Resolving Decompositions for Polynomial Modules

Mario Albert(✉) and Werner M. Seiler

Institut für Mathematik, Universität Kassel, 34132 Kassel, Germany
{albert,seiler}@mathematik.uni-kassel.de

Abstract. We introduce the novel concept of a resolving decomposition of a polynomial module as a combinatorial structure that allows for the effective construction of free resolutions. It provides a unifying framework for recent results of the authors for different types of bases.

1 Introduction

The determination of free resolutions for polynomial modules is a fundamental task in computational commutative algebra and algebraic geometry. Free resolutions are needed for derived functors like Ext and Tor and many important homological invariants like the projective dimension or the Castelnuovo-Mumford regularity are defined via the minimal resolution. Furthermore, the Betti numbers contain much geometric and topological information.

Unfortunately, it is rather expensive to compute a resolution. As a rough rule of thumb one may say that computing a resolution of length ℓ corresponds to computing ℓ Gröbner bases. In many cases one needs only partial information about the resolution like the Betti numbers simply measuring its size. However, all classical algorithms require to determine always a full resolution.

In [1] we provided a novel approach by combining the theory of Pommaret bases and algebraic discrete Morse theory (together with an implementation in the CoCoALib). It allows for the first time to determine Betti numbers—even individual ones—without computing a full resolution and thus is for most problems much faster than classical approaches (see [1,3] for detailed benchmarks). Furthermore, it scales much better and can be easily parallelised.

Because of these good properties it is of great interest to generalise this approach to other situations. In [3], we extended it to Janet bases. While the proofs remained essentially the same, the use of another involutive division required the adaption of a number of technical points. Currently we are working on extensions to some alternative bases which do not necessarily come from an involutive division but provide similar combinatorial decompositions. Again this would require a number of minor modifications of the same proofs. In a different line of work [2], we introduced recently modules marked on quasi-stable submodules. Again one obtains combinatorial decompositions based on multiplicative variables defined by the Pommaret division, but this time no term order is used. Nevertheless, we could show that many results of [1] remain true.

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The main objective of the current paper is the development of an axiomatic framework that unifies all the above works. We introduce the novel concept of a *resolving decomposition* which is defined via several direct sum decompositions. It implies in particular the existence of standard representations and normal forms. We then show that such a decomposition allows for the explicit determination of a free resolution and of Betti numbers.

The point of such a unification is *not* that it leads to any new algorithms. Indeed, we will not present a general algorithm for the construction of resolving decomposition. Instead one should see our results as a "meta-machinery" which given any concept of a basis that induces a resolving decomposition delivers automatically an effective syzygy theory for this kind of basis. For the concrete case of the resolving decompositions induced by Janet or Pommaret bases, an implementation of this effective theory is described (together with benchmarks) in [1,3]. For other types of underlying bases only fairly trivial modifications of this implementation would be required.

2 Resolving Decompositions

Let $\Bbbk$ be an algebraic closed field and $\mathcal{P} = \Bbbk[\mathbf{x}]$ with $\mathbf{x} = (x_0, \ldots, x_n)$. Let $\mathcal{P}^m_{\mathbf{d}} = \bigoplus_{i=1}^m \mathcal{P}(-d_i)\mathbf{e}_i$ be a finitely generated free $\mathcal{P}$-module with grading $\mathbf{d} = (d_1, \ldots, d_m)$ and free generators $\mathbf{e}_1, \ldots, \mathbf{e}_m$. A module $U \subseteq \mathcal{P}^m_{\mathbf{d}}$ is called *monomial module* if it is of the form $\oplus_{k=1}^m J^{(k)}\mathbf{e}_k$ with $J^{(k)}$ is a monomial ideal in $\mathcal{P}$. A *module term (with index i)* is a term of the form $x^\mu \mathbf{e}_i$. For a monomial ideal $J \subseteq \mathcal{P}$ we define $\mathcal{N}(J) \subseteq \mathbb{T}$ as the set of terms in $\mathbb{T}$ not belonging to J. For a monomial module U we define $\mathcal{N}(U) = \bigcup_{k=1}^m \mathcal{N}(J^{(k)})\mathbf{e}_k$. For an element $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ we define $\operatorname{supp}(\mathbf{f})$ as the set of module terms appearing in $\mathbf{f}$ with a non-zero coefficient: $\mathbf{f} = \sum_{x^\alpha \mathbf{e}_{i_\alpha} \in \operatorname{supp}(\mathbf{f})} c_\alpha x^\alpha \mathbf{e}_{i_\alpha}$. If $\mathcal{B}$ is a set of homogeneous elements of degree s in $\mathcal{P}^m_{\mathbf{d}}$, we write $\langle \mathcal{B} \rangle$ for the $\Bbbk$-vector space generated by $\mathcal{B}$ in $(\mathcal{P}^m_{\mathbf{d}})_s$. For a module $U \subseteq \mathcal{P}^m_{\mathbf{d}}$ we denote by $\operatorname{pd}(U)$ the *projective dimension* and by $\operatorname{reg}(U)$ the *(Castelnuovo-Mumford) regularity* of U.

Let $U \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded submodule and $\mathcal{B} = \{\mathbf{h}_1, \ldots, \mathbf{h}_s\}$ a finite homogeneous generating set of U. For every $\mathbf{h}_i \in \mathcal{B}$ we choose a term $x^{\mu_i}\mathbf{e}_{k_i} \in \operatorname{supp} \mathbf{h}_i$ denoted by $\operatorname{hm} \mathbf{h}_i$ and call it *head module term*. In addition to that we define the *head module terms of* $\mathcal{B}$, $\operatorname{hm}(\mathcal{B}) := \{\operatorname{hm}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}\}$ and the *head module of* U, $\operatorname{hm}(U) = \langle \operatorname{hm} \mathcal{B} \rangle$. Note that $\operatorname{hm}(U)$ depends on the choice of $\mathcal{B}$ and on the choice of the head module terms in $\mathcal{B}$.

Definition 1. *We define a* resolving decomposition *of the submodule U as a quadruple $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ with the following five properties:*

(i) $U = \langle \mathcal{B} \rangle$.
(ii) *Let $\mathbf{h} \in \mathcal{B}$ be an arbitrary generator. Then, for every module term $x^\mu \mathbf{e}_k \in \operatorname{supp}(\mathbf{h}) \setminus \{\operatorname{hm}(\mathbf{h})\}$, we have $x^\mu \mathbf{e}_k \notin \operatorname{hm}(U)$.*
(iii) *We assign a set of* multiplicative variables $X_{\mathcal{B}}(\mathbf{h}) \subseteq \mathbf{x}$ *to every head module term* $\operatorname{hm}(\mathbf{h})$ *with $\mathbf{h} \in \mathcal{B}$ such that we have direct sum decompositions of both*

the head module

$$\mathrm{hm}(U) = \bigoplus_{\mathbf{h}\in\mathcal{B}} \Bbbk[X_{\mathcal{B}}(\mathbf{h})] \cdot \mathrm{hm}(\mathbf{h}) \tag{1}$$

and of the module itself

$$U = \bigoplus_{\mathbf{h}\in\mathcal{B}} \Bbbk[X_{\mathcal{B}}(\mathbf{h})] \cdot \mathbf{h}\,. \tag{2}$$

