Chapter 2
Infinite-Dimensional Monte Carlo Integration

2.1 Introduction

In mathematics, Monte Carlo integration is a technique for numerical integration using random numbers and a a particular Monte Carlo method numerically computes the Riemann integral. Whereas other algorithms usually evaluate the integrand at a regular grid, Monte Carlo randomly chooses points at which the integrand is evaluated. This method is particularly useful for higher-dimensional integrals. There are different methods to perform a Monte Carlo integration, such as uniform sampling, stratified sampling, and importance sampling. In this chapter we describe a certain technique for numerical calculation of infinite-dimensional integrals by using methods of the theory of uniform distribution modulo (u.d.mod) 1. Development of this theory for one-dimensional Riemann integrals was begun by Hermann Weyl's [W] celebrated theorem.

Theorem 2.1.1 ([KN], Theorem 1.1, p. 2) *The sequence* $(x_n)_{n \in N}$ *of real numbers is u.d. mod 1 if and only if for every real-valued continuous function* f *defined on the closed unit interval* $\overline{I} = [0, 1]$ *we have*

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} f(\{x_n\})}{N} = \int_{\overline{I}} f(x)dx, \tag{2.1.1}$$

where $\{\cdot\}$ *denotes the fractional part of the real number.*

Main corollaries of this theorem were used successfully in Diophantine approximations and have applications to Monte Carlo integration (see, e.g., [H1, H2, KN]). During the last decades the methods of the theory of uniform distribution modulo one have been intensively used in various branches of mathematics as diverse as number theory, probability theory, mathematical statistics, functional analysis, topological algebra, and so on.

G. Pantsulaia, *Applications of Measure Theory to Statistics*,
DOI 10.1007/978-3-319-45578-5_2

In [P2], the concept of increasing families of finite subsets uniformly distributed in infinite-dimensional rectangles has been introduced and a certain infinite generalization of the Theorem 2.1.1 has been obtained as follows.

Theorem 2.1.2 ([P2], Theorem 3.5, p. 339) *Let $(Y_n)_{n\in N}$ be an of* $[0,1]^\infty$. *Then $(Y_n)_{n\in N}$ is uniformly distributed in the infinite-dimensional rectangle* $[0,1]^\infty$ *if and only if for every Riemann integrable function f on* $[0,1]^\infty$ *the following equality*

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \int_{[0,1]^\infty} f(x)d\lambda(x) \tag{2.1.2}$$

holds true, where λ denotes the infinite-dimensional "Lebesgue measure" [B1].

The purpose of the present chapter is to consider some corollaries and applications of Theorem 2.1.2. More precisely, we elaborate Monte Carlo integration for real-valued functions of infinitely many variables.

This chapter is organized as follows.

In Sect. 2.2, in terms of the "Lebesgue measure" λ [B1], we consider concepts of the uniform distribution and Riemann integrability in infinite-dimensional rectangles in R^∞ and prove infinite-dimensional versions of Lebesgue's [N] and Weyl's [W] famous results, respectively. In this section we show that if $(\alpha_n^{(k)})_{n\in N}$ is an infinite sequence of different integer numbers for every $k \in N$, then a set of all sequences $(x_k)_{k\in N}$ in R^∞ for which a sequence of increasing sets $(Y_n((x_k)_{k\in N}))_{n\in N}$ is not λ uniformly distributed on the $\prod_{k\in N}[a_k, b_k]$), where

$$Y_n((x_k)_{k\in N}) = \prod_{k=1}^{n}((\cup_{j=1}^{n}\{< \alpha_j^{(k)} x_k > (b_k - a_k)\}) + a_k) \times \prod_{k\in N\setminus\{1,\dots,n\}}\{a_k\}$$

and λ is the "Lebesgue measure" constructed by R. Baker in 1991, and is of λ measure zero, and hence shy in R^∞.

In Sect. 2.3, a Monte Carlo algorithm for estimating the value of infinite-dimensional Riemann integrals over infinite-dimensional rectangles in R^∞ is described. Furthermore, we introduce Riemann integrability for real-valued functions with respect to product measures in R^∞ and give some sufficient conditions under which a real-valued function of infinitely many real variables is Riemann integrable. We describe a Monte Carlo algorithm for computing infinite-dimensional Riemann integrals for such functions.

In Sect. 2.4, we consider some interesting applications of Monte Carlo algorithms for computing infinite-dimensional Riemann integrals described in Sect. 2.3.

2.2 Uniformly Distributed Sequences of an Increasing Family of Finite Sets in Infinite-Dimensional Rectangles

Definition 2.2.1 A bounded sequence $s_1, s_2, s_3, \dots$ of real numbers is said to be equidistributed or uniformly distributed on an interval $[a, b]$ if for any subinterval $[c, d]$ of the $[a, b]$ we have

$$\lim_{n\to\infty} \frac{\#(\{s_1, s_2, s_3, \dots, s_n\} \cap [c, d])}{n} = \frac{d-c}{b-a},$$

where # denotes a counting measure.

Remark 2.2.1 For $a \le c < d \le b$, let $][c, d][$ denote a subinterval of the $[a, b]$ that has one of the following forms $[c, d]$, $[c, d[$, $]c, d[$ or $]c, d]$. Then it is obvious to show that a bounded sequence $s_1, s_2, s_3, \dots$ of real numbers is equidistributed or uniformly distributed on an interval $[a, b]$ iff, for any subinterval$][c, d][$ of the $[a, b]$, we have

$$\lim_{n\to\infty} \frac{\#(\{s_1, s_2, s_3, \dots, s_n\}\cap][c, d][)}{n} = \frac{d-c}{b-a}.$$

Definition 2.2.2 (*Weyl* [W]) The sequence $s_1, s_2, s_3, \dots$ is said to be equidistributed modulo 1 or uniformly distributed modulo 1 if the sequence $(s_n - [s_n])_{n\in N}$ of the fractional parts of the $(s_n)_{n\in N}$'s is equidistributed (equivalently, uniformly distributed) in the interval [0, 1].

Example 2.2.1 ([KN], *Exercise 1.12, p. 16*) The sequence of all multiples of an irrational α

$$0, \alpha, 2\alpha, 3\alpha, \dots$$

is uniformly distributed modulo 1.

Example 2.2.2 ([KN], Exercise 1.13, p. 16) The sequence

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \dots, \frac{0}{k}, \dots, \frac{k-1}{k}, \dots$$

is uniformly distributed modulo 1.

Example 2.2.3 The sequence of all multiples of an irrational α by successive prime numbers

$$2\alpha, 3\alpha, 5\alpha, 7\alpha, 11\alpha, \dots$$

is equidistributed modulo 1. This is a famous theorem of analytic number theory, proved by I. M. Vinogradov in 1935 (see [V]).

Agreement In the sequel, unlike N. Bourbaki's well-known notion, under N we understand a set $\{1, 2, \ldots\}$.

Remark 2.2.2 If $(s_k)_{k\in N}$ is uniformly distributed modulo 1, then $((s_k - [s_k])(b - a) + a)_{k\in N}$ is uniformly distributed in an interval $[a, b)$.

The following assertion contains an interesting application of uniformly distributed sequences for a calculation of Riemann integrals.

Lemma 2.2.1 (Weyl [W]) *The following two conditions are equivalent.*
(i) $(a_n)_{n\in N}$ *is equidistributed modulo* 1.
(ii) for every Riemann integrable function f *on* $[0, 1]$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(a_j) = \int_{[0,1]} f(x)dx.$$

Remark 2.2.3 Let $s_1, s_2, s_3, \ldots$ be uniformly distributed in an interval $[a, b]$. Setting $Y_n = \{s_1, s_2, s_3, \ldots, s_n\}$ for $n \in N$, the $(Y_n)_{n\in N}$ will be an increasing sequence of finite subsets of the $[a, b]$ such that, for any subinterval $[c, d]$ of the $[a, b]$, the following equality

$$\lim_{n\to\infty} \frac{\#(Y_n \cap [c, d])}{\#(Y_n)} = \frac{d-c}{b-a}$$

will be valid.

Remark 2.2.3 gives rise to the following definition.

Definition 2.2.3 An increasing sequence $(Y_n)_{n\in N}$ of finite subsets of the $[a, b]$ is said to be equidistributed or uniformly distributed in an interval $[a, b]$ if, for any subinterval $[c, d]$ of the $[a, b]$, we have

$$\lim_{n\to\infty} \frac{\#(Y_n \cap [c, d])}{\#(Y_n)} = \frac{d-c}{b-a}.$$

Definition 2.2.4 Let $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$. A set U is called an elementary rectangle in the $\prod_{k\in N}[a_k, b_k]$ if it admits the representation:

$$U = \prod_{k=1}^{m}][c_k, d_k][\times \prod_{k\in N\setminus\{1,\ldots,m\}} [a_k, b_k],$$

where $a_k \le c_k < d_k \le b_k$ for $1 \le k \le m$.

It is obvious that

$$\lambda(U) = \prod_{k=1}^{m}(d_k - c_k) \times \prod_{k=m+1}^{\infty}(b_k - a_k),$$

for the elementary rectangle U.

Definition 2.2.5 An increasing sequence $(Y_n)_{n\in N}$ of finite subsets of the infinite-dimensional rectangle $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$ is said to be uniformly distributed in the $\prod_{k\in N}[a_k, b_k]$ if for every elementary rectangle U in the $\prod_{k\in N}[a_k, b_k[$ we have

$$\lim_{n\to\infty}\frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{k\in N}[a_k, b_k[)}.$$

Theorem 2.2.1 ([P2], Theorem 3.1, p. 4) *Let* $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$. *Let* $(x_n^{(k)})_{n\in N}$ *be uniformly distributed in the interval* $[a_k, b_k]$ *for* $k \in N$. *We set*

$$Y_n = \prod_{k=1}^{n}(\cup_{j=1}^{n} x_j^{(k)}) \times \prod_{k\in N\setminus\{1,\dots,n\}}\{x_1^{(k)}\}.$$

Then $(Y_n)_{n\in N}$ *is uniformly distributed in the rectangle* $\prod_{k\in N}[a_k, b_k]$.

