Chapter 2
Existence and Uniqueness for the Cauchy Problem

In this chapter, we will present some results concerning the existence, uniqueness and dependence on data of the solutions to the Cauchy problem for ODEs and systems of ODEs. From a mathematical point of view, this is a fundamental issue in the theory of differential equations. If we view a differential equation as a mathematical model of a physical theory, the existence of solutions to the Cauchy problem represents one of the first means of testing the validity of the model and, ultimately, of the physical theory. An existence result highlights the states and the minimal physical parameters that determine the evolution of a process and, often having a constructive character, it leads to numerical procedures for approximating the solutions. Basic references for this chapter are [1, 6, 9, 11, 18].

2.1 Existence and Uniqueness for First-Order ODEs

We begin by investigating the existence and uniqueness of solutions to a Cauchy problem in a special case, namely that of the scalar ODE (1.2) defined in a rectangle centered at $(t_0, x_0) \in \mathbb{R}^2$. In other words, we consider the Cauchy problem

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

where f is a real-valued function defined in the domain

$$\Delta := \Big\{ (t, x) \in \mathbb{R}^2; \quad |t - t_0| \leq a, \quad |x - x_0| \leq b \Big\}. \tag{2.2}$$

The central existence result for problem (2.1) is stated in our next theorem.

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Theorem 2.1 *Assume that the following hold:*

(i) *The function f is continuous on Δ.*
(ii) *The function f satisfies the Lipschitz condition in the variable x, that is, there exists an $L > 0$ such that*

$$\big|\, f(t,x) - f(t,y)\,\big| \le L|x-y|, \quad \forall (t,x),\ (t,y) \in \Delta. \tag{2.3}$$

Then there exists a unique solution $x = x(t)$ to the Cauchy problem (2.1) defined on the interval $|t - t_0| \le \delta$, where

$$\delta := \min\left(a, \frac{b}{M}\right), \quad M := \sup_{(t,x)\in\Delta} |f(t,x)|. \tag{2.4}$$

Proof We begin by observing that problem (2.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)ds. \tag{2.5}$$

Indeed, if the continuous function $x(t)$ satisfies (2.5) on an interval I, then it is clearly a C^1-function and satisfies the initial condition $x(t_0) = x_0$. The equality $x'(t) = f\big(t, x(t)\big)$ is then an immediate consequence of the Fundamental Theorem of Calculus. Conversely, any solution of (2.1) is also a solution of (2.5). Hence, to prove the theorem it suffices to show that (2.5) has a unique continuous solution on the interval $I := [t_0 - \delta, t_0 + \delta]$.

We will rely on the method of successive approximations used by many mathematicians, starting with Newton, to solve algebraic and transcendental equations. For the problem at hand, this method was successfully pioneered by *E. Picard* (1856–1941).

Consider the sequence of functions $x_n : I \to \mathbb{R}$, $n = 0, 1, \dots$, defined iteratively as follows

$$\begin{aligned} x_0(t) &= x_0, \quad \forall t \in I, \\ x_{n+1}(t) &= x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \quad \forall t \in I, \quad \forall n = 0, 1, \dots . \end{aligned} \tag{2.6}$$

It is easy to see that the functions x_n are continuous and, moreover,

$$\big|\, x_n(t) - x_0 \,\big| \le M\delta \le b, \quad \forall t \in I, \quad n = 1, 2, \dots . \tag{2.7}$$

This proves that the sequence $\{x_n\}_{n\ge 0}$ is well defined. We will prove that this sequence converges uniformly to a solution of (2.5). Using (2.6) and the Lipschitz condition (2.3), we deduce that

$$\begin{aligned}\big|\, x_n(t) - x_{n-1}(t)\,\big| &\leq \int_{t_0}^{t} \big|\, f(s, x_{n-1}(s)) - f(s, x_{n-2}(s)\,)\big| ds \\ &\leq L \left| \int_{t_0}^{t} \big|\, x_{n-1}(s) - x_{n-2}(s)\,\big| ds \right|.\end{aligned} \tag{2.8}$$

Iterating (2.8) and using (2.7), we have

$$\big|\, x_n(t) - x_{n-1}(t)\,\big| \leq \frac{ML^{n-1}}{n!}|t - t_0|^n \leq \frac{ML^{n-1}\delta^n}{n!}, \quad \forall n, \quad \forall t \in I. \tag{2.9}$$

Observe that the sequence $\{x_n\}_{n\geq 0}$ is uniformly convergent on I if and only if the telescopic series

$$\sum_{n\geq 1} \big(\, x_n(t) - x_{n-1}(t)\,\big)$$

is uniformly convergent on this interval. The uniform convergence of this series follows from (2.9) by invoking Weierstrass' M-test: the above series is majorized by the convergent numerical series

$$\sum_{n\geq 1} \frac{ML^{n-1}\delta^n}{n!}.$$

Hence the limit

$$x(t) = \lim_{n\to\infty} x_n(t)$$

exists uniformly on the interval I. The function $x(t)$ is continuous, and from the uniform continuity of the function $f(t, x)$ we deduce that

$$f\big(\,t, x(t)\,\big) = \lim_{n\to\infty} f\big(\,t, x_n(t)\,\big),$$

uniformly in $t \in I$. We can pass to the limit in the integral that appears in (2.6) and we deduce that

$$x(t) = x_0 + \int_{t_0}^{t} f\big(\,s, x(s)\big) ds, \quad \forall t \in I. \tag{2.10}$$

In other words, $x(t)$ is a solution of (2.5).

To prove the uniqueness, we argue by contradiction and assume that $x(t)$, $y(t)$ are two solutions of (2.5) on I. Thus

$$\big|x(t) - y(t)\big| = \left| \int_{t_0}^{t} f\big(s, x(s)\big) - f\big(s, y(s)\big) ds \right| \leq L \left| \int_{t_0}^{t} |x(s) - y(s)| ds \right|, \ \forall t \in I.$$

Using Gronwall's Lemma 1.1 with $\varphi \equiv 0$ and $\psi \equiv L$, we deduce $x(t) = y(t)$, $\forall t \in I$. $\square$

Remark 2.1 In particular, the Lipschitz condition (2.3) is satisfied if the function f has a partial derivative $\frac{\partial f}{\partial x}$ that is continuous on the rectangle Δ, or more generally, that is bounded on this rectangle.

Remark 2.2 We note that Theorem 2.1 is a local existence and uniqueness result for the Cauchy problem (2.1), that is the existence and uniqueness was proved on an interval $[t_0 - \delta, t_0 + \delta]$ which is smaller than the interval $[t_0 - a, t_0 + a]$ of the definition for the functions $t \to f(t, x)$.

2.2 Existence and Uniqueness for Systems of First-Order ODEs

Consider the differential system

$$x_i' = f_i(t, x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{2.11}$$

together with the initial conditions

$$x_i(t_0) = x_i^0, \quad i = 1, \ldots, n, \tag{2.12}$$

where the functions f_i are defined on a parallelepiped

$$\Delta := \{(t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1};\ |t - t_0| \le a,\ |x_i - x_i^0| \le b,\ i = 1, \ldots, n\}. \tag{2.13}$$

Theorem 2.1 generalizes to differential systems of type (2.11).

Theorem 2.2 *Assume that the following hold:*

(i) *The functions f_i are continuous on Δ for any $i = 1, \ldots, n$.*
(ii) *The functions f_i are Lipschitz in $\boldsymbol{x} = (x_1, \ldots, x_n)$ on Δ, that is, there exists an $L > 0$ such that*

$$\big| f_i(t, x_1, \ldots, x_n) - f_i(t, y_1, \ldots, y_n) \big| \le L \max_{1 \le k \le n} |x_k - y_k|, \tag{2.14}$$

for any $i = 1, \ldots, n$ and any $(t, x_1, \ldots, x_n), (t, y_1, \ldots, y_n) \in \Delta$.

Then there exists a unique solution $x_i = \varphi_i(t)$, $i = 1, \ldots, n$, of the Cauchy problem (2.11) and (2.12) *defined on the interval*

$$I := [t_0 - \delta, t_0 + \delta], \quad \delta := \min\left(a, \frac{b}{M}\right), \tag{2.15}$$

where $M := \max\big\{ |f_i(t, \boldsymbol{x})|; \quad (t, \boldsymbol{x}) \in \Delta, \quad i = 1, \ldots, n \big\}$.

Proof The proof of Theorem 2.2 is based on an argument very similar to the one used in the proof of Theorem 2.1. For this reason, we will only highlight the main steps.

We observe that the Cauchy problem (2.11) and (2.12) is equivalent to the system of integral equations

$$x_i(t) = x_i^0 + \int_a^t f_i\big(s, x_1(s), \ldots, x_n(s)\big)ds, \quad i = 1, \ldots, n. \tag{2.16}$$

To construct a solution to this system, we again use successive approximations

$$\begin{aligned} &x_i^k(t) = x_i^0 + \int_a^t f_i\big(s, x_1^{k-1}(s), \ldots, x_n^{k-1}(s)\big)ds, \; i = 1, \ldots, n, \; k \geq 1, \\ &x_i^0(s) \equiv x_i^0, \quad i = 1, \ldots, n. \end{aligned} \tag{2.17}$$

Arguing as in the proof of Theorem 2.1, we deduce that the functions $t \mapsto x_i^k(t)$ are well defined and continuous on the interval I. An elementary argument based on the Lipschitz condition yields the following counterpart of (2.9)

$$\max_{1 \leq i \leq n} |x_i^k(t)| \leq \frac{ML^{k-1}\delta^k}{k!}, \quad \forall k \geq 1, \quad t \in I.$$

Invoking as before the Weierstrass M-test, we deduce that the limits

$$\varphi_i(t) = \lim_{k \to \infty} x_i^k(t), \quad 1 \leq i \leq n,$$

exist and are uniform on I. Letting $k \to \infty$ in (2.17), we deduce that $(\varphi_1, \ldots, \varphi_n)$ is a solution of system (2.16), and thus also a solution of the Cauchy problem (2.11) and (2.12).

The uniqueness follows from Gronwall's Lemma (Lemma 1.1) via an argument similar to the one in the proof of Theorem 2.1. □

Both the statement and the proof of the existence and uniqueness theorem for systems do not seem to display meaningful differences when compared to the scalar case. Once we adopt the vector notation, we will see that there are not even formal differences between these two cases.

Consider the vector space $\mathbb{R}^n$ of vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ equipped with the norm (see Appendix A)

$$\|\boldsymbol{x}\| := \max_{1 \leq i \leq n} |x_i|, \quad \boldsymbol{x} = (x_1, \ldots, x_n). \tag{2.18}$$

On the space $\mathbb{R}^n$ equipped with the above (or any other) norm, we can develop a differential and integral calculus similar to the familiar one involving scalar functions.

Given an interval I, we define a vector-valued function $\boldsymbol{x} : I \to \mathbb{R}$ of the form

$$\boldsymbol{x}(t) = \big(x_1(t), \ldots, x_n(t)\big),$$

where $x_i(t)$ are scalar functions defined on I. The function $\boldsymbol{x} : I \to \mathbb{R}^n$ is called *continuous* if all its components $\{x_i(t);\ i = 1, \dots, n\}$ are continuous. The function $\boldsymbol{x}$ is called *differentiable* at t_0 if all its components x_i have this property. The derivative of $\boldsymbol{x}(t)$ at the point t, denoted by $\boldsymbol{x}'(t)$, is the vector

$$\boldsymbol{x}'(t) := \big(x_1'(t), \dots, x_n'(t) \big).$$

We can define the integral of the vector function in a similar fashion. More precisely,

$$\int_a^b \boldsymbol{x}(t)dt := \left(\int_a^b x_1(t)dt, \dots, \int_a^b x_n(t)dt \right) \in \mathbb{R}^n.$$

The sequence $\{\boldsymbol{x}^\nu\}$ of vector-valued functions

$$\boldsymbol{x}^\nu : I \to \mathbb{R}^n, \quad \nu = 0, 1, 2, \dots,$$

is said to *converge uniformly* (respectively *pointwisely*) to $\boldsymbol{x} : I \to \mathbb{R}^n$ as $\nu \to \infty$ if each component sequence has these properties. The space of continuous functions $x : I \to \mathbb{R}^n$ is denoted by $C(I; \mathbb{R}^n)$.

All the above notions have an equivalent formulation involving the norm $\| - \|$ of the space $\mathbb{R}^n$; see Appendix A. For example, the continuity of $\boldsymbol{x} : I \to \mathbb{R}^n$ at $t_0 \in I$ is equivalent to

$$\lim_{t \to t_0} \|\boldsymbol{x}(t) - \boldsymbol{x}(t_0)\| = 0.$$

The derivative, integral and the concept of convergence can be defined along similar lines.

Returning to the differential system (2.11), observe that, if we denote by $\boldsymbol{x}(t)$ the vector-valued function

$$\boldsymbol{x}(t) = (\, x_1(t), \dots, x_n(t)\,)$$

and by $\boldsymbol{f} : \varDelta \to \mathbb{R}^n$ the function

$$\boldsymbol{f}(t, \boldsymbol{x}) = \big(f_1(t, \boldsymbol{x}), \dots, f_n(t, \boldsymbol{x}) \big),$$

then we can rewrite (2.11) as

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \tag{2.19}$$

while the initial condition (2.12) becomes

$$\boldsymbol{x}(t_0) = \boldsymbol{x}^0 := (x_1^0, \dots, x_n^0). \tag{2.20}$$

In vector notation, Theorem 2.2 can be rephrased as follows.

Theorem 2.3 *Assume that the following hold:*

(i) *The function* $\boldsymbol{f} : \Delta \to \mathbb{R}^n$ *is continuous.*
(ii) *The function* $\boldsymbol{f}$ *is Lipschitz in the variable* $\boldsymbol{x}$ *on* Δ.

Then there exists a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ *of the system* (2.19) *satisfying the initial condition* (2.20) *and defined on the interval*

$$I := [t_0 - \delta, t_0 + \delta], \quad \delta := \min\left(a, \frac{b}{M}\right), \quad M := \sup_{(t,\boldsymbol{x})\in\Delta} \|\boldsymbol{f}(t, \boldsymbol{x})\|.$$

In this formulation, Theorem 2.3 can be proved by following word for word the proof of Theorem 2.1, with one obvious exception: where appropriate, we need to replace the absolute value $|-|$ with the norm $\|-\|$. In the sequel, we will systematically use the vector notation when working with systems of differential equations.

2.3 Existence and Uniqueness for Higher Order ODEs

Consider the differential equation of order n,

$$x^{(n)} = g(t, x, x', \ldots, x^{(n-1)}), \tag{2.21}$$

together with the Cauchy condition (see Sect. 1.1)

$$x(t_0) = x_0^0, \quad x'(t_0) = x_1^0, \ldots, \quad x^{(n-1)}(t_0) = x_{n-1}^0, \tag{2.22}$$

where $(t_0, x_0^0, x_1^0, \ldots, x_{n-1}^0) \in \mathbb{R}^{n+1}$ is fixed and the function g satisfies the following conditions.

(I) The function g is defined and continuous on the set

$$\Delta = \left\{ (t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}; \quad |t - t_0| \le a, \quad |x_i - x_{i-1}^0| \le b, \quad \forall i = 1, \ldots, n \right\}.$$

(II) There exists an $L > 0$ such that

$$|g(t, \boldsymbol{x}) - g(t, \boldsymbol{y})| \le L\|\boldsymbol{x} - \boldsymbol{y}\|, \quad \forall (t, \boldsymbol{x}), \; (t, \boldsymbol{y}) \in \Delta. \tag{2.23}$$

Theorem 2.4 *Assume that conditions* (I) *and* (II) *above hold. Then the Cauchy problem* (2.21) *and* (2.22) *admits a unique solution on the interval*

$$I := [t_0 - \delta, t_0 + \delta], \quad \delta := \min\left(a, \frac{b}{M}\right),$$

where

$$M := \sup_{(t,x)\in\Delta} \max\left\{ |g(t,\boldsymbol{x})|, |x_2|, \ldots, |x_n| \right\}.$$

Proof As explained before (see (1.10) and (1.11)), using the substitutions

$$x_1 := x, \quad x_2 = x', \ldots, x_n := x^{(n-1)},$$

the differential equation (2.21) reduces to the system of ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ \vdots &\;\vdots\; \vdots \\ x_n' &= g(t, x_1, \ldots, x_n), \end{aligned} \tag{2.24}$$

while the Cauchy condition becomes

$$x_i(t_0) = x_{i-1}^0, \quad \forall i = 1, \ldots, n. \tag{2.25}$$

In view of (I) and (II), Theorem 2.4 becomes a special case of Theorem 2.2. □

2.4 Peano's Existence Theorem

We will prove an existence result for the Cauchy problem due to *G. Peano* (1858–1932). Roughly speaking, it states that the continuity of f alone suffices to guarantee that the Cauchy problem (2.11) and (2.12) has a solution in a neighborhood of the initial point. Beyond its theoretical significance, this result will offer us the opportunity to discuss another important technique for investigating and approximating the solutions of an ODE. We are talking about the polygonal method, due essentially to *L. Euler* (1707–1783).

Theorem 2.5 *Let $f : \Delta \to \mathbb{R}^n$ be a continuous function defined on*

$$\Delta := \left\{ (t, \boldsymbol{x}) \in \mathbb{R}^{n+1}; \quad |t - t_0| \le a, \quad \|\boldsymbol{x} - \boldsymbol{x}_0\| \le b \right\}.$$

Then the Cauchy problem (2.11) and (2.12) admits at least one solution on the interval

$$I := [t_0 - \delta, t_0 + \delta], \quad \delta := \min\left(a, \frac{b}{M}\right), \quad M := \sup_{(t,\boldsymbol{x})\in\Delta} \|\boldsymbol{f}(t, \boldsymbol{x})\|.$$

Proof We will prove the existence on the interval $[t_0, t_0 + \delta]$. The existence on $[t_0 - \delta, t_0]$ follows by a similar argument.