(iv) $(\mathcal{P}^m_{\mathbf{d}})_r = U_r \oplus \langle \mathcal{N}(\mathrm{hm}(U))_r \rangle$ *for all* $r \geq 0$*.*

(v) Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ *denote the standard basis of the free module* $\mathcal{P}^s$*. Given an arbitrary term* $x^\delta \in \mathbb{T}$ *and an arbitrary generator* $\mathbf{h}_\alpha \in \mathcal{B}$*, we find for every term* $x^\epsilon \mathbf{e}_i \in \mathrm{supp}(x^\delta \mathbf{h}_\alpha) \cap \mathrm{hm}(U)$ *a unique* $\mathbf{h}_\beta \in \mathrm{hm}(\mathcal{B})$ *such that* $x^\epsilon \mathbf{e}_i = x^{\delta'}\,\mathrm{hm}(\mathbf{h}_\beta)$ *with* $x^{\delta'} \in \Bbbk[X_{\mathcal{B}}(\mathbf{h}_\beta)]$ *by (iii). Then the term order* $\prec_{\mathcal{B}}$ *on* $\mathcal{P}^s$ *must satisfy* $x^\delta \mathbf{f}_\alpha \succeq_{\mathcal{B}} x^{\delta'} \mathbf{f}_\beta$*.*

In the sequel, we will always assume that $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ is a resolving decomposition of the finitely generated module $U = \langle \mathcal{B} \rangle \subseteq \mathcal{P}^m_{\mathbf{d}}$. In addition to the multiplicative variables, we define for $\mathbf{h} \in \mathcal{B}$ the *non-multiplicative variables* as $\overline{X}_{\mathcal{B}}(\mathbf{h}) = \{x_0, \ldots, x_n\} \setminus X_{\mathcal{B}}(\mathbf{h})$.

Remark 1. Resolving decompositions may be considered as a refinement of Stanley decompositions. Indeed, (1) gives us a Stanley decomposition of the head module of U and (2) of U itself. Consequently, it is easy to compute the Hilbert functions of $\mathrm{hm}(U)$ and of U, respectively. Because of the identical structure of the two decompositions, these two Hilbert functions are trivially the same (which may be considered as a term order free version of Macaulay's theorem in the theory of Gröbner bases). In addition, (iii) gives us for every $\mathbf{f} \in U$ a unique *standard representation*

$$\mathbf{f} = \sum_{\alpha=1}^{s} P_\alpha \mathbf{h}_\alpha$$

with $P_\alpha \in \Bbbk[X_{\mathcal{B}}(\mathbf{h}_\alpha)]$. Condition (iv) implies the existence of unique *normal forms* for all homogeneous elements $\mathbf{f} \in \mathcal{P}^m$. Due to this condition, we find unique $P_\alpha \in \Bbbk[X_{\mathcal{B}}(\mathbf{h}_\alpha)]$ for every $\mathbf{h}_\alpha \in \mathcal{B}$ such that $\mathbf{f}' = \mathbf{f} - \sum_{\alpha=1}^{s} P_\alpha \mathbf{h}_\alpha$ and $\mathbf{f}' \in \langle \mathcal{N}(\mathrm{hm}(U)) \rangle$. Another important consequence of the definition of a resolving decomposition is that (1) implies that every generators in $\mathcal{B}$ has a different head module term.

While for the purposes of this work the mere existence of normal forms is sufficient, we note that (v) implies that they can be effectively computed. The choice of head terms and multiplicative variables in a resolving decomposition induces a natural reduction relation. If $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ contains a term $x^\epsilon \mathbf{e}_i \in \mathrm{hm}\,(U)$, then there exists a unique generator $\mathbf{h} \in \mathrm{hm}(\mathcal{B})$ such that $x^\epsilon \mathbf{e}_i = x^\delta\,\mathrm{hm}(\mathbf{h})$ with $x^\delta \in \Bbbk[X_{\mathcal{B}}(\mathbf{h})]$ and we have a possible reduction $\mathbf{f} \xrightarrow{\mathcal{B}} \mathbf{f} - c x^\delta \mathbf{h}$ for a suitably chosen coefficient $c \in \Bbbk$.

Lemma 1. *For any resolving decomposition* $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ *the transitive closure* $\xrightarrow{\mathcal{B}}{}^*$ *of* $\xrightarrow{\mathcal{B}}$ *is Noetherian and confluent.*

Proof. It is sufficient to prove that for every term $x^\gamma \mathbf{e}_k$ in $\operatorname{hm}(U)$, there is a unique $g \in \mathcal{P}^m_\mathbf{d}$ such that $x^\gamma \mathbf{e}_k \xrightarrow{\mathcal{B}}{}^* g$ and $g \in \langle \mathcal{N}(\operatorname{hm}(U)) \rangle$.

Since $x^\gamma \mathbf{e}_k \in \operatorname{hm}(U)$, there exists a unique $x^\delta \mathbf{h}_\alpha \in U$ such that $x^\delta \operatorname{hm}(\mathbf{h}_\alpha) = x^\gamma \mathbf{e}_k$ and $x^\delta \in X_\mathcal{B}(\mathbf{h}_\alpha)$. Hence, $x^\gamma \mathbf{e}_k \xrightarrow{\mathcal{B}} x^\gamma \mathbf{e}_k - c x^\delta \mathbf{h}_\alpha$ for a suitably chosen coefficient $c \in \Bbbk$. Denoting again the standard basis of $\mathcal{P}^s$ by $\{\mathbf{f}_1, \dots, \mathbf{f}_s\}$, we associated the term $x^\delta \mathbf{f}_\alpha$ with this reduction step. If we could proceed infinitely with further reduction steps, then the reduction process would induce a sequence of terms in $\mathcal{P}^s$ containing an infinite chain which, by condition (v) of Definition 1, is strictly descending for $\prec_\mathcal{B}$. But this is impossible, since $\prec_\mathcal{B}$ is a well-ordering. Hence $\xrightarrow{\mathcal{B}}{}^*$ is Noetherian. Confluence is immediate by the uniqueness of the element that is used at each reduction step. □

Furthermore, every resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ induces naturally a directed graph. Its vertices are given by the elements in $\mathcal{B}$. If $x_j \in \overline{X}_\mathcal{B}(\mathbf{h})$ for some $\mathbf{h} \in \mathcal{B}$, then, by definition, $\mathcal{B}$ contains a unique generator $\mathbf{h}'$ such that $x_j \operatorname{hm} \mathbf{h} = x^\mu \operatorname{hm} \mathbf{h}'$ with $x^\mu \in \Bbbk[X_\mathcal{B}(\mathbf{h}')]$. In this case we include a directed edge from $\mathbf{h}$ to $\mathbf{h}'$. We call the thus defined graph the $\mathcal{B}$-graph.