Proof Let U be an elementary rectangle in the $\prod_{k\in N}[a_k, b_k]$.

Because $(x_n^{(k)})_{n\in N}$ is uniformly distributed in the interval $[a_k, b_k]$ for $k \in N$, we get

$$\lim_{n\to\infty}\frac{\#(\{x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\}\cap][c_k, d_k][)}{n} = \frac{d_k - c_k}{b_k - a_k}.$$

Therefore we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\#(Y_n \cap U)}{\#(Y_n)} &= \lim_{n\to\infty}\prod_{k=1}^{m}\frac{\#(\{x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\}\cap][c_k, d_k][)}{n} \\ &= \prod_{k=1}^{m}\lim_{n\to\infty}\frac{\#(\{x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\}\cap][c_k, d_k][)}{n} \\ &= \prod_{k=1}^{m}\frac{d_k - c_k}{b_k - a_k} = \frac{\lambda(U)}{\lambda(\prod_{k\in N}[a_k, b_k])}. \end{aligned} \tag{2.2.1}$$

The theorem is proved.

Remark 2.2.4 In the context of Theorem 2.2.1, it is natural to ask whether there exists an increasing sequence of finite subsets $(Y_n)_{n\in N}$ such that

$$\lim_{n\to\infty}\frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{k\in N}[a_k, b_k])}$$

for every infinite-dimensional rectangle $U = \prod_{k\in N} X_k \subset \prod_{k\in N}[a_k, b_k]$, where, for each $k \in N$, X_k is a finite sum of pairwise disjoint intervals in $[a_k, b_k]$.

Let us show that the answer to this question is no.

Indeed, assume the contrary and let $(Y_n)_{n\in N}$ be such an increasing sequence of finite subsets in $\prod_{k\in N}[a_k, b_k[$. Then we have

$$\cup_{n\in N} Y_n = \{(x_i^{(k)})_{i\in N} : k \in N\}.$$

For $k \in N$, we set $X_k = [a_k, b_k] \setminus x_k^{(k)}$. Then, it is clear that

$$\lambda\Big(\prod_{k\in N} X_k\Big) = \lambda\Big(\prod_{k\in N}[a_k, b_k]\Big)$$

and

$$\frac{\#(Y_n \cap \prod_{k\in N} X_k)}{\#(Y_n)} = 0$$

for $k \in N$, which follows

$$\lim_{n\to\infty}\frac{\#(Y_n \cap \prod_{k\in N} X_k)}{\#(Y_n)} = 0 < 1 = \frac{\lambda(\prod_{k\in N} X_k)}{\lambda(\prod_{k\in N}[a_k, b_k])}.$$

Definition 2.2.6 Let $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$. A family of pairwise disjoint elementary rectangles $\tau = (U_k)_{1\le k\le n}$ of the $\prod_{k\in N}[a_k, b_k]$ is called the Riemann partition of the $\prod_{k\in N}[a_k, b_k]$ if $\cup_{1\le k\le n} U_k = \prod_{k\in N}[a_k, b_k]$.

Definition 2.2.7 Let $\tau = (U_k)_{1\le k\le n}$ be the Riemann partition of the $\prod_{k\in N}[a_k, b_k]$. Let $\ell(Pr_i(U_k))$ be a length of the ith projection $Pr_i(U_k)$ of the U_k for $i \in N$. We set

$$d(U_k) = \sum_{i\in N}\frac{\ell(Pr_i(U_k))}{2^i(1 + \ell(Pr_i(U_k)))}.$$

It is obvious that $d(U_k)$ is a diameter of the elementary rectangle U_k for $k \in N$ with respect to the Tikhonov metric ρ defined as

$$\rho((x_k)_{k\in N}, (y_k)_{k\in N}) = \sum_{k\in N}\frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}$$

for $(x_k)_{k\in N}, (y_k)_{k\in N} \in \mathbf{R}^\infty$.

A number $d(\tau)$, defined by

$$d(\tau) = \max\{d(U_k) : 1 \leq k \leq n\}$$

is called the mesh or norm of the Riemann partition τ.

Definition 2.2.8 Let $\tau_1 = (U_i^{(1)})_{1 \leq i \leq n}$ and $\tau_2 = (U_j^{(2)})_{1 \leq j \leq m}$ be Riemann partitions of the $\prod_{k \in N}[a_k, b_k]$. We say that $\tau_2 \leq \tau_1$ iff

$$(\forall j)((1 \leq j \leq m) \rightarrow (\exists i_0)(1 \leq i_0 \leq n \ \& \ U_j^{(2)} \subseteq U_{i_0}^{(1)})).$$

Definition 2.2.9 Let f be a real-valued bounded function defined on the $\prod_{i \in N}[a_i, b_i]$. Let $\tau = (U_k)_{1 \leq k \leq n}$ be the Riemann partition of the $\prod_{k \in N}[a_k, b_k]$ and $(t_k)_{1 \leq k \leq n}$ be a sample such that, for each k, $t_k \in U_k$. Then

(i) A sum $\sum_{k=1}^{n} f(t_k)\lambda(U_k)$ is called the Riemann sum of the f with respect to Riemann partition $\tau = (U_k)_{1 \leq k \leq n}$ together with sample $(t_k)_{1 \leq k \leq n}$.

(ii) A sum $S_\tau = \sum_{k=1}^{n} M_k \lambda(U_k)$ is called the upper Darboux sum with respect to Riemann partition τ, where $M_k = \sup_{x \in U_k} f(x)(1 \leq k \leq n)$.

(iii) A sum $s_\tau = \sum_{k=1}^{n} m_k \lambda(U_k)$ is called the lower Darboux sum with respect to Riemann partition τ, where $m_k = \inf_{x \in U_k} f(x)(1 \leq k \leq n)$.

Definition 2.2.10 Let f be a real-valued bounded function defined on $\prod_{i \in N}[a_i, b_i[$. We say that the f is Riemann integrable on $\prod_{i \in N}[a_i, b_i]$ if there exists a real number s such that for every positive real number ε there exists a real number $\delta > 0$ such that, for every Riemann partition $(U_k)_{1 \leq k \leq n}$ of the $\prod_{k \in N}[a_k, b_k]$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1 \leq k \leq n}$, we have

$$\Big| \sum_{k=1}^{n} f(t_k)\lambda(U_k) - s \Big| < \varepsilon.$$

The number s is called the Riemann integral and is denoted by

$$(R) \int_{\prod_{k \in N}[a_k, b_k]} f(x) d\lambda(x).$$

Definition 2.2.11 A function f is called a step function on $\prod_{k \in N}[a_k, b_k]$ if it can be written as

$$f(x) = \sum_{k=1}^{n} c_k \mathscr{X}_{U_k}(x),$$

where $\tau = (U_k)_{1 \leq k \leq n}$ is any Riemann partition of the $\prod_{k \in N}[a_k, b_k]$, $c_k \in R$ for $1 \leq k \leq n$ and $\mathscr{X}_A$ is the indicator function of the A.

Theorem 2.2.2 *Let f be a continuous function on $\prod_{k\in N}[a_k, b_k]$ with respect to the Tikhonov metric ρ. Then f is Riemann integrable on $\prod_{k\in N}[a_k, b_k]$.*

Proof It is obvious that, for every Riemann partition $\tau = (U_k)_{1\le k\le n}$ of the $\prod_{k\in N}[a_k, b_k]$ and for every sample $(t_k)_{1\le k\le n}$ with $t_k \in U_k(1 \le k \le n)$, we have

$$s_\tau \le \sum_{k=1}^{n} f(t_k)\lambda(U_k) \le S_\tau.$$

Note that if τ_1 and τ_2 are two Riemann partitions of the $\prod_{k\in N}[a_k, b_k]$ such that $\tau_2 \le \tau_1$, then

$$s_{\tau_1} \le s_{\tau_2} \le \sum_{k=1}^{n} f(t_k)\lambda(U_k) \le S_{\tau_2} \le S_{\tau_1}.$$

Let us show the validity of the condition

$$(\forall\varepsilon)(\varepsilon > 0 \to (\exists r)(\forall\tau)(d(\tau) < r \to S_\tau - s_\tau < \varepsilon)),$$

which yields that $\inf_\tau S_\tau = \sup_\tau s_\tau$.

Following Tikhonov's theorem, the $\prod_{k\in N}[a_k, b_k]$ is a compact set in the Polish group $\mathbf{R}^\infty$ equipped with the Tikhonov metric ρ.

Following Cantor's well-known result, the function f is uniformly continuous on the $\prod_{k\in N}[a_k, b_k]$. Thus for $\varepsilon > 0$, there exists $r > 0$ such that

$$(\forall x, y)\Big(x, y \in \prod_{k\in N}[a_k, b_k] \& \rho(x, y) < r \to |f(x) - f(y)| \le \frac{\varepsilon}{\lambda(\prod_{k\in N}[a_k, b_k])}\Big).$$

Thus, for every Riemann partition $\tau = (U_k)_{1\le k\le n}$ with $d(\tau) < r$ we get

$$S_\tau - s_\tau \le \frac{\varepsilon}{\lambda(\prod_{k\in N}[a_k, b_k[)} \times \sum_{1\le k\le n} \lambda(U_k) = \varepsilon.$$

Thus, $\inf_\tau S_\tau = \sup_\tau s_\tau$.