Fix $\varepsilon > 0$. Since f is uniformly continuous on Δ, there exists an $\eta(\varepsilon) > 0$ such that

$$\|f(t, \boldsymbol{x}) - f(s, \boldsymbol{y})\| \leq \varepsilon,$$

for any $(t, \boldsymbol{x}), (s, \boldsymbol{y}) \in \Delta$ such that

$$|t - s| \leq \eta(\varepsilon), \quad \|\boldsymbol{x} - \boldsymbol{y}\| \leq \eta(\varepsilon).$$

Consider the uniform subdivision $t_0 < t_1 < \cdots < t_{N(\varepsilon)} = t_0 + \delta$, where $t_j = t_0 + jh_\varepsilon$, for $j = 0, \ldots, N(\varepsilon)$, and $N(\varepsilon)$ is chosen large enough so that

$$h_\varepsilon = \frac{\delta}{N(\varepsilon)} \leq \min\left(\eta(\varepsilon), \frac{\eta(\varepsilon)}{M}\right). \tag{2.26}$$

We consider the polygonal line, that is, the piecewise linear function $\boldsymbol{\varphi}_\varepsilon : [t_0, t_0 + \delta] \to \mathbb{R}^{n+1}$ defined by

$$\begin{aligned} &\boldsymbol{\varphi}_\varepsilon(t) = \boldsymbol{\varphi}_\varepsilon(t_j) + (t - t_j)\boldsymbol{f}\big(t, \boldsymbol{\varphi}_\varepsilon(t_j)\big), \; t_j < t \leq t_{j+1} \\ &\boldsymbol{\varphi}_\varepsilon(t_0) = \boldsymbol{x}_0. \end{aligned} \tag{2.27}$$

Notice that if $t \in [t_0, t_0 + \delta]$, then

$$\|\boldsymbol{\varphi}_\varepsilon(t) - \boldsymbol{x}_0\| \leq M\delta \leq b.$$

Thus $(t, \boldsymbol{\varphi}_\varepsilon(t)) \in \Delta$, $\forall t \in [t_0, t_0 + \delta]$, so that equalities (2.27) are consistent. Equalities (2.27) also imply the estimates

$$\|\boldsymbol{\varphi}_\varepsilon(t) - \boldsymbol{\varphi}_\varepsilon(s)\| \leq M|t - s|, \quad \forall t, x \in [t_0, t_0 + \delta]. \tag{2.28}$$

In particular, inequality (2.28) shows that the family of functions $(\boldsymbol{\varphi}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded and equicontinuous on the interval $[t_0, t_0 + \delta]$. Arzelà's theorem (see Appendix A.3) shows that there exist a continuous function $\boldsymbol{\varphi} : [t_0, t_0 + \delta] \to \mathbb{R}^n$ and a subsequence $(\boldsymbol{\varphi}_{\varepsilon_\nu})$, $\varepsilon_\nu \searrow 0$, such that

$$\lim_{\nu\to\infty} \boldsymbol{\varphi}_{\varepsilon_\nu}(t) = \boldsymbol{\varphi}(t) \;\; \text{uniformly on } [t_0, t_0 + \delta]. \tag{2.29}$$

We will prove that $\boldsymbol{\varphi}(t)$ is a solution of the Cauchy problem (2.11) and (2.12).

With this goal in mind, we consider the sequence of functions

$$g_{\varepsilon_\nu}(t) := \begin{cases} \boldsymbol{\varphi}'_{\varepsilon_\nu}(t) - \boldsymbol{f}(t, \boldsymbol{\varphi}_{\varepsilon_\nu}(t)), & t \neq t_j^\nu \\ 0, & t = t_j^\nu, \quad j = 0, 1, \ldots, N(\varepsilon_\nu), \end{cases} \tag{2.30}$$

where t_j^ν, $j = 0, 1, \ldots, N(\varepsilon_\nu)$, are the nodes of the subdivision corresponding to ε_ν. Equality (2.27) implies that

$$\varphi'_{\varepsilon_\nu}(t) = \boldsymbol{f}\big(t, \ \ \varphi_{\varepsilon_\nu}(t_j^\nu)\big), \ \ \forall t \in]t_j^\nu, t_{j+1}^\nu[,$$

and thus, invoking (2.26), we deduce that

$$\|g_{\varepsilon_\nu}(t)\| \le \varepsilon_\nu, \ \ \forall t \in [t_0, t_0 + \delta]. \tag{2.31}$$

On the other hand, the function g_{ε_ν}, though discontinuous, is Riemann integrable on $[t_0, t_0 + \delta]$ since its set of discontinuity points, $\{t_j^\nu\}_{0 \le j \le N(\varepsilon_\nu)}$, is finite. Integrating both sides of (2.30), we get

$$\varphi_{\varepsilon_\nu}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}\big(s, \varphi_{\varepsilon_\nu}(s)\big)ds + \int_{t_0}^{t} g_{\varepsilon_\nu}(s)ds, \ \ \forall t \in [t_0, t_0 + \delta]. \tag{2.32}$$

Since $\boldsymbol{f}$ is continuous on Δ and $\varphi_{\varepsilon_\nu}$ converge uniformly on $[t_0, t_0 + \delta]$, we have

$$\lim_{\nu \to \infty} \boldsymbol{f}\big(s, \varphi_{\varepsilon_\nu}(s)\big) = \boldsymbol{f}\big(s, \varphi(s)\big) \ \text{ uniformly in } s \in [t_0, t_0 + \delta].$$

Invoking (2.31), we can pass to the limit in (2.32) and we obtain the equality

$$\varphi(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}\big(s, \varphi(s)\big)ds.$$

In other words, the function $\varphi(t)$ is a solution of the Cauchy problem (2.11) and (2.12). This completes the proof of Theorem 2.5. □

An alternative proof. Consider a sequence $\boldsymbol{f}_\varepsilon : \Delta \to \mathbb{R}^n$ of continuously differentiable functions such that

$$\|\boldsymbol{f}_\varepsilon(t, \boldsymbol{x}) - \boldsymbol{f}_\varepsilon(t, \boldsymbol{y})\| \le L_\varepsilon \|\boldsymbol{x} - \boldsymbol{y}\|, \ \ \forall\, (t, \boldsymbol{x}), (t, \boldsymbol{y}) \in \Delta, \tag{2.33}$$

$$\lim_{\varepsilon \to 0} \|\boldsymbol{f}_\varepsilon(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{x})\| = 0, \ \ \text{uniformly on } \Delta. \tag{2.34}$$

An example of such an approximation $\boldsymbol{f}_\varepsilon$ of f is

$$\boldsymbol{f}_\varepsilon(t, \boldsymbol{x}) = \frac{1}{\varepsilon^n} \int_{[\boldsymbol{y};\ \|\boldsymbol{y} - \boldsymbol{x}_0\| \le b]} \boldsymbol{f}(t, \boldsymbol{y})\rho\left(\frac{\boldsymbol{x} - \boldsymbol{y}}{\varepsilon}\right) d\boldsymbol{y}, \ \ \forall\, (t, \boldsymbol{x}) \in \Delta,$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function such that $\int_{\mathbb{R}^n} \rho(\boldsymbol{x})d\boldsymbol{x} = 1, \ \rho(\boldsymbol{x}) = 0$ for $\|\boldsymbol{x}\| \ge 1$.) Then, by Theorem 2.3, the Cauchy problem

$$\begin{aligned} &\frac{d\boldsymbol{x}}{dt}(t) = \boldsymbol{f}_\varepsilon(t, \boldsymbol{x}(t)), \ t \in [t_0 = \delta, t_0 + \delta), \\ &\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \end{aligned} \tag{2.35}$$

has a unique solution $\boldsymbol{x}_\varepsilon$ on the interval $[t_0 - \delta, t_0 + \delta)$. By (2.34) and (2.35), it follows that

$$\left\| \frac{d}{dt} \boldsymbol{x}_\varepsilon(t) \right\| \le C, \quad \forall\, t \in [t_0 - \delta, t_0 + \delta], \quad \forall\, \varepsilon > 0,$$

where C is independent of ε. This implies that the family of functions $\{\boldsymbol{x}_\varepsilon\}$ is uniformly bounded and equicontinuous on $[t_0 - \delta, t_0 + \delta]$ and so, by Arzelà's theorem, there is a subsequence $\{\boldsymbol{x}_{\varepsilon_n}\}$ which is uniformly convergent for $\{\varepsilon_n\} \to 0$ to a continuous function $\boldsymbol{x} : [t_0 - \delta, t_0 + \delta]$. Then, by (2.35), it follows that

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))ds, \quad \forall\, t \in [t_0 - \delta, t_0 + \delta]$$

and so $\boldsymbol{x}$ is a solution to the Cauchy problem (2.11) and (2.12).

Remark 2.3 (*Nonuniqueness in the Cauchy problem*) We cannot deduce the uniqueness of the solution from the above proof since the family $(\boldsymbol{\varphi}_\varepsilon)_{\varepsilon>0}$ may contain several uniform convergent subsequences, each with its own limit. In general, assuming only the continuity of $\boldsymbol{f}$, we cannot expect uniqueness in the Cauchy problem. One example of nonuniqueness is offered by the Cauchy problem

$$x' = x^{\frac{1}{3}}, \quad x(0) = 0. \tag{2.36}$$

This equation has an obvious solution $x(t) = 0, \forall t$. On the other hand, as easily seen, the function

$$\varphi(t) = \begin{cases} \left(\dfrac{2t}{3}\right)^{\frac{3}{2}}, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

is also a solution of (2.36).

Remark 2.4 (*Numerical approximations*) If, in Theorem 2.5, we assume that $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$ is Lipschitz in $\boldsymbol{x}$, then according to Theorem 2.3 the Cauchy problem (2.19)–(2.20) has a *unique* solution. Thus, necessarily,

$$\lim_{\varepsilon \searrow 0} \boldsymbol{\varphi}_\varepsilon(t) = \boldsymbol{\varphi}(t) \ \text{ uniformly on } [t_0 - \delta, t_0 + \delta], \tag{2.37}$$

because any sequence of the family $(\boldsymbol{\varphi}_\varepsilon)_{\varepsilon>0}$ contains a subsequence that converges uniformly to $\boldsymbol{\varphi}(t)$. Thus, the above procedure leads to a numerical approximation scheme for the solution of the Cauchy problem (2.19)–(2.20), or equivalently (2.11) and (2.12). If h is fixed, $h = \frac{\delta}{N}$, and

$$t_j := t_0 + jh, \quad j = 0, 1, \ldots, N,$$

then we compute the approximations of the values of $\varphi(t)$ at the nodes t_j using (2.27), that is,

$$\varphi_{j+1} = \varphi_j + h\boldsymbol{f}\big(t_j, \varphi_j\big), \quad j = 0, 1, \ldots, N-1. \tag{2.38}$$

The iterative formulae (2.38) are known in numerical analysis as the *Euler scheme* and they form the basis of an important class of numerical methods for solving the Cauchy problem. Equalities (2.38) are also known as *difference equations*.

2.5 Global Existence and Uniqueness

We consider the system of differential equations described in vector notation by

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \tag{2.39}$$

where the function $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is continuous on the open subset $\Omega \subset \mathbb{R}^{n+1}$. Additionally, we will assume that $\boldsymbol{f}$ is *locally Lipschitz* in $\boldsymbol{x}$ on Ω, that is, for any compact set $K \subset \Omega$, there exists an $L_K > 0$ such that

$$\|\, \boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\,\| \le L_K \|\boldsymbol{x} - \boldsymbol{y}\|, \quad \forall (t, \boldsymbol{x}), (t, \boldsymbol{y}) \in K. \tag{2.40}$$

If $A, B \subset \mathbb{R}^m$, then the distance between them is defined by

$$\operatorname{dist}(A, B) = \inf\big\{\, \|\boldsymbol{a} - \boldsymbol{b}\|; \quad \boldsymbol{a} \in A, \quad \boldsymbol{b} \in B\,\big\}.$$

It is useful to remark that, if K is a compact subset of Ω, then the distance $\operatorname{dist}(K, \partial\Omega)$ from K to the boundary $\partial\Omega$ of Ω is strictly positive. Indeed, suppose that $(\boldsymbol{x}_\nu)$ is a sequence in K and $(\boldsymbol{y}_\nu)$ is a sequence in $\partial\Omega$ such that

$$\lim_{\nu\to\infty} \|\boldsymbol{x}_\nu - \boldsymbol{y}_\nu\| = \operatorname{dist}(K, \partial\Omega). \tag{2.41}$$

Since K is compact, the sequence $(\boldsymbol{x}_\nu)$ is bounded. Using (2.41), we deduce that the sequence $(\boldsymbol{y}_\nu)$ is also bounded. The Bolzano–Weierstrass theorem now implies that there exist subsequences $(\boldsymbol{x}_{\nu_k})$ and $(\boldsymbol{y}_{\nu_k})$ converging to $\boldsymbol{x}_0$ and respectively $\boldsymbol{y}_0$. Since both K and $\partial\Omega$ are closed, we deduce that $\boldsymbol{x}_0 \in K$, $\boldsymbol{y}_0 \in \partial\Omega$, and

$$\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| = \lim_{k\to\infty} \|\boldsymbol{x}_{\nu_k} - \boldsymbol{y}_{\nu_k}\| = \operatorname{dist}(K, \partial\Omega).$$

Since $K \cap \partial\Omega = \emptyset$, we conclude that $\operatorname{dist}(K, \partial\Omega) > 0$.

Returning to the differential system (2.39), consider $(t_0, \boldsymbol{x}_0) \in \Omega$ and a parallelepiped $\Delta \subset \Omega$ of the form

$$\Delta = \Delta_{a,b} := \Big\{ (t, \boldsymbol{x}) \in \mathbb{R}^{n+1}; \quad |t - t_0| \le a, \quad \|\boldsymbol{x} - \boldsymbol{x}_0\| \le b \Big\}.$$

(Since Ω is open, $\Delta_{a,b} \subset \Omega$ if a and b are sufficiently small.)

Applying Theorem 2.3 to system (2.39) restricted to Δ, we deduce the existence and uniqueness of a solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ satisfying the initial condition $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}_0$ and defined on an interval $[t_0 - \delta, t_0 + \delta]$, where

$$\delta = \min\left(a, \frac{b}{M}\right), \quad M = \sup_{(t,\boldsymbol{x}) \in \Delta} \|\boldsymbol{f}(t, \boldsymbol{x})\|.$$

In other words, we have the following local existence result.

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and assume that the function $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x}) : \Omega \to \mathbb{R}^n$ is continuous and locally Lipschitz as a function of $\boldsymbol{x}$. Then for any $(t_0, \boldsymbol{x}_0) \in \Omega$ there exists a unique solution $\boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ of (2.39) defined on a neighborhood of t_0 and satisfying the initial condition*

$$\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)\Big|_{t=t_0} = \boldsymbol{x}_0.$$

We must emphasize the local character of the above result. As mentioned earlier in Remark 2.2, both the existence and the uniqueness of the Cauchy problem take place in a neighborhood of the initial moment t_0. However, we expect the uniqueness to have a global nature, that is, if two solutions $\boldsymbol{x} = \boldsymbol{x}(t)$ and $\boldsymbol{y} = \boldsymbol{y}(t)$ of (2.39) are equal at a point t_0, then they should coincide on the common interval of existence. (Their equality on a neighborhood of t_0 follows from the local uniqueness result.)

The next theorem, which is known in the literature as the *global uniqueness theorem*, states that global uniqueness holds under the assumptions of Theorem 2.6.

Theorem 2.7 *Assume that $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ satisfies the assumptions in Theorem 2.6. If $\boldsymbol{x}$, $\boldsymbol{y}$ are two solutions of (2.39) defined on the open intervals I and J, respectively, and if $\boldsymbol{x}(t_0) = \boldsymbol{y}(t_0)$ for some $t_0 \in I \cap J$, then $\boldsymbol{x}(t) = \boldsymbol{y}(t)$, $\forall t \in I \cap J$.*

Proof Let $(t_1, t_2) = I \cap J$. We will prove that $\boldsymbol{x}(t) = \boldsymbol{y}(t), \forall t \in [t_0, t_2)$. The equality to the left of t_0 is proved in a similar fashion. Let

$$\mathcal{T} := \Big\{ \tau \in [t_0, t_2); \quad \boldsymbol{x}(t) = \boldsymbol{y}(t); \quad \forall t \in [t_0, \tau] \Big\}.$$

Then $\mathcal{T} \neq \emptyset$ and we set $T := \sup \mathcal{T}$. We claim that $T = t_2$.

To prove the claim, we argue by contradiction. Assume that $T < t_2$. Then $\boldsymbol{x}(t) = \boldsymbol{y}(t)$, $\forall t \in [t_0, T]$, and since $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are both solutions of (2.39), we deduce from Theorem 2.6 that there exists a $\varepsilon > 0$ such that $\boldsymbol{x}(t) = \boldsymbol{y}(t)$, $\forall t \in [T, T + \varepsilon]$. This contradicts the maximality of T and concludes the proof of the theorem. □

Remark 2.5 If the function $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is of class C^{k-1} on the domain Ω, then, obviously, the local solution of system (2.39) is of class C^k on the interval it is defined. Moreover, if $\boldsymbol{f}$ is real analytic on Ω, that is, it is C^∞, and the Taylor series of $\boldsymbol{f}$ at any point $(t_0, \boldsymbol{x}_0) \in \Omega$ converges to $\boldsymbol{f}$ in a neighborhood of that point, then any solution $\boldsymbol{x}$ of (2.39) is also real analytic.

