1}^{-1}(\mathfrak{q})N(\mathfrak{q})^{i+1}) \times$$

$$\prod_{i=0}^{n-1-r_2} L_{\mathfrak{c}}(-i, \chi_{i,1}^{-1}\theta^{n+i-1}) \in p^n\mathcal{O}.$$

But now the congruences follow from the existence of the Cassou-Nogues, Deligne–Ribet p-adic measure since the above congruences can be understood as convolution (which we denote as product below) of the measures

$$\left(\prod_{\substack{i=0 \\ i+n\equiv 1 \bmod 2}}^{n-1} \mathcal{N}^{i+1}\mu_{F,\mathfrak{q}} \right) \star \left(\prod_{i=0}^{n-1-r_2} \mathcal{N}^{i+1}\mu_{F,\theta^{n+i-1}} \right) \star P,$$

where P is the measure in the Iwasawa algebra represented by the polynomial $P_{\tau_1} \times P_{\tau_2} \in R$, where the Iwasawa algebra. The rest of the proof is entirely identical where of course we need to replace the quantities $A^+(\chi)$ and $B^+(\chi)$ with $A^-(\chi)$ and $B^-(\chi)$ respectively.

9 The Values of the p-adic Measures

We now obtain a result regarding the values of the p-adic measures constructed above. We show the following theorem.

Theorem 9.1 *Write μ for any of the measures constructed in Theorems 8.2, 8.3, 8.4 and 8.5. Define the normalized measure*

$$\mu' := \left(\tau(\psi_1\theta^{n^2})\right)^{-\rho} i^{-n\sum_{v\in\mathbf{a}} p_v}\mu,$$

where the p_v's are defined as in Theorem 5.2. Assume that one of the cases of Theorem 5.2 occurs. Then μ' is W-valued, where W is the field appearing in the Theorem 5.2.

Proof By comparing the interpolation properties of the measure μ' and the reciprocity law shown in Theorem 5.3, we need only to establish that the Gauss sums $\tau(\chi_1)$ and $\tau(\chi)$ have the same reciprocity properties, namely $\left(\frac{\tau(\chi)}{\tau(\chi_1)}\right)^{\sigma} = \frac{\tau(\chi^{\sigma})}{\tau(\chi_1^{\sigma})}$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/W)$, and any character χ of G. For the proof we follow the strategy sketched in [17, p. 33] and [28, p. 105].

We first recall a property (see [26, p. 36]) of the transfer map,

$$det(\rho) = \theta \cdot \chi \circ Ver = \theta \cdot \chi_1,$$

where $\rho := Ind_F^K(\chi)$ is the two-dimensional representation induced from K to F, and for the second equakity we used the fact that the restriction $F_{\mathbb{A}}^{\times} \hookrightarrow K_{\mathbb{A}}^{\times}$ on the automorphic side is the transfer map (Ver) on the Galois side. We note here that the result in [26] is more general but we have applied it to our special case (i.e. χ is a one-dimensional representation and the extension K/F is quadratic). Recalling that the gauss sum attached to a character is closely related to the Deligne–Langlands epsilon factor attached to the same character, we have that

$$\tau(det(\rho)) = \tau(\theta\chi_1) = \pm\tau(\chi_1)\tau(\theta),$$

where we have used the fact that K/F is unramified above p, χ_1 can be ramified only above p, θ is a quadratic character, and the property [32, p. 15, Eq. (3.4.6)]. Now we note that by [13, p. 330, Eq. 5.5.1 and 5.5.2] we have that

$$\left(\frac{\tau(\rho)}{\tau(det(\rho))}\right)^{\sigma} = \frac{\tau(\rho^{\sigma})}{\tau(det(\rho^{\sigma}))}$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We note here that we write $\tau(\rho)$ for the Deligne–Langlands epsilon factor associated to the representation ρ. In particular since $\tau(\theta) \in W$ we have that

$$\left(\frac{\tau(\rho)}{\tau(\chi_1)}\right)^{\sigma} = \frac{\tau(\rho^{\sigma})}{\tau(\chi_1^{\sigma})},$$

and now using the fact that also $\tau(\rho) = \tau(\chi)$ up to elements in $W^{\times}$ we conclude that

$$\left(\frac{\tau(\chi)}{\tau(\chi_1)}\right)^{\sigma} = \frac{\tau(\chi^{\sigma})}{\tau(\chi_1^{\sigma})}, \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/W),$$