Lemma 2. *The* $\mathcal{B}$*-graph of a resolving decomposition* $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ *is always acyclic.*

Proof. Assume the $\mathcal{B}$-graph was cyclic. Then we find generators $\mathbf{h}_{k_1}, \dots, \mathbf{h}_{k_t} \in \mathcal{B}$, which are pairwise distinct, variables $x_{i_1}, \dots, x_{i_t}$ such that $x_{i_j} \in \overline{X}_\mathcal{B}(\operatorname{hm}(\mathbf{h}_{k_j}))$ for all $j \in \{1, \dots, t\}$ and terms $x^{\mu_1}, \dots, x^{\mu_t}$ such that $x^{\mu_j} \in \Bbbk[X_\mathcal{B}(\operatorname{hm}(\mathbf{h}_{k_j}))]$ for all $j \in \{1, \dots, t\}$ satisfying:

$$\begin{aligned} x_{i_1} \operatorname{hm}(\mathbf{h}_{k_1}) &= x^{\mu_2} \operatorname{hm}(\mathbf{h}_{k_2}), \\ x_{i_2} \operatorname{hm}(\mathbf{h}_{k_2}) &= x^{\mu_3} \operatorname{hm}(\mathbf{h}_{k_3}), \\ &\vdots \\ x_{i_t} \operatorname{hm}(\mathbf{h}_{k_t}) &= x^{\mu_1} \operatorname{hm}(\mathbf{h}_{k_1}). \end{aligned}$$

Multiplying with some variables, we obtain the following chain of equations:

$$\begin{aligned} x_{i_1} \cdots x_{i_t} \operatorname{hm}(\mathbf{h}_{k_1}) &= x_{i_2} \cdots x_{i_t} x^{\mu_2} \operatorname{hm}(\mathbf{h}_{k_2}) \\ &= x_{i_3} \cdots x_{i_t} x^{\mu_2} x^{\mu_3} \operatorname{hm}(\mathbf{h}_{k_3}) \\ &\vdots \\ &= x_{i_t} x^{\mu_2} \cdots x^{\mu_t} \operatorname{hm}(\mathbf{h}_{k_t}) \\ &= x^{\mu_1} \cdots x^{\mu_t} \operatorname{hm}(\mathbf{h}_{k_1}) \end{aligned}$$

which implies that $x_{i_1} \cdots x_{i_t} = x^{\mu_1} \cdots x^{\mu_t}$. Furthermore, condition (v) of Definition 1 implies in $\mathcal{P}^s$ the following chain:

$$x_{i_1} \cdots x_{i_t} \mathbf{f}_{k_1} \succeq_\mathcal{B} x_{i_2} \cdots x_{i_t} x^{\mu_2} \mathbf{f}_{k_2} \succeq_\mathcal{B} \cdots \succeq_\mathcal{B} x^{\mu_1} \cdots x^{\mu_t} \mathbf{f}_{k_1} \,.$$

Because of $x_{i_1} \cdots x_{i_t} = x^{\mu_1} \cdots x^{\mu_t}$, we must have throughout equality entailing that $k_1 = \cdots = k_t$ which contradicts our assumptions. $\square$

Example 1. Let $\prec$ be a term order on the free module $\mathcal{P}^m_{\mathbf{d}}$, L a continuous involutive division ([7, Definition 2.1]) and $\mathcal{B}$ a finite, L-involutively autoreduced set ([7, Definition 5.8]) which is a strong L-involutive basis ([7, Definition 5.1]) of the submodule $U \subseteq \mathcal{P}^m_{\mathbf{d}}$ it generates. Then $\mathcal{B}$ induces a resolving decomposition of U with $\mathrm{hm}(\mathcal{B}) = \{\mathrm{lt}(\mathbf{h}_1), \dots, \mathrm{lt}(\mathbf{h}_s)\}$. The multiplicative variables $X_{\mathcal{B}}$ are assigned according to the involutive division L and we take as $\prec_{\mathcal{B}}$ the Schreyer order induced by $\mathcal{B}$ and $\prec$. Condition (i) of Definition 1 follows from [7, Corollary 5.5], condition (ii) is a consequence of the fact that $\mathcal{B}$ is involutively autoreduced and condition (iii) follows from [7, Lemma 5.12]. According to [7, Proposition 5.13] every $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ possesses a unique normal form. In Remark 1 we have seen that this is equivalent to the fourth condition in our definition. Finally, (v) is satisfied because of [8, Lemma 5.5] and the existence of an L-ordering. Hence an autoreduced involutive basis always induces a resolving decomposition.

Example 2. Another example for resolving decompositions are the marked modules introduced in [2]. Marked modules are only defined for quasi-stable modules. The construction of a marked basis is a bit different from the usual construction of Gröbner bases. We start with a quasi-stable monomial module $V \subseteq \mathcal{P}^m_{\mathbf{d}}$ which is generated by a monomial Pommaret basis $\mathcal{H} = \{x^{\mu_1}\mathbf{e}_{k_1}, \dots x^{\mu_s}\mathbf{e}_{k_s}\}$. Then we define a marked basis $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ such that $\mathrm{hm}(\mathbf{h}_i) = x^{\mu_i}\mathbf{e}_{k_i}$ and $\mathrm{supp}(\mathbf{h}_i - x^{\mu_i}\mathbf{e}_{k_i}) \subseteq \langle \mathcal{N}(V)_{\deg(x^{\mu_i}\mathbf{e}_{k_i})} \rangle$. Furthermore it is required that $\mathcal{N}(V)_r$ induces a $\Bbbk$-basis of $(\mathcal{P}^m_{\mathbf{d}})_r / \langle \mathcal{B} \rangle_r$ for all degrees r, which implies that $(\mathcal{P}^m_{\mathbf{d}})_r = \langle \mathcal{B} \rangle_r \oplus \langle \mathcal{N}(V)_r \rangle$ for all r. The multiplicative variables $X_{\mathcal{B}}$ are assigned according to the multiplicative variables of the Pommaret basis $\mathcal{H}$ (for a detailed treatment see section two in [2]). We see immediately that the conditions (i), (ii) and (iv) are satisfied. The first part of condition (iii) follows from the fact that $\mathcal{H} = \mathrm{hm}(\mathcal{B})$ is a Pommaret basis and the second part follows from the uniqueness of the reduction process [2, Lemma 5.1]. Finally, we take for $\prec_{\mathcal{B}}$ the TOP lift of the lexicographic order; condition (v) then follows from [2, Lemma 3.6].

