

Chapter 2

The Poisson Integral

This chapter deals with harmonic Hardy spaces h^p , $1 \leq p \leq \infty$, in the unit disk \mathbb{D} . We review the Poisson and Poisson-Stieltjes integrals, the nontangential limits, and the maximal functions.

Since this chapter is intended as a preparation for the study of Hardy spaces H^p , $0 < p \leq \infty$, of analytic functions in \mathbb{D} , which is our main purpose, our presentation is somewhat brief. In some cases proofs are omitted. For the missing proofs and other information the reader is referred to the classical sources such as [1–3], or [4].

The reader should be aware that much of the material presented in this chapter and in the following one extends to more general settings (see [5–10]).

2.1 The Poisson Integral of a Continuous Function

The Disk Automorphisms. We first introduce a class of simple maps which is extremely useful in function theory: the group of disk automorphisms.

By a *disk automorphism* we shall mean a bijective holomorphic map of \mathbb{D} onto itself. These transformations form a group with respect to the composition of mappings, called the *automorphism group* and denoted by $Aut(\mathbb{D})$.

For an arbitrary point $a \in \mathbb{D}$, the following special Möbius transformation defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D},$$

is readily verified to be a disk automorphism which exchanges the points 0 and a . It is not difficult to verify the standard properties:

$$\varphi'_a(z) = -\frac{1 - |a|^2}{(1 - \bar{a}z)^2}, \quad 1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}. \quad (2.1)$$

The second formula readily shows that φ_a maps the disk into itself.

Moreover, φ_a is an *involution* in the sense that $\varphi_a(\varphi_a(z)) = z$ for all z in the disk. This property immediately tells us that φ_a maps \mathbb{D} onto itself.

As is well known, every disk automorphism can be represented in the form $\varphi(z) = e^{i\alpha} \varphi_a(z)$, for some $\alpha \in \mathbb{R}$ and some $a \in \mathbb{D}$.

Green's Formula. We first state the following variant of Green's theorem; see [11, Chapter 1].

Theorem 2.A (Green's formula) *If Ω is a finitely connected domain in the plane, bounded by analytic Jordan curves, then*

$$\int_{\Omega} (u \Delta v - v \Delta u) dm = \int_{\partial \Omega} \left(u(z) \frac{\partial v}{\partial n}(z) - v(z) \frac{\partial u}{\partial n}(z) \right) |dz|, \quad u, v \in C^2(\bar{\Omega}),$$

where $\frac{\partial}{\partial n}$ is the outer normal derivative.

In particular, if F is a C^2 function on $\mathbb{D}_R(0)$, taking $u = 1$ and $v = F$ we obtain

$$\frac{d}{dr} \left(\frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) d\theta \right) = \frac{1}{2\pi r} \int_{|z| < r} \Delta F(z) dm(z). \quad (2.2)$$

Now if we integrate (2.2), we obtain

$$\int_0^r \frac{d}{d\rho} \left(\frac{1}{2\pi} \int_0^{2\pi} F(\rho e^{i\theta}) d\theta \right) d\rho = \int_0^r \left(\frac{1}{2\pi \rho} \int_{|z| < \rho} \Delta F(z) dm(z) \right) d\rho$$

and consequently by using Fubini's theorem, we find that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) d\theta - F(0) &= \frac{1}{2\pi} \int_{\mathbb{D}} \Delta F(z) dm(z) \int_0^r \chi_{\mathbb{D}_\rho(0)}(z) \frac{d\rho}{\rho} \\ &= \frac{1}{2\pi} \int_{|z| < r} \Delta F(z) \log \frac{r}{|z|} dm(z), \end{aligned} \quad (2.3)$$

where $0 < r < R$.

Harmonic Functions. For a complex-valued function u , defined on a domain $\Omega \subset \mathbb{C}$ and of class C^2 there, its *Laplacian* is defined in the usual way:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}, \quad \text{where } z = x + iy.$$

A function u is said to be *harmonic* in Ω if $\Delta u = 0$ in Ω .