Finally, setting $\delta = r$ and $s = \inf_\tau S_\tau$, we deduce that for every Riemann partition $(U_k)_{1\le k\le n}$ of the $\prod_{k\in N}[a_k, b_k]$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1\le k\le n}$ with $t_k \in U_k(1 \le k \le n)$, we have

$$\Big|\sum_{k=1}^{n} f(t_k)\lambda(U_k) - s\Big| \le S_\tau - s_\tau \le \varepsilon.$$

This ends the proof of Theorem 2.2.2.

We have the following infinite-dimensional version of the Lebesgue theorem (see [N], Lebesgue Theorem, p. 359).

Theorem 2.2.3 *Let f be a bounded real-valued function on $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$. Then f is Riemann integrable on $\prod_{k\in N}[a_k, b_k]$ if and only if f is λ almost continuous on $\prod_{k\in N}[a_k, b_k]$.*

Proof (*Necessity*) Let f be a Riemann integrable function on $\prod_{k\in N}[a_k, b_k] \in \mathscr{R}$.

Then, for every $\varepsilon > 0$ and $\mu > 0$, there exists a Riemann partition $\tau = (U_k)_{1\le k\le n}$ such that

$$\varepsilon \times \mu \ge S_\tau - s_\tau \ge \sum_{1\le k\le n}(M_k - m_k)\lambda(U_k)$$
$$\ge \sum_{k\in I_1}(M_k - m_k)\lambda(U_k) \ge \mu \sum_{k\in I_1}\lambda(U_k), \qquad (2.2.1)$$

where $I_1 = \{k : 1 \le k \le n \;\&\; U_k$ contains at least one inner point p belonging to the set $E_\mu\}$, where

$$E_\mu = \Big\{x : x \in \prod_{k\in N}[a_k, b_k] \;\&\; \omega(f, x) \ge \mu\Big\}$$

and

$$\omega(f, x) = \lim_{\delta\to 0} \sup_{x', x'' \in V(x,\delta)\cap\prod_{k\in N}[a_k,b_k]} \big|f(x') - f(x'')\big|.$$

Here, for $x \in \mathbf{R}^\infty$ and $\delta > 0$, $V(x, \delta)$ is denoted by

$$V(x, \delta) = \Big\{y : y \in \prod_{k\in N}[a_k, b_k] \;\&\rho(x, y) \le \delta\Big\}.$$

Because, for $k \in I_1$, p is an inner point of the U_k, there exists $V(p, \delta(k, p))$ such that $V(p, \delta(k, p)) \subseteq U_k$.

Inasmuch as $\omega(f, p) \ge \mu$, we have

$$M_k - m_k \ge M_p - m_p \ge \omega(f, p) \ge \mu,$$

where

$$M_p = \sup_{x\in V(p,\delta(k,p))} f(x), \;\; m_\delta = \inf_{x\in V(p,\delta(k,p))} f(x).$$

From (2.2.1), we get

$$\varepsilon \geq \sum_{k \in I_1} \lambda(U_k).$$

Other points of the E_μ, which are not inner points of elements of the partition τ, may be placed on the boundary of elements of the τ, whose λ measure is zero.

Thus, for $\mu > 0$, we have

$$\lambda(E_\mu) \leq \sum_{k \in I_1} \lambda(U_k) + \lambda(\cup_{1 \leq k \leq n} \partial(U_k)) \leq \frac{\varepsilon}{\mu},$$

which yields that $\lambda(E_\mu) = 0$. Because a set E of all points of discontinuity of the f admits the representation $E = \cup_{k=1}^{\infty} E_{\frac{1}{k}}$, we deduce that $\lambda(E) = 0$.

This ends Necessity.

Proof of the sufficiency. Let, for $K \in \mathbf{R}^+$, us have $|f(x)| \leq K$ whenever $x \in \prod_{k \in N}[a_k, b_k]$.

Suppose that f is λ almost continuous on $\prod_{k \in N}[a_k, b_k]$.

For $\varepsilon > 0$, let μ be such a positive number that

$$4\mu\lambda\Big(\prod_{k \in N}[a_k, b_k]\Big) < \varepsilon.$$

Because, for a set E of all points of discontinuity of the f on $\prod_{k \in N}[a_k, b_k]$ we have $\lambda(E) = 0$, we easily claim that $\lambda(E_\mu) = 0$. Because E_μ is closed in $\prod_{k \in N}[a_k, b_k]$, we claim that E_μ is compact. Hence, for $\varepsilon > 0$, there exists a finite family of open elementary rectangles in $\prod_{k \in N}[a_k, b_k]$ whose union covers E_μ such that

$$\lambda(\cup_{1 \leq k \leq n} U_k) < \frac{\varepsilon}{4K}.$$

Finally, we have

$$\prod_{k \in N}[a_k, b_k] = \cup_{1 \leq k \leq n} U_k \cup F,$$

where F is a compact subset in $\prod_{k \in N}[a_k, b_k]$.

It is obvious that, for every point $x \in F$, we have $\omega(f, x) < \mu$. Because F is compact, we can choose $\delta > 0$ such that for every $x, x^{'} \in F$ the condition $\rho(x, x^{'}) < \delta$ yields $\left|f(x) - f(x^{'})\right| < 2\lambda$.

F is a finite union of elementary rectangles in $\prod_{k \in N}[a_k, b_k]$ (this follows from the fact that the class of all elementary rectangles in $\prod_{k \in N}[a_k, b_k]$ is a ring), therefore there exists a partition $\tau_1 = (F_i)_{2 \leq i \leq m}$ of the F such that, for i with $2 \leq i \leq m$, F_i is an elementary rectangle in $\prod_{k \in N}[a_k, b_k]$ with $d(F_i) < \delta$. Then $\tau = \{\cup_{1 \leq k \leq n} U_k, F_2, \ldots, F_m\}$ will be a Riemann partition of the $\prod_{k \in N}[a_k, b_k]$ such that

$$S_\tau - s_\tau = (M_1 - m_1)\lambda(\cup_{1\le k\le n} U_k) + \sum_{1\le i\le m} (M_i - m_i)\lambda(F_k)$$
$$\le \frac{\varepsilon}{2} + 2\mu\lambda(\prod_{k\in N}[a_k, b_k])) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The theorem is proved.

Remark 2.2.5 Note that Theorem 2.2.2 can be considered as a simple consequence of Theorem 2.2.3. Now, by using Theorem 2.2.3, one can extend the concept of the Riemann integrability theory for functions defined in the topological vector space $\mathbf{R}^\infty$ of all real-valued sequences equipped with Tikhonov topology.

In the sequel we need some important notions and well-known results from general topology and measure theory.

Definition 2.2.12 A topological Hausdorff space X is called normal if given any disjoint closed sets E and F, there are neighborhoods U of E and V of F that are also disjoint.

Lemma 2.2.2 (Urysohn [Ur]) *A topological space X is normal if and only if any two disjoint closed sets can be separated by a function. That is, given disjoint closed sets E and F, there is a continuous function f from X to* $[0, 1]$ *such that the preimages of* 0 *and* 1 *under f are E and F, respectively.*

Remark 2.2.6 Because all compact Hausdorff spaces are normal, we deduce that $\prod_{k\in N}[a_k, b_k]$ equipped with Tikhonov topology, is normal. By Urysohn's lemma we deduce that any two disjoint closed sets in $\prod_{k\in N}[a_k, b_k]$ can be separated by a function.

Definition 2.2.13 A Borel measure μ, defined on a Hausdorff topological space X, is called a Radon if

$$(\forall Y)(Y \in \mathscr{B}(X) \ \& \ 0 \le \mu(Y) < +\infty \to \mu(Y) = \sup_{\substack{K\subseteq Y \\ \text{K is compact in X}}} \mu(K)).$$

Lemma 2.2.3 (Ulam [Ul]) *Every probability Borel measure defined on Polish metric space is a Radon.*

In the sequel we denote by $\mathscr{C}(\prod_{k\in N}[a_k, b_k])$ a class of all continuous (with respect to Tikhonov topology) real-valued functions on $\prod_{k\in N}[a_k, b_k]$.

Theorem 2.2.4 *For $\prod_{i\in N}[a_i,b_i]\in\mathscr{R}$, let $(Y_n)_{n\in N}$ be an increasing family of its finite subsets. Then $(Y_n)_{n\in N}$ is uniformly distributed in the $\prod_{k\in N}[a_k,b_k]$ if and only if for every $f\in\mathscr{C}(\prod_{k\in N}[a_k,b_k])$ the equality*

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}=\frac{(R)\int_{\prod_{k\in N}[a_k,b_k]}f(x)d\lambda(x)}{\lambda\left(\prod_{i\in N}[a_i,b_i]\right)}$$

holds.

Proof **Necessity**. Let $(Y_n)_{n\in N}$ be a uniformly distributed in the $\prod_{k\in N}[a_k,b_k]$ and let $f(x)=\sum_{k=1}^{m}c_k\mathscr{X}_{U_k}(x)$ be a step function. Then we have

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}=\lim_{n\to\infty}\frac{\sum_{y\in Y_n}\sum_{k=1}^{m}c_k\mathscr{X}_{U_k}(y)}{\#(Y_n)}$$

$$=\lim_{n\to\infty}\frac{\sum_{k=1}^{m}c_k\#(U_k\cap Y_n)}{\#(Y_n)}=\sum_{k=1}^{m}c_k\lim_{n\to\infty}\frac{\#(U_k\cap Y_n)}{\#(Y_n)}$$

$$=\sum_{k=1}^{m}c_k\frac{\lambda(U_k)}{\lambda(\prod_{i\in N}[a_i,b_i])}=\frac{(R)\int_{\prod_{k\in N}[a_k,b_k]}f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i,b_i])}.$$

Now, let $f\in\mathscr{C}(\prod_{k\in N}[a_k,b_k])$. By Theorem 2.2.3 we deduce that f is Riemann integrable. From the definition of the Riemann integral we deduce that, for every positive ε, there exist two step functions f_1 and f_2 on $\prod_{i\in N}[a_i,b_i]$ such that

$$f_1(x)\le f(x)\le f_2(x)$$

and

$$(R)\int_{\prod_{i\in N}[a_i,b_i]}(f_1(x)-f_2(x))d\lambda(x)<\varepsilon.$$

Then we have

$$(R)\int_{\prod_{i\in N}[a_i,b_i]}f(x)d\lambda(x)-\varepsilon\le(R)\int_{\prod_{i\in N}[a_i,b_i]}f_1(x)d\lambda(x)$$

$$=\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f_1(y)}{\#(Y_n)}\le\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\underline{\lim}_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}$$

$$\le\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\overline{\lim}_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}\le\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f_2(y)}{\#(Y_n)}$$

$$\leq \lambda\Big(\prod_{i\in N}[a_i,b_i]\Big) \times (R)\int_{\prod_{i\in N}[a_i,b_i]} f_2(x)d\lambda(x) \leq (R)\int_{\prod_{i\in N}[a_i,b_i]} f(x)d\lambda(x) + \varepsilon.$$

The latter relation yields an existence of the limit $\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)}$ such that

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{k\in N}[a_k,b_k]} f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i,b_i])}.$$

This ends the proof of Necessity.