Example 3. Even in the case of a monomial module, not every Stanley decomposition can be extended to a resolving decomposition. For $m = 1$, $n = 4$ and the standard grading, we take as U the homogeneous maximal ideal in $\mathcal{P}$. A Stanley decomposition of U is then given by the set

$$\begin{aligned}\mathcal{B} = \{&\mathbf{h}_1 = x_0, \mathbf{h}_2 = x_1, \mathbf{h}_3 = x_2, \mathbf{h}_4 = x_3, \mathbf{h}_5 = x_4, \mathbf{h}_6 = x_0x_1x_3, \mathbf{h}_7 = x_0x_2x_3,\\ &\mathbf{h}_8 = x_0x_2x_4, \mathbf{h}_9 = x_1x_2x_4, \mathbf{h}_{10} = x_1x_3x_4, \mathbf{h}_{11} = x_0x_1x_2x_3x_4\}\end{aligned}$$

with multiplicative variables

$$\begin{aligned} X_{\mathcal{B}}(\mathbf{h}_1) &= \{x_0, x_1, x_2\}, & X_{\mathcal{B}}(\mathbf{h}_2) &= \{x_1, x_2, x_3\} \\ X_{\mathcal{B}}(\mathbf{h}_3) &= \{x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_4) &= \{x_0, x_3, x_4\} \\ X_{\mathcal{B}}(\mathbf{h}_5) &= \{x_0, x_1, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_6) &= \{x_0, x_1, x_2, x_3\} \\ X_{\mathcal{B}}(\mathbf{h}_7) &= \{x_0, x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_8) &= \{x_0, x_1, x_2, x_4\} \\ X_{\mathcal{B}}(\mathbf{h}_9) &= \{x_1, x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_{10}) &= \{x_0, x_1, x_3, x_4\} \\ X_{\mathcal{B}}(\mathbf{h}_{11}) &= \{x_0, x_1, x_2, x_3, x_4\}\,. \end{aligned}$$

It is not possible to find a term order $\prec_{\mathcal{B}}$ which makes this Stanley decomposition to a resolving one, as the corresponding $\mathcal{B}$-graph contains a cycle (note that here obviously $\operatorname{hm}(\mathbf{h}_i) = \mathbf{h}_i$):

$$x_3\mathbf{h}_1 = x_0\mathbf{h}_4\,, \quad x_1\mathbf{h}_4 = x_3\mathbf{h}_2\,, \quad x_0\mathbf{h}_2 = x_1\mathbf{h}_1\,.$$

3 Syzygy Resolutions via Resolving Decompositions

Let $\mathcal{P}^m_{\mathbf{d}_0}$ be a graded free polynomial module with standard basis $\{\mathbf{e}^{(0)}_1, \dots, \mathbf{e}^{(0)}_m\}$ and grading $\mathbf{d}_0 = (d^{(0)}_1, \dots d^{(0)}_m)$. Furthermore, let $(\mathcal{B}^{(0)}, \operatorname{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ be a resolving decomposition of a finitely generated graded module $U \subseteq \mathcal{P}^m_{\mathbf{d}_0}$ with $\mathcal{B}^{(0)} = \{\mathbf{h}_1, \dots, \mathbf{h}_{s_1}\}$. Our first goal is now to construct a resolving decomposition of the syzygy module $\operatorname{Syz}(\mathcal{B}^{(0)}) \subseteq \mathcal{P}^{s_1}$ which may be considered as a refined version of the well-known Schreyer theorem for Gröbner bases.

For every non-multiplicative variable x_k of a generator $\mathbf{h}_\alpha$, we have a standard representation $x_k\mathbf{h}_\alpha = \sum_{\beta=1}^{s_1} P^{(\alpha;k)}_\beta \mathbf{h}_\beta$ and thus a syzygy

$$\mathbf{S}_{\alpha;k} = x_k\mathbf{e}^{(1)}_\alpha - \sum_{\beta=1}^{s_1} P^{(\alpha;k)}_\beta \mathbf{e}^{(1)}_\beta \tag{3}$$

where $\{\mathbf{e}^{(1)}_1, \dots, \mathbf{e}^{(1)}_{s_1}\}$ denotes the standard basis of the free module $\mathcal{P}^{s_1}_{\mathbf{d}_1}$ with grading $\mathbf{d}_1 = (\deg(\mathbf{h}_1), \dots, \deg(\mathbf{h}_s))$. Let $\mathcal{B}^{(1)}$ be the set of all these syzygies.

Lemma 3. *Let $S = \sum_{l=1}^{s_1} S_l\mathbf{e}^{(1)}_l$ be an arbitrary syzygy of $\mathcal{B}^{(0)}$ with coefficients $S_l \in \mathcal{P}$. Then $S_l \in \Bbbk[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$ for all $1 \leq l \leq s_1$ if and only if $S = 0$.*

Proof. If $S \in \operatorname{Syz}(\mathcal{B}^{(0)})$, then $\sum_{l=1}^{s_1} S_l\mathbf{h}_l = 0$. Each $\mathbf{f} \in U$ can be uniquely written in the form $\mathbf{f} = \sum_{l=1}^{s_1} P_l\mathbf{h}_l$ with $\mathbf{h}_l \in \mathcal{B}^{(0)}$ and $P_l \in \Bbbk[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$. In particular, this holds for $0 \in U$. Thus $0 = S_l \in \Bbbk[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$ for all l and hence $S = 0$. □

For $\mathbf{h}_\alpha \in \mathcal{B}^{(0)}$ we denote the non-multiplicative variables by $\{x_{i^\alpha_1}, \dots, x_{i^\alpha_{r_\alpha}}\}$ with $i^\alpha_1 < \cdots < i^\alpha_{r_\alpha}$. Thus $\mathcal{B}^{(1)} = \cup_{j=1}^{s_1}\{\mathbf{S}_{j;i^j_k} \mid 1 \leq k \leq i^j_{r_j}\}$.

Theorem 1. *For every syzygy $S_{\alpha;i_k^\alpha} \in \mathcal{B}^{(1)}$ we set*

$$\mathrm{hm}(S_{\alpha;i_k^\alpha}) = x_{i_k^\alpha}\mathbf{e}_\alpha^{(1)}$$

and

$$X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha;i_k^\alpha}) = \{x_0,\dots x_n\} \setminus \{x_{i_1^\alpha},\dots,x_{i_{k-1}^\alpha}\}\,.$$

Furthermore, we define $\prec_{\mathcal{B}^{(1)}}$ as the Schreyer order associated to $\mathcal{B}^{(0)}$ and $\prec_{\mathcal{B}^{(0)}}$. Then the quadruple $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ is a resolving decomposition of the syzygy module $\mathrm{Syz}(\mathcal{B}^{(0)})$.

Proof. We first show that $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ is a resolving decomposition of $\langle\mathcal{B}^{(1)}\rangle$. In a second step, we show that $\langle\mathcal{B}^{(1)}\rangle = \mathrm{Syz}(\mathcal{B}^{(0)})$.

The first condition of Definition 1 is trivially satisfied. By construction it is obvious to see that

$$\mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle) = \bigoplus_{i=1}^{s_1}\langle\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_i)\rangle\mathbf{e}_i^{(1)}. \tag{4}$$

A term $x^\mu\mathbf{e}_l^{(1)} \in \mathrm{supp}(\mathbf{S}_{\alpha;k} - x_k\mathbf{e}_\alpha^{(1)})$ must satisfy by (3) that $x^\mu \in \Bbbk[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$ and hence $x^\mu\mathbf{e}_l^{(1)} \notin \mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle)$ which implies condition (ii). The first part of condition (iii) is again easy to see. It is obvious that

$$\langle\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)\rangle\mathbf{e}_\alpha^{(1)} = \bigoplus_{k=1}^{r_\alpha}\Bbbk[X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha,i_k^\alpha})]x_{i_k^\alpha}\mathbf{e}_\alpha^{(1)}.$$

If we combine this equation with (4) the first part of the third condition follows.