This follows by direct computation from equations (2.39), and, using the fact that a real function $g = g(x_1, \dots, x_m)$ defined on a domain D of $\mathbb{R}^m$ is real analytic if and only if, for any compact set $K \subset D$ and any positive integer k, there exists a positive constant $M(k)$ such that for any multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$ we have

$$\left| \frac{\partial^{|\alpha|} f(\boldsymbol{x})}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}} \right| \leq M\big(|\alpha|\big)^{|\alpha|} \alpha!, \quad \forall \boldsymbol{x} = (x_1, \dots, x_m) \in K,$$

where

$$|\alpha| := \alpha_1 + \cdots + \alpha_m, \quad \alpha! := \alpha_1! \cdots \alpha_m!$$

A solution $\boldsymbol{x} = \varphi(t)$ of (2.39) defined on the interval $I = [a, b]$ is called *extendible* if there exists a solution $\psi(t)$ of (2.39) defined on an interval $J \supsetneq I$ such that $\varphi = \psi$ on I. The solution φ is called *right-extendible* if there exists $b' > b$ and a solution ψ of (2.39), defined on $[a, b']$, such that $\psi = \varphi$ on $[a, b]$. The notion of *left-extendible* solutions is defined analogously. A solution that is not extendible is called *saturated*. In other words, a solution φ defined on an interval I is saturated if I is its maximal domain of existence. Similarly, a solution that is not right-extendible (respectively left-extendible) is called *right-saturated* (respectively *left-saturated*).

Theorem 2.6 implies that a maximal interval on which a saturated solution is defined must be an open interval. If a solution φ is right-saturated, then the interval on which it is defined is open on the right. Similarly, if a solution φ is left-saturated, then the interval on which it is defined is open on the left.

Indeed, if $\varphi : [a, b) \to \mathbb{R}^n$ is a solution of (2.39) defined on an interval that is not open on the left, then Theorem 2.6 implies that there exists a solution $\widetilde{\varphi}(t)$ defined on an interval $[a - \delta, a + \delta]$ satisfying the initial condition $\widetilde{\varphi}(a) = \varphi(a)$. The local uniqueness theorem implies that $\widetilde{\varphi} = \varphi$ on $[a, a + \delta]$ and thus the function

$$\widehat{\varphi}_0(t) = \begin{cases} \varphi(t), & t \in [a, b), \\ \widetilde{\varphi}(t), & t \in [a - \delta, a], \end{cases}$$

is a solution of (2.39) on $[a - \delta, b)$ that extends φ, showing that φ is not left-saturated.

As an illustration, consider the ODE

$$x' = x^2 + 1,$$

with the initial condition $x(t_0) = x_0$. This is a separable ODE and we find that

$$x(t) = \tan\big(t - t_0 + \arctan x_0\big).$$

It follows that, on the right, the maximal existence interval is $[t_0,\ t_0 + \frac{\pi}{2} - \arctan x_0)$, while on the left, the maximal existence interval is $(t_0 - \frac{\pi}{2} -$

$\arctan x_0, t_0]$. Thus, the saturated solution is defined on the interval $(t_0 - \frac{\pi}{2} - \arctan x_0, t_0 + \frac{\pi}{2} - \arctan x_0)$.

Our next result characterizes the right-saturated solutions. In the remainder of this section, we will assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open subset and $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is a continuous map that is also locally Lipschitz in the variable $\boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2.8 *Let* $\varphi : [t_0, t_1) \to \mathbb{R}^n$ *be a solution to system* (2.39). *Then the following are equivalent.*

(i) *The solution* φ *is right-extendible.*
(ii) *The graph of* φ*,*

$$\Gamma := \big\{ (t, \varphi(t)); \quad t \in [t_0, t_1) \big\},$$

is contained in a compact subset of Ω.

Proof (i) $\Rightarrow$ (ii). Assume that φ is right-extendible. Thus, there exists a solution $\psi(t)$ of (2.39) defined on an interval $[t_0, t_1 + \delta)$, $\delta > 0$, and such that

$$\psi(t) = \varphi(t), \quad \forall t \in [t_0, t_1).$$

In particular, it follows that Γ is contained in $\widehat{\Gamma}$, the graph of the restriction of ψ to $[t_0, t_1]$. Now, observe that $\widehat{\Gamma}$ is a compact subset of Ω because it is the image of the compact interval $[t_0, t_1]$ via the continuous map $t \mapsto (t, \psi(t))$.

(ii) $\Rightarrow$ (i) Assume that $\Gamma \subset K$, where K is a compact subset of Ω. We will prove that $\varphi(t)$ can be extended to a solution of (2.39) on an interval of the form $[t_0, t_1 + \delta]$, for some $\delta > 0$.

Since $\varphi(t)$ is a solution, we have

$$\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} \boldsymbol{f}\big(s, \varphi(s)\big)ds, \quad \forall t \in [t_0, t_1).$$

We deduce that

$$\|\varphi(t) - \varphi(t')\| \le \left| \int_{t'}^{t} \big\| \boldsymbol{f}\big(s, \varphi(s)\big) \big\| ds \right| \le M_K |t - t'|, \quad \forall t, t' \in [t_0, t_1),$$

where $M_K := \sup_{(s,\boldsymbol{x})\in K} \|\boldsymbol{f}(s, \boldsymbol{x})\|$. Cauchy's characterization of convergence now shows that $\varphi(t)$ has a (finite) limit as $t \nearrow t_1$ and we set

$$\varphi(t_1) := \lim_{t \nearrow t_1} \varphi(t).$$

We have thus extended φ to a continuous function on $[t_0, t_1]$ that we continue to denote by φ. The continuity of $\boldsymbol{f}$ implies that

$$\varphi'(t_1 - 0) = \lim_{t \nearrow t_1} \varphi'(t) = \lim_{t \nearrow t_1} \boldsymbol{f}(t, \varphi(t)) = \boldsymbol{f}(t_1, \varphi(t_1)). \tag{2.42}$$

On the other hand, according to Theorem 2.6, there exists a solution $\psi(t)$ of (2.39) defined on an interval $[t_1 - \delta, t_1 + \delta]$ and satisfying the initial condition $\psi(t_1) = \varphi(t_1)$. Consider the function

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in [t_0, t_1], \\ \psi(t), & t \in (t_1, t_1 + \delta]. \end{cases}$$

Obviously,

$$\widetilde{\varphi}'(t_1 + 0) = \psi'(t_1) = \boldsymbol{f}(t_1, \psi(t_1)) = \boldsymbol{f}(t_1, \varphi(t_1)) \stackrel{(2.42)}{=} \varphi'(t_1 - 0).$$

This proves that $\widetilde{\varphi}$ is C^1, and satisfies the differential equation (2.39). Clearly, $\widetilde{\varphi}$ extends φ to the right. □

The next result shows that any solution can be extended to a saturated solution.

Theorem 2.9 *Any solution φ of (2.39) admits a unique extension to a saturated solution.*

Proof The uniqueness is a consequence of Theorem 2.7 on global uniqueness. To prove the extendibility to a saturated solution, we will limit ourselves to proving the extendibility to a right-saturated solution.

We denote by $\mathcal{A}$ the set of all solutions ψ of (2.39) that extend φ to the right. The set $\mathcal{A}$ is totally ordered by the inclusion of the domains of definition of the solutions ψ and, as such, the set $\mathcal{A}$ has an upper bound, $\widetilde{\varphi}$. This is a right-saturated solution of (2.39). □

We will next investigate the behavior of the saturated solutions of (2.39) in a neighborhood of the boundary $\partial\Omega$ of the domain Ω where (2.39) is defined. For simplicity, we only discuss the case of right-saturated solutions. The case of left-saturated solutions is identical.

Theorem 2.10 *Let $\varphi(t)$ be a right-saturated solution of (2.39) defined on the interval $[t_0, T)$. Then any limit point as $t \nearrow T$ of the graph*

$$\Gamma := \left\{ (t, \varphi(t)); \quad t_0 \le t < T \right\}$$

is either the point at infinity of $\mathbb{R}^{n+1}$, or a point on $\partial\Omega$.

Proof The theorem states that, if (τ_ν) is a sequence in $[t_0, T)$ such that the limit $\lim_{\nu\to\infty}(\tau_\nu, \varphi(\tau_\nu))$ exists, then

(i) either $T = \infty$,
(ii) or $T < \infty$, $\lim_{\nu\to\infty} \|\varphi(\tau_\nu)\| = \infty$,
(iii) or $T < \infty$, $\boldsymbol{x}^* = \lim_{\nu\to\infty} \varphi(\tau_\nu) \in \mathbb{R}^n$ and $(T, \boldsymbol{x}^*) \in \partial\Omega$.

We argue by contradiction. Assume that all three options are violated. Since (i), (ii) do not hold, we deduce that $T < \infty$ and that the limit $\lim_{\nu\to\infty} \varphi(\tau_\nu)$ exists and is

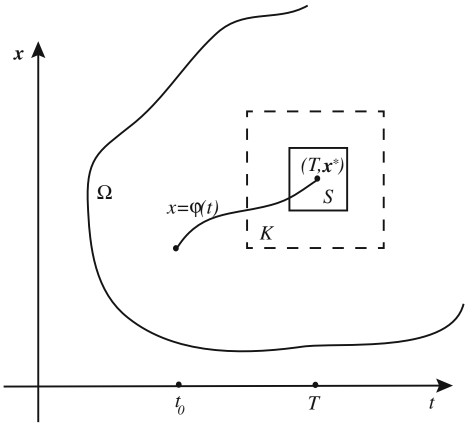


Fig. 2.1 The behavior of a right-saturated solution

a point $\boldsymbol{x}^* \in \mathbb{R}^n$. Since (iii) is also violated, we deduce that $(T, \boldsymbol{x}^*) \in \Omega$. Thus, for $r > 0$ sufficiently small, the closed ball

$$S := \left\{ (t, \boldsymbol{x}) \in \mathbb{R}^{n+1}; \quad |t - T| \leq r, \quad \|\boldsymbol{x} - \boldsymbol{x}^*\| \leq r \right\}$$

is contained in Ω; see Fig. 2.1.

If $\eta := \operatorname{dist}(S, \partial\Omega) > 0$, we deduce that for any $(s_0, \boldsymbol{y}_0) \in S$ the parallelepiped

$$\Delta := \left\{ (t, \boldsymbol{x}) \in \mathbb{R}^{n+1}; \quad |t - s_0| \leq \frac{\eta}{4}, \quad \|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \frac{\eta}{4} \right\} \tag{2.43}$$

is contained in the compact subset of Ω,

$$K = \left\{ (t, \boldsymbol{x}) \in \mathbb{R}^{n+1}; \quad |t - T| \leq r + \frac{\eta}{2}, \quad \|\boldsymbol{x} - \boldsymbol{x}^*\| \leq r + \frac{\eta}{2} \right\}.$$

(See Fig. 2.1.) We set

$$\delta := \min\left\{ \frac{\eta}{4}, \frac{\eta}{4M} \right\}, \quad M := \sup_{(t,\boldsymbol{x})\in K} \|\boldsymbol{f}(t, \boldsymbol{x})\|.$$

Appealing to the existence and uniqueness theorem (Theorem 2.3), where Δ is defined in (2.43), it follows that, for any $(s_0, \boldsymbol{y}_0) \in S$, there exists a unique solution $\boldsymbol{\psi}_{s_0,\boldsymbol{y}_0}(t)$ of (2.39) defined on the interval $[s_0 - \delta, s_0 + \delta]$ and satisfying the initial condition $\boldsymbol{\psi}(s_0) = \boldsymbol{y}_0$.

Fix ν sufficiently large so that

$$(\tau_\nu, \boldsymbol{\varphi}(\tau_\nu)) \in S \quad \text{and} \quad |\tau_\nu - T| \leq \frac{\delta}{2},$$

and define $\boldsymbol{y}_\nu := \boldsymbol{\varphi}(\tau_\nu)$,

$$\widetilde{\varphi}(t) := \begin{cases} \varphi(t), & t_0 \leq t \leq \tau_\nu, \\ \psi_{\tau_\nu, \boldsymbol{y}_\nu}(t), & \tau_\nu < t \leq \tau_\nu + \delta. \end{cases}$$

Then $\widetilde{\varphi}(t)$ is a solution of (2.39) defined on the interval $[t_0, \tau_\nu + \delta]$. This interval strictly contains the interval $[t_0, T]$ and $\widetilde{\varphi} = \varphi$ on $[t_0, T)$. This contradicts our assumption that φ is a right-saturated solution, and completes the proof of Theorem 2.10. □

Theorem 2.11 *Let* $\Omega = \mathbb{R}^{n+1}$ *and* $\varphi(t)$ *be a right-saturated solution of* (2.39) *defined on* $[0, T)$. *Then only the following two options are possible:*

(i) either $T = \infty$,

(ii) or $T < \infty$ and $\lim_{t \nearrow T} \|\varphi(t)\| = \infty$.

Proof From Theorem 2.10 it follows that any limit point as $t \nearrow T$ on the graph Γ of φ is the point at infinity. If $T < \infty$, then necessarily

$$\lim_{t \nearrow T} \|\varphi(t)\| = \infty. \qquad \square$$

Theorems 2.10 and 2.11 are useful in determining the maximal existence interval of a solution. Loosely speaking, Theorem 2.11 states that a solution φ is either defined on the whole positive semi-axis, or it "blows up" in finite time. This phenomenon is commonly referred to as the finite-time *blowup* phenomenon.

To illustrate Theorem 2.11, we depict in Fig. 2.2 the graph of the saturated solution of the Cauchy problem

$$x' = x^2 - 1, \quad x(0) = 2.$$

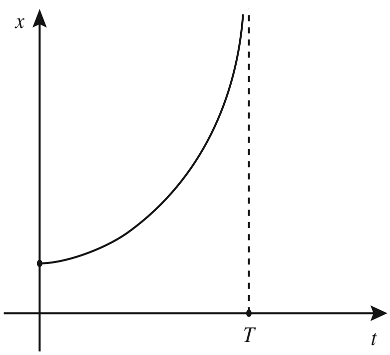
Its maximal existence interval on the right is $[0, T)$, $T = \frac{1}{2}\log 3$.

In the following examples, we describe other applications of these theorems.

Example 2.1 Consider the scalar ODE

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2.44}$$

Fig. 2.2 A finite-time blowup



where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and satisfies

$$(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \le \gamma_1 \|\boldsymbol{x}\|_{\mathbf{e}}^2 + \gamma_2, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \tag{2.45}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$. (Here $(-,-)$ is the Euclidean scalar product on $\mathbb{R}^n$ and $\|-\|_{\mathbf{e}}$ is the Euclidean norm.) According to the existence and uniqueness theorem, for any $(t_0, \boldsymbol{x}_0) \in \mathbb{R}^2$ there exists a unique solution $\varphi(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ of (2.44) satisfying $\varphi(t_0) = \boldsymbol{x}_0$ and defined on a maximal interval $[t_0, T)$. We want to prove that under the above assumptions we have $T = \infty$.

To show this, we multiply scalarly both sides of (2.44) by $\varphi(t)$. Using (2.45), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|_{\mathbf{e}}^2 = (\varphi(t), \varphi'(t)) = (\boldsymbol{f}(\varphi(t)), \varphi(t)) \le \gamma_1 \|\varphi(t)\|_{\mathbf{e}}^2 + \gamma_2, \ \forall t \in [t_0, T),$$

and, therefore,

$$\|\varphi(t)\|_{\mathbf{e}}^2 \le \|\varphi(t_0)\|_{\mathbf{e}}^2 + \gamma_1 \int_0^t \|\varphi(s)\|_{\mathbf{e}}^2 \, ds + \gamma_2 T, \quad \forall t \in [t_0, T).$$

Then, by Gronwall's lemma (Lemma 1.1), we get

$$\|\varphi(t)\|_{\mathbf{e}}^2 \le (\|\varphi(t_0)\|_{\mathbf{e}}^2 + \gamma_2 T) \exp(\gamma_1 T), \quad \forall \, t \in (0, T).$$

Thus, the solution $\varphi(t)$ is bounded, and so there is no blowup, $T = \infty$.

It should be noted that, in particular, condition (2.45) holds if f is globally Lipschitz on $\mathbb{R}^n$.

Example 2.2 Consider the Riccati equation

$$x' = a(t)x + b(t)x^2 + c(t), \tag{2.46}$$

where $a, b, c : [0, \infty) \to \mathbb{R}$ are continuous functions. We associate with (2.46) the Cauchy condition

$$x(t_0) = x_0, \tag{2.47}$$

where $t_0 = 0$. We will prove the following result.

If $x_0 \ge 0$ *and*

$$b(t) \le 0, \quad c(t) \ge 0, \quad \forall t \ge 0,$$

then the Cauchy problem (2.46)–(2.47) *admits a unique solution* $x = \varphi(t)$ *defined on the semi-axis* $[t_0, \infty)$. *Moreover,* $\varphi(t) \ge 0, \forall t \ge t_0$.

We begin by proving the result under the stronger assumption

$$c(t) > 0, \quad \forall t \ge 0.$$

Let $\varphi(t)$ denote the right-saturated solution of (2.46) and (2.47). It is defined on a maximal interval $[t_0, T)$. We will first prove that

$$\varphi(t) \geq 0, \quad \forall t \in [t_0, T). \tag{2.48}$$

Note that, if $x_0 = 0$, then $\varphi'(t_0) = c(t_0) > 0$, so $\varphi(t) > 0$ for t in a small interval $[t_0, t_0 + \delta]$, $\delta > 0$. This reduces the problem to the case when the initial condition is positive. Assume, therefore, that $x_0 > 0$. There exists a maximal interval $[t_0, T_1) \subset [t_0, T)$ on which $\varphi(t)$ is nonnegative. Clearly, either $T_1 = T$, or $T_1 < T$ and $\varphi(T_1) = 0$. If $T_1 < T$, then arguing as above we can extend φ past T_1 while keeping it nonnegative. This contradicts the maximality of T_1, thus proving (2.48).

To prove that $T = \infty$, we will rely on Theorem 2.11 and we will show that $\varphi(t)$ cannot blow up in finite time. Using the equality

$$\varphi'(t) = a(t)\varphi(t) + b(t)\varphi(t)^2 + c(t), \quad \forall t \in [t_0, T)$$

and the inequalities $b(t) \leq 0$, $\varphi(t) \geq 0$, we deduce that

$$|\varphi(t)| = \varphi(t) \leq \underbrace{\varphi(t_0) + \int_{t_0}^{t} c(s)ds}_{=:\beta(t)} + \int_{t_0}^{t} |a(s)|\,|\varphi(s)|ds.$$

We can invoke Gronwall's lemma to conclude that

$$|\varphi(t)| \leq \beta(t) + \int_{t_0}^{t} \beta(s)|a(s)| \exp\left(\int_{s}^{t} |a(\tau)|d\tau\right) ds.$$

The function in the right-hand-side of the above inequality is continuous on $[t_0, \infty)$, showing that $\varphi(t)$ cannot blow up in finite time. Hence $T = \infty$.