Sufficiency. Assume that $(Y_n)_{n\in N}$ is an increasing sequence of subsets of the $\prod_{k\in N}[a_k,b_k]$ such that for every $f \in \mathscr{C}(\prod_{k\in N}[a_k,b_k])$ the equality

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{k\in N}[a_k,b_k[} f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i,b_i])}$$

holds.

Let U be any elementary rectangle in $\prod_{i\in N}[a_i,b_i]$.

For $\varepsilon > 0$, by Ulam's lemma we can choose such a compact set

$$F \subset \prod_{k\in N}[a_k,b_k] \setminus [U]_T,$$

that $\lambda((\prod_{k\in N}[a_k,b_k]\setminus[U]_T)\setminus F) < \frac{\varepsilon}{2}$, where $[U]_T$ denotes completion of the set U by Tikhonov topology in $\prod_{k\in N}[a_k,b_k]$. Then, by Urysohn's lemma we deduce that there is a continuous function g_2 from $\prod_{k\in N}[a_k,b_k]$ to $[0,1]$ such that the preimages of 0 and 1 under g_2 are F and $[U]_T$, respectively. Then, for $x \in \prod_{k\in N}[a_k,b_k]$, we have

$$\mathscr{X}_U(x) \leq g_2(x)$$

and

$$(R)\int_{\prod_{k\in N}[a_k,b_k]} (g_2(x) - \mathscr{X}_U(x))d\lambda(x) \leq \frac{\varepsilon}{2},$$

where $\mathscr{X}_U$ is an indicator of the U defined on the $\prod_{k\in N}[a_k,b_k]$.

Now let us consider a set $[\prod_{k\in N}[a_k,b_k] \setminus U]_T$. Using Ulam's lemma, we can choose such a compact set

$$F_1 \subset \prod_{k\in N}[a_k,b_k] \setminus \Big[\prod_{k\in N}[a_k,b_k] \setminus U\Big]_T$$

that

$$\lambda\Big(\Big(\prod_{k\in N}[a_k,b_k]\setminus\Big[\prod_{k\in N}[a_k,b_k]\setminus U\Big]_T\Big)\setminus F_1\Big)<\frac{\varepsilon}{2}.$$

Then, by Urysohn's lemma we deduce that there is a continuous function g_1 from $\prod_{k\in N}[a_k,b_k]$ to $[0,1]$ such that the preimages of 0 and 1 under g_1 are $[\prod_{k\in N}[a_k,b_k]\setminus U]_T$ and F_1, respectively. Then, for $x\in\prod_{k\in N}[a_k,b_k]$, we have

$$g_1(x)\le\mathscr{X}_U(x)$$

and

$$(R)\int_{\prod_{k\in N}[a_k,b_k]}(\mathscr{X}_U(x)-g_1(x))d\lambda(x)\le\frac{\varepsilon}{2}.$$

Now, we deduce that for every elementary rectangle U in $\prod_{i\in N}[a_i,b_i]$ there exist two continuous functions g_1 and g_2 on the $\prod_{i\in N}[a_i,b_i]$ such that

$$g_1(x)\le\mathscr{X}_U(x)\le g_2(x)$$

and

$$(R)\int_{\prod_{i\in N}[a_i,b_i]}(g_2(x)-g_1(x))d\lambda(x)\le\varepsilon.$$

Then we have

$$\lambda(U)-\varepsilon\le(R)\int_{\prod_{i\in N}[a_i,b_i]}g_2(x)d\lambda(x)-\varepsilon\le(R)\int_{\prod_{i\in N}[a_i,b_i]}g_1(x)d\lambda(x)$$

$$=\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\lim_{n\to\infty}\frac{\sum_{y\in Y_n}g_1(y)}{\#(Y_n)}\le\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\underline{\lim}_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)}$$

$$\le\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\overline{\lim}_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)}\le\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\lim_{n\to\infty}\frac{\sum_{y\in Y_n}g_2(y)}{\#(Y_n)}$$

$$=(R)\int_{\prod_{i\in N}[a_i,b_i]}g_2(x)d\lambda(x)\le(R)\int_{\prod_{i\in N}[a_i,b_i]}g_1(x)d\lambda(x)+\varepsilon\le\lambda(U)+\varepsilon.$$

Because ε was taken as arbitrary, we deduce that

$$\lambda\Big(\prod_{i\in N}[a_i,b_i]\Big)\times\lim_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)}=\lambda(U).$$

This ends the proof of Theorem 2.2.4.

Now by the scheme used in the proof of Theorem 2.2.4, one can get the validity of an infinite-dimensional analogue of Lemma 2.2.1. In particular, the following assertion is valid.

Theorem 2.2.5 *For $\prod_{i \in N}[a_i, b_i] \in \mathscr{R}$, let $(Y_n)_{n \in N}$ be an increasing family of its finite subsets. Then $(Y_n)_{n \in N}$ is uniformly distributed in the $\prod_{k \in N}[a_k, b_k]$ if and only if for every Riemann integrable function f on $\prod_{k \in N}[a_k, b_k]$ the equality*

$$\lim_{n \to \infty} \frac{\sum_{y \in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{k \in N}[a_k, b_k]} f(x) d\lambda(x)}{\lambda(\prod_{i \in N}[a_i, b_i])}$$

holds.

Definition 2.2.14 (*Weyl* [W]) A sequence $s_1, s_2, s_3, \dots$ is said to be equidistributed modulo 1 or uniformly distributed modulo 1 if the sequence $(< s_n >)_{n \in N}$ of the fractional parts of $(s_n)_{n \in N}$'s is equidistributed (equivalently, uniformly distributed) on the interval [0, 1].

Lemma 2.2.4 *Let $\prod_{k \in N}[a_k, b_k] \in \mathscr{R}$. Let $(x_n^{(k)})_{n, k \in N}$ be a double sequence of elements of $\prod_{k \in N}[a_k, b_k]$. We set*

$$Y_n = \prod_{k=1}^{n} (\cup_{j=1}^{n} \{x_j^{(k)}\}) \times \prod_{k \in N \setminus \{1, \dots, n\}} \{x_1^{(k)}\}.$$

Then $(Y_n)_{n \in N}$ is uniformly distributed in the rectangle $\prod_{k \in N}[a_k, b_k]$ if and only if $(x_n^{(k)})_{n \in N}$ is uniformly distributed on the interval $[a_k, b_k]$ for $k \in N$.

Proof (*Sufficiency*) Because $(Y_n)_{n \in N}$ is uniformly distributed in the rectangle $\prod_{k \in N}[a_k, b_k]$, for an elementary rectangle $U = \prod_{i=1}^{k-1}[a_i, b_i] \times]\![c, d]\![\times \prod_{i=k+1}^{+\infty}[a_i, b_i]$ with $]\![c, d]\![\subseteq [a_k, b_k]$, we have

$$\frac{d - c}{b_k - a_k} = \frac{\lambda(U)}{\lambda(\prod_{i \in N}[a_i, b_i[)} = \lim_{n \to \infty} \frac{\#(Y_n \cap U)}{\#(Y_n)} = \lim_{n \to \infty} \frac{\#(\{x_1^{(k)}, \dots, x_n^{(k)}\} \cap U)}{n}.$$

The latter relation means that $(x_n^{(k)})_{n \in N}$ is uniformly distributed on the interval $[a_k, b_k]$ for $k \in N$.

Necessity. See Theorem 2.2.1.

Lemma 2.2.5 ([KN], Theorem 4.1, p. 42) *Let $(\alpha_n)_{n \in N}$ be a sequence of different integer numbers. Then a sequence of real numbers $(\alpha_n x)_{n \in N}$ is u.d. mod 1 for l_1 almost points of R.*

Definition 2.2.15 Following Brian R. Hunt, Tim Sauer, and James A. Yorke (cf. [HSY]), a set X is called shy if it is a subset of a Borel set X' for which $\mu(X' + v) = 0$ for every $v \in B$ and some Borel probability measure μ such that $\mu(K) = \mu(B)$ for some compact K.

Definition 2.2.16 A Borel measure μ in V is called a generator (of shy sets) in V, if

$$(\forall X)(\overline{\mu}(X) = 0 \rightarrow X \in S(V)),$$

where $\overline{\mu}$ denotes a usual completion of the Borel measure μ.