The second part of this condition is a bit harder to prove. We take an arbitrary $\mathbf{f} \in \langle\mathcal{B}^{(1)}\rangle$ and construct a standard representation for this module element. We construct this representation according to $\mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle)$. We take the biggest term $x^\mu\mathbf{e}_\alpha^{(1)} \in \mathrm{supp}(\mathbf{f}) \cap \mathrm{hm}(\mathcal{B}^{(1)})$ with respect to the order $\prec_{\mathcal{B}^{(0)}}$. There must be a syzygy $\mathbf{S}_{\alpha;i}$, such that $x_i \mid x^\mu$ and $x^\mu/x_i \in \Bbbk[X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha;i})]$. We reduce $\mathbf{f}$ by this element and get

$$\mathbf{f}' = \mathbf{f} - c\frac{x^\mu}{x_i}\mathbf{S}_{\alpha;i}$$

for a suitable constant $c \in \Bbbk$ such that the term $x^\mu\mathbf{e}_\alpha^{(1)}$ is no longer in the support of $\mathbf{f}'$. Every term $x^\lambda\mathbf{e}_\beta^{(1)}$ newly introduced by $\frac{x^\mu}{x_i}\mathbf{S}_{\alpha;i}$ which also lies in $\mathrm{hm}(\mathcal{B}^{(1)})$ is strictly less than $x^\mu\mathbf{e}_\alpha^{(1)}$ according to condition (v) of Definition 1 and Eq. (3) defining the syzygies $\mathbf{S}_{\alpha;i}$. Now we repeat this procedure until we arrive at an $\mathbf{f}''$ such that $\mathrm{supp}(\mathbf{f}'') \cap \mathrm{hm}\left(\langle\mathcal{B}^{(1)}\rangle\right) = \emptyset$. It is clear that we reach such an $\mathbf{f}''$ in a finite number of steps, since the terms during the reduction decrease with respect to $\prec_{\mathcal{B}^{(0)}}$ which is a well-order. We know that now all $x^\epsilon\mathbf{e}_\alpha^{(1)} \in \mathrm{supp}(\mathbf{f}'')$ have the property that $x^\epsilon \in X_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$. Therefore we get that $\mathbf{f}'' = 0$ due to Lemma 3, which finishes the proof of this condition.

The above procedure provides us with an algorithm to compute arbitrary normal forms and hence condition (iv) of Definition 1 follows immediately.

For the last condition we note that now each head term $x_i\mathbf{e}_\alpha^{(1)}$ is actually the leading term of $\mathbf{S}_{\alpha;i}$ with respect to the order $\prec_{\mathcal{B}^{(0)}}$. Hence the corresponding Schreyer order satisfies the last condition of Definition 1. $\square$

As with the usual Schreyer theorem, we can iterate this construction and derive this way a free resolution of U. By contrast to the classical situation, it is however now possible to make precise statements about the *shape* of the resolution (even if we do not obtain explicit formulae for the differentials).

Theorem 2. *Let $\beta_{0,j}^{(k)}$ be the number of generators $\mathbf{h} \in \mathcal{B}^{(0)}$ of degree j having k multiplicative variables and set $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$. Then U possesses a finite free resolution*

$$0 \to \bigoplus \mathcal{P}(-j)^{r_{n+1-d,j}} \to \cdots \to \bigoplus \mathcal{P}(-j)^{r_{1,j}} \to \bigoplus \mathcal{P}(-j)^{r_{0,j}} \to U \to 0 \quad (5)$$

of length $n+1-d$ where the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=1}^{n+1-i} \binom{n+1-k}{i} \beta_{0,j-i}^{(k)}.$$

Proof. According to Theorem 1, $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ is a resolving decomposition for the module $\mathrm{Syz}_1(U)$. Applying the theorem again, we can construct a resolving decomposition of the second syzygy module $\mathrm{Syz}_2(U)$ and so on. Recall that for every index $1 \leq l \leq m$ and for every non-multiplicative variable $x_k \in \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_{\alpha(l)})$ we have $|\overline{X}_{\mathcal{B}^{(1)}}(S_{l;k})| < |\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_{\alpha(l)})|$.

If D is the minimal number of multiplicative variables for a head module term in $\mathcal{B}^{(0)}$, then the minimal number of multiplicative variables for a head term in $\mathcal{B}^{(1)}$ is $D+1$. This observation yields the length of the resolution (5). Furthermore $\deg(\mathbf{S}_{k;i}) = \deg(\mathbf{h}_k) + 1$, e. g. from the jth to the $(j+1)$th module the degree from the basis element to the corresponding syzygies grows by one.

The ranks of the modules follow from a rather straightforward combinatorial calculation. Let $\beta_{i,j}^{(k)}$ denote the number of generators of degree j of the i-th syzygy module $\mathrm{Syz}_i(U)$ with k multiplicative variables according to the head module terms. By definition of the generators, we find

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-1} \beta_{i-1,j-1}^{(n+1-t)}$$

as each generator with less multiplicative variables and degree $j-1$ in the resolving decomposition of $\mathrm{Syz}_i(\mathcal{B}^{(0)})$ contributes one generator with k multiplicative variables. A lengthy induction allows us to express $\beta_{i,j}^{(k)}$ in terms of $\beta_{0,j}^{(k)}$:

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-i} \binom{k-l-1}{i-1} \beta_{0,j-i}^{(t)}.$$

Now we are able to compute the ranks of the free modules via

$$r_{i,j} = \sum_{k=1}^{n+1} \beta_{i,j}^{(k)} = \sum_{k=1}^{n+1} \sum_{t=1}^{k-i} \binom{k-t-1}{i-1} \beta_{0,j-i}^{(t)} = \sum_{k=1}^{n+1-i} \binom{n+1-k}{i} \beta_{0,j-i}^{(k)}.$$

The last equality follows from a classical identity for binomial coefficients. □

Theorem 2 allows us to construct recursively resolving decompositions for the higher syzygy modules. In the sequel, we denote the corresponding resolving decomposition of the syzygy module $\mathrm{Syz}_j(U)$ by $(\mathcal{B}^{(j)}, \mathrm{hm}(\mathcal{B}^{(j)}), X_{\mathcal{B}^{(j)}}, \prec_{\mathcal{B}^{(j)}})$. To define an element of $\mathcal{B}^{(j)}$, we consider for each generator $\mathbf{h}_\alpha \in \mathcal{B}^{(0)}$ all ordered integer sequences $\mathbf{k} = (k_1, \dots, k_j)$ with $0 \le k_1 < \cdots < k_j \le n$ of length $|\mathbf{k}| = j$ such that $x_{k_i} \in \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ for all $1 \le i \le j$. We denote for any $1 \le i \le j$ by $\mathbf{k}_i$ the sequence obtained by eliminating k_i from $\mathbf{k}$. Then the generator $\mathbf{S}_{\alpha;\mathbf{k}}$ arises recursively from the standard representation of $x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j}$ according to the resolving decomposition $(\mathcal{B}^{(j-1)}, \mathrm{hm}(\mathcal{B}^{(j-1)}), X_{\mathcal{B}^{(j-1)}}, \prec_{\mathcal{B}^{(j-1)}})$:

$$x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j} = \sum_{\beta=1}^{s_1} \sum_{\mathbf{l}} P_{\beta;\mathbf{l}}^{(\alpha;\mathbf{k})} \mathbf{S}_{\beta;\mathbf{l}}. \tag{6}$$