To deal with the general case, when $c(t) \geq 0$, $\forall t \geq 0$, we consider the equation

$$x' = a(t)x + b(t)x^2 + c(t) + \varepsilon, \tag{2.49}$$

where $\varepsilon > 0$. According to the results proven so far, this equation has a unique solution $x_\varepsilon(t; t_0, x_0)$ satisfying (2.47) and defined on $[t_0, \infty)$.

Denote by $x(t; t_0, x_0)$ the right-saturated solution of (2.46) and (2.47), defined on a maximal interval $[t_0, T)$. According to the forthcoming Theorem 2.15, we have

$$\lim_{\varepsilon \searrow 0} x_\varepsilon(t; t_0, x_0) = x(t; t_0, x_0), \quad \forall t \in [t_0, T).$$

We conclude that $x(t; t_0, x_0) \geq 0$, $\forall t \in [t_0, T)$. Using Gronwall's lemma as before, we deduce that $x(t; t_0, x_0)$ cannot blow up in finite time, and thus $T = \infty$.

Example 2.3 Let A be a real $n \times n$ matrix and Q, X_0 be two real symmetric, nonnegative definite $n \times n$ matrices. We recall that a symmetric $n \times n$ matrix S is called *nonnegative definite* if

$$(S\boldsymbol{v}, \boldsymbol{v}) \geq 0, \quad \forall \boldsymbol{v} \in \mathbb{R}^n,$$

where $(-, -)$ denotes the canonical scalar product on $\mathbb{R}^n$. The symmetric matrix S is called *positive definite* if

$$(S\boldsymbol{v}, \boldsymbol{v}) > 0, \quad \forall \boldsymbol{v} \in \mathbb{R}^n \setminus \{0\}.$$

We denote by A^* the adjoint (transpose) of S and we consider the matrix differential equation

$$X'(t) + A^* X(t) + X(t)A + X(t)^2 = Q, \tag{2.50}$$

together with the initial condition

$$X(t_0) = X_0. \tag{2.51}$$

By a solution of Eq. (2.50), we understand a matrix-valued map

$$X : I \to \mathbb{R}^{n^2}, \quad t \mapsto X(t) = \big(x_{ij}(t)\big)_{1 \leq i, j \leq n}$$

of class C^1 that satisfies (2.50) everywhere on I. Thus (2.50) is a system of ODEs involving n^2 unknown functions. When $n = 1$, Eq. (2.50) reduces to (2.46). Equation (2.50) is called the *matrix-valued Riccati type* equation and it plays an important role in the theory of control systems with quadratic cost functions. In such problems, one is interested in finding global solutions $X(t)$ of (2.50) such that $X(t)$ is symmetric and nonnegative definite for any t. (See Eq. (5.114).)

Theorem 2.12 *Under the above assumptions, the Cauchy problem* (2.50) and (2.51) *admits a unique solution $X = X(t)$ defined on the semi-axis* $[t_0, \infty)$. *Moreover, $X(t)$ is symmetric and nonnegative definite for any $t \geq t_0$.*

Proof From Theorem 2.3, we deduce the existence and uniqueness of a right-saturated solution of this Cauchy problem defined on a maximal interval $[t_0, T)$. Taking the adjoints of both sides of (2.50) and using the fact that X_0 and Q are symmetric matrices, we deduce that $X^*(t)$ is also a solution of the same Cauchy problem (2.50) and (2.51). This proves that $X(t) = X^*(t)$, that is, $X(t)$ is symmetric for any $t \in [t_0, T)$.

Let us prove that

(i) the matrix $X(t)$ is also nonnegative definite for any $t \in [t_0, T)$, and
(ii) $T = \infty$.

We distinguish two cases.

1. The matrix Q is positive definite. Recall that $X(t_0) = X_0$ is nonnegative definite. We set

$$T' := \sup\{\tau \in [t_0, T); \quad X(t) \geq 0, \quad \forall t \in [t_0, \tau)\Big\}.$$

We have to prove that $T' = T$. If $T' < T$, then $X(T')$ is nonnegative definite and there exist sequences (t_k) in (T', T) and $(\boldsymbol{v}_k)$ in $\mathbb{R}^n$ with the following properties

- $\lim_{k\to\infty} t_k = T'$.
- $\|\boldsymbol{v}_k\|_{\mathbf{e}} = 1$, $\big(X(t_k)\boldsymbol{v}_k, \boldsymbol{v}_k \big) < 0$, $\forall k$, where $\| - \|_{\mathbf{e}}$ is the standard Euclidean norm on $\mathbb{R}^n$.
- $\exists \boldsymbol{v}^* \in \mathbb{R}^n$ such that $\boldsymbol{v}^* = \lim_{k\to\infty} \boldsymbol{v}_k, \quad X(T')\boldsymbol{v}^* = 0$.

For each $\boldsymbol{v} \in \mathbb{R}^n$, we define the functions

$$\varphi_{\boldsymbol{v}}, \quad \psi_{\boldsymbol{v}} : [t_0, T) \to \mathbb{R}, \quad \varphi_{\boldsymbol{v}}(t) = \big(X(t)\boldsymbol{v}, \boldsymbol{v} \big), \quad \psi_{\boldsymbol{v}}(t) = \big(X(t)\boldsymbol{v}, A\boldsymbol{v} \big).$$

Since $X(t)$ is symmetric, from (2.50) we see that $\varphi_{\boldsymbol{v}}(t)$ satisfies the ODE

$$\varphi'_{\boldsymbol{v}}(t) = -2\psi_{\boldsymbol{v}}(t) - \|X(t)\boldsymbol{v}\|^2_{\mathbf{e}} + (Q\boldsymbol{v}, \boldsymbol{v}). \tag{2.52}$$

Moreover, we have

$$\varphi_{\boldsymbol{v}^*}(T') = \psi_{\boldsymbol{v}^*}(T') = 0,$$

and

$$\varphi'_{\boldsymbol{v}^*}(T') = (Q\boldsymbol{v}^*, \boldsymbol{v}^*) > 0. \tag{2.53}$$

Using the mean value theorem, we deduce that for any k there exists an $s_k \in (T', t_k)$ such that

$$\varphi'_{\boldsymbol{v}_k}(s_k) = \frac{\varphi_{\boldsymbol{v}_k}(t_k) - \varphi_{\boldsymbol{v}_k}(T')}{t_k - T'}.$$

We note that, by definition of T', $\varphi_{\boldsymbol{v}_k}(T') \geq 0$. Since $\varphi_{\boldsymbol{v}_k}(t_k) < 0$, we deduce that $\varphi'_{\boldsymbol{v}_k}(s_k) > 0$. Observing that

$$\lim_{k\to\infty} \varphi'_{\boldsymbol{v}_k}(s_k) = \lim_{k\to\infty} \big(X'(s_k)\boldsymbol{v}_k, \boldsymbol{v}_k\big) = \big(X'(T')\boldsymbol{v}^*, \boldsymbol{v}^* \big) = \varphi'_{\boldsymbol{v}^*}(T'),$$

we deduce that $\varphi'_{\boldsymbol{v}^*}(T') \leq 0$.

This contradicts (2.53) and proves that $X(t) \geq 0$, $\forall t \in [t_0, T')$.

According to Theorem 2.11, to prove that $T = \infty$ it suffices to show that for any $\boldsymbol{v} \in \mathbb{R}^n$ there exists a continuous function $f_{\boldsymbol{v}} : [t_0, \infty) \to \mathbb{R}$ such that

$$\varphi_{\boldsymbol{v}}(t) \leq f(t), \quad \forall t \in [t_0, T).$$

Fix $\boldsymbol{v} \in \mathbb{R}^n$. Using the Cauchy–Schwarz inequality (Lemma A.4),

$$|\psi_{\boldsymbol{v}}(t)| = \Big| \big(X(t)\boldsymbol{v}, A\boldsymbol{v} \big) \Big| \leq \underbrace{\|X(t)\boldsymbol{v}\|_{\mathbf{e}}}_{=:g_{\boldsymbol{v}}(t)} \cdot \underbrace{\|A\boldsymbol{v}\|_{\mathbf{e}}}_{=:C_{\boldsymbol{v}}}.$$

Using this in (2.52), we get

$$\varphi_v'(t) \leq 2C_{\boldsymbol{v}} g_{\boldsymbol{v}}(t) - (g_{\boldsymbol{v}}(t))^2 + (Q\boldsymbol{v}, \boldsymbol{v})$$

and, therefore,

$$\begin{aligned} \varphi_{\boldsymbol{v}}(t) \leq f_{\boldsymbol{v}}(t) &:= \varphi_{\boldsymbol{v}}(t_0) + (t - t_0)(Q\boldsymbol{v}, \boldsymbol{v}) \\ &\quad + \int_{t_0}^{t} \Big(2C_{\boldsymbol{v}} g_{\boldsymbol{v}}(s) - (g_{\boldsymbol{v}}(s))^2 \Big) ds, \end{aligned} \tag{2.54}$$

$\forall t \in [t_0, T)$. This proves that $T = \infty$.

2. The matrix Q is only nonnegative definite. For any $\varepsilon > 0$, we set $Q_\varepsilon := Q + \varepsilon \mathbb{1}_n$, where $\mathbb{1}_n$ denotes the identity $n \times n$ matrix. Denote by $X_\varepsilon(t)$ the right-saturated solution of the Cauchy problem

$$X'(t) + A^* X(t) + X(t)A + X(t)^2 = Q_\varepsilon, \quad X_\varepsilon(t_0) = X_0.$$

According to the previous considerations, $X_\varepsilon(t)$ is defined on $[t_0, \infty)$ and it is nonnegative definite on this interval. Moreover, for any $\boldsymbol{v} \in \mathbb{R}^n$, any $\varepsilon > 0$ and any $t \geq t_0$, we have

$$\begin{aligned} (X_\varepsilon(t)\boldsymbol{v}, \boldsymbol{v}) \leq f_{\boldsymbol{v}}^\varepsilon(t) &:= (X_0 \boldsymbol{v}, \boldsymbol{v}) + (t - t_0)(Q\boldsymbol{v}, \boldsymbol{v}) \\ &\quad + \int_{t_0}^{t} \Big(2C_{\boldsymbol{v}} g_{\boldsymbol{v}}^\varepsilon(s) - g_{\boldsymbol{v}}(s)^2 \Big) ds, \\ g_{\boldsymbol{v}}^\varepsilon(t) &:= \|X_\varepsilon(t)\boldsymbol{v}\|_{\mathbf{e}}. \end{aligned} \tag{2.55}$$

From Theorem 2.15, we deduce that

$$\lim_{\varepsilon \searrow 0} X_\varepsilon(t) = X(t), \quad \forall t \in [t_0, T).$$

If we now let $\varepsilon \to 0$ in (2.55), we deduce that

$$(X_\varepsilon(t)\boldsymbol{v}, \boldsymbol{v}) \leq f_{\boldsymbol{v}}(t) \;\; \forall t \in [t_0, T),$$

where $f_{\boldsymbol{v}}(t)$ is defined in (2.54). This implies that $T = \infty$. □

Example 2.4 (*Dissipative systems of ODEs*) As a final application, we consider *dissipative, autonomous differential systems*, that is, systems of ordinary differential equations of the form

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2.56}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map satisfying the *dissipativity* condition

$$\big(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y}), \, \boldsymbol{x} - \boldsymbol{y} \big) \leq 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \tag{2.57}$$

where, as usual, $(-,-)$ denotes the canonical Euclidean scalar product on $\mathbb{R}^n$. We associate with (2.56) the initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{2.58}$$

where $(t_0, \boldsymbol{x}_0)$ is a given point in $\mathbb{R}^{n+1}$.

Mathematical models of a large class of physical phenomena, such as diffusion, lead to dissipative differential systems. In the case $n = 1$, the monotonicity condition (2.57) is equivalent to the requirement that f be monotonically nonincreasing. For dissipative systems, we have the following interesting existence and uniqueness result.

Theorem 2.13 *If the continuous map* $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is dissipative, then for any* $(t_0, \boldsymbol{x}_0) \in \mathbb{R}^{n+1}$ *the Cauchy problem* (2.56) and (2.57) *admits a unique solution* $\boldsymbol{x} = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ *defined on* $[t_0, \infty)$. *Moreover, the map*

$$S : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, \boldsymbol{x}_0) \mapsto S(t)\boldsymbol{x}_0 := \boldsymbol{x}(t; 0, \boldsymbol{x}_0),$$

satisfies the following properties.

$$S(0)\boldsymbol{x}_0 = \boldsymbol{x}_0, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n, \tag{2.59}$$

$$S(t+s)\boldsymbol{x}_0 = S(t)S(s)\boldsymbol{x}_0, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n, \quad t, s \geq 0, \tag{2.60}$$

$$\|S(t)\boldsymbol{x}_0 - S(t)\boldsymbol{y}_0\|_{\mathbf{e}} \leq \|\boldsymbol{x}_0 - \boldsymbol{y}_0\|_{\mathbf{e}}, \quad \forall t \geq 0, \quad \boldsymbol{x}_0, \boldsymbol{y}_0 \in \mathbb{R}^n. \tag{2.61}$$

Proof According to Peano's theorem, for any $(t_0, \boldsymbol{x}_0) \in \mathbb{R}^n$ there exists a solution $\boldsymbol{x} = \varphi(t)$ to the Cauchy problem (2.56) and (2.57) defined on a maximal interval $[t_0, T)$. To prove its uniqueness, we argue by contradiction and assume that this Cauchy problem admits another solution $\boldsymbol{x} = \widetilde{\varphi}(t)$. On their common domain of existence $[t_0, t_1)$, the functions φ and $\widetilde{\varphi}$ satisfy the differential system

$$\big(\varphi(t) - \widetilde{\varphi}(t)\big)' = \boldsymbol{f}(\varphi(t)) - \boldsymbol{f}(\widetilde{\varphi}(t)). \tag{2.62}$$

Taking the scalar product of both sides of (2.62) with $\varphi(t) - \widetilde{\varphi}(t)$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\varphi(t) - \widetilde{\varphi}(t)\|_{\mathbf{e}}^2 &\overset{\text{Lemma } A.6}{=} \Big(\big(\varphi(t) - \widetilde{\varphi}(t)\big)', \ \varphi(t) - \widetilde{\varphi}(t) \Big) \\ &= \big(\boldsymbol{f}(\varphi(t)) - \boldsymbol{f}(\widetilde{\varphi}(t)), \varphi(t) - \widetilde{\varphi}(t) \big) \overset{(2.57)}{\leq} 0, \quad \forall t \in [t_0, t_1). \end{aligned} \tag{2.63}$$

Thus

$$\|\varphi(t) - \widetilde{\varphi}(t)\|_{\mathbf{e}}^2 \leq \|\varphi(t_0) - \widetilde{\varphi}(t_0)\|_{\mathbf{e}}^2, \quad \forall t \in [t_0, t_1). \tag{2.64}$$

This proves that $\varphi = \widetilde{\varphi}$ on $[t_0, t_1)$ since $\widetilde{\varphi}(t_0) = \varphi(t_0)$.

To prove that φ is defined on the entire semi-axis $[t_0, \infty)$ we first prove that it is bounded on $[t_0, T)$. To achieve this, we take the scalar product of

$$\varphi'(t) = \boldsymbol{f}(\varphi(t))$$

with $\varphi(t)$ and we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|_{\mathbf{e}}^2 &= (\boldsymbol{f}(\varphi(t)), \varphi(t)) \\ &= \big(\boldsymbol{f}(\varphi(t)) - \boldsymbol{f}(0), \varphi(t)\big) + \big(\boldsymbol{f}(0), \varphi(t)\big) \\ &\overset{(2.57)}{\leq} \|\boldsymbol{f}(0)\|_{\mathbf{e}} \cdot \|\varphi(t)\|_{\mathbf{e}}, \ \forall t \in [t_0, T).\end{aligned}$$

Integrating this inequality on $[t_0, t]$ and setting $u(t) := \|\varphi(t)\|_{\mathbf{e}}$, $C = \|\boldsymbol{f}(0)\|_{\mathbf{e}}$, we deduce that

$$\frac{1}{2}u(t)^2 \leq \frac{1}{2}\|\boldsymbol{x}_0\|_{\mathbf{e}}^2 + C\int_{t_0}^{t} u(s)ds, \quad \forall t \in [t_0, T).$$

From Proposition 1.2 we deduce that

$$\|\varphi(t)\|_{\mathbf{e}} = u(t) \leq \|\boldsymbol{x}_0\|_{\mathbf{e}} + \|\boldsymbol{f}(0)\|_{\mathbf{e}}(t - t_0), \quad \forall t \in [t_0, T). \tag{2.65}$$

Since we have not assumed that the function $\boldsymbol{f}$ is locally Lipschitz, we cannot invoke Theorems 2.10 or 2.11 directly. However, inequality (2.65) implies in a similar fashion the equality $T = \infty$. Here are the details.

We argue by contradiction and we assume that $T < \infty$. Inequality (2.65) implies that there exists an increasing sequence (t_k) and $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\lim_{k\to\infty} t_k = T, \quad \lim_{k\to\infty} \varphi(t_k) = \boldsymbol{v}.$$

According to the facts established so far, there exists a solution ψ of (2.56) defined on $[T - \delta, T + \delta]$ and satisfying the initial condition $\psi(T) = \boldsymbol{v}$.