Lemma 2.2.6 ([P4], Example 2.1, p. 242) *"Lebesgue measure" [B1] is a generator of shy sets.*

Theorem 2.2.6 *Let $(\alpha_n^{(k)})_{n\in N}$ be an infinite sequence of different integer numbers for every $k \in N$. Then a set of all sequences $(x_k)_{k\in N}$ in R^∞ for which a sequence of increasing sets $(Y_n((x_k)_{k\in N}))_{n\in N}$ is not λ u.d. on the $\prod_{k\in N}[a_k, b_k]$, where*

$$Y_n((x_k)_{k\in N}) = \prod_{k=1}^{n}(\cup_{j=1}^{n}\{< \alpha_j^{(k)} x_k > (b_k - a_k)\}) \times \prod_{k\in N\setminus\{1,\dots,n\}}\{a_k\},$$

is of λ measure zero, where $< \cdot >$ denotes the fractal part of the number.

Proof For $k \in N$, we denote D_k by

$D_k = \{x_k : x_k \in R \; \& \; (< \alpha_j^{(k)} x_k > (b_k - a_k))_{j\in N}$ is l_1 u.d. on $[a_k, b_k]\}$.

By Lemma 2.2.4 we have that $l_1(R \setminus D_k) = 0$ for $k \in N$.

We set $D = \prod_{k\in N} D_k$. It is clear that $\lambda(R^\infty \setminus D) = 0$. For $(x_k)_{k\in N} \in D$, we have that $(< \alpha_j^{(k)} x_k > (b_k - a_k))_{j\in N}$ is l_1 u.d. on $[a_k, b_k]$ for every $k \in N$. By Lemma 2.2.4 we claim that $(Y_n((x_k)_{k\in N}))_{n\in N}$ is λ u.d. on the $\prod_{k\in N}[a_k, b_k]$ for $(x_k)_{k\in N} \in D$. Inasmuch as $\lambda(R^\infty \setminus D) = 0$, Theorem 2.2.6 is proved.

Remark 2.2.7 Following Lemma 2.2.6, λ is the generator of shy sets. The latter relation means that every set of λ measure zero is shy in R^∞. Following Theorem 2.2.6, we claim that a set of all sequences $(x_j)_{j\in N}$ in R^∞ for which a sequence of increasing sets $(Y_n((x_j)_{j\in N}))_{n\in N}$ is λ u.d. on the $\prod_{k\in N}[a_k, b_k]$) is the prevalent set.

2.3 Monte Carlo Algorithm for Estimating the Value of Infinite-Dimensional Riemann Integrals

Now we give some basic definitions that help us define more precisely what we mean by a Riemann integral with respect to product measure in $\mathbf{R}^\infty$. Then we give some conditions for the existence of the Riemann integral with respect to product measure in $\mathbf{R}^\infty$ and go through a certain algorithm useful in computing this integral.

Let $(F_k)_{k\in\mathbf{N}}$ be a sequence of strictly increasing continuous distribution functions on $\mathbf{R}$. Let μ_k be a Borel probability measure in $\mathbf{R}$ defined by F_k for $k \in \mathbf{N}$. Let us denote by $\prod_{k\in\mathbf{N}} \mu_k$ the product of measures $(\mu_k)_{k\in\mathbf{N}}$.

For $-\infty < c < d < +\infty$, let $][c, d][$ denote a subinterval of the real axis $(-\infty, +\infty)$ which has one of the forms $[c, d]$, $[c, d[$, $]c, d[$ or $]c, d]$. If $c = -\infty$ and $d \neq +\infty$, then $][c, d][$ denotes a subinterval of the real axis $(-\infty, +\infty)$ which has one of the forms $]c, d]$ or $]c, d]$. Similarly, if $c \neq -\infty$ and $d = +\infty$, then $][c, d][$ denotes a subinterval of the real axis $(-\infty, +\infty)$ which has one of the forms $]c, d[$ or $[c, d[$. Finally, if $c = -\infty$ and $d = +\infty$, then $][c, d][$ denotes the whole real axis $(-\infty, +\infty)$.

Definition 2.3.1 A set U^* is called an elementary rectangle in $\mathbf{R}^\infty$ if it admits the following representation.

$$U^* = \prod_{k=1}^{m}][c_k, d_k][\times \mathbf{R}^{\mathbf{N}\setminus\{1,\cdots,m\}}, \tag{2.3.1}$$

where $-\infty \le c_k < d_k \le +\infty$ for $1 \le k \le m$.

Definition 2.3.2 A family of pairwise disjoint elementary rectangles $\tau = (U_k^*)_{1\le k\le n}$ in $\mathbf{R}^\infty$ is called the Riemann partition of the $\mathbf{R}^\infty$ if $\cup_{1\le k\le n} U_k^* = \mathbf{R}^\infty$.

Definition 2.3.3 Let $\tau^* = (U_k^*)_{1\le k\le n}$ be the Riemann partition of $\mathbf{R}^\infty$ and $\ell_1(F_i^{-1}(Pr_i(U_k^*)))$ the length of the preimage of the ith projection $Pr_i(U_k^*)$ of the U_k^* under mapping F_i for $i \in N$. We set

$$d^*(U_k^*) = \sum_{i\in N} \frac{\ell_1(F_i(Pr_i(U_k^*)))}{2^i(1 + \ell_1(F_i(Pr_i(U_k^*))))}. \tag{2.3.2}$$

It is obvious that $d^*(U_k^*)$ is a diameter of the elementary rectangle U_k^* for $k \in N$ with respect to metric ρ defined as

$$\rho((x_k)_{k\in N}, (y_k)_{k\in N}) = \sum_{k\in N} \frac{|F_k(x_k) - F_k(y_k)|}{2^k(1 + |F_k(x_k) - F_k(y_k)|)} \tag{2.3.3}$$

for $(x_k)_{k\in N}, (y_k)_{k\in N} \in \mathbf{R}^\infty$.

Remark 2.3.1 Note that metrics ρ and ρ_T are equivalent provided that

$$\rho((x_k)_{k\in N}, (y_k)_{k\in N}) = 0$$

if and only if

$$\rho_T((x_k)_{k\in N}, (y_k)_{k\in N}) = 0.$$

Note also that both topologies induced by these metrics coincide.

Definition 2.3.4 A number $d^*(\tau)$, defined by

$$d^*(\tau) = \max\{d^*(U_k) : 1 \leq k \leq n\} \tag{2.3.4}$$

is called the mesh or norm of the Riemann partition τ^* of the $\mathbf{R}^\infty$.

Definition 2.3.5 Let $\tau_1^* = (U_i^{*(1)})_{1 \leq i \leq n}$ and $\tau_2^* = (U_j^{*(2)})_{1 \leq j \leq m}$ be Riemann partitions of the $\mathbf{R}^\infty$. We say that $\tau_2^* \leq \tau_1^*$ iff

$$(\forall j)((1 \leq j \leq m) \to (\exists i_0)(1 \leq i_0 \leq n \ \& \ U_j^{*(2)} \subseteq U_{i_0}^{*(1)})). \tag{2.3.5}$$

Definition 2.3.6 A function f is called a step function on $\mathbf{R}^\infty$ if it can be written as

$$f(x) = \sum_{k=1}^{n} c_k \chi_{U_k^*}(x), \tag{2.3.6}$$

where $\tau^* = (U_k^*)_{1 \leq k \leq n}$ is any Riemann partition of the $\mathbf{R}^\infty$, $c_k \in R$ for $1 \leq k \leq n$ and χ_A is the indicator function of the set A.

Definition 2.3.7 Let f be a real-valued bounded function defined on $\mathbf{R}^\infty$. Let $\tau^* = (U_k^*)_{1 \leq k \leq n}$ be the Riemann partition of $\mathbf{R}^\infty$ and $(t_k^*)_{1 \leq k \leq n}$ be a sample such that, for each k, $t_k^* \in U_k^*$. Then

(i) A sum $\sum_{k=1}^n f(t_k^*)(\prod_{i \in \mathbf{N}} \mu_i)(U_k^*)$ is called the Riemann sum of the f with respect to the Riemann partition $\tau^* = (U_k^*)_{1 \leq k \leq n}$ together with sample $(t_k)_{1 \leq k \leq n}$.

(ii) A sum $S_{\tau^*} = \sum_{k=1}^n M_k(\prod_{i \in \mathbf{N}} \mu_i)(U_k^*)$ is called the upper Darboux sum with respect to the Riemann partition τ^*, where $M_k = \sup_{x \in U_k^*} f(x)(1 \leq k \leq n)$.

(iii) A sum $s_{\tau^*} = \sum_{k=1}^n m_k(\prod_{i \in \mathbf{N}} \mu_i)(U_k^*)$ is called the lower Darboux sum with respect to the Riemann partition τ^*, where $m_k = \inf_{x \in U_k^*} f(x)(1 \leq k \leq n)$.

Definition 2.3.8 Let f be a real-valued bounded function defined on $\mathbf{R}^\infty$. We say that the f is Riemann integrable with respect to product measure $\prod_{i \in \mathbf{N}} \mu_i$ on $\mathbf{R}^\infty$ if there exists a real number s such that for every positive real number ε there exists a real number $\delta > 0$ such that, for every Riemann partition $(U_k^*)_{1 \leq k \leq n}$ of the $\mathbf{R}^\infty$ with $d^*(\tau^*) < \delta$ and for every sample $(t_k^*)_{1 \leq k \leq n}$, we have

$$\Big| \sum_{k=1}^{n} f(t_k^*) \left(\prod_{i \in \mathbf{N}} \mu_i \right) (U_k^*) - s \Big| < \varepsilon. \tag{2.3.7}$$

The number s is called the Riemann integral of f over $\mathbf{R}^\infty$ and is denoted by

$$(R) \int_{\mathbf{R}^\infty} f(x) d \left(\prod_{i \in \mathbf{N}} \mu_i \right) (x). \tag{2.3.8}$$

In this section we present some conditions that help us determine whether the Riemann integral of a certain function over $\mathbf{R}^\infty$ exists.

