Lyubeznik Numbers of Local Rings and Linear Strands of Graded Ideals

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Abstract We report recent work on the study of Lyubeznik numbers and their relation to invariants coming from the study of linear strands of free resolutions.

The aim of this paper is to put in the same spotlight two different sets of invariants that apparently come from completely different approaches. On one hand we have the *Lyubeznik numbers* that are a set of invariants of local rings coming from the study of injective resolutions of local cohomology modules. On the other hand the ν*-numbers* that are defined in terms of the acyclycity of the linear strands of free resolutions of $\mathbb{Z}$-graded ideals. It turns out that both sets of invariants satisfy analogous properties and, in the particular framework of Stanley–Reisner theory, they are equivalent.

This note will survey some recent work done by the author with his collaborators Vahidi and Yanagawa in [1–3].

1 Lyubeznik Numbers of Local Rings

Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field $\mathbb{K}$, and I an ideal of R. Consider a minimal injective resolution of a local cohomology module $H_I^{n-i}(R)$

$$\mathbb{E}_\bullet(H_I^{n-i}(R)):\ 0 \longrightarrow H_I^{n-i}(R) \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots, \qquad (1)$$

where the p-th term is of the form

$$E_p = \bigoplus_{\mathfrak{p}\in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_p(\mathfrak{p}, H_I^{n-i}(R))},$$

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with the invariants $\mu_p(\mathfrak{p}, H_I^{n-i}(R))$ being the so-called *Bass numbers*. Since local cohomology modules are not finitely generated R-modules, it shouldn't be clear that this injective resolution is finite or that the Bass numbers are finite. These properties were proved to be true in the seminal works of Huneke–Sharp [6] and Lyubeznik [7] for the cases where the characteristic of the field $\mathbb{K}$ is positive and zero, respectively. Relying on these facts, Lyubeznik [7] proved that the Bass numbers

$$\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R)))$$

are numerical invariants of the local ring R/I. These invariants are nowadays known as *Lyubeznik numbers* and, despite its algebraic nature, encode interesting geometrical and topological information.

1.1 Properties of Lyubeznik Numbers

It was already proved in [7] that these invariants satisfy the following properties:

(i) *Vanishing*: $\lambda_{p,i}(R/I) \neq 0$ implies $0 \leq p \leq i \leq d$, where $d = \dim R/I$;
(ii) *Highest Lyubeznik number*: $\lambda_{d,d}(R/I) \neq 0$.

Therefore, we can collect them in the so-called *Lyubeznik table*:

$$\Lambda(R/I) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

and we say that the Lyubeznik table is *trivial* if $\lambda_{d,d} = 1$ and the rest of these invariants vanish. In [2, 3] we used the Grothendieck's spectral sequence

$$E_2^{p,n-i} = H_{\mathfrak{m}}^p(H_I^{n-i}(R)) \Longrightarrow H_{\mathfrak{m}}^{p+n-i}(R)$$

to extract some further constraints for the shape of the Lyubeznik table:

(iii) *Euler characteristic*: $\sum_{0 \leq p,i \leq d}(-1)^{p-i}\lambda_{p,i}(A) = 1$.
(iv) *Consecutiveness of trivial diagonals*: Let

$$\rho_j(R/I) = \sum_{i=0}^{d-j} \lambda_{i,i+j}(R/I)$$

denote the sum of the entries of the diagonals of the Lyubeznik table. Then,

(1) if $\rho_1(R/I) = 0$, then $\rho_0(R/I) = 1$;
(2) if $\rho_0(R/I) = 1$ and $\rho_2(R/I) = 0$, then $\rho_1(R/I) = 0$;
(3) if $\rho_{j-1}(R/I) = 0$ and $\rho_{j+1}(R/I) = 0$, then $\rho_j(R/I) = 0$ for $2 \leq j \leq d-1$.

1.2 Some Examples

Very little is known about the possible configurations of Lyubeznik tables. The first example one may think is when there is only one local cohomology module different from zero. Then, using Grothendieck's spectral sequence, we obtain a trivial Lyubeznik table. This situation is achieved, among others, in the following cases:

- R/I is Cohen–Macaulay and contains a field of positive characteristic.
- R/I is Cohen–Macaulay and I is a square-free monomial ideal in any characteristic.

In characteristic zero we may find examples of Cohen–Macaulay rings with non-trivial Lyubeznik table, e.g., the ideals generated by the 2×2 minors of a 2×3 matrix. We also point out that replacing the Cohen–Macaulay condition by sequentially Cohen–Macaulay in the cases considered above, the result still holds true; see [2].