On the interval $[T - \delta, T)$ we have

$$\varphi'(t) - \psi'(t) = \boldsymbol{f}(\varphi(t)) - \boldsymbol{f}(\psi(t)).$$

Taking the scalar product of this equality with $\varphi(t) - \psi(t)$ and using the dissipativity condition (2.57), we deduce as before that

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t) - \psi(t)\|_{\mathbf{e}}^2 \leq 0, \quad \forall t \in [T - \delta, T).$$

Hence

$$\|\varphi(t) - \psi(t)\|_{\mathbf{e}}^2 \leq \|\varphi(t_k) - \psi(t_k)\|_{\mathbf{e}}^2, \quad \forall t \in [T - \delta, t_k].$$

Since $\lim_{k\to\infty}\|\varphi(t_k)-\psi(t_k)\|_{\mathbf{e}}=0$, we conclude that $\varphi=\psi$ on $[T-\delta,T)$. In other words, ψ is a proper extension of the solution φ. This contradicts the maximality of the interval $[t_0,T)$. Thus $T=\infty$.

To prove (2.60), we observe that both functions

$$\boldsymbol{y}_1(t)=S(t+s)\boldsymbol{x}_0 \text{ and } \boldsymbol{y}_2(t)=S(t)S(s)\boldsymbol{x}_0$$

satisfy equations (2.56) and have identical values at $t=0$. The uniqueness of the Cauchy problems for (2.56) now implies that $\boldsymbol{y}_1(t)=\boldsymbol{y}_2(t)$, $\forall t\geq 0$.

Inequality (2.61) now follows from (2.64) where $\widetilde{\varphi}(t)=\boldsymbol{x}(t;0,\boldsymbol{y}_0)$. □

Remark 2.6 A family of maps $S(t):\mathbb{R}^n\to\mathbb{R}^n$, $t\geq 0$, satisfying (2.59), (2.60), (2.61) is called a *continuous semigroup of contractions* on the space $\mathbb{R}^n$. The function $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^n$ is called the *generator* of the semigroups $S(t)$.

2.6 Continuous Dependence on Initial Conditions and Parameters

We now return to the differential system (2.39) defined on the open subset $\Omega\subset\mathbb{R}^{n+1}$. We will assume as in the previous section that the function $\boldsymbol{f}:\Omega\to\mathbb{R}^n$ is continuous in the variables $(t,\boldsymbol{x})$, and locally Lipschitz in the variable $\boldsymbol{x}$. Theorem 2.6 shows that for any $(t_0,\boldsymbol{x}_0)\in\Omega$ there exists a unique solution $\boldsymbol{x}=\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)$ of system (2.39) that satisfies the initial condition $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$. The solution $\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)$, which we will assume to be saturated, is defined on an interval typically dependent on the point $(t_0,\boldsymbol{x}_0)$. For simplicity, we will assume the initial moment t_0 to be fixed.

It is reasonable to expect that, as $\boldsymbol{v}$ varies in a neighborhood of $\boldsymbol{x}_0$, the corresponding solution $\boldsymbol{x}(t;t_0,\boldsymbol{v})$ will not stray too far from the solution $\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)$. The next theorem confirms that this is the case, in a rather precise form. To state this result, let us denote by $B(\boldsymbol{x}_0,\eta)$ the ball of center $\boldsymbol{x}_0$ and radius η in $\mathbb{R}^n$, that is,

$$B(\boldsymbol{x}_0,\eta):=\big\{\boldsymbol{v}\in\mathbb{R}^n;\quad \|\boldsymbol{v}-\boldsymbol{x}_0\|\leq\eta\big\}.$$

Theorem 2.14 (Continuous dependence on initial data) *Let* $[t_0,T)$ *be the maximal interval of existence on the right of the solutions* $\boldsymbol{x}(t;t_0,\boldsymbol{x}_0)$ *of* (2.39). *Then, for any* $T'\in[t_0,T)$*, there exists an* $\eta=\eta(T')>0$ *such that, for any* $\boldsymbol{v}\in S(\boldsymbol{x}_0,\eta)$*, the solution* $\boldsymbol{x}(t;t_0,\boldsymbol{v})$ *is defined on the interval* $[t_0,T']$. *Moreover, the correspondence*

$$B(\boldsymbol{x}_0,\eta)\ni\boldsymbol{v}\mapsto\boldsymbol{x}(t;t_0,\boldsymbol{v})\in C\big([t_0,T'];\mathbb{R}^n\big)$$

is a continuous map from the ball $S(\boldsymbol{x}_0,\eta)$ *to the space* $C\big([t_0,T'];\mathbb{R}^n\big)$ *of continuous maps from* $[t_0,T']$ *to* $\mathbb{R}^n$. *In other words, for any sequence* $(\boldsymbol{v}_k)$ *in* $B(\boldsymbol{x}_0,\eta)$ *that*

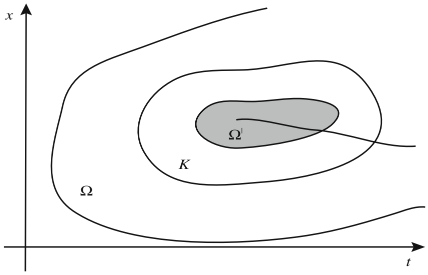


Fig. 2.3 Isolating a compact portion of an integral curve

converges to $\boldsymbol{v} \in B(\boldsymbol{x}_0, \eta)$*, the sequence of functions* $\boldsymbol{x}(t; t_0, \boldsymbol{v}_k)$ *converges uniformly on* $[t_0, T']$ *to* $\boldsymbol{x}(t; t_0, \boldsymbol{v})$.

Proof Fix $T' \in [t_0, T)$. The restriction to $[t_0, T']$ of $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ is continuous and, therefore, the graph of this restriction is compact. We can find an open set Ω' whose closure $\bar{\Omega}'$ is compact and contained in Ω and such that

$$\big\{(t, \boldsymbol{x}(t; t_0, \boldsymbol{x}_0));\ t_0 \leq t \leq T'\big\} \subset \bar{\Omega}', \quad \operatorname{dist}\big(\bar{\Omega}', \partial\Omega\big) =: \delta > 0. \tag{2.66}$$

We denote by K the compact subset of Ω defined by (see Fig. 2.3)

$$K := \left\{ (t, \boldsymbol{x}) \in \Omega;\ \ \operatorname{dist}(\,(t, \boldsymbol{x}), \bar{\Omega}'\,) \leq \frac{\delta}{2} \right\}. \tag{2.67}$$

For any $(t_0, \boldsymbol{v}) \in \Omega'$, there exists a maximal $\tilde{T} \in (t_0, T']$ such that the solution $\boldsymbol{x}(t; t_0, \boldsymbol{v})$ exists for all $t \in [t_0, \tilde{T}]$ and $\big\{(t, \boldsymbol{x}(t; t_0, \boldsymbol{v}));\ t_0 \leq t \leq \tilde{T}\big\} \subset K$. On the interval $[t_0, T']$, we have the equality

$$\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) - \boldsymbol{x}(t; t_0, \boldsymbol{v}) = \int_{t_0}^{t} \Big(\boldsymbol{f}\big(s, \boldsymbol{x}(s; t_0, \boldsymbol{x}_0)\big) - \boldsymbol{f}\big(s, \boldsymbol{x}(s; t_0, \boldsymbol{v})\big) \Big) ds.$$

Because the graphs of $\boldsymbol{x}(s; t_0, \boldsymbol{x}_0)$ and $\boldsymbol{x}(s; t_0, \boldsymbol{v})$ over $[t_0, T']$ are contained in the compact set K, the locally Lipschitz assumption implies that there exists a constant $L_K > 0$ such that

$$\big\|\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) - \boldsymbol{x}(t; t_0, \boldsymbol{v})\big\| \leq \|\boldsymbol{x}_0 - \boldsymbol{v}\| + L_K \int_{t_0}^{t} \big\|\, \boldsymbol{x}(s; t_0, \boldsymbol{x}_0) - \boldsymbol{x}(s; t_0, \boldsymbol{v})\, \big\| ds.$$

Gronwall's lemma now implies

$$\big\|\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) - \boldsymbol{x}(t; t_0, \boldsymbol{v})\big\| \leq e^{L_K(t - t_0)}\|\boldsymbol{x}_0 - \boldsymbol{v}\|, \quad \forall t \in [t_0, \tilde{T}]. \tag{2.68}$$

We can now prove that, given $T' \in [t_0, T)$, there exists an $\eta = \eta(T') > 0$ such that, for any $\boldsymbol{v} \in B(\boldsymbol{x}_0, \eta)$,

(a) the solution $\boldsymbol{x}(t; t_0, \boldsymbol{v})$ is defined on $[t_0, T']$, and
(b) the graph of this solution is contained in K.

We argue by contradiction. We can find a sequence $(\boldsymbol{v}_j)_{j\geq 1}$ in $\mathbb{R}^n$ such that

- $\|\boldsymbol{x}_0 - \boldsymbol{v}_j\| \leq \frac{1}{j}$, $\forall j \geq 1$, and
- the maximal closed interval $[t_0, T_j]$, with the property that the graph of $\boldsymbol{x}(t; t_0, \boldsymbol{v}_j)$ is contained in K, is a subinterval of the half-open interval $[t_0, T')$.

Using (2.68), we deduce that

$$\big\|\boldsymbol{x}(t; t_0, \boldsymbol{x}_0) - \boldsymbol{x}(t; t_0, \boldsymbol{v}_j)\big\| \leq \frac{e^{L_K(t-t_0)}}{j}, \quad \forall t_0 \leq t \leq T_j. \tag{2.69}$$

Thus, if

$$j \geq \frac{\delta e^{L_K(t-t_0)}}{4},$$

then the distance between the graph of $\boldsymbol{x}(t; t_0, \boldsymbol{v}_j)$ and the graph of $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ over $[t_0, T_j]$ is $\leq \frac{\delta}{4}$. Conditions (2.66) and (2.67) imply that

$$\operatorname{dist}((t, \boldsymbol{x}(t; t_0, \boldsymbol{v}_j)), \partial K) \geq \frac{\delta}{4}, \quad \forall t \in [t_0, T_j].$$

We conclude that the function $\boldsymbol{x}(t; t_0, \boldsymbol{v}_j)$ can be extended slightly to the right of T_j as a solution of (2.39) so that its graph continues to be inside K. This violates the maximality of T_j. This proves the existence of $\eta(T')$ with the postulated properties (a) and (b) above.

Consider now two solutions $\boldsymbol{x}(t; t_0, \boldsymbol{u})$, $\boldsymbol{x}(t; t_0, \boldsymbol{v})$, where $\boldsymbol{u}, \boldsymbol{v} \in B(\boldsymbol{x}_0, \eta(T'))$. For $t \in [t_0, T']$, we have

$$\boldsymbol{x}(t; t_0, \boldsymbol{u}) - \boldsymbol{x}(t; t_0, \boldsymbol{v}) = \boldsymbol{u} - \boldsymbol{v} + \int_{t_0}^{t} \Big(\boldsymbol{f}\big(s, \boldsymbol{x}(s; t_0, \boldsymbol{u})\big) - \boldsymbol{f}\big(s, \boldsymbol{x}(s; t_0, \boldsymbol{v})\big)\Big) ds.$$

Using the local Lipschitz condition, we deduce as before that

$$\|\boldsymbol{x}(t; t_0, \boldsymbol{u}) - \boldsymbol{x}(t; t_0, \boldsymbol{v})\| \leq \|\boldsymbol{u} - \boldsymbol{v}\| + L_k \int_{t_0}^{t} \|\boldsymbol{x}(s; t_0, \boldsymbol{u}) - \boldsymbol{x}(s; t_0, \boldsymbol{v})\| ds \tag{2.70}$$

and, invoking Gronwall's lemma again, we obtain

$$\|\boldsymbol{x}(t; t_0, \boldsymbol{u}) - \boldsymbol{x}(t; t_0, \boldsymbol{v})\| \leq e^{L_K(t-t_0)} \|\boldsymbol{u} - \boldsymbol{v}\|, \quad \forall t \in [t_0, T]. \tag{2.71}$$

The last inequality proves the continuity of the mapping $\boldsymbol{v} \mapsto \boldsymbol{x}(t; t_0, \boldsymbol{v})$ on the ball $B(\boldsymbol{x}_0, \eta(T'))$. This completes the proof of Theorem 2.14. □

Let us now consider the special case when system (2.39) is autonomous, that is, the map $\boldsymbol{f}$ is independent of t.

More precisely, we assume that $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. One should think of $\boldsymbol{f}$ as a vector field on $\mathbb{R}^n$.

For any $\boldsymbol{y} \in \mathbb{R}^n$, we set

$$S(t)\boldsymbol{u} := \boldsymbol{x}(t; 0, \boldsymbol{u}),$$

where $\boldsymbol{x}(t; 0, u)$ is the unique saturated solution of the system

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \tag{2.72}$$

satisfying the initial condition $\boldsymbol{x}(0) = \boldsymbol{u}$. Theorem 2.14 shows that for any $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists a $T > 0$ and a neighborhood $U_0 = B(\boldsymbol{x}_0, \eta)$ of $\boldsymbol{x}_0$ such that $S(t)\boldsymbol{u}$ is well defined for any $\boldsymbol{u} \in U_0$ and any $|t| \leq T$. Moreover, the resulting maps

$$U_0 \ni \boldsymbol{u} \mapsto S(t)\boldsymbol{u} \in \mathbb{R}^n$$

are continuous for any $|t| \leq T$. From the local existence and uniqueness theorem, we deduce that the family of maps $S(t) : U_0 \to \mathbb{R}^n$, $-T \leq t \leq T$, has the following properties

$$S(0)\boldsymbol{u} = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in U_0, \tag{2.73}$$

$$\begin{aligned} S(t+s)\boldsymbol{u} = S(t)S(s)\boldsymbol{u},\ \forall s, t \in [-T, T] \\ \text{such that } |t+s| \leq T,\ S(s)\boldsymbol{u} \in U_0, \end{aligned} \tag{2.74}$$

$$\lim_{t \to 0} S(t)\boldsymbol{u} = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in U_0. \tag{2.75}$$

The family of applications $\{S(t)\}_{|t| \leq T}$ is called the *local flow* or the *continuous local one-parameter group* generated by the vector field $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. From the definition of $S(t)$, we deduce that

$$\boldsymbol{f}(\boldsymbol{u}) = \lim_{t \to 0} \frac{1}{t}\big(S(t)\boldsymbol{u} - \boldsymbol{u} \big), \quad \forall \boldsymbol{u} \in U_0. \tag{2.76}$$

Consider now the differential system

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \lambda), \quad \lambda \in \Lambda \subset \mathbb{R}^m, \tag{2.77}$$

where $\boldsymbol{f} : \Omega \times \Lambda \to \mathbb{R}^n$ is a continuous function, Ω is an open subset of $\mathbb{R}^{n+1}$, and Λ is an open subset of $\mathbb{R}^m$. Additionally, we will assume that $\boldsymbol{f}$ is locally Lipschitz in $(\boldsymbol{x}, \lambda)$ on $\Omega \times \Lambda$. In other words, for any compact sets $K_1 \subset \Omega$ and $K_2 \subset \Lambda$ there exists a positive constant L such that

$$\begin{aligned} \|\boldsymbol{f}(t, \boldsymbol{x}, \lambda) - \boldsymbol{f}(t, \boldsymbol{y}, \mu)\| \leq L\big(\|\boldsymbol{x} - \boldsymbol{y}\| + \|\lambda - \mu\| \big), \\ \forall (t, \boldsymbol{x}), (t, \boldsymbol{y}) \in K_1, \quad \lambda, \mu \in K_2. \end{aligned} \tag{2.78}$$

Above, we denoted by the same symbol the norms $\| - \|$ on $\mathbb{R}^m$ and $\mathbb{R}^n$.

For any $(t_0, \boldsymbol{x}_0) \in \Omega$, and $\lambda \in \Lambda$, the system (2.77) admits a unique solution $\boldsymbol{x} = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0, \lambda)$ satisfying the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Loosely speaking, our next result states that the correspondence $\lambda \mapsto \boldsymbol{x}(-; t_0, \boldsymbol{x}_0, \lambda)$ is continuous.

Theorem 2.15 (Continuous dependence on parameters) *Fix a point* $(t_0, \boldsymbol{x}_0, \lambda_0) \in \Omega \times \Lambda$. *Let* $[t_0, T)$ *be the maximal interval of existence on the right of the solution* $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0, \lambda_0)$. *Then, for any* $T' \in [t_0, T)$, *there exists an* $\eta = \eta(T') > 0$ *such that for any* $\lambda \in B(\lambda_0, \eta)$ *the solution* $\boldsymbol{x}(t; t_0, \boldsymbol{x}_0, \lambda)$ *is defined on* $[t_0, T']$. *Moreover, the application*

$$B(\lambda_0, \eta) \ni \lambda \mapsto \boldsymbol{x}(-; t_0, \boldsymbol{x}_0, \lambda) \in C\big([t_0, T'], \mathbb{R}^n\big)$$

is continuous.

Proof The above result is a special case of Theorem 2.14 on the continuous dependence on initial data.

Indeed, if we denote by z the $(n+m)$-dimensional vector $(\boldsymbol{x}, \lambda) \in \mathbb{R}^{n+m}$, and we define

$$\widetilde{\boldsymbol{f}} : \Omega \times \Lambda \to \mathbb{R}^{n+m}, \quad \widetilde{\boldsymbol{f}}(t, \boldsymbol{x}, \lambda) = \big(\boldsymbol{f}(t, \boldsymbol{x}, \lambda), 0\big) \in \mathbb{R}^n \times \mathbb{R}^m,$$

then system (2.77) can be rewritten as

$$z'(t) = \widetilde{\boldsymbol{f}}\big(t, z(t)\big), \tag{2.79}$$

while the initial condition becomes

$$z(t_0) = z_0 := (\boldsymbol{x}_0, \lambda). \tag{2.80}$$

We have thus reduced the problem to investigating the dependence of the solutions $z(t)$ of (2.79) on the initial data. Our assumptions on $\boldsymbol{f}$ show that $\widetilde{\boldsymbol{f}}$ satisfies the assumptions of Theorem 2.14. □

2.7 Differential Inclusions

One of the possible extensions of the concept of a differential equation is to consider instead of the function $\boldsymbol{f} : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ a set-valued, or multi-valued map

$$F : \Omega \to 2^{\mathbb{R}^n},$$

where we recall that for any set S we denote by 2^S the collection of its subsets. In this case, system (2.39) becomes a *differential inclusion*

$$\boldsymbol{x}'(t) \in F\big(t, \boldsymbol{x}(t)\big), \; t \in I, \tag{2.81}$$

to which we associate the initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2.82}$$

In general, we cannot expect the existence of a continuously differentiable solution of the Cauchy problem (2.81) and (2.82). Consider, for example, the differential inclusion

$$x' \in \operatorname{Sign} x, \tag{2.83}$$

where Sign : $\mathbb{R} \to 2^{\mathbb{R}^n}$ is given by

$$\operatorname{Sign}(x) = \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0. \end{cases}$$

Note that, if $x_0 > 0$, then the function

$$x(t) = \begin{cases} t - t_0 + x_0, & t \geq -x_0 + t_0, \\ 0, & t < -x_0 + t_0, \end{cases}$$

is the unique solution of (2.83) on $\mathbb{R} \setminus \{t_0 - x_0\}$. However, it is not a C^1-function since its derivative has a discontinuity at $t_0 - x_0$. Thus, the above function is not a solution in the sense we have adopted so far.