The space $h(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ of all functions harmonic on \mathbb{D} and continuous on $\bar{\mathbb{D}}$, will be denoted by hC . The norm on hC is defined by $\|u\|_\infty = \|u\|_{hC} = \sup_{z \in \mathbb{D}} |u(z)|$.

It should be noted that every function $u \in hC$ has the *mean value property*:

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta. \quad (2.4)$$

This is an easy consequence of (2.3).

The Poisson Kernel and Integral Formula. By applying (2.4) to the function $u \circ \varphi_a \in hC$, where φ_a is a disk automorphism as introduced at the beginning of this chapter, we get

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\varphi_a(e^{i\theta})) d\theta.$$

Now we introduce the substitution $e^{it} = \varphi_a(e^{i\theta})$ (equivalently, $e^{i\theta} = \varphi_a(e^{it})$). Since $d\theta = \frac{1 - |a|^2}{|1 - \bar{a}e^{it}|^2} dt$, we have

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|e^{it} - a|^2} u(e^{it}) dt. \quad (2.5)$$

This is the *Poisson integral formula*. The function

$$P_a(e^{i\theta}) := P_a(\theta) = \frac{1 - |a|^2}{|e^{i\theta} - a|^2}$$

is called the *Poisson kernel* for the point $a \in \mathbb{D}$. Note that $P_{re^{it}}(\theta) = P_{re^{i\theta}}(t)$.

It is also useful to note that, as a special case of (2.5), we get

$$\frac{1}{2\pi} \int_0^{2\pi} P_z(t) dt = 1, \quad z \in \mathbb{D}. \quad (2.6)$$

Solution to the Dirichlet Problem. It is well known that the Poisson integral can be used to solve explicitly the *Dirichlet problem* with continuous boundary data for the disk \mathbb{D} . We recall the precise statement and include a proof.

Theorem 2.1.1 *Let φ be continuous on \mathbb{T} and define u (the Poisson integral of φ) on \mathbb{D} by*

$$u(z) = P[\varphi](z) := \frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) \varphi(e^{i\theta}) d\theta \quad \text{for } z \in \mathbb{D}$$

and $u(z) = \varphi(z)$ for $z \in \mathbb{T}$. Then u is continuous on $\overline{\mathbb{D}}$ and harmonic in \mathbb{D} .

Proof Notice that the Poisson kernel $P_z(\theta)$ also has the form

$$P_z(\theta) = \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z},$$

so that, for $e^{i\theta}$ fixed, $P_z(\theta)$ is a harmonic function of $z \in \mathbb{D}$. Hence the function u is harmonic in \mathbb{D} .

To prove that $u \in C(\overline{\mathbb{D}})$, fix a point $e^{i\theta}$ on the unit circle and $\varepsilon > 0$. Choose $\delta > 0$ so that $|\varphi(e^{it}) - \varphi(e^{i\theta})| < \varepsilon$ whenever $|t - \theta| < \delta$. For $z \in \mathbb{D}$, using (2.6) and some obvious estimates, we have

$$\begin{aligned} |u(z) - u(e^{i\theta})| &= \left| \frac{1}{2\pi} \int_0^{2\pi} P_z(t)(\varphi(e^{it}) - \varphi(e^{i\theta}))dt \right| \\ &\leq \frac{1}{2\pi} \int_{|t-\theta| \leq \delta} P_z(t)|\varphi(e^{it}) - \varphi(e^{i\theta})|dt + \\ &\quad \frac{1}{2\pi} \int_{|t-\theta| > \delta} P_z(t)|\varphi(e^{it}) - \varphi(e^{i\theta})|dt \\ &\leq \varepsilon + \frac{\max_{e^{it} \in \mathbb{T}} |\varphi(e^{it})|}{\pi} \int_{|t-\theta| > \delta} P_z(t)dt. \end{aligned}$$

The last term tends to 0 when $z \rightarrow e^{i\theta}$ proving that u is continuous at $e^{i\theta}$. \square

As was shown above, (2.5) holds for functions $u \in hC$. Therefore the solution of the Dirichlet problem for the unit disk given in Theorem 2.1.1 is unique.