Another case that has received a lot of attention is when all local cohomology modules $H_I^{n-i}(R)$ have dimension zero for $i \neq d$. In this case the Lyubeznik numbers of R/I satisfy $\lambda_{p,i}(R/I) = 0$ if $p \neq 0$ or $i \neq d$. Moreover, $\lambda_{0,i}(R/I) = \lambda_{d-i+1,d}(R/I)$ for $2 \leq i < d$, and $\lambda_{0,1}(R/I) = \lambda_{d,d}(R/I) - 1$. On top of that, these Lyubeznik numbers can be described in terms of certain singular cohomology groups in characteristic zero or étale cohomology groups in positive characteristic; see [4, 5].

2 Linear Strands of Graded Ideals

Now we turn our attention to $\mathbb{Z}$-graded ideals I in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. Consider a minimal $\mathbb{Z}$-graded free resolution of I,

$$\mathbb{L}_\bullet(I)\colon\ 0 \longrightarrow L_n \xrightarrow{d_n} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow I \longrightarrow 0, \qquad (2)$$

where the i-th term is of the form

$$L_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(I)},$$

and the invariants $\beta_{i,j}(I)$ are the so-called *Betti numbers*. Given $r \in \mathbb{N}$, we also consider the *r-linear strand* of $L_\bullet(I)$,

$$\mathbb{L}_\bullet^{<r>}(I)\colon\ 0 \longrightarrow L_n^{<r>} \xrightarrow{d_n^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0,$$

where $L_i^{<r>} = R(-i-r)^{\beta_{i,i+r}(I)}$, and the differential $d_i^{<r>}$ is the corresponding component of d_i.

In Àlvarez-Montaner–Yanagawa [3] we introduced a new set of invariants measuring the acyclicity of the linear strands. Namely, given the field of fractions $Q(R)$ of R, we introduce the ν-numbers:

$$\nu_{i,j}(I) := \dim_{Q(R)}[H_i(\mathbb{L}_\bullet^{<j-i>}(I) \otimes_R Q(R))].$$

2.1 Properties of ν-Numbers

Quite nicely, these invariants satisfy analogous properties to those satisfied by Lyubeznik numbers. However, mimicking the construction of the *Betti table*, we will consider the following table for ν-numbers:

$\nu_{i,i+r}$	0	1	$2\cdots$
0	$\nu_{0,0}$	$\nu_{1,1}$	$\nu_{2,2}$ $\cdots$
1	$\nu_{0,1}$	$\nu_{1,2}$	$\nu_{2,3}$ $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Given a $\mathbb{Z}$-graded ideal I, we denote $I_i = \{f \in I \mid \deg(f) = i\}$. Then we have:

(i) *highest ν-number*: $\nu_{0,l}(I) \neq 0$ where $l := \min\{i \mid I_i \neq 0\}$; in particular, we will say that I has a trivial ν-table when $\nu_{0,l}(I) \neq 1$ and the rest of these invariants are zero;
(ii) *Euler characteristic*: $\sum_{i,j\in\mathbb{N}}(-1)^i \nu_{i,j}(I) = 1$;
(iii) *consecutiveness of trivial columns*: with $\nu_i(I) = \sum_{j\in\mathbb{N}} \nu_{i,j}(I)$ denoting the sum of entries of the columns of the ν-table, we have

(1) if $\nu_1(I) = 0$, then $\nu_0(I) = 1$;
(2) if $\nu_0(I) = 1$ and $\nu_2(I) = 0$, then $\nu_1(I) = 0$;
(3) if $\nu_{j-1}(I) = 0$ and $\nu_{j+1}(I) = 0$, then $\nu_j(I) = 0$ for $2 \leq j \leq d-1$.

We also obtain the following property which, in general, is not known for the case of Lyubeznik numbers:

(iv) *Thom–Sebastiani type formula*: let I, J be $\mathbb{Z}$-graded ideals in two disjoint sets of variables, say $I \subseteq R = \mathbb{K}[x_1, \ldots, x_m]$ and $J \subseteq S = \mathbb{K}[y_1, \ldots, y_n]$. The ν-numbers of $IT + JT$, where $T = R \otimes_{\mathbb{K}} S = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ satisfy:

- if $I_1 \neq 0$ or $J_1 \neq 0$ then $IT + JT$ has trivial ν-table;
- if $I_1 = 0$ and $J_1 = 0$ then we have the equality

$$\nu_{i,j}(IT + JT) = \nu_{i,j}(IT) + \nu_{i,j}(JT) + \sum_{\substack{k+k'=i-1\\ l+l'=j}} \nu_{k,l}(IT)\nu_{k',l'}(JT).$$

2.2 Some Examples

So far we cannot present too many examples of possible configurations of ν-tables, except for the case of monomial ideals via the correspondence given in Theorem 1. However, using similar arguments to those considered in [2] to prove that sequentially Cohen–Macaulay rings have trivial Lyubeznik numbers, one can prove that *componentwise linear* ideals have trivial ν-table; see [3] for details.