Theorem 2.3.1 *(Riemann necessary and sufficient condition for integrability). Consider the bounded function $f : R^\infty \to R$. f is Riemann integrable in $\mathbf{R}^\infty$ with respect to product measure $\prod_{i\in\mathbf{N}} \mu_i$ if and only if for arbitrary positive ε there is a Riemann partition τ^* of $\mathbf{R}^\infty$ such that $S_{\tau^*} - s_{\tau^*} < \varepsilon$.*

The proof of Theorem 2.3.1 can be obtained by the standard scheme.

Example 2.3.1 Define $u((x_k)_{k\in\mathbf{N}}) = sin(x_1^{-1})$ for $(x_k)_{k\in N} \in (0,1)^\infty$. Then u is bounded (by 1) and continuous on $(0,1)^\infty$, but it is neither uniformly continuous nor continuously extendable to $[0,1]^\infty$.

In the context of Example 2.3.1 the following lemma is of some interest.

Lemma 2.3.1 *Let f be any bounded and uniformly continuous function on* $(0,1)^\infty$. *Then f has a unique continuous extension $\overline{f}$ to whole* $[0,1]^\infty$.

Proof For any $x \in [0,1]^\infty$, find a sequence $(x_n) \in (0,1)^\infty$ such that $\lim_{n\to\infty} x_n = x$.

Step 1. Because $(x_n)_{n\in\mathbf{N}}$ is Cauchy, and f is uniformly continuous, we deduce that $(f(x_n))_{n\in\mathbf{N}}$ is Cauchy.

Assume the contrary and that $(f(x_n))_{n\in\mathbf{N}}$ is not a Cauchy sequence. Then for some $\varepsilon > 0$ and for each natural number m there are two natural numbers $n_1^{(m)} > m$ and $n_2^{(m)} > m$ such that $|f(x_{n_1^{(m)}}) - f(x_{n_2^{(m)}})| > \varepsilon$.

Let us consider a set $\{x_{n_1^{(m)}}, x_{n_2^{(m)}} : m \in \mathbf{N}\}$.

Because f is a uniformly continuous function on $(0,1)^\infty$, for $\varepsilon/2$ there exists $\delta > 0$ such that if $x, y \in (0,1)^\infty$ and $\rho_T(x,y) < \varepsilon/2$ then $|f(x) - f(y)| < \varepsilon/2$. Inasmuch as $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence we can choose such $m \in \mathbf{N}$ that $\rho_T(x_{n_1^{(m_0)}}, x_{n_2^{(m_0)}}) < \delta$. But $|f(x_{n_1^{(m_0)}}) - f(x_{n_2^{(m_0)}})| > \varepsilon$ and we get the contradiction.

Step 2. Define $\overline{f}(x) = \lim_{n\to\infty} f(x_n)$.

Step 3. Let us show that this definition is independent of the choice of the sequence $(x_n)_{n\in\mathbf{N}}$.

Indeed, we have another sequence $(y_n)_{n\in\mathbf{N}}$ of elements of $(0,1)^\infty$ which tends to x. Let us show that $\lim_{n\to\infty} f(y_n) = f(x)$. For $\varepsilon > 0$ there is $n(\varepsilon)$ such that for each $n \geq n(\varepsilon)$ we get $|f(x_n) - f(x)| < \varepsilon/2$.

Because f is uniformly continuous on $(0,1)^\infty$ for $\varepsilon/2$ there is $\delta(\varepsilon, f) > 0$ such that if $\rho_T(w,z) < \delta(\varepsilon, f)$ then $|f(w) - f(z)| < \varepsilon/2$. Because $(y_n)_{n\in\mathbf{N}}$ and $(x_n)_{n\in\mathbf{N}}$ tend to x, for $\delta(\varepsilon, f)/2$ there exists a natural number $n(\delta(\varepsilon, f))$ such that $\rho_T(y_n, x) < \delta(\varepsilon, f)/2$ and $\rho_T(x_n, x) < \delta(\varepsilon, f)/2$ for $n \geq n(\delta(\varepsilon, f))$. Then for $n \geq n(\delta(\varepsilon, f))$ we get

$$\rho_T(x_n, y_n) \leq \rho_T(x_n, x) + \rho_T(x, y_n) < \delta(\varepsilon, f)/2 + \delta(\varepsilon, f)/2 = \delta(\varepsilon, f) \quad (2.3.9)$$

which implies $|f(x_n) - f(y_n)| < \varepsilon/2$.

Then for $n \geq \max\{n(\varepsilon), n(\delta(\varepsilon, f))\}$ we get

$$|f(x) - f(y_n)| = |f(x) - f(x_n) + f(x_n) - f(y_n)| \leq$$

$$|f(x) - f(x_n)| + |f(x_n) - f(y_n)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{2.3.10}$$

Note that $\overline{f}$ is an extension of f (i.e., it coincides with f on $(0, 1)^\infty$) because of Step 3.

Uniqueness holds because any continuous extension of f must satisfy the equality of Step 2; that is, if g is another continuous extension of f, then for any $(x_n)_{n\in\mathbf{N}}$ as above $g(x) = \lim_{n\to\infty} f(x_n) = \overline{f}(x)$. As for boundedness, it again follows from Step 2. If $|f(y)| \leq M$ for all $y \in (0, 1)^\infty$, then $|\overline{f}(x)| = \lim_{n\to\infty} |f(x_n)| \leq M$ as well.

Let $f : \mathbf{R}^\infty \to \mathbf{R}$ be a real-valued function. We set $f_{(F_i)_{i\in\mathbf{N}}} : (0, 1)^\infty \to \mathbf{R}$ as follows. $f_{(F_i)_{i\in\mathbf{N}}}((x_k)_{k\in\mathbf{N}}) = f((F_k^{-1}(x_k))_{k\in\mathbf{N}})$ if $(x_k)_{k\in\mathbf{N}} \in (0, 1)^\infty$.

Now it is not hard to prove the following assertion.

Theorem 2.3.2 *Let f be a real-valued bounded function on $\mathbf{R}^\infty$ such that $f_{(F_i)_{i\in\mathbf{N}}}$ admits the Riemann integrable (with respect to the infinite-dimensional "Lebesgue measure" in $[0, 1]^\infty$) extension $\overline{f}_{(F_i)_{i\in N}}$ from $(0, 1)^\infty$ to whole $[0, 1]^\infty$. Then f is Riemann integrable w.r.t. the product measure $\prod_{i\in\mathbf{N}} \mu_i$ and the following equality,*

$$(R)\int_{\mathbf{R}^\infty} f(x)d\left(\prod_{i\in\mathbf{N}} \mu_i\right)(x) = (R)\int_{[0,1]^\infty} \overline{f}_{(F_i)_{i\in\mathbf{N}}}(x)d\lambda(x), \tag{2.3.11}$$

holds true.

Theorem 2.3.3 *If f is a real-valued bounded uniformly continuous function on $\mathbf{R}^\infty$ then f is Riemann integrable w.r.t. the product measure $\prod_{i\in\mathbf{N}} \mu_i$ and the following equality,*

$$(R)\int_{\mathbf{R}^\infty} f(x)d\left(\prod_{i\in\mathbf{N}} \mu_i\right)(x) = (R)\int_{[0,1]^\infty} \overline{f}_{(F_i)_{i\in\mathbf{N}}}(x)d\lambda(x), \tag{2.3.12}$$

holds true, where $\overline{f}_{(F_i)_{i\in\mathbf{N}}}$ is a continuous extension of $f_{(F_i)_{i\in\mathbf{N}}}$ from $(0, 1)^\infty$ to whole $[0, 1]^\infty$ defined by Lemma 2.3.1.

Proof Because f is bounded and uniformly continuous on $\mathbf{R}^\infty$ with respect to metric ρ we claim that $f_{(F_i)_{i\in\mathbf{N}}}$ is bounded and uniformly continuous on $(0, 1)^\infty$ with respect to metric ρ_T. By Lemma 2.3.1, we know that $f_{(F_i)_{i\in\mathbf{N}}}$ has a unique bounded continuous extension $\overline{f}_{(F_i)_{i\in\mathbf{N}}}$ on $[0, 1]^\infty$. By Theorem 2.3.2 we know that $\overline{f}_{(F_i)_{i\in\mathbf{N}}}$ is Riemann integrable on $[0, 1]^\infty$ w.r.t. λ. This means that there exists a real number s such that for every positive real number ε there exists a real number $\delta > 0$ such that for every

Riemann partition $(U_k)_{1 \le k \le n}$ of the $[0,1]^\infty$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1 \le k \le n}$, we have

$$\Big|\sum_{k=1}^{n} \overline{f}_{(F_i)_{i\in\mathbf{N}}}(t_k)\lambda(U_k) - s\Big| < \varepsilon. \tag{2.3.13}$$

The latter relation implies that for every Riemann partition $(U_k)_{1 \le k \le n}$ of the $[0,1]^\infty$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1 \le k \le n}$ for which $t_k \in U_k \cap (0,1)^\infty (1 \le k \le n)$, we have

$$\Big|\sum_{k=1}^{n} f_{(F_i)_{i\in\mathbf{N}}}(t_k)\lambda(U_k \cap (0,1)^\infty) - s\Big| < \varepsilon. \tag{2.3.14}$$

We have to show that s is a real number such that for every positive real number ε, δ is a number such that for every Riemann partition $\tau^* = (U_k^*)_{1 \le k \le n}$ of the $\mathbf{R}^\infty$ with $d^*(\tau^*) < \delta$ and for every sample $(t_k^*)_{1 \le k \le n}$ with $t_k^* \in U_k^* (1 \le k \le n)$, we have

$$\Big|\sum_{k=1}^{n} f(t_k^*)\left(\prod_{i\in\mathbf{N}} \mu_i\right)(U_k^*) - s\Big| < \varepsilon. \tag{2.3.15}$$

We set $\mathbf{F}((x_k)_{k\in\mathbf{N}}) = (F_k(x_k))_{k\in\mathbf{N}}$ for $(x_k)_{k\in\mathbf{N}} \in \mathbf{R}^\infty$.