Theorem 2.1.1 and the maximum principle for harmonic functions (see Proposition 3.2.3 for a more general statement) yield the following:

Theorem 2.1.2 *The Poisson integral acts as an isometric isomorphism from $C(\mathbb{T})$ onto hC .*

As is usual, by $C(\mathbb{T})$ we denote the space of all continuous functions on \mathbb{T} equipped with the standard norm:

$$\|\varphi\|_\infty := \|\varphi\|_{C(\mathbb{T})} = \max_{\xi \in \mathbb{T}} |\varphi(\xi)|.$$

2.2 Borel Measures and the Space h^1

2.2.1 The Poisson Integral of a Measure

For $e^{i\theta}$ fixed, $P_z(\theta)$ is a harmonic function of $z \in \mathbb{D}$, hence the function defined by

$$P[u](z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(\theta)u(e^{i\theta})d\theta \quad (2.7)$$

is harmonic in \mathbb{D} whenever $u \in L^1(\mathbb{T})$.

Since $P_z(\theta)$ is also continuous function of θ , we obtain a harmonic function from (2.7) if we replace $u(e^{i\theta})d\theta$ by a complex measure $d\mu(e^{i\theta})$ on \mathbb{T} . This function, the *Poisson integral of the measure μ* , will be written as follows

$$P[\mu](z) = \frac{1}{2\pi} \int_{\mathbb{T}} P_z(e^{i\theta}) d\mu(e^{i\theta}). \quad (2.8)$$

Let $M(\mathbb{T})$ denote the space of all complex Borel measures on \mathbb{T} , with the norm given by $\|\mu\| = \frac{1}{2\pi} |\mu|(\mathbb{T})$. Here, as before, $|\mu|$ denotes the total variation of the measure μ (see [12, Chapter 6]).

Recall that $M(\mathbb{T})$ becomes a Banach space when equipped with the total variation norm. By the Riesz representation theorem, if we identify $\mu \in M(\mathbb{T})$ with the linear functional Λ_μ on $C(\mathbb{T})$ given by $\Lambda_\mu(f) = \frac{1}{2\pi} \int_{\mathbb{T}} f d\mu$, then $M(\mathbb{T})$ is isometrically isomorphic to the dual space of $C(\mathbb{T})$. We write $C(\mathbb{T})' = M(\mathbb{T})$.

Proposition 2.2.1 *The Poisson integral is an injective linear operator from $M(\mathbb{T})$ into $h(\mathbb{D})$.*

Proof Obviously the Poisson integral is a linear operator on $M(\mathbb{T})$. To prove the injectivity assume $P[\mu] = 0$. If ϕ is continuous on \mathbb{T} , then Fubini's theorem and identity $P_{re^{it}}(\theta) = P_{re^{i\theta}}(t)$ give

$$\begin{aligned} \int_{\mathbb{T}} \phi d\mu &= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{\mathbb{T}} \left(\int_{-\pi}^{\pi} P_{re^{i\theta}}(t) \phi(e^{it}) dt \right) d\mu(e^{i\theta}) \\ &= \lim_{r \rightarrow 1} \int_{-\pi}^{\pi} \phi(e^{it}) \frac{1}{2\pi} \int_{\mathbb{T}} P_{re^{it}}(\theta) d\mu(e^{i\theta}) d\theta \\ &= \lim_{r \rightarrow 1} \int_{-\pi}^{\pi} \phi(e^{it}) P[\mu](re^{it}) dt = 0. \end{aligned}$$

Thus, $\int_{\mathbb{T}} \phi d\mu = 0$ for all continuous ϕ and the Riesz representation theorem gives $\mu = 0$. \square

2.2.2 The Banach-Alaoglu Theorem

The Weak-* Topology. We recall briefly the concept of weak-* topology. Let X be a Banach space and X' its (complex) dual space, that is, the space of all continuous linear functionals on X . Obviously, X' is equipped with the natural norm induced by the norms of functionals. In addition to this metric topology on X' , it is often also useful or necessary to consider a weaker topology.