This simple example suggests the need to extend the concept of solution.

Definition 2.1 The function $\boldsymbol{x} : [t_0, T] \to \mathbb{R}^n$ is called a *Carathéodory solution* of the differential inclusion (2.81) if the following hold.

(i) The function $\boldsymbol{x}(t)$ is absolutely continuous on $[t_0, T]$.
(ii) There exists a negligible set $N \subset [t_0, T]$ such that, for any $t \in [t_0, T] \setminus N$, the function $x(t)$ is differentiable at t and $\boldsymbol{x}'(t) \in F(t, \boldsymbol{x}(t))$.

According to Lebesgue's theorem (see e.g. [12, Sect. 33]), an absolutely continuous function $\boldsymbol{x} : [t_0, T] \to \mathbb{R}^n$ is almost everywhere differentiable on the interval $[t_0, T]$.

Differential inclusions naturally appear in the modern theory of variational calculus and of control systems. An important source of differential inclusion is represented by differential equations with a discontinuous right-hand side. More precisely, if $f = f(t, x)$ is discontinuous in x, then the Cauchy problem (2.1) does not have a Carathéodory solution, but this might happen if we extend f to a multi-valued mapping $(t, x) \to F(t, x)$. (This happens for Eq. (2.83), where the discontinuous function $\frac{x}{|x|}$ was extended to Sign x.) In this section, we will investigate a special class of differential inclusions known as *evolution variational inequalities*. They were introduced in mathematics, in a more general context, by *G. Stampacchia* (1922–1978) and *J.L. Lions* (1928–2001). To state and solve such problems, we need to make a brief digression into (finite-dimensional) convex analysis.

Recall that a subset $C \subset \mathbb{R}^n$ is *convex* if

$$(1-t)\boldsymbol{x} + t\boldsymbol{y} \in C, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in C, \quad \forall t \in [0,1].$$

Geometrically, this means that for any two points in C the line segment connecting them is entirely contained in C. Given a closed convex set $C \subset \mathbb{R}^n$ and $\boldsymbol{x}_0 \in C$, we set

$$N_C(\boldsymbol{x}_0) := \left\{ \boldsymbol{w} \in \mathbb{R}^n; \quad (\boldsymbol{w}, \boldsymbol{y} - \boldsymbol{x}_0) \le 0, \quad \forall \boldsymbol{y} \in C \right\}. \tag{2.84}$$

The set $N_C(\boldsymbol{x}_0)$ is a closed convex cone called the (outer) *normal cone* of C at the point $\boldsymbol{x}_0 \in C$. We extend N_C to a multi-valued map $N_C : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ by setting

$$N_C(\boldsymbol{x}) = \emptyset, \quad \forall \boldsymbol{x} \in \mathbb{R}^n \setminus C.$$

Example 2.5 (a) If C is a convex domain in $\mathbb{R}^n$ with smooth boundary and $\boldsymbol{x}_0$ is a point on the boundary, then $N_C(\boldsymbol{x}_0)$ is the cone spanned by the unit outer normal to ∂C at $\boldsymbol{x}_0$. If $\boldsymbol{x}_0$ is in the interior of C, then $N_C(\boldsymbol{x}_0) = \{0\}$.
(b) If $C \subset \mathbb{R}^n$ is a vector subspace, then for any $\boldsymbol{x} \in C$ we have $N_C(\boldsymbol{x}) = C^{\perp}$, the orthogonal complement of C in $\mathbb{R}^n$.

To any closed convex set $C \subset \mathbb{R}^n$ there corresponds a projection

$$P_C : \mathbb{R}^n \to C$$

that associates to each $\boldsymbol{x} \in \mathbb{R}^n$ the point in C closest to $\boldsymbol{x}$ with respect to the Euclidean distance. The next result makes this precise.

Lemma 2.1 *Let C be a closed convex subset of $\mathbb{R}^n$. Then the following hold.*
(a) *For any $\boldsymbol{x} \in \mathbb{R}^n$ there exists a unique point $\boldsymbol{y} \in C$ such that*

$$\|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{e}} = \operatorname{dist}(\boldsymbol{x}, C) := \inf_{z \in C} \|\boldsymbol{x} - \boldsymbol{z}\|_{\mathbf{e}}. \tag{2.85}$$

We denote by $P_C(\boldsymbol{x})$ this unique point in C, and we will refer to the resulting map $P_C : \mathbb{R}^n \to C$ as the orthogonal projection onto C.
(b) *The map $P_C : \mathbb{R}^n \to C$ satisfies the following properties:*

$$\begin{gathered} \boldsymbol{x} - P_C\boldsymbol{x} \in N_C\big(P_C\boldsymbol{x}\big), \quad \textit{that is,} \\ \big(\boldsymbol{x} - P_C\boldsymbol{x}, \boldsymbol{y} - P_C\boldsymbol{x}\big) \le 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{y} \in C. \end{gathered} \tag{2.86}$$

$$\|P_C\boldsymbol{x} - P_C\boldsymbol{z}\|_{\mathbf{e}} \le \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{e}}, \quad \forall \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n. \tag{2.87}$$

Proof (a) There exists a sequence $(\boldsymbol{y}_\nu)$ in C such that

$$\operatorname{dist}(\boldsymbol{x}, C) \le \|\boldsymbol{x} - \boldsymbol{y}_\nu\|_{\mathbf{e}} \le \operatorname{dist}(\boldsymbol{x}, C) + \frac{1}{\nu}. \tag{2.88}$$

The sequence $(\boldsymbol{y}_\nu)$ is obviously bounded and thus it has a convergent subsequence $(\boldsymbol{y}_{\nu_k})$. Its limit $\boldsymbol{y}$ is a point in C since C is closed. Moreover, inequalities (2.88) imply that

$$\|\boldsymbol{x}-\boldsymbol{y}\|_{\mathbf{e}} = \operatorname{dist}(\boldsymbol{x}, C).$$

This completes the proof of the existence part of (a).

Let us prove the uniqueness statement. Let $\boldsymbol{y}_1, \boldsymbol{y}_2 \in C$ such that

$$\|\boldsymbol{x}-\boldsymbol{y}_1\|_{\mathbf{e}} = \|\boldsymbol{x}-\boldsymbol{y}_2\|_{\mathbf{e}} = \operatorname{dist}(\boldsymbol{x}, C).$$

Since C is convex, we deduce that

$$y_0 := \frac{1}{2}(y_1+y_2) \in C.$$

From the triangle inequality we deduce that

$$\begin{aligned}\operatorname{dist}(\boldsymbol{x}, C) &\leq \|\boldsymbol{x}-\boldsymbol{y}_0\|_{\mathbf{e}}\\ &\leq \frac{1}{2}\Big(\|\boldsymbol{x}-\boldsymbol{y}_1\|_{\mathbf{e}} + \|\boldsymbol{x}-\boldsymbol{y}_2\|_{\mathbf{e}}\Big)\\ &= \operatorname{dist}(\boldsymbol{x}, C).\end{aligned}$$

Hence

$$\|\boldsymbol{x}-\boldsymbol{y}_0\|_{\mathbf{e}} = \|\boldsymbol{x}-\boldsymbol{y}_1\|_{\mathbf{e}} = \|\boldsymbol{x}-\boldsymbol{y}_2\|_{\mathbf{e}} = \operatorname{dist}(\boldsymbol{x}, C). \tag{2.89}$$

One the other hand, we have the parallelogram identity

$$\left\|\frac{1}{2}(\boldsymbol{y}+\boldsymbol{z})\right\|_{\mathbf{e}}^2 + \left\|\frac{1}{2}(\boldsymbol{y}-\boldsymbol{z})\right\|_{\mathbf{e}}^2 = \frac{1}{2}\Big(\|\boldsymbol{y}\|_{\mathbf{e}}^2 + \|\boldsymbol{z}\|_{\mathbf{e}}^2\Big),\quad \forall \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n.$$

If in the above equality we let $\boldsymbol{y} = \boldsymbol{x}-\boldsymbol{y}_1$, $\boldsymbol{z} = \boldsymbol{x}-\boldsymbol{y}_2$, then we conclude from (2.89) that

$$\|\boldsymbol{y}_1-\boldsymbol{y}_2\|_{\mathbf{e}}^2 = 0.$$

This completes the proof of the uniqueness.

(b) To prove (2.86), we start with the defining inequality

$$\|\boldsymbol{x}-P_C\boldsymbol{x}\|_{\mathbf{e}}^2 \leq \|\boldsymbol{x}-\boldsymbol{y}\|^2,\quad \forall \boldsymbol{y} \in C.$$

Consider now the function

$$f_{\boldsymbol{y}} : [0,1] \to \mathbb{R},\quad f_{\boldsymbol{y}}(t) = \|\boldsymbol{x}-\boldsymbol{y}_t\|^2 - \|\boldsymbol{x}-P_C\boldsymbol{x}\|_{\mathbf{e}}^2,$$

where

$$y_t = (1-t)P_C\boldsymbol{x} + t\boldsymbol{y} = P_C(x) + t(\boldsymbol{y} - P_C\boldsymbol{x}).$$

We have $f_{\boldsymbol{y}}(t) \geq 0, \forall t \geq 0$ and $f_{\boldsymbol{y}}(0) = 0$. Thus

$$f'_{\boldsymbol{y}}(0) \geq 0.$$

Observing that

$$\begin{aligned} f_{\boldsymbol{y}}(t) &= \|(\boldsymbol{x} - P_C\boldsymbol{x}) - t(\boldsymbol{y} - P_C\boldsymbol{x})\|_{\mathbf{e}}^2 - \|\boldsymbol{x} - P_C\boldsymbol{x}\|_{\mathbf{e}}^2 \\ &= t^2\big\|\boldsymbol{y} - P_C\boldsymbol{x}\big\|_{\mathbf{e}}^2 - 2t\big(\boldsymbol{x} - P_C\boldsymbol{x},\ \boldsymbol{y} - P_C\boldsymbol{x}\big), \end{aligned}$$

we deduce that

$$f'_{\boldsymbol{y}}(0) = -2\big(\boldsymbol{x} - P_C\boldsymbol{x},\ \boldsymbol{y} - P_C\boldsymbol{x}\big) \geq 0, \quad \forall \boldsymbol{y} \in C.$$

This proves (2.86).

To prove (2.87), let $\boldsymbol{z} \in \mathbb{R}^n$ and set

$$\boldsymbol{u} := \boldsymbol{x} - P_C\boldsymbol{x}, \quad \boldsymbol{v} := \boldsymbol{z} - P_C\boldsymbol{z}, \quad w := P_C\boldsymbol{z} - P_C\boldsymbol{x}.$$

From (2.86) we deduce that

$$\boldsymbol{u} \in N_C(P_C\boldsymbol{x}), \quad \boldsymbol{v} \in N_C(P_C\boldsymbol{z}),$$

so that $(\boldsymbol{u}, \boldsymbol{w}) \leq 0 \leq (\boldsymbol{v}, \boldsymbol{w})$ and thus

$$(\boldsymbol{w}, \boldsymbol{u} - \boldsymbol{w}) \leq 0. \tag{2.90}$$

On the other hand, we have $\boldsymbol{x} - \boldsymbol{z} = \boldsymbol{u} - \boldsymbol{w} - \boldsymbol{v}$, so that

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{z}\|_{\mathbf{e}}^2 &= \|(\boldsymbol{u} - \boldsymbol{v}) - \boldsymbol{w}\|^2 \\ &= \|\boldsymbol{w}\|_{\mathbf{e}}^2 + \|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbf{e}}^2 - 2(\boldsymbol{w}, \boldsymbol{u} - \boldsymbol{w}) \\ &\overset{(2.90)}{\geq} \|\boldsymbol{w}\|_{\mathbf{e}}^2 \\ &= \big\|P_C\boldsymbol{x} - P_C\boldsymbol{z}\big\|_{\mathbf{e}}^2. \end{aligned}$$

□

Suppose that K is a closed convex subset of $\mathbb{R}^n$. Fix real numbers $t_0 < T$, a continuous function $\boldsymbol{g} : [t_0, T] \to \mathbb{R}^n$ and a (globally) Lipschitz map $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. We want to investigate the differential inclusion

$$\begin{aligned} \boldsymbol{x}'(t) \in \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{g}(t) - N_K\big(\boldsymbol{x}(t)\big), \ \text{a.e. } t \in (t_0, T) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \end{aligned} \tag{2.91}$$

This differential inclusion can be rewritten as an evolution variational inequality

$$\boldsymbol{x}(t) \in K, \quad \forall t \in [0, T], \tag{2.92}$$

$$\big(\boldsymbol{x}'(t) - \boldsymbol{f}\big(\boldsymbol{x}(t)\big) - \boldsymbol{g}(t), \boldsymbol{y} - \boldsymbol{x}(t)\big) \geq 0, \ \text{a.e. } t \in (t_0, T), \ \forall \boldsymbol{y} \in K, \tag{2.93}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2.94}$$

Theorem 2.16 *Suppose that $\boldsymbol{x}_0 \in K$ and $g : [t_0, T] \to \mathbb{R}^n$ is a continuously differentiable function. Then the initial value problem (2.91) admits a unique Carathéodory solution $\boldsymbol{x} : [t_0, T] \to \mathbb{R}^n$. Moreover,*

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(t) - P_{N_K(\boldsymbol{x}(t))}(\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(t)), \ \textit{a.e. } t \in (t_0, T). \tag{2.95}$$

Proof For simplicity, we denote by P the orthogonal projection P_K onto K defined in Lemma 2.1. Define the map

$$\Gamma : \mathbb{R}^n \to \mathbb{R}^n, \quad \Gamma \boldsymbol{x} = P\boldsymbol{x} - \boldsymbol{x}.$$

Note that $-\Gamma \boldsymbol{x} \in N_K(P\boldsymbol{x})$ and $\|\Gamma \boldsymbol{x}\|_{\mathbf{e}} = \operatorname{dist}(\boldsymbol{x}, K)$. Moreover, Γ is dissipative, that is,

$$\big(\Gamma \boldsymbol{x} - \Gamma \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y}\big) \leq 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{2.96}$$

Indeed,

$$\begin{aligned} \big(\Gamma \boldsymbol{x} - \Gamma \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y}\big) &= \big(P\boldsymbol{x} - P\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y}\big) - \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{e}}^2 \\ &\leq \|P\boldsymbol{x} - P\boldsymbol{y}\|_{\mathbf{e}} \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{e}} - \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{e}}^2 \overset{(2.87)}{\leq} 0. \end{aligned}$$

We will obtain the solution of (2.91) as the limit of the solutions $\{\boldsymbol{x}_\varepsilon\}_{\varepsilon>0}$ of the approximative Cauchy problem

$$\begin{aligned} \boldsymbol{x}_\varepsilon'(t) = \boldsymbol{f}\big(\boldsymbol{x}_\varepsilon(t)\big) + \boldsymbol{g}(t) + \frac{1}{\varepsilon}\Gamma \boldsymbol{x}_\varepsilon(t), \\ \boldsymbol{x}_\varepsilon(t_0) = \boldsymbol{x}_0. \end{aligned} \tag{2.97}$$

For any $\varepsilon > 0$, the map $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, $F_\varepsilon(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \frac{1}{\varepsilon}\Gamma \boldsymbol{x}$ is Lipschitz. Hence, the Cauchy problem (2.97) has a unique right-saturated solution $\boldsymbol{x}_\varepsilon(t)$ defined on an interval $[t_0, T_\varepsilon)$. Since F_ε is globally Lipschitz, it follows that $\boldsymbol{x}_\varepsilon$ is defined over $[t_0, T]$. (See Example 2.1.)