The second sum is over all ordered integer sequences $\mathbf{l}$ of length $j-1$ such that for all entries ℓ_i the variables x_{ℓ_i} is non-multiplicative for the generator $\mathbf{h}_\beta \in \mathcal{B}^{(0)}$. Denoting the free generators of the free module which contains the jth syzygy module by $\mathbf{e}_{\alpha,\mathbf{l}}^{(j)}$, such that $\alpha \in \{1, \dots, s_1\}$ and $\mathbf{l}$ is an ordered subset of $\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ of length $j-1$ we get the following representation for $\mathbf{S}_{\alpha,\mathbf{k}}$:

$$\mathbf{S}_{\alpha;\mathbf{k}} = x_{k_j}\mathbf{e}_{\alpha;\mathbf{k}_j}^{(j)} - \sum_{\beta=1}^{s_1} \sum_{\mathbf{l}} P_{\beta;\mathbf{l}}^{(\alpha;\mathbf{k})} \mathbf{e}_{\beta;\mathbf{l}}^{(j)}.$$

Corollary 1. *In the situation of Theorem 2, set* $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$ *and* $q = \deg(\mathcal{B}^{(0)}) = \max\{\deg(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}^{(0)}\}$. *Then we obtain the following bounds for the projective dimension, the Castelnuovo-Mumford regularity and the depth, respectively, of the submodule* U*:*

$$\mathrm{pd}(U) \le n+1-d\,, \qquad \mathrm{reg}(U) \le q\,, \qquad \mathrm{depth}(U) \ge d\,.$$

Proof. The first estimate follows immediately from the resolution (5) induced by the resolving decomposition $(\mathcal{B}^{(0)}, \mathrm{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ of U. The last estimate is a simple consequence of the first one and the graded form of the Auslander-Buchsbaum formula. Finally, the ith module of this resolution is obviously generated by elements of degree less than or equal to $q+i$. This observation implies that U is q-regular and thus the second estimate. □

Remark 2. The resolving decomposition $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ constructed in Theorem 1 is always a Janet basis of the first syzygy module with respect to the term order $\prec_{\mathcal{B}^{(0)}}$. This is simply due to the fact that the choice of the multiplicative variables in the resolving decomposition of the syzygy module made in Theorem 1 is actually inspired by what happens for the Janet division. Hence in the special case that the resolving decomposition is induced by a Pommaret or a Janet basis, it is easy to see that also the resolving decompositions of the higher syzygy modules are actually induced by Pommaret or Janet bases for a Schreyer order constructed as in Theorem 1. Since a Janet basis which only consists of variables is simultaneously an involutive basis for the alex division (see [5] for the definition), the same is true for resolving decompositions induced by alex bases.

At this point, one can also see some advantages of our general framework. Our previous results require that the used involutive division is of Schreyer type. This assumption ensures that we obtain at each step again an L-involutive basis for the syzygy module with respect to a Schreyer order. In our new approach, we automatically obtain Janet basis, as we can choose the head terms and the multiplicative variables as we like. Consequently, we can now use an involutive basis $\mathcal{B}$ for an arbitrary involutive division L as starting point for the construction of a resolution, provided its L-graph is acyclic (which is always the case if L is continuous). The construction will not necessarily lead to L-involutive bases of the syzygy modules, but for most applications this fact is irrelevant.

4 An Explicit Formula for the Differential

As in Sect. 3 let $\mathcal{P}^m_{\mathbf{d}_0}$ be a graded free module with free generators $\mathbf{e}^{(0)}_1, \dots \mathbf{e}^{(0)}_m$ and grading $\mathbf{d}_0 = (d^{(0)}_1, \dots d^{(0)}_m)$. We always work with a finitely generated graded module $U \in \mathcal{P}^m_{\mathbf{d}_0}$ with a resolving decomposition $(\mathcal{B}^{(0)}, \mathrm{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ where $\mathcal{B}^{(0)} = \{\mathbf{h}_1, \dots, \mathbf{h}_{s_1}\}$.

First we give an alternative description of the complex underlying the resolution (5). Let $\mathcal{W} = \bigoplus_{\alpha=1}^{s_1} \mathcal{P}\mathbf{w}_\alpha$ and $\mathcal{V} = \bigoplus_{i=0}^{n} \mathcal{P}\mathbf{v}_i$ be two free $\mathcal{P}$-modules whose ranks are given by the size of the resolving decomposition $(\mathcal{B}^{(0)}, \mathrm{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ and by the number of variables in $\mathcal{P}$, respectively. Then we set $\mathcal{C}_i = \mathcal{W} \otimes_{\mathcal{P}} \Lambda_i \mathcal{V}$ where $\Lambda_\bullet$ denotes the exterior product. A $\mathcal{P}$-linear basis of $\mathcal{C}_i$ is provided by the elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ where $\mathbf{v}_\mathbf{k} = \mathbf{v}_{k_1} \wedge \cdots \wedge \mathbf{v}_{k_i}$ for an ordered sequence $\mathbf{k} = (k_1, \dots, k_i)$ with $0 \le k_1 < \cdots < k_i \le n$. Then the free subcomplex $\mathcal{S}_\bullet \subset \mathcal{C}_\bullet$ generated by all elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ with $\mathbf{k} \subseteq \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ corresponds to (5) upon the identification $\mathbf{e}^{(i+1)}_{\alpha;\mathbf{k}} \leftrightarrow \mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$. Let $k_{i+1} \in \overline{X}_{\mathcal{B}^0}(\mathbf{h}_\alpha) \setminus \mathbf{k}$, then the differential comes from (6),

$$d_{\mathcal{S}}(\mathbf{w}_\alpha \otimes \mathbf{v}_{\mathbf{k},k_{i+1}}) = x_{k_{i+1}} \mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k} - \sum_{\beta,\mathbf{l}} P^{(\alpha;\mathbf{k},k_{i+1})}_{\beta;\mathbf{l}} \mathbf{w}_\beta \otimes \mathbf{v}_\mathbf{l} \,,$$

and thus requires the explicit determination of all the higher syzygies (6).

In this section we present a method to directly compute the differential without computing higher syzygies. It is based on ideas of Sköldberg [9,10] and generalises the theory which we developed in [1,3] for the special case of a resolution induced by a Pommaret or a Janet basis for a given term order.