3 The Case of Stanley–Reisner Rings

Let I be a square-free monomial ideal in the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$. In this case, I coincides with the *Stanley–Reisner ideal* I_Δ of a simplicial complex Δ in n vertices, and it is known that R/I_Δ reflects topological properties of the geometric realization $|\Delta|$ of Δ in several ways.

The jewel of the paper [1] is the following correspondence between the Lyubeznik numbers of a Stanley–Reisner ring R/I_Δ and the ν-numbers of the Alexander dual ideal $I_{\Delta^\vee}$ of I_Δ.

Theorem 1 (Àlvarez-Montaner–Vahidi, [1, Corollary 4.2]) *Consider a square-free monomial ideal $I_\Delta \subseteq R = \mathbb{K}[x_1, \dots, x_n]$. We have $\lambda_{p,i}(R/I_\Delta) = \nu_{i-p,n-p}(I_{\Delta^\vee})$.*

In this setting, another interesting result is that Lyubeznik numbers of Stanley–Reisner rings are not only algebraic invariants but also topological invariants. Namely,

Theorem 2 (Àlvarez-Montaner–Yanagawa, [3, Theorem 5.3]) *Consider a square-free monomial ideal $I_\Delta \subseteq R = \mathbb{K}[x_1, \dots, x_n]$. Then, $\lambda_{p,i}(R/I_\Delta)$ depends only on the homeomorphism class of $|\Delta|$ and* $\mathrm{char}(\mathbb{K})$.

Of course, we may find examples of simplicial complexes Δ and Δ' with homeomorphic geometric realizations but R/I_Δ being non-isomorphic to $R/I_{\Delta'}$. It is also well know that local cohomology modules as well as free resolutions depend on the characteristic of the base field, the most recurrent example being the Stanley–Reisner ideal associated to a minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$. Namely, for the ideal

$$I = (x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5,\ x_2x_4x_5,\ x_3x_4x_5,\ x_2x_3x_6,\ x_1x_4x_6,\ x_3x_4x_6,\ x_1x_5x_6,\ x_2x_5x_6)$$

in $R = \mathbb{K}[x_1, \ldots, x_6]$, the Lyubeznik table in characteristics zero and two are

$$\Lambda_{\mathbb{Q}}(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \Lambda_{\mathbb{Z}/2\mathbb{Z}}(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix},$$

respectively.

4 Open Questions

There are still a lot of open questions concerning Lyubeznik numbers and, by analogy, one can also formulate similar questions for ν-numbers. To name a few:

(i) What can we say about the vanishing of Lyubeznik numbers and possible configuration of Lyubeznik tables? In particular, is there an homological characterization of rings having trivial Lyubeznik table?
(ii) Recent developments in the study of local cohomology of determinantal ideals suggest that this would be a good set of examples where one can try to compute their Lyubeznik table; in general, it would be interesting to have other families of examples.
(iii) What kind of topological information is provided by Lyubeznik numbers? Is it possible to extend the results of [4, 5] to other situations?
(iv) It follows from Theorem 1 that Lyubeznik numbers of Stanley–Reisner rings satisfy a Thom–Sebastiani type formula. Does this property hold in general?
(v) Consider the local ring at the vertex of the affine cone for some embedding of a projective variety in a projective space. Zhang [8] proved that its Lyubeznik numbers depend only on the projective variety but not on the embedding when the base field has positive characteristic. However, it is still open whether this result is true in characteristic zero.

References

1. J. Àlvarez-Montaner, A. Vahidi, Lyubeznik numbers of monomial ideals. Trans. Am. Math. Soc. **366**, 1829–1855 (2014)
2. J. Àlvarez-Montaner, Lyubeznik table of sequentially Cohen-Macaulay rings. Commun. Algebra **43**, 3695–3704 (2015)
3. J. Àlvarez-Montaner, K. Yanagawa, Lyubeznik numbers of local rings and linear strands of graded ideals. arXiv:1409.6486
4. M. Blickle, R. Bondu, Local cohomology multiplicities in terms of étale cohomology. Ann. Inst. Fourier **55**, 2239–2256 (2005)
5. R. García, C. Sabbah, Topological computation of local cohomology multiplicities. Collect. Math. **49**, 317–324 (1998)

6. C. Huneke, R.Y. Sharp, Bass numbers of local cohomology modules. Trans. Am. Math. Soc. **339**(2), 765–779 (1993)
7. G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). Invent. Math. **113**, 41–55 (1993)
8. W. Zhang, Lyubeznik numbers of projective schemes. Adv. Math. **228**, 575–616 (2011)

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