Taking the scalar product of (2.97) with $\boldsymbol{x}_\varepsilon(t) - \boldsymbol{x}_0$, and observing that $\Gamma \boldsymbol{x}_0 = 0$, $\boldsymbol{x}_0 \in K$, we deduce from (2.96) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2 &\leq \big(\,\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t))+\boldsymbol{g}(t),\ \boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\,\big)\\ &\leq \|\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t))-\boldsymbol{f}(\boldsymbol{x}_0)\|_{\mathbf{e}}\cdot\|\boldsymbol{x}_\varepsilon-\boldsymbol{x}_0\|_{\mathbf{e}}\\ &\quad+\big(\|\boldsymbol{f}(\boldsymbol{x}_0)\|_{\mathbf{e}}+\|\boldsymbol{g}(t)\|_{\mathbf{e}}\big)\cdot\|\boldsymbol{x}_\varepsilon-\boldsymbol{x}_0\|_{\mathbf{e}}.\end{aligned}$$

Hence, if L denotes the Lipschitz constant of $\boldsymbol{f}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2 \leq L\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2+\frac{1}{2}\big(\|\boldsymbol{f}(\boldsymbol{x}_0)\|_{\mathbf{e}}+\|\boldsymbol{g}(t)\|_{\mathbf{e}}\big)^2\\ +\frac{1}{2}\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2.\end{aligned}$$

We set

$$M:=\sup_{t\in[t_0,T]}\Big(\|\boldsymbol{f}(\boldsymbol{x}_0)\|_{\mathbf{e}}+\|\boldsymbol{g}(t)\|_{\mathbf{e}}\Big)^2,\ L'=2L+1,$$

and we get

$$\frac{d}{dt}\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2\leq L'\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2+M.$$

Gronwall's lemma now yields the following ε-*independent* upper bound

$$\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_0\|_{\mathbf{e}}^2\leq Me^{L'(t-t_0)},\quad \forall t\in[t_0,T]. \tag{2.98}$$

Using (2.97), we deduce that

$$\begin{aligned}\frac{d}{dt}\big(\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\big)=\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t+h))-\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t))+\boldsymbol{g}(t+h)-\boldsymbol{g}(t)\\ +\frac{1}{\varepsilon}\big(\varGamma\boldsymbol{x}_\varepsilon(t+h)-\varGamma\boldsymbol{x}_\varepsilon(t)\big).\end{aligned}$$

Let $h>0$. Taking the scalar product with $\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)$ of both sides of the above equality and using the dissipativity of $\varGamma$, we deduce that for any $t\in[t_0,T]$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2\leq\big(\,\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t+h))-\boldsymbol{f}(\boldsymbol{x}_\varepsilon(t)),\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\,\big)\\ +\big(\,\boldsymbol{g}(t+h)-\boldsymbol{g}(t),\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\,\big)\\ \leq L\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2+\frac{1}{2}\Big(\|\boldsymbol{g}(t+h)-\boldsymbol{g}(t)\|_{\mathbf{e}}^2+\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2\Big),\end{aligned}$$

so that, setting again $L'=2L+1$, we obtain by integration

$$\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2 \le \|\boldsymbol{x}_\varepsilon(h)-\boldsymbol{x}_0\|_{\mathbf{e}}^2 + \int_{t_0}^t \|\boldsymbol{g}(s+h)-\boldsymbol{g}(s)\|_{\mathbf{e}}^2 ds$$
$$+L' \int_{t_0}^t \|\boldsymbol{x}_\varepsilon(s+h)-\boldsymbol{x}_\varepsilon(s)\|_{\mathbf{e}}^2 ds,$$

for all $t \in [t_0, T-h]$. Since g is a C^1-function, there exists a $C_0 > 0$ such that

$$\|\boldsymbol{g}(s+h)-\boldsymbol{g}(s)\|_{\mathbf{e}} \le C_0 h, \quad \forall t \in [t_0, T-h].$$

Hence, $\forall t \in [t_0, T-h]$, we have

$$\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2 \le \|\boldsymbol{x}_\varepsilon(h)-\boldsymbol{x}_0\|_{\mathbf{e}} + C_0^2 h^2 (T-t_0)$$
$$+ L' \int_{t_0}^t \|\boldsymbol{x}_\varepsilon(s+h)-\boldsymbol{x}_\varepsilon(s)\|_{\mathbf{e}}^2 ds.$$

Using Gronwall's lemma once again, we deduce that

$$\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}}^2 \le \Big(\|\boldsymbol{x}_\varepsilon(h)-\boldsymbol{x}_0\|_{\mathbf{e}} + C_0^2 h^2 (T-t_0)\Big) e^{L'(t-t_0)}$$
$$\overset{(2.98)}{\le} \Big(M e^{L'h} + C_0^2 h^2 (T-t_0)\Big) e^{L'(t-t_0)}.$$

Thus, for some constant $C_1 > 0$, *independent of* ε *and* h, we have

$$\|\boldsymbol{x}_\varepsilon(t+h)-\boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}} \le C_1 h, \quad \forall t \in [t_0, T-h].$$

Thus

$$\|\boldsymbol{x}_\varepsilon'(t)\| \le C_1, \quad \forall t \in [0, T]. \tag{2.99}$$

From the equality

$$\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_\varepsilon(s) = \int_s^t \boldsymbol{x}_\varepsilon'(\tau) d\tau$$

we find that

$$\|\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_\varepsilon(s)\|_{\mathbf{e}} \le C_1 |t-s|, \quad \forall t, s \in [t_0, T]. \tag{2.100}$$

This shows that the family $\{\boldsymbol{x}_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded and equicontinuous on $[t_0, T]$. Arzelà's theorem now implies that there exists a subsequence (for simplicity, denoted by ε) and a continuous function $\boldsymbol{x} : [t_0, T] \to \mathbb{R}^n$ such that $\boldsymbol{x}_\varepsilon(t)$ converges uniformly to $\boldsymbol{x}(t)$ on $[t_0, T]$ as $\varepsilon \to 0$.

Passing to the limit in (2.100), we deduce that the limit function $\boldsymbol{x}(t)$ is Lipschitz on $[t_0, T]$. In particular, $\boldsymbol{x}(t)$ is absolutely continuous and almost everywhere differentiable on this interval. From (2.97) and (2.99), it follows that there exists a constant $C_2 > 0$, *independent of* ε, such that

$$\mathrm{dist}(\boldsymbol{x}_\varepsilon(t), K) = \|\Gamma \boldsymbol{x}_\varepsilon(t)\|_{\mathbf{e}} \le C_2\varepsilon, \quad \forall t \in [t_0, T].$$

This proves that $\mathrm{dist}(\boldsymbol{x}(t), K) = 0$, $\forall t$, that is, $\boldsymbol{x}(t) \in K, \quad \forall t \in [t_0, T]$.

We can now prove inequality (2.93). To do this, we fix a point t where the function $\boldsymbol{x}$ is differentiable (we saw that this happens for almost any $t \in [t_0, T]$). From (2.97) and (2.86), we deduce that for almost all $s \in [t_0, T]$ and any $\boldsymbol{z} \in K$ we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}_\varepsilon(s) - \boldsymbol{z}\|_{\mathbf{e}}^2 \le \big(\boldsymbol{f}(\boldsymbol{x}_\varepsilon(s)) + \boldsymbol{g}(s),\ \boldsymbol{x}_\varepsilon(s) - \boldsymbol{z}\big).$$

Integrating from t to $t+h$, we deduce that

$$\begin{aligned}&\frac{1}{2}(\|\boldsymbol{x}_\varepsilon(t+h) - \boldsymbol{z}\|_{\mathbf{e}}^2 - \|\boldsymbol{x}_\varepsilon(t) - \boldsymbol{z}\|_{\mathbf{e}}^2)\\ &\qquad\qquad \le \int_t^{t+h} (\boldsymbol{f}(\boldsymbol{x}_\varepsilon(s)) + \boldsymbol{g}(s), \boldsymbol{x}_\varepsilon(s) - \boldsymbol{z})ds, \ \forall \boldsymbol{z} \in K.\end{aligned}$$

Now, let us observe that, for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we have

$$\frac{1}{2}\big(\|\boldsymbol{u} + \boldsymbol{v}\|_{\mathbf{e}}^2 - \|\boldsymbol{v}\|_{\mathbf{e}}^2\big) \ge (\boldsymbol{u}, \boldsymbol{v}).$$

Using this inequality with $\boldsymbol{u} = \boldsymbol{x}_\varepsilon(t+h) - \boldsymbol{x}_\varepsilon(t), \quad \boldsymbol{v} = \boldsymbol{x}_\varepsilon(t) - \boldsymbol{z}$, we get

$$\frac{1}{h}\big(\boldsymbol{x}_\varepsilon(t+h) - \boldsymbol{x}_\varepsilon(t),\ \boldsymbol{x}_\varepsilon(t) - \boldsymbol{z}\big) \le \frac{1}{2h}\big(\|\boldsymbol{x}_\varepsilon(t+h) - \boldsymbol{z}\|_{\mathbf{e}}^2 - \|\boldsymbol{x}_\varepsilon(t) - \boldsymbol{z}\|_{\mathbf{e}}^2\big)$$

$$\le \frac{1}{h}\int_t^{t+h} \big(\boldsymbol{f}(\boldsymbol{x}_\varepsilon(s)) + \boldsymbol{g}(s),\ \boldsymbol{x}_\varepsilon(s) - \boldsymbol{z}\big)ds, \quad \forall \boldsymbol{z} \in K.$$

Letting $\varepsilon \to 0$, we find

$$\frac{1}{h}\big(\boldsymbol{x}(t+h) - \boldsymbol{x}(t),\ \boldsymbol{x}(t) - \boldsymbol{z}\big) \le \frac{1}{h}\int_t^{t+h} \big(\boldsymbol{f}(\boldsymbol{x}(s)) + \boldsymbol{g}(s),\ \boldsymbol{x}(s) - \boldsymbol{z}\big)ds, \quad \forall \boldsymbol{z} \in K.$$

Finally, letting $h \to 0$, we obtain

$$\big(\boldsymbol{x}'(t) - \boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{g}(t), \boldsymbol{x}(t) - \boldsymbol{z}) \le 0, \quad \forall \boldsymbol{z} \in K.$$

This is precisely (2.93).

The uniqueness of the solution now follows easily. Suppose that $\boldsymbol{x}, \boldsymbol{y}$ are solutions of (2.69)–(2.94). We obtain from (2.93) that

$$\big(\boldsymbol{x}'(t) - \boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{g}(t), \boldsymbol{x}(t) - \boldsymbol{y}(t)) \le 0,$$

$$\big(\boldsymbol{y}'(t) - \boldsymbol{f}(\boldsymbol{y}(t)) - \boldsymbol{g}(t), \boldsymbol{y}(t) - \boldsymbol{x}(t)) \le 0,$$

so that

$$\big(\boldsymbol{x}'(t) - \boldsymbol{y}'(t) - (\boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{f}(\boldsymbol{y}(t))), \boldsymbol{x}(t) - \boldsymbol{y}(t)\big) \leq 0$$

which finally implies

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|_{\mathbf{e}}^2 \leq L\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|_{\mathbf{e}}^2,$$

for almost all $t \in [t_0, T]$. Integrating and using the fact that $t \mapsto \|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|_{\mathbf{e}}^2$ is Lipschitz, we deduce that

$$\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|_{\mathbf{e}}^2 \leq 2L\int_{t_0}^{t} \|\boldsymbol{x}(s) - \boldsymbol{y}(s)\|_{\mathbf{e}}^2 ds.$$

Gronwall's lemma now implies $\boldsymbol{x}(t) = \boldsymbol{y}(t), \forall t$. We have one last thing left to prove, namely, (2.95). Let us observe that (2.91) implies

$$\frac{d}{ds}\boldsymbol{x}(t+s) - \boldsymbol{f}(\boldsymbol{x}(t+s)) - \boldsymbol{g}(t+s) \in -N_K(\boldsymbol{x}(t+s))$$

for almost all t, s. On the other hand, using (2.84), we deduce that

$$\big(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{x}(t+s) - \boldsymbol{x}(s)\big) \geq 0, \quad \forall \boldsymbol{u} \in N_K(\boldsymbol{x}(t+s)), \quad \boldsymbol{v} \in N_K(\boldsymbol{x}(t)).$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|\boldsymbol{x}(t+s) - \boldsymbol{x}(t)\|_{\mathbf{e}}^2 &= \Big(\frac{d}{ds}\boldsymbol{x}(t+s),\ \boldsymbol{x}(t+s) - \boldsymbol{x}(t)\Big)\\ &\leq \big(-\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}(t+s)) + \boldsymbol{g}(t+s),\ \boldsymbol{x}(t+s) - \boldsymbol{x}(t)\big),\end{aligned}$$

$\forall \boldsymbol{v} \in N_K(\boldsymbol{x}(t))$. Integrating with respect to s on $[0, h]$, we deduce that

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\|_{\mathbf{e}}^2 &\leq \int_0^h (-\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}(t+s)) + \boldsymbol{g}(t+s), \boldsymbol{x}(t+s) - \boldsymbol{x}(t))ds\\ &\leq \int_0^h \|-\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}(t+s)) + \boldsymbol{g}(t+s)\|_{\mathbf{e}} \cdot \|\boldsymbol{x}(t+s) - \boldsymbol{x}(t)\|_{\mathbf{e}}\, ds.\end{aligned}$$

Using Proposition 1.2, we conclude that

$$\|\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\|_{\mathbf{e}} \leq \int_0^h \|-\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}(t+s)) + \boldsymbol{g}(t+s)\|_{\mathbf{e}} ds, \quad \forall h, \quad \forall \boldsymbol{v} \in N_K(\boldsymbol{x}(t)).$$

Dividing by $h > 0$ and letting $h \to 0$, we deduce that

$$\|\boldsymbol{x}'(t)\|_{\mathbf{e}} \leq \|-\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(t)\|_{\mathbf{e}}, \quad \forall \boldsymbol{v} \in N_K(\boldsymbol{x}(t)).$$

This means that $\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(t) - \boldsymbol{x}'(t)$ is the point in $N_K(\boldsymbol{x}(t))$ closest to $\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(t)$. This is precisely the statement (2.95). □

Remark 2.7 If the solution $\varphi(t)$ of the Cauchy problem

$$\varphi' = \boldsymbol{f}(\varphi) + \boldsymbol{g}, \quad \forall t \in [t_0, T], \quad \varphi(t_0) = \boldsymbol{x}_0, \tag{2.101}$$

stays in K for all $t \in [t_0, T]$, then φ coincides with the unique solution $\boldsymbol{x}(t)$ of (2.92)–(2.94).

Indeed, if we subtract (2.101) from (2.91) and we take the scalar product with $\boldsymbol{x}(t) - \varphi(t)$, then we obtain the inequality

$$\frac{1}{2}\|\boldsymbol{x}(t) - \varphi(t)\|_{\mathbf{e}}^2 \leq L\|\boldsymbol{x}(t) - \varphi(t)\|_{\mathbf{e}}^2, \quad \forall t \in [t_0, T].$$

Gronwall's lemma now implies $\boldsymbol{x} \equiv \varphi$.

Example 2.6 Consider a particle of unit mass that is moving in a planar domain $K \subset \mathbb{R}^2$ under the influence of a homogeneous force field $\boldsymbol{F}(t)$. We assume that K is convex; see Fig. 2.4.

If we denote by $\boldsymbol{g}(t)$ an antiderivative of $\boldsymbol{F}(t)$, and by $\boldsymbol{x}(t)$ the position of the particle at time t, then, intuitively, the motion ought to be governed by the differential equations

$$\begin{aligned} \boldsymbol{x}'(t) &= \boldsymbol{g}(t), \ \text{ if } \ \boldsymbol{x}(t) \in \operatorname{int} K \\ \boldsymbol{x}'(t) &= \boldsymbol{g}(t) - P_{N(\boldsymbol{x}(t))}\boldsymbol{g}(t), \ \text{ if } \ \boldsymbol{x}(t) \in \partial K, \end{aligned} \tag{2.102}$$

where $N(\boldsymbol{x}(t))$ is the half-line starting at $\boldsymbol{x}(t)$, normal to ∂K and pointing towards the exterior of K; see Fig. 2.4. (If ∂K is not smooth, then $N_K(\boldsymbol{x}(t))$ is a cone pointed at $\boldsymbol{x}(t)$.) Thus $\boldsymbol{x}(t)$ is the solution of the evolution variational inequation

$$\begin{aligned} \boldsymbol{x}(t) &\in K, \ \forall t \geq 0, \\ \boldsymbol{x}'(t) &\in \boldsymbol{g}(t) - N(\boldsymbol{x}(t)), \ \forall t > 0. \end{aligned}$$

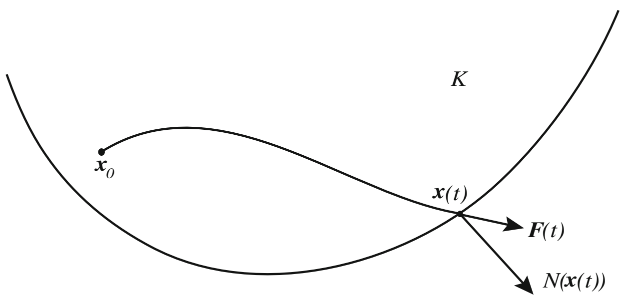


Fig. 2.4 Motion of a particle confined to a convex region

Theorem 2.16 confirms that the motion of the particle is indeed the one described above. Moreover, the proof of Theorem 2.16 offers a way of approximating its trajectory.

Example 2.7 Let us have another look at the radioactive disintegration model we discussed in Sect. 1.3.1. If $x(t)$ denotes the quantity of radioactive material, then the evolution of this quantity is governed by the ODE

$$x'(t) = -\alpha x(t) + g(t), \tag{2.103}$$

where $g(t)$ denotes the amount of radioactive material that is added or extracted per unit of time at time t. Clearly, $x(t)$ is a solution of (2.103) only for those t such that $x(t) > 0$. That is why it is more appropriate to assume that x satisfies the following equations

$$\begin{aligned} x(t) &\geq 0, \quad \forall t \geq 0, \\ x'(t) &= -\alpha x(t) + g(t), \quad \forall t \in E_x \\ x'(t) &= \max\{g(t), 0\}, \quad \forall t \in [0, \infty) \setminus E_x, \end{aligned} \tag{2.104}$$

where the set

$$E_x := \big\{ t \geq 0; \quad x(t) > 0 \big\}$$

is also one of the unknowns in the above problem. This is a so-called "free boundary" problem. Let us observe that (2.104) is equivalent to the variational inequality (2.93) with

$$K := \big\{ x \in \mathbb{R}; \quad x \geq 0 \big\}.$$

More precisely, (2.104) is equivalent to

$$\big(x'(t) + \alpha x(t) - g(t) \big) \cdot \big(x(t) - y \big) \leq 0, \quad \forall y \in K, \tag{2.105}$$

for almost all $t \geq 0$.

In formulation (2.105), the set E_x has disappeared, but we have to pay a price, namely, the new equation is a differential inclusion.