If $(U_k^*)_{1 \le k \le n}$ is a Riemann partition of $\mathbf{R}^\infty$ with $d^*(\tau^*) < \delta$, then $\tau = (U_k)_{1 \le k \le n} := (\mathbf{F}(U_k^*))_{1 \le k \le n}$ will be a Riemann partition of $(0,1)^\infty$ with $d(\tau) < \delta$ and $(t_k)_{1 \le k \le n} = (\mathbf{F}(^*t_k))_{1 \le k \le n}$ will sample from the partition τ such that

$$\Big|\sum_{k=1}^{n} f(t_k^* \times \left(\prod_{i\in\mathbf{N}} \mu_i\right)(U_k^*)) - s\Big| = \Big|\sum_{k=1}^{n} f_{(F_i)_{i\in\mathbf{N}}}(t_k)\lambda(U_k) - s\Big| < \varepsilon. \tag{2.3.16}$$

The latter relation means that

$$(R)\int_{\mathbf{R}^\infty} f(x) d\left(\prod_{i\in\mathbf{N}} \mu_i\right)(x) = s. \tag{2.3.17}$$

On the other hand we have that

$$(R)\int_{[0,1]^\infty} \overline{f}_{(F_i)_{i\in\mathbf{N}}}(x) d\lambda(x) = s. \tag{2.3.18}$$

This ends the proof of the theorem.

The following corollary shows us how the Riemann integral can be computed with respect to the product measure in $\mathbf{R}^{\infty}$.

Corollary 2.3.1 *Let f be a bounded uniformly continuous real-valued function on $\mathbf{R}^{\infty}$. Let $(Y_n)_{n\in N}$ be an increasing family of uniformly distributed finite subsets in $[0,1]^{\infty}$. Then the equality*

$$(R)\int_{\mathbf{R}^{\infty}} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = \lim_{n\to\infty}\frac{\sum_{y\in Y_n}\overline{f}_{(F_i)_{i\in\mathbf{N}}}(y)}{\#(Y_n)} \tag{2.3.19}$$

holds true.

Proof By Theorem 2.3.3 we know that

$$(R)\int_{\mathbf{R}^{\infty}} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = (R)\int_{[0,1]^{\infty}}\overline{f}_{(F_i)_{i\in\mathbf{N}}}(x)d\lambda(x). \tag{2.3.20}$$

By Theorem 2.2.5 we have

$$(R)\int_{[0,1]^{\infty}}\overline{f}_{(F_i)_{i\in\mathbf{N}}}(x)d\lambda(x) = \lim_{n\to\infty}\frac{\sum_{y\in Y_n}\overline{f}_{(F_i)_{i\in\mathbf{N}}}(y)}{\#(Y_n)}. \tag{2.3.21}$$

This ends the proof of the corollary.

Remark 2.3.2 Let f be a bounded uniformly continuous real-valued function on $\mathbf{R}^{\infty}$. It is not hard to show that there is an increasing family of uniformly distributed finite subsets $(Y_n)_{n\in N}$ in $[0,1]^{\infty}$ such that $Y_n \subseteq (0,1)^{\infty}$ for each $n \in N$. Then the equality

$$(R)\int_{\mathbf{R}^{\infty}} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = \lim_{n\to\infty}\frac{\sum_{(y_i)_{i\in N}\in Y_n} f((F_i^{-1}(y_i))_{i\in\mathbf{N}})}{\#(Y_n)} \tag{2.3.22}$$

holds true.

The following example can be considered as a certain application of the Remark 2.3.2 in mathematical analysis.

Example 2.3.2 The following equality

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\cdots,i_n)\in\{1,\cdots,n\}^n}\sum_{k=1}^{n}\frac{\{i_k\omega\}^{\alpha}}{2^k}}{n^n} = \frac{1}{1+\alpha} \tag{2.3.23}$$

holds true for all irrational numbers ω and positive real numbers α.

Let $f : \mathbf{R}^\infty \to \mathbf{R}$ be defined by $f((x_k)_{k\in\mathbf{N}}) = \sum_{k\in\mathbf{N}} F_k^\alpha(x_k)/2^k$, where $\alpha > 0$. Then

$$f((F_k^{-1}(y_k))_{k\in\mathbf{N}}) = \sum_{k\in\mathbf{N}} \frac{F_k^\alpha(F_k^{-1}(y_k))}{2^k} = \sum_{k\in\mathbf{N}} \frac{y_k^\alpha}{2^k} \tag{2.3.24}$$

for $(y_k)_{k\in\mathbf{N}} \in (0,1)^\infty$.

Let ω be an arbitrary irrational number. Let $Y_n = \{\{\omega\},\{2\omega\},\dots,\{n\omega\}\}^n \times (\{\omega\},\{\omega\},\dots)$ for $n \in \mathbf{N}$. Then by virtue of Remark 2.3.2 we have

$$(R)\int_{\mathbf{R}^\infty} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = \lim_{n\to\infty}\frac{\sum_{(y_i)_{i\in N}\in Y_n} f((F_i^{-1}(y_i))_{i\in\mathbf{N}})}{\#(Y_n)} =$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\cdots,i_n)\in\{1,\cdots,n\}^n}(\sum_{k=1}^n\frac{\{i_k\omega\}^\alpha}{2^k}+\sum_{k>n}\frac{\{\omega\}^\alpha}{2^k})}{n^n} =$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\cdots,i_n)\in\{1,\cdots,n\}^n}\sum_{k=1}^n\frac{\{i_k\omega\}^\alpha}{2^k}}{n^n}. \tag{2.3.25}$$

On the other hand we have

$$(R)\int_{\mathbf{R}^\infty} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = (R)\int_{[0,1]^\infty}\sum_{k\in\mathbf{N}}\frac{x_k^\alpha}{2^k}d\lambda(x) =$$

$$\sum_{k\in\mathbf{N}}\frac{1}{2^k}(R)\int_{[0,1]^\infty} x_k^\alpha d\lambda(x) = \frac{1}{1+\alpha}\sum_{k\in\mathbf{N}}\frac{1}{2^k} = \frac{1}{1+\alpha}. \tag{2.3.26}$$

Finally, we get the identity:

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n}\sum_{k=1}^n\frac{\{i_k\omega\}^\alpha}{2^k}}{n^n} = \frac{1}{1+\alpha}. \tag{2.3.27}$$

2.4 Applications to Statistics

In probability theory, there exist several different notions of convergence of random variables. The convergence of sequences of random variables to some limit random variable is an important concept in probability theory. Almost sure convergence is called the *strong law* because random variables that converge strongly (almost surely) are guaranteed to converge weakly (in probability) and in distribution (see, e.g., [Sh], Theorem 2, p. 272). Theorems that establish almost sure convergence of such sequences to some limit random variable are called *strong law type theorems* and they have interesting applications to statistics and stochastic processes. The purpose of the present section is to establish the validity of essentially new and interesting strong law type theorems in an infinite-dimensional case by using Monte Carlo algorithms elaborated in Sect. 2.3.

Theorem 2.4.1 *Let* $(\Omega, \mathbf{F}, P)$ *be a probability space and* $(\xi_k)_{k\in N}$ *be a sequence of independent real-valued random variables uniformly distributed on the interval* $[0, 1]$ *such that* $0 \le \xi_k(\omega) \le 1$. *Let* $f : [0, 1]^\infty \to R$ *be a Riemann integrable real-valued function. Then the equality:*

$$P\left\{\omega : \lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n} f(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \dots, \xi_{i_n}(\omega), \xi_1(\omega), \xi_1(\omega), \dots)}{n^n} = \int_{[0,1]^\infty} f(x)d\lambda(x)\right\} = 1 \tag{2.4.1}$$

holds true.

Proof Without loss of generality we can assume that

$$(\mathbf{\Omega}, \mathbf{F}, P) = ([0, 1]^\infty, \mathbf{B}([0, 1]^\infty), \ell_1^\infty), \tag{2.4.2}$$

where ℓ_1 is the Lebesgue measure in $(0, 1)$ and $\xi_k((\omega_i)_{i\in N}) = \omega_k$ for each $k \in N$ and $(\omega_i)_{i\in N} \in [0, 1]^\infty$. Let S be a set of all uniformly distributed sequences on $(0, 1)$. By Lemma 1.2.4 we know that $\ell_1^N(S) = 1$; equivalently, $\lambda(S) = 1$, where λ denotes the infinite-dimensional "Lebesgue measure". The latter relation means that

$$P\{\omega : (\xi_k(\omega))_{k\in N} \text{ is uniformly distributed on } (0, 1)\} = 1. \tag{2.4.3}$$

We put

$$Y_n(\omega) = (\cup_{j=1}^n \{\xi_j(\omega)\})^n \times (\xi_1(\omega), \xi_1(\omega), \dots) \tag{2.4.4}$$

for each $n \in N$.

Note that if the sequence of real numbers $(\xi_k(\omega))_{k\in N}$ is uniformly distributed in the interval $[0, 1]$ then by Theorem 2.2.1, $(Y_n(\omega))_{n\in N}$ will be uniformly distributed in the rectangle $[0, 1]^\infty$ which according to Theorem 2.2.5 implies that

$$\int_{[0,1]^\infty} f(x)d\lambda(x) =$$

$$\lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\ldots,i_n)\in\{1,\ldots,n\}^n} f(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \ldots, \xi_{i_n}(\omega), \xi_1(\omega), \xi_1(\omega), \ldots)}{n^n}. \quad (2.4.5)$$

But the set of all ω points for which the latter equality holds true contains the set S for which $P(S) = 1$.

This ends the proof of the theorem.