Definition 2. *A graded polynomial module U has* head linear syzygies*, if it possesses a finite presentation*

$$0 \longrightarrow \ker\eta \longrightarrow \mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \xrightarrow{\eta} U \longrightarrow 0 \tag{7}$$

with a finite generating set $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_t\}$ of $\ker\eta$ *where one can choose for each generator $\mathbf{h}_\alpha \in \mathcal{H}$ a head module term* $\operatorname{hm}(\mathbf{h}_\alpha)$ *of the form $x_i\mathbf{w}_\alpha$.*

Sköldberg's construction begins with the following two-sided Koszul complex $(\mathcal{F}, d_\mathcal{F})$ defining a free resolution of U. Let $\mathcal{V}$ be a $\Bbbk$-linear space with basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ (with $n+1$ still the number of variables in $\mathcal{P}$) and set $\mathcal{F}_j = \mathcal{P} \otimes \Lambda_j\mathcal{V} \otimes \ U$ which obviously yields a free $\mathcal{P}$-module. Choosing a $\Bbbk$-linear basis $\{m_a \mid a \in A\}$ of U, a $\mathcal{P}$-linear basis of $\mathcal{F}_j$ is given by the elements $1 \otimes v_\mathbf{k} \otimes m_a$ with ordered sequences $\mathbf{k}$ of length j. The differential is now defined by

$$d_\mathcal{F}(1 \otimes \mathbf{v}_\mathbf{k} \otimes m_a) = \sum_{i=1}^{j} (-1)^{i+1}\big(x_{k_i} \otimes \mathbf{v}_{\mathbf{k}_i} \otimes m_a - 1 \otimes \mathbf{v}_{\mathbf{k}_i} \otimes x_{k_i}m_a\big). \tag{8}$$

Here it should be noted that the second term on the right hand side is not yet expressed in the chosen $\Bbbk$-linear basis of U. For notational simplicity, we will drop in the sequel the tensor sign $\otimes$ and leading factors 1 when writing elements of $\mathcal{F}_\bullet$.

Sköldberg uses a specialisation of head linear terms. He requires that for a given term order $\prec$ the leading module of $\ker\eta$ in the presentation (7) must be generated by terms of the form $x_i\mathbf{w}_\alpha$. In this case he says that U has *initially linear syzygies*. Our definition is term order free.

Under the assumption that the module U has initially linear syzygies via a presentation (7), Sköldberg [10] constructs a Morse matching leading to a smaller resolution $(\mathcal{G}, d_\mathcal{G})$. He calls the variables

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j\mathbf{w}_\alpha \in \operatorname{lt}\ \ker\eta\}\,;$$

critical for the generator $\mathbf{w}_\alpha$; the remaining *non-critical* ones are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$. Then a $\Bbbk$-linear basis of U is given by all elements $x^\mu\mathbf{h}_\alpha$ with $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$ and $x^\mu \in \Bbbk[\operatorname{ncrit}(\mathbf{w}_\alpha)]$.

According to [9] we define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as the free submodule generated by those vertices $\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha$ where the ordered sequences $\mathbf{k}$ are of length j and such that every entry k_i is critical for $\mathbf{w}_\alpha$. In particular $\mathcal{W} \cong \mathcal{G}_0$ with an isomorphism induced by $\mathbf{w}_\alpha \mapsto \mathbf{v}_\emptyset\mathbf{h}_\alpha$.

The description of the differential $d_\mathcal{G}$ is based on reduction paths in the associated Morse graph (for a detailed treatment of these notions, see [1,9] or [6]) and expresses the differential as a triple sum. If we assume that, after expanding the right hand side of (8) in the chosen $\Bbbk$-linear basis of U, the differential of the complex $\mathcal{F}_\bullet$ can be expressed as

$$d_\mathcal{F}(\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha) = \sum_{\mathbf{m},\mu,\gamma} Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_\mathbf{m}(x^\mu\mathbf{h}_\gamma)\,,$$

then $d_{\mathcal{G}}$ is defined by

$$d_{\mathcal{G}}(\mathbf{v_k h}_\alpha) = \sum_{\mathbf{l},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \rho_p\big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma} \mathbf{v_m}(x^\mu \mathbf{h}_\gamma)\big) \tag{9}$$

where the first sum ranges over all ordered sequences $\mathbf{l}$ which consists entirely of critical indices for $\mathbf{w}_\beta$. Moreover the second sum may be restricted to all values such that a polynomial multiple of $\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)$ effectively appears in $d_{\mathcal{F}}(\mathbf{v_k h}_\alpha)$ and the third sum ranges over all reduction paths p going from $\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)$ to $\mathbf{v_l h}_\beta$. Finally ρ_p is the reduction associated with the reduction path p satisfying

$$\rho_p\big(\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)\big) = q_p \mathbf{v_l h}_\beta$$

for some polynomial $q_p \in \mathcal{P}$.

It turns out that Sköldberg uses the term order $\prec$ only for distinguishing the critical and non-critical variables. Therefore it is straightforward to see that his construction also works for modules which have head linear syzygies. We simply replace the definition of critical and non-critical variables. We define

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j \mathbf{w}_\alpha \in \operatorname{hm}(\mathcal{H})\},$$

where $\mathcal{H}$ is chosen as in Definition 2. Again the remaining variables are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$.

In the sequel we will show that for a finitely generated graded module U with resolving decomposition $(\mathcal{B}^{(0)}, \operatorname{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ the resolution constructed by Sköldberg's method is isomorphic to the resolution which is induced by the resolving decomposition if we choose the head linear syzygies properly. Firstly we obtain the following trivial assertion.

Lemma 4. *If the graded submodule $U \subseteq \mathcal{P}^{s_1}_{\mathbf{d}_0}$ possesses a resolving decomposition $(\mathcal{B}^{(0)}, \operatorname{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$, then it has head linear syzygies. More precisely, we can set $\operatorname{crit}(\mathbf{w}_\alpha) = \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$, i.e. the critical variables of the generator $\mathbf{w}_\alpha$ are the non-multiplicative variables of $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$.*

The lemmata which we subsequently cite from [1] are formulated for a Pommaret basis, which is an involutive basis. Nevertheless we can apply them directly in our setting, if not stated otherwise, because their proofs remain applicable for resolving decompositions. The reason for this is that they only need the existence of unique standard representations and the division of variables into multiplicative and non-multiplicative ones. Some of the proofs in [1] explicitly use the class of a generator in $\mathcal{B}^{(0)}$, a notion arising in the context of Pommaret bases. When working with resolving decompositions, one has to replace it by the maximal index of a multiplicative variable.

The reduction paths can be divided into elementary ones of length two. There are essentially three types of reductions paths [1, Sect. 4]. The elementary reductions of *type 0* are not of interest [1, Lemma 4.5]. All other elementary reductions paths are of the form

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\Big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\Big) \longrightarrow \mathbf{v_l}(x^\nu \mathbf{h}_\beta).$$

Here $\mathbf{k}\cup i$ is the ordered sequence which arises when i is inserted into $\mathbf{k}$; likewise $\mathbf{k}\setminus i$ stands for the removal of an index $i\in\mathbf{k}$.