which concludes the proof of the theorem. □

References

1. Barsky, D.: Fonctions zêta p-adiques d'une classe de rayon des corps de nombres totalement réels. In: Amice, Y., Barskey, D., Robba, P. (eds.) Groupe d'Etude d'Analyse Ultramétrique (5e année) (1977/78)
2. Böcherer, S., Schmidt, C.-G.: p-adic measures attached to Siegel modular forms. Annales de l' institut Fourier, tome **50**(5), 1375–1443 (2000)
3. Bouganis, Th: On special L-values attached to Siegel modular forms. In: Bouganis, Th., Venjakob, O. (eds.) Iwasawa Theory 2012-state of the art and recent developments. Springer (2014)
4. Bouganis, Th: Non-abelian p-adic L-functions and Eisenstein series of unitary groups; the CM method. Ann. Inst. Fourier (Grenoble) **64**(2), 793–891 (2014)
5. Bouganis, Th.: On special L-values attached to metaplectic modular forms (submitted)
6. Bouganis, Th: On the algebraicity of special L-values of Hermitian modular forms. Documenta Mathematica **20**, 1293–1329 (2015)
7. Bouganis, Th.: On p-adic measures for Hermitian and Siegel modular forms (in preparation)
8. Bump, D.: Hecke Algebras (Notes available on-line)
9. Bump, D.: Automorphic Forms and Representations. Cambridge Stud. Adv. Math. **53**, CUP (1998)
10. Cassou-Noguès, P.: Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. Inventiones Mathematicae **51**(1), 29–59 (1979)
11. Coates, J., Schmidt, C.-G.: Iwasawa Theory for the symmetric square of an elliptic curve. J. Reine Angew. Math. 104–156
12. Courtieu, M., Panchishkin, A.: Non-Archimedean L-functions and Arithmetical Siegel Aodular Forms. 2nd edn. Lecture Notes in Mathematics, 1471. Springer, Berlin (2004)
13. Deligne, P.: Valeurs de Fonctions L et Périodes d'Intégrales. In: Proceedings of Symposia in Pure Mathematics, vol. 33, part 2, pp. 313–346 (1979)
14. Deligne, P., Ribet, K.: Values of abelian L-functions at negative integers over totally real fields. Inventiones Mathematicae **59**(3) (1980)
15. Eischen, E.: A p-adic Eisenstein measure for unitary groups. J. Reine Angew. Math. **699**, 111–142 (2015)
16. Eischen, E.: p-adic differential operators on automorphic forms on unitary groups. Ann. Inst. Fourier (Grenoble) **62**(1), 177–243 (2012)
17. Harder, G., Schappacher, N.: Special values of Hecke L-functions and abelian integrals. Springer Lecture Notes in Mathematics, vol. 1111 (1985)
18. Harris, M.: L-functions and periods of polarized regular motives. J. reine angew. Math. **483**, 75–161 (1997)
19. Harris, M.: A simple proof of rationality of Siegel–Weil Eisenstein series, Eisenstein series and applications, pp. 149–185, Progr. Math., 258, Birkhäuser Boston, Boston, MA (2008)
20. Harris, M., Li, J.-S., Skinner, C.: The Rallis inner product formula and p-adic L-functions. In: Rallis. S. (eds.) Automorphic Representations, L-Functions and Applications: Progress and Prospects. de Gruyter, Berlin, pp. 225–255 (2005)
21. Harris, M., Li, J.-S., Skinner, C.: p-adic L-functions for unitary Shimura varieties. In: Construction of the Eisenstein measure, Documenta Math., Extra Volume: John H.Coates' Sixtieth Birthday pp. 393–464 (2006)
22. Hida, H.: Elementary theory of L-functions and Eisenstein series. London Mathematical Society, Student Texts 26, CUP (1993)
23. Katsurada, H.: On the period of the Ikeda lift for $U(m, n)$. http://arxiv.org/abs/1102.4393
24. Katz, N.: p-adic L-functions for CM fields. Inventiones Math. **49**, 199–297 (1978)
25. Klosin, K.: Maass spaces on U(2,2) and the Bloch-Kato conjecture for the symmetric square motive of a modular form. J. Math. Soc. Jpn. **67**(2), 797–860 (2015)
26. Martinet, J.: Character theory and Artin L-functions. In: Fröhlich, A. (ed.) Algebraic Number Fields, Proceedings of LMS Symposium Durham, Academic Press (1977)
27. Panchishkin, A.: Non-Archimedean L-functions of Siegel and Hilbert Modular Forms. 1st edn. Lecture Notes in Mathematics, 1471. Springer, Berlin (1991)

28. Schappacher, N.: Periods of Hecke Characters. Lecture Notes in Mathematics, Springer, 1301 (1988)
29. Shimura, G.: Euler Products and Eisenstein Series. In: CBMS Regional Conference Series in Mathematics, No. 93. American Mathematical Society (1997)
30. Shimura, G.: Arithmeticity in the Theory of Automorphic Forms, Mathematical Surveys and Monographs, vol. 82. American Mathematical Society (2000)
31. Sturm, J.: The critical values of zeta functions associated to the symplectic group. Duke Math. J. **48**(2) (1981)
32. Tate, J.: Number Theoretic Background. In: Proceedings of Symposia in Pure Mathematics, vol. 33, part 2, pp. 3–26 (1979)

Springer

http://www.springer.com/978-3-319-45031-5

Elliptic Curves, Modular Forms and Iwasawa Theory
In Honour of John H. Coates' 70th Birthday, Cambridge, UK, March 2015
Loeffler, D.; Zerbes, S.L. (Eds.)
2016, VIII, 492 p. 14 illus., 1 illus. in color., Hardcover
ISBN: 978-3-319-45031-5