As a simple consequence of Theorem 2.4.1, we get the validity of the strong law of large numbers for a sequence of independent real-valued random variables uniformly distributed on the interval [0, 1] as follows.

Corollary 2.4.1 *Let* $(\Omega, \mathbf{F}, P)$ *be a probability space and* $(\xi_k)_{k\in N}$ *be a sequence of independent real-valued random variables uniformly distributed on the interval* $[0, 1]$ *such that* $0 \le \xi_k(\omega) \le 1$. *Then the condition*

$$P\left\{\omega : \lim_{n\to\infty} \frac{\sum_{k=1}^n \xi_k(\omega)}{n} = 1/2\right\} = 1 \quad (2.4.6)$$

holds true.

Proof Let $f : [0, 1]^\infty \to R$ be defined by $f(x_1, x_2, \ldots) = x_1$. By Theorem 2.4.1 we have

$$P\{\omega : \int_{[0,1]^\infty} f(x)d\lambda(x) =$$

$$\lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\ldots,i_n)\in\{1,\ldots,n\}^n} f(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \ldots, \xi_{i_n}(\omega), \xi_1(\omega), \xi_1(\omega), \ldots)}{n^n}\} = 1. \quad (2.4.7)$$

Note that

$$\int_{[0,1]^\infty} f(x)d\lambda(x) = 1/2 \quad (2.4.8)$$

and

$$\lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\ldots,i_n)\in\{1,\ldots,n\}^n} f(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \ldots, \xi_{i_n}(\omega), \xi_1(\omega), \xi_1(\omega), \ldots)}{n^n} =$$

$$\lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\ldots,i_n)\in\{1,\ldots,n\}^n} \xi_{i_1}(\omega)}{n^n} =$$

$$\lim_{n\to\infty} \frac{n^{n-1}\sum_{k=1}^{n}\xi_k(\omega)}{n^n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\xi_k(\omega)}{n}. \tag{2.4.9}$$

This ends the proof of Corollary 2.4.1.

The next corollary also being a simple consequence of Theorem 2.4.1 gives interesting but well-known information for statisticians regarding whether the value of m-dimensional Riemann integrals over the m-dimensional rectangle $[0, 1]^m$ can be estimated by using infinite samples.

Corollary 2.4.2 *Let* $(\boldsymbol{\Omega}, \mathbf{F}, P)$ *be a probability space and* $(\xi_k)_{k\in N}$ *be a sequence of independent real-valued random variables uniformly distributed on the interval* $[0, 1]$ *such that* $0 \le \xi_k(\omega) \le 1$. *Let* $f : [0, 1]^m \to R$ *be a Riemann integrable real-valued function. Then the equality*

$$P\{\omega : \lim_{n\to\infty} \frac{\sum_{(i_1,i_2,\dots,i_m)\in\{1,\dots,n\}^m} f(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \dots, \xi_{i_m}(\omega))}{n^m} =$$

$$\int_{[0,1]^m} f(x_1, \dots, x_m)dx_1 \dots dx_m\} = 1 \tag{2.4.10}$$

holds true.

Proof For $(x_k)_{k\in N} \in [0, 1]^\infty$ we put $\overline{f}((x_k)_{k\in N}) = f(x_1, \dots, x_m)$. Without loss of generality we can assume that

$$(\boldsymbol{\Omega}, \mathbf{F}, P) = ([0, 1]^\infty, \mathbf{B}([0, 1]^\infty), \ell_1^\infty), \tag{2.4.11}$$

where ℓ_1 is the Lebesgue measure in $(0, 1)$ and $\xi_k((\omega_i)_{i\in N}) = \omega_k$ for each $k \in N$ and $(\omega_i)_{i\in N} \in [0, 1]^\infty$. Let S be a set of all uniformly distributed sequences on $(0, 1)$. By Lemma 1.2.4 we know that $P(S) = 1$. The latter relation means that

$$P\{\omega : (\xi_k(\omega))_{k\in N} \text{ is uniformly distributed on the interval } (0, 1)\} = 1. \tag{2.4.12}$$

We put

$$Y_n(\omega) = (\cup_{j=1}^{n}\{\xi_j(\omega)\})^m \times (\xi_1(\omega), \xi_1(\omega), \dots) \tag{2.4.13}$$

for each $n \in N$.

Note that if $(\xi_k(\omega))_{k\in N}$ is uniformly distributed on the interval $(0, 1)$ then by Theorem 2.2.1, $(Y_n(\omega))_{n\in N}$ will be uniformly distributed in the rectangle $[0, 1]^\infty$, which according to Theorem 2.2.5 implies that

$$\int_{[0,1]^m} f(x_1, \dots, x_m)dx_1 \dots dx_m = \int_{[0,1]^\infty} \overline{f}(x)d\lambda(x) =$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n}\overline{f}(\xi_{i_1}(\omega),\xi_{i_2}(\omega),\dots,\xi_{i_n}(\omega),\xi_1(\omega),\xi_1(\omega),\dots)}{n^n}=$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n}f(\xi_{i_1}(\omega),\xi_{i_2}(\omega),\dots,\xi_{i_m}(\omega))}{n^n}=$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_m)\in\{1,\dots,n\}^m}n^{n-m}f(\xi_{i_1}(\omega),\xi_{i_2}(\omega),\dots,\xi_{i_m}(\omega))}{n^n}=$$

$$\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_m)\in\{1,\dots,n\}^m}f(\xi_{i_1}(\omega),\xi_{i_2}(\omega),\dots,\xi_{i_m}(\omega))}{n^m}. \tag{2.4.14}$$

A set of all points ω for which the latter equality holds true, contains the set S for which $P(S)=1$.

This ends the proof of Corollary 2.4.2.

Corollary 2.4.3 *Let $(\mathbf{\Omega},\mathbf{F},P)$ be a probability space and $(\xi_k)_{k\in N}$ be a sequence of independent real-valued random variables such that the distribution function F_k defined by ξ_k is strictly increasing and continuous. Let f be a real-valued bounded function on $\mathbf{R}^\infty$ such that $f_{(F_i)_{i\in\mathbf{N}}}$ admits such an extension $\overline{f}_{(F_i)_{i\in N}}$ from $(0,1)^\infty$ to whole $[0,1]^\infty$ that $\overline{f}_{(F_i)_{i\in N}}$ is Riemann integrable with respect to the infinite-dimensional Lebesgue measure λ in $[0,1]^\infty$. Then f is Riemann integrable w.r.t. product measure $\prod_{i\in\mathbf{N}}\mu_i$ and the condition*

$$P\left\{\omega:\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n}f(\xi_{i_1}(\omega),\xi_{i_2}(\omega),\dots,\xi_{i_n}(\omega),\xi_1(\omega),\xi_1(\omega),\dots)}{n^n}=\right.$$

$$\left.(R)\int_{\mathbf{R}^\infty}f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x)\right\}=1 \tag{2.4.15}$$

holds true.

Proof Without loss of generality we can assume that

$$(\mathbf{\Omega},\mathbf{F},P)=\left(\mathbf{R}^\infty,\mathbf{B}(\mathbf{R}^\infty),\prod_{i\in\mathbf{N}}\mu_i\right), \tag{2.4.16}$$

and $\xi_k((\omega_i)_{i\in N})=\omega_k$ for each $k\in N$ and $(\omega_i)_{i\in N}\in\mathbf{R}^\infty$.

Let ω be an element of the $\mathbf{\Omega}$ such that $(F_k(\xi_k(\omega))_{k\in N}$ is a uniformly distributed sequence on $(0,1)$. Note that all such points ω constitute a set D_0 for which $(\prod_{i\in\mathbf{N}}\mu_i)(D_0)=1$.

According to Theorem 2.3.2, f is Riemann integrable with respect to the product measure $\prod_{i\in\mathbf{N}}\mu_i$ and the equality

$$(R)\int_{\mathbf{R}^\infty} f(x)d\left(\prod_{i\in\mathbf{N}}\mu_i\right)(x) = (R)\int_{[0,1]^\infty}\overline{f}_{(F_i)_{i\in N}}(x)d\lambda(x) \tag{2.4.17}$$

holds true. For $\omega\in D_0$ we have

$$\begin{aligned}
&(R)\int_{[0,1]^\infty}\overline{f}_{(F_i)_{i\in N}}(x)d\lambda(x)\\
&=\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n}\overline{f}_{(F_i)_{i\in N}}(F_1(\xi_{i_1}(\omega)),\dots,F_n(\xi_{i_n}(\omega)),F_{n+1}(\xi_1(\omega)),F_{n+2}(\xi_1(\omega)),\dots)}{n^n}\\
&=\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n} f_{(F_i)_{i\in N}}(F_1(\xi_{i_1}(\omega)),\dots,F_n(\xi_{i_n}(\omega)),F_{n+1}(\xi_1(\omega)),F_{n+2}(\xi_1(\omega)),\dots)}{n^n}\\
&=\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\dots,n\}^n} f(F_1^{-1}(F_1(\xi_{i_1}(\omega)),\dots,F_n^{-1}(F_n(\xi_{i_n}(\omega)),F_{n+1}^{-1}(F_{n+1}(\xi_1(\omega)),\dots)}{n^n}\\
&=\lim_{n\to\infty}\frac{\sum_{(i_1,i_2,\dots,i_n)\in\{1,\cdots,n\}^n} f(\xi_{i_1}(\omega),\dots,\xi_{i_n}(\omega),\xi_1(\omega),\xi_1(\omega),\dots)}{n^n}.
\end{aligned} \tag{2.4.18}$$

This ends the proof of Corollary 2.4.3.

Remark 2.4.1 Main results of Sect. 2.3–2.4 were obtained in [P3].

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http://www.springer.com/978-3-319-45577-8

Applications of Measure Theory to Statistics
Pantsulaia, G.
2016, XV, 134 p., Hardcover
ISBN: 978-3-319-45577-8