The *weak-* topology* is the weakest topology on X' such that for each $x \in X$ the function $\phi_x : X' \rightarrow \mathbb{C}$, defined by $\phi_x(F) = F(x)$, is continuous. This means that a sequence $\{F_n\}_n$ in X' converges weak-* to F if and only if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x), \quad \text{for all } x \in X.$$

The Banach-Alaoglu Theorem. A basic result in this context is the *Banach-Alaoglu theorem* which says that the closed unit ball in X' is weak-* compact. In

other words, for each sequence of linear functionals $\{F_n\}_n$ on X for which $\|F_n\| \leq 1$ for all $n \in \mathbb{N}$, there exists a linear functional F on X with $\|F\| \leq 1$ and such that some subsequence $\{F_{n_k}\}_k$ converges pointwise to F :

$$\lim_{k \rightarrow \infty} F_{n_k}(x) = F(x), \quad \text{for all } x \in X.$$

A concrete situation where this result is frequently used is the following. If $\{\mu_n\}_n$ is a sequence of positive Borel measures on the compact topological space K and $\mu_n(K) \leq 1$ for all $m \in \mathbb{N}$ then there exists a finite measure μ and a subsequence $\{\mu_{n_k}\}_k$ such that

$$\lim_{k \rightarrow \infty} \int_K f d\mu_{n_k} = \int_K f d\mu, \quad \text{for all } f \in C(K).$$

Here $C(K)$, as is usual, denotes the space of all continuous functions on the compact set K , equipped with the supremum (maximum) norm.

Also, in the context of Riemann-Stieltjes integrals of functions of bounded variation over a finite closed interval $[a, b]$, the special case of the Banach-Alaoglu theorem is known as the *Helly selection theorem*; see [2, Chapter 1].

2.2.3 The Riesz-Herglotz Theorem

Harmonic Hardy Spaces. The *harmonic Hardy space* h^p , $0 < p \leq \infty$, is defined as

$$h^p = \left\{ u \in h(\mathbb{D}) : \|u\|_p = \sup_{0 < r < 1} M_p(r, u) < \infty \right\},$$

where

$$M_p(r, u) = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right)^{1/p}$$

is the integral mean of order p as defined in Sect. 1.1. In the case $p = \infty$, the right-hand side of the above equality is to be interpreted as the supremum:

$$M_\infty(r, u) = \sup_{\theta \in [-\pi, \pi]} |u(re^{i\theta})|.$$

The Riesz-Herglotz Representation. The Poisson integral of a measure $\mu \in M(\mathbb{T})$ belongs to h^1 because $\|P[\mu]\|_1 \leq \|\mu\|$. This is easily deduced from the fact that

$$\frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) d\theta = 1, \quad z \in \mathbb{D}.$$

The converse is true as well. Every function belonging to h^1 is equal to the Poisson integral of some (unique) measure. A more precise statement is as follows. (The result is often also called the *Herglotz representation formula*.)

Theorem 2.2.2 (Riesz-Herglotz) *The Poisson integral acts as an isometric isomorphism from $M(\mathbb{T})$ onto h^1 .*

Proof First we show that the transformation $\mu \mapsto P[\mu]$ maps $M(\mathbb{T})$ onto h^1 .

Suppose $u \in h^1$. By the assumption, the family $\{u_r : r \in [0, 1)\}$ is norm-bounded in $L^1(\mathbb{T})$, and the family of measures $d\mu_r(e^{i\theta}) = u(re^{i\theta})d\theta$ is bounded in $M(\mathbb{T})$, which is the dual of a separable space $C(\mathbb{T})$. Now, we can apply the Banach-Alaoglu theorem to conclude that there exists a sequence $r_n \rightarrow 1$ such that μ_{r_n} converges in the weak-* topology to some $\mu \in M(\mathbb{T})$. Next, we show that $u = P[\mu]$.