Example 2.8 Consider a factory consisting of n production units, each generating only one type of output. We denote by $x_i(t)$ the size of the output of unit i at time t, by $c_i(t)$ the demand for the product i at time t, and by $p_i(t)$ the rate at which the output i is produced. The demands and stocks define, respectively, the vector-valued maps

$$\boldsymbol{c}(t) := \begin{bmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{bmatrix}, \quad \boldsymbol{x}(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

and we will assume that the demand vector depends linearly on the stock vector, that is,

$$\boldsymbol{c}(t) = C\boldsymbol{x}(t) + \boldsymbol{d}(t),$$

where C is an $n \times n$ matrix, and $\boldsymbol{d} : [0, T] \to \mathbb{R}^n$ is a differentiable map. For $i = 1, \dots, n$, define

$$E_{x_i} := \big\{ t \in [0, T]; \quad x_i(t) > 0 \big\}.$$

Obviously, the functions x_i satisfy the following equations

$$\begin{aligned} x_i(t) \geq 0, \ \ \forall t \in [0, T], \\ x_i'(t) = p_i(t) - \big(C\boldsymbol{x}(t)\big)_i - d_i(t), \ \ t \in E_{x_i}, \\ x_i'(t) - p_i(t) + \big(C\boldsymbol{x}(t)\big)_i + d_i(t) \geq 0, \ \ \forall t \in [0, T] \setminus E_{x_i}. \end{aligned} \tag{2.106}$$

We can now see that the solutions of the variational problem (2.92) and (2.93) with

$$\boldsymbol{f}(\boldsymbol{x}) = -C\boldsymbol{x}, \quad \boldsymbol{g}(t) = \boldsymbol{p}(t) - \boldsymbol{d}(t),$$

and

$$K = \big\{ \boldsymbol{x} \in \mathbb{R}^n; \quad x_i \geq 0, \quad \forall i = 1, \dots, n \big\},$$

are also solutions of (2.106).

Remark 2.8 Theorem 2.16 extends to differential inclusions of the form

$$\begin{aligned} &\boldsymbol{x}'(t) \in \boldsymbol{f}(\boldsymbol{x}(t)) + \phi(\boldsymbol{x}(t)) + \boldsymbol{g}(t), \quad \text{a.e. } t \in (0, T), \\ &\boldsymbol{x}(0) = \boldsymbol{x}_0, \end{aligned} \tag{2.107}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and $\phi : D \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a maximal dissipative mapping, that is,

$$(v_1 - v_2, u_1 - u_2) \leq 0, \ \forall v_i \in \phi(u_i), \ i = 1, 2,$$

and the range of the map $u \to u + \lambda\phi(u)$ is all of $\mathbb{R}^n$ for $\lambda > 0$. The method of proof is essentially the same as that of Theorem 2.16. Namely, one approximates (2.107) by

$$\begin{aligned} &\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \phi_\varepsilon(\boldsymbol{x}(t)) + \boldsymbol{g}(t), \ t \in (0, T), \\ &\boldsymbol{x}(0) = \boldsymbol{x}_0, \end{aligned} \tag{2.108}$$

where ϕ_ε is the Lipschitz mapping $\frac{1}{\varepsilon}((I - \varepsilon\phi)^{-1}\boldsymbol{x} - \boldsymbol{x})$, $\varepsilon > 0$, $\boldsymbol{x} \in \mathbb{R}^n$.

Then, one obtains the solution to (2.107) as $\boldsymbol{x}(t) = \lim_{\varepsilon \to 0} \boldsymbol{x}_\varepsilon(t)$, where $\boldsymbol{x}_\varepsilon \in C^1([0, T]; \mathbb{R}^n)$ is the solution to (2.108). We refer to [2] for details and more general results. We note, however, that this result applies to the Cauchy problem with discontinuous monotone functions ϕ. For instance, if ϕ is discontinuous in $\boldsymbol{x}^0$, then one fills the jump at $\boldsymbol{x}^0$ by redefining ϕ as

$$\widetilde{\phi}(\boldsymbol{x}) = \begin{cases} \phi(\boldsymbol{x}) & \text{for } \boldsymbol{x} \neq \boldsymbol{x}^0, \\ \lim\limits_{\boldsymbol{y} \to \boldsymbol{x}^0} \phi(\boldsymbol{y}) & \text{for } \boldsymbol{x} = \boldsymbol{x}^0. \end{cases}$$

Clearly, $\widetilde{\phi}$ is maximal dissipative.

Problems

2.1 Find the maximal interval of existence for the solution of the Cauchy problem

$$\begin{aligned} x' &= -x^2 + t + 1, \\ x(0) &= 1, \end{aligned}$$

and then find the first three Picard iterations of this problem.

2.2 Consider the Cauchy problem

$$\begin{aligned} x' &= f(t, x), \\ x(t_0) &= x_0, \quad (t_0, x_0) \in \Omega \subset \mathbb{R}^2, \end{aligned} \tag{2.109}$$

where the function f is continuous in (t, x) and locally Lipschitz in x. Prove that if $x_0 \geq 0$ and $f(t, 0) > 0$ for any $t \geq t_0$, then the saturated solution $x(t; t_0, x_0)$ is nonnegative for any $t \geq t_0$ in the existence interval.

2.3 Consider the system

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(t, \boldsymbol{x}), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \quad t_0 \geq 0, \end{aligned}$$

where the function $\boldsymbol{f} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t, \boldsymbol{x})$, locally Lipschitz in $\boldsymbol{x}$ and satisfies the condition

$$\big(\boldsymbol{f}(t, \boldsymbol{x}), P\boldsymbol{x} \big) \leq 0, \quad \forall t \geq 0, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{2.110}$$

where P is a real, symmetric and positive definite $n \times n$ matrix. Prove that any right-saturated solution of the system is defined on the semi-axis $[t_0, \infty)$.

Hint. Imitate the argument used in the proof of Theorem 2.12. Another approach is to replace the scalar product of $\mathbb{R}^n$ by

$$\langle u, v \rangle = (u, Pv), \ \forall u, v \in \mathbb{R}^n,$$

and argue as in the proof of Theorem 2.13.

2.4 Consider the Cauchy problem

$$x'' + ax + f(x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = x_1, \tag{2.111}$$

where a is a positive constant, and $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying

$$yf(y) \geq 0, \quad \forall y \in \mathbb{R}.$$

Prove that any right-saturated solution of (2.111) is defined on the semi-axis $[t_0, \infty)$.

Hint. Multiplying (2.111) by x' we deduce

$$\frac{1}{2}\frac{d}{dt}|x'(t)|^2 + a|x'(t)|^2 \leq 0, \ \forall t \geq t_0.$$

Conclude that the functions $x(t)$ and $x'(t)$ are bounded and then use Theorem 2.11.

2.5 In the anisotropic theory of relativity due to V.G. Boltyanski, the propagation of light in a neighborhood of a mass m located at the origin of $\mathbb{R}^3$ is described by the equation

$$\boldsymbol{x}' = -\frac{m\gamma}{\|\boldsymbol{x}\|_{\mathbf{e}}^3}\boldsymbol{x} + \boldsymbol{u}(t), \tag{2.112}$$

where γ is a positive constant, $\boldsymbol{u} : [0, \infty) \to \mathbb{R}^3$ is a continuous and *bounded* function, that is,

$$\exists C > 0; \quad \|\boldsymbol{u}(t)\|_{\mathbf{e}} \leq C, \quad \forall t \geq 0,$$

and $\boldsymbol{x}(t) \in \mathbb{R}^3$ is the location of the photon at time t. Prove that there exists an $r > 0$ such that all the trajectories of (2.112), which start at $t = 0$ in the ball

$$B_r := \left\{\boldsymbol{x} \in \mathbb{R}^3; \quad \|\boldsymbol{x}\|_{\mathbf{e}} < r\right\},$$

will stay inside the ball as long as they are defined. (Such a ball is called a *black hole* in astrophysics.)

Hint. Take the scalar product of (2.112) with $\boldsymbol{x}(t)$ to obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}(t)\|_{\mathbf{e}}^2 = -\frac{m\gamma}{\|\boldsymbol{x}(t)\|_{\mathbf{e}}} + \big(\boldsymbol{u}(t), \boldsymbol{x}(t)\big) \leq -\frac{m\gamma}{\|\boldsymbol{x}(t)\|_{\mathbf{e}}} + C\|\boldsymbol{x}(t)\|_{\mathbf{e}}.$$

Use this differential inequality to obtain an upper estimate for $\|\boldsymbol{x}(t)\|_{\mathbf{e}}$.

2.6 (Wintner's extendibility test) Prove that, if the continuous function $f = f(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in x and satisfies the inequality

$$|f(t, x)| \leq \mu(|x|), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{2.113}$$

where $\mu : (0, \infty) \to (0, \infty)$ satisfies

$$\int_0^\infty \frac{dr}{\mu(r)} < \infty, \tag{2.114}$$

then all the solutions of $x' = f(t, x)$ are defined on the whole axis $\mathbb{R}$.

Hint. According to Theorem 2.11, it suffices to show that all the solutions of $x' = f(t, x)$ are bounded. To prove this, we conclude from (2.113) that

$$\left| \int_{x_0}^{x(t)} \frac{dr}{\mu(r)} \right| \leq |t - t_0|, \quad \forall t,$$

and then invoke (2.114).

2.7 Prove that the saturated solution of the Cauchy problem

$$\begin{aligned} x' &= e^{-x^2} + t^2, \\ x(0) &= 1, \end{aligned} \tag{2.115}$$

is defined on the interval $\left[0, \frac{1}{2}\right]$. Use Euler's method with step size $h = 10^{-2}$ to find an approximation of this solution at the nodes $t_j = jh$, $j = 1, \ldots, 50$.

2.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nonincreasing function. Consider the Cauchy problem

$$\begin{aligned} x'(t) &= f(x), \quad \forall t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{2.116}$$

According to Theorem 2.13, this problem has a unique solution $x(t)$ which exists on $[0, \infty)$.

(a) Prove that, for any $\lambda > 0$, the function

$$\mathbb{1} - \lambda f : \mathbb{R} \to \mathbb{R}, \ x \mapsto x - \lambda f(x),$$

is bijective. For any integer $n > 0$, we set

$$(\mathbb{1} - \lambda f)^{-n} := \underbrace{(\mathbb{1} - \lambda f)^{-1} \circ \cdots \circ (\mathbb{1} - \lambda f)^{-1}}_{n}.$$

(b) Prove that $x(t)$ is given by the formula

$$x(t) = \lim_{n \to \infty} \left(\mathbb{1} - \frac{t}{n} f \right)^{-n} x_0, \quad \forall t \geq 0. \tag{2.117}$$

Hint. Fix $t > 0$, $n > 0$, set

$$h_n := \frac{t}{n}$$

and define iteratively

$$x_0^n = x_0, \quad \frac{x_i^n - x_{i-1}^n}{h_n} = f(x_i^n), \quad i = 1, \dots n,$$

that is,

$$x_i^n = \left(\mathbb{1} - \frac{t}{n} f\right)^{-1} x_{i-1}^n = \left(\mathbb{1} - \frac{t}{n} f\right)^{-i} x_0. \tag{2.118}$$

Let $x^n : [0, t] \to \mathbb{R}$ be the unique continuous function which is linear on each of the intervals $[(i-1)h_n, ih_n]$ and satisfies

$$x^n(ih_n) = x_i^n, \quad \forall i = 0, \dots, n.$$

Argue as in the proof of Peano's theorem to show that x^n converges uniformly to x on $[0, t]$ as $n \to \infty$. Equality (2.117) now follows from (2.118).

2.9 Consider the Cauchy problem

$$\begin{aligned} x' &= f(x), \quad t \geq 0 \\ x(0) &= x_0, \end{aligned} \tag{2.119}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function. Let $x = \varphi(t)$ be a solution of (2.119). Prove that, if the set

$$F := \left\{ y \in \mathbb{R}; \quad f(y) = 0 \right\}$$

is nonempty, then the following hold.

(i) The function $t \mapsto |x'(t)|$ is nonincreasing on $[0, \infty)$.
(ii) $\lim_{t\to\infty} |x'(t)| = 0$.
(iii) There exists an $x_\infty \in F$ such that $\lim_{t\to\infty} x(t) = x_\infty$.

Hint. Since f is nonincreasing we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big(x(t+h) - x(t)\big)^2 &= \big(x'(t+h) - x'(t)\big)\big(x(t+h) - x(t)\big) \\ &= \big(f(x(t+h)) - f(x(t)\big)\big(x(t+h) - x(t)\big) \leq 0. \end{aligned}$$

Hence, for any $h \geq 0$ and $t_2 \geq t_1 \geq 0$, we have

$$\big| x(t_2 + h) - x(t_1) \big| \leq \big| x(t_1 + h) - x(t_1) \big|.$$

This proves (i). To prove (ii) multiply both sides of (2.119) by $x(t) - y_0$, where $y_0 \in F$. Conclude, similarly, that

$$\frac{d}{dt}\big(x(t) - y_0\big)^2 \le 0,$$

showing that $\lim_{t\to\infty}(x(t) - y_0)^2$ exists. Next multiply the equation by $x'(t)$ to obtain

$$\int_0^t |x'(s)|^2 ds = g(x(t)) - g(x_0),$$

where g is an antiderivative of f. We deduce that

$$\int_0^\infty |x'(t)|^2 dt < \infty,$$

which when combined with (i) yields (ii). Next, pick a subsequence $t_n \to \infty$ such that $x(t_n) \to y_0$. From (i) and (ii), it follows that $y_0 \in F$. You can now conclude that

$$\lim_{t\to\infty} x(t) = y_0.$$

2.10 Prove that the conclusions of Problem 2.9 remain true for the system

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}), \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a dissipative and continuous mapping of the form $f = \nabla \boldsymbol{g}$, where $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and $\boldsymbol{g} \ge 0$.

Hint. One proceeds as above by taking into account that

$$\frac{d}{dt}\boldsymbol{g}(\boldsymbol{x}(t)) = (\boldsymbol{x}'(t), \boldsymbol{f}(\boldsymbol{x}(t))),\ \forall t \ge 0.$$

2.11 Consider the system

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}) - \lambda\boldsymbol{x} + \boldsymbol{f}_0, \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0, \end{aligned}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and dissipative, $\lambda > 0$ and $\boldsymbol{f}_0, \boldsymbol{x}_0 \in \mathbb{R}^n$. Prove that

(a) $\lim_{t\to\infty} \boldsymbol{x}(t)$ exists (call this limit $\boldsymbol{x}_\infty$),
(b) $\lim_{t\to\infty} \boldsymbol{x}'(t) = 0$,
(c) $\lambda\boldsymbol{x}_\infty - \boldsymbol{f}(\boldsymbol{x}_\infty) = f_0$.

Hint. For each $h > 0$, one has

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\|_{\mathbf{e}}^2 + \lambda\|\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\|_{\mathbf{e}}^2 \le 0,$$

which implies that (a) holds and that

$$\|\boldsymbol{x}'(t)\|_{\mathbf{e}} \leq e^{-\lambda t}\|\boldsymbol{x}'(0)\|_{\mathbf{e}}, \quad \forall t \geq 0.$$

2.12 Prove that (2.117) remains true for solutions $\boldsymbol{x}$ to the Cauchy problem (2.116), where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and dissipative.

Hint. By Problem 2.11(c), the function $\mathbb{1} - \lambda \boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and the Euler scheme (2.38) is equivalent to

$$\boldsymbol{x}_i^n = \left(\mathbb{1} - \frac{t}{k}\,\boldsymbol{f}\right)^{-i} \boldsymbol{x}_0 \ \text{ for } t \in (ih_k, (i+1)h_k).$$

Then, by the convergence of this scheme, one obtains

$$\boldsymbol{x}(t) = \lim_{k\to\infty}\left(\mathbb{1} - \frac{t}{k}\right)^{-k} \boldsymbol{x}_0, \quad \forall t \geq 0.$$

2.13 Consider the equation $x' = f(t, x)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a function continuous in (t, x) and locally Lipschitz in x, and satisfies the growth constraint

$$\big|\, f(t, x) \,\big| \leq \alpha(t)|x|, \quad \forall (t, x) \in \mathbb{R}^2,$$

where

$$\int_{t_0}^{\infty} \alpha(t)dt < \infty.$$

(a) Prove that any solution of the equation has finite limit as $t \to \infty$.

(b) Prove that if, additionally, f satisfies the Lipschitz condition

$$\big|\, f(t, x) - f(t, y) \,\big| \leq \alpha(t)|x - y|, \quad \forall t \in \mathbb{R}, \quad x, y \in \mathbb{R},$$

then there exists a bijective correspondence between the initial values of the solutions and their limits at ∞.

Hint. Use Theorem 2.11 as in Example 2.1.

2.14 Prove that the maximal existence interval of the Cauchy problem

$$\begin{aligned} x' &= ax^2 + t^2, \\ x(0) &= x_0, \end{aligned} \tag{2.120}$$

(a is a positive constant) is bounded from above. Compare this with the situation encountered in Example 2.2.

Hint. Let $x_0 \geq 0$ and $[0, T)$ be the maximal interval of definition on the right. Then, on this interval,

$$x(t; 0, x_0) \geq x_0 + \frac{t^3}{3}, \quad \frac{1}{x(t; 0, x_0)} \leq \frac{1}{x_0} - at.$$

Hence, $T = (ax_0)^{-1}$.

2.15 Consider the Volterra integral equation

$$\boldsymbol{x}(t) = \boldsymbol{g}(t) + \int_a^t \boldsymbol{f}(s, \boldsymbol{x}(s))ds, \quad t \in [a, b],$$

where $\boldsymbol{g} \in C([a, b]; \mathbb{R}^n)$, $\boldsymbol{f} : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and

$$\|\boldsymbol{f}(s, \boldsymbol{x}) - \boldsymbol{f}(s, \boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|, \forall s \in [a, b], \boldsymbol{x} \in \mathbb{R}^n.$$

Prove that there is a unique solution $\boldsymbol{x} \in C([a, b]; \mathbb{R}^n)$.

Hint. One proceeds as in the proofs of Theorems 2.1 and 2.4 by the method of successive approximations

$$\boldsymbol{x}_{n+1}(t) = \boldsymbol{g}(t) + \int_0^t f(s, \boldsymbol{x}_n(s))ds, \quad t \in [a, b],$$

and proving that $\{\boldsymbol{x}_n\}$ is uniformly convergent.

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