Type 1: Here $\mathbf{l}=(\mathbf{k}\cup i)\setminus j$, $x^\nu=\frac{x^\mu}{x_i}$ and $\beta=\alpha$. Note that $i=j$ is allowed. We define $\epsilon(i;\mathbf{k})=(-1)^{|\{j\in\mathbf{k}\mid j>i\}|}$. Then the corresponding reduction is

$$\rho(\mathbf{v_k}x^\mu\mathbf{h}_\alpha)=\epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)x_j\mathbf{v}_{(\mathbf{k}\cup i)\setminus j}\big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\big)\,.$$

Type 2: Now $\mathbf{l}=(\mathbf{k}\cup i)\setminus j$ and $x^\nu\mathbf{h}_\beta$ appears in the involutive standard representation of $\frac{x^\mu x_j}{x_i}\mathbf{h}_\alpha$ with the coefficient $\lambda_{j,i,\alpha,\mu,\nu,\beta}\in\Bbbk$. In this case, by construction of the Morse matching, we have $i\neq j$. The reduction is

$$\rho(\mathbf{v_k}x^\mu\mathbf{h}_\alpha)=-\epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)\lambda_{j,i,\alpha,\mu,\nu,\beta}\mathbf{v}_{(\mathbf{k}\cup i)\setminus j}(x^\nu\mathbf{h}_\beta)\,.$$

These reductions follow from the differential (8): The summands appearing there are either of the form $x_{k_i}\mathbf{v}_{\mathbf{k}_i}m_a$ or of the form $\mathbf{v}_{\mathbf{k}_i}(x_{k_i}m_a)$. For each of these summands, we have a directed edge in the Morse graph $\Gamma^A_{\mathcal{F}_\bullet}$. Thus for an elementary reduction path

$$\mathbf{v_k}(x^\mu\mathbf{h}_\alpha)\longrightarrow\mathbf{v}_{\mathbf{k}\cup i}\big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\big)\longrightarrow\mathbf{v_l}(x^\nu\mathbf{h}_\beta)\,,$$

the second edge can originate from summands of either form. For the first form we then have an elementary reduction path of type 1 and for the second form we have type 2.

To show that the resolution induced by a resolving decomposition is isomorphic to the resolution constructed via Sköldberg's method we need a classical theorem concerning the uniqueness of free resolutions.

Theorem 3. [4, Theorem 1.6] *Let U be a finitely generated graded $\mathcal{P}^m_{\mathbf{d}}$-module. If $\mathcal{F}$ is the graded minimal free resolution of U and $\mathcal{G}$ an arbitrary graded free resolution of U, then $\mathcal{G}$ is isomorphic to the direct sum of $\mathcal{F}$ and a trivial complex.*

Assume that we have two graded free resolutions $\mathcal{F}$, $\mathcal{G}$ of the same module U with the same shape (which means that the homogeneous components of the free modules in the two resolutions have always the same dimensions: $\dim\ (\mathcal{F}_i)_j=\dim\ (\mathcal{G}_i)_j$). Then Theorem 3 implies that the two resolutions are isomorphic. For the next theorem, we note the following important observation. The bases of the free modules in the resolution $\mathcal{G}$ of Sköldberg are given by the generators $\mathbf{v_k}\mathbf{h}_\alpha$ with $\mathbf{k}\subseteq\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$.

Theorem 4. *Let $\mathcal{F}$ be the graded free resolution which is induced by the resolving decomposition $(\mathcal{B}^{(0)},\operatorname{hm}(\mathcal{B}^{(0)}),X_{\mathcal{B}^{(0)}},\prec_{\mathcal{B}^{(0)}})$ and $\mathcal{G}$ the graded free resolution which is constructed by the method of Sköldberg when the head linear syzygies are chosen such that $\operatorname{crit}(\mathbf{h}_\alpha)=\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ for every $\mathbf{h}_\alpha\in\mathcal{B}^{(0)}$. Then the resolutions $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. According to the observation made above, it is obvious that the two resolutions $\mathcal{F}$ and $\mathcal{G}$ have the same shape. Together with Theorem 3, the claim follows then immediately. □

For completeness, we repeat some simple results from [1]. They will show us, that the differentials of both resolutions are very similar. In fact we show for the resolution constructed via Sköldberg's method, that we can find head module terms in the higher syzygies which are equal to the head module terms of the resolving decompositions of the higher syzygies of the induced free resolution.

Lemma 5. [1, Lemma 4.3] *For a non-multiplicative index*[1] $i \in \operatorname{crit}(\mathbf{h}_\alpha)$ *let* $x_i\mathbf{h}_\alpha = \sum_{\beta=1}^{s_1} P_\beta^{(\alpha;i)}\mathbf{h}_\beta$ *be the standard representation. Then we have* $d_{\mathcal{G}}(\mathbf{v}_i\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset\mathbf{h}_\alpha - \sum_{\beta=1}^{s_1} P_\beta^{(\alpha;i)}\mathbf{v}_\emptyset\mathbf{h}_\beta$.

The next result states that if one starts at a vertex $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha)$ with certain properties and follows through all possible reduction paths in the graph, one will never get to a point where one must calculate an involutive standard representation. If there are no critical (i. e. non-multiplicative) variables present at the starting point, then this will not change throughout any reduction path. In order to generalise this lemma to higher homological degrees, one must simply replace the conditions $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ by ordered sequences $\mathbf{k}, \mathbf{l}$ with $\mathbf{k} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\mathbf{l} \subseteq \operatorname{ncrit}(\mathbf{h}_\beta)$.

Lemma 6. [1, Lemma 4.4] *Assume that* $i \cup \operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$. *Then for any reduction path* $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu\mathbf{h}_\beta)$ *we have* $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$. *In particular, in this situation there is no reduction path* $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_k\mathbf{h}_\beta$ *with* $k \in \operatorname{crit}(\mathbf{h}_\beta)$.

The next corollary asserts that we can choose in Sköldberg's resolution head module terms in such a way that there is a one-to-one correspondence to the head terms of the syzygies contained in the free resolution induced by the resolving decomposition. This corollary is a direct consequence of Lemma 6.

Corollary 2. *Let* $(k_1, \ldots, k_j) = \mathbf{k} \subseteq \operatorname{crit}\mathbf{h}_\alpha$, *then*

$$x_{k_l}\mathbf{v}_{\mathbf{k}\setminus k_l}\mathbf{h}_\alpha \in \operatorname{supp}(d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha)).$$

In [1,3] we show a method to effectively compute graded Betti numbers via the induced free resolution of Janet and Pommaret bases and the method of Sköldberg. We show that we can compute the graded Betti numbers with computing only the constant part of the resolution. With this method it is also possible to compute only a single Betti number without compute the complete constant part of the free resolution. The reason for that is that Sköldberg's formula allows to compute a differential in the free resolution independently of the rest of the free resolution. Furthermore the theorem about the induced free resolution gives us a formula to compute the ranks of these resolution. These methods are also applicable for an arbitrary resolving decomposition due to the fact that we proved Theorem 2 and the form of the differential (9).

[1] For notational simplicity, we will identify sets X of variables with sets of the corresponding indices and thus simply write $i \in X$ instead of $x_i \in X$.

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