Fix $a \in \mathbb{D}$. Since the functions $z \rightarrow u(r_j z)$ are harmonic in \mathbb{D} , we have

$$u(r_j a) = \frac{1}{2\pi} \int_0^{2\pi} P_a(\theta) u(r_j e^{i\theta}) d\theta, \quad \text{for each } j. \quad (2.9)$$

Now let $j \rightarrow \infty$. By continuity, the left-hand side of (2.9) converges to $u(a)$. On the other hand, since $P_a(\cdot) \in C(\mathbb{T})$, the right-hand side of (2.9) converges to $P[\mu](a)$. Therefore, $u(a) = P[\mu](a)$, and thus $u = P[\mu]$.

Because of the weak convergence we have

$$\|\mu\| \leq \liminf_{j \rightarrow \infty} \|\mu_{r_j}\| = \liminf_{j \rightarrow \infty} \|u_{r_j}\|_1 \leq \|u\|_1,$$

which together with the inequality $\|u\|_1 = \|P[\mu]\|_1 \leq \|\mu\|$ implies that $\|u\|_1 = \|P[\mu]\|_1 = \|\mu\|$. \square

If $d\mu(e^{i\theta}) = \phi(e^{i\theta})d\theta$ with $\phi \in L^1(\mathbb{T})$, then $\|\mu\| = \|\phi\|_{L^1(\mathbb{T})}$, which means in particular that $M(\mathbb{T})$ contains an isometric copy of $L^1(\mathbb{T})$.

2.2.4 The Poisson-Stieltjes Integral

Instead of dealing with measures, it is sometimes more convenient to work with functions of bounded variations. Due to lack of space we cannot review here the functions of bounded variations and the concept of the Riemann-Stieltjes integral. For the functions of bounded variation, we refer the reader to [12, Chapter 8] or [13, Chapter 5] and for the Riemann-Stieltjes integral to [14, Chapter 6], for example.

Let $BV[a, b]$ denote the set of functions of bounded variation on $[a, b]$. The *Poisson-Stieltjes integral* of a function $\gamma \in BV[-\pi, \pi]$ is defined as the (harmonic) function

$$PS[\gamma](re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) d\gamma(t).$$

Here, $P(r, t) := P_{reit}(1) = (1 - r^2)/(1 - 2r \cos t + r^2)$.

It is convenient to recall that every function of bounded variation is a difference of two increasing functions. In view of the well-known connection between Borel measures and functions of bounded variation, it can be deduced from the Riesz-Herglotz theorem that a function $u \in h(\mathbb{D})$ belongs to h^1 if and only if u is equal to the Poisson-Stieltjes integral of a function $\gamma \in BV[-\pi, \pi]$ and a positive harmonic function in \mathbb{D} is equal to the Poisson-Stieltjes integral of an increasing function in $BV[-\pi, \pi]$.

2.3 The Poisson Integral, $L^p(\mathbb{T})$, and h^p Spaces

When $1 < p \leq \infty$

2.3.1 The Poisson Integral of a Function

A slight modification of Theorem 2.2.2 shows that the following is true:

Theorem 2.3.1 *Let $1 < p \leq \infty$. Then the Poisson integral acts as an isometric isomorphism from $L^p(\mathbb{T})$ onto h^p .*

If $\varphi \in C(\mathbb{T})$ and $u = P[\varphi]$, we know that $u_r \rightarrow \varphi$ in $C(\mathbb{T})$ as $r \rightarrow 1$. This fact and Theorem 2.3.1 enable us to prove the following result about L^p -convergence.

Proposition 2.3.2 *Suppose $1 \leq p < \infty$. If $\varphi \in L^p(\mathbb{T})$ and $u = P[\varphi]$, then $\|u_r - \varphi\|_p \rightarrow 0$, as $r \rightarrow 1$.*

Proof In order to simplify the notation, we shall often write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\mathbb{T})}$.

Fix $\varepsilon > 0$ and choose $\phi \in C(\mathbb{T})$ with $\|\varphi - \phi\|_p < \varepsilon$. Setting $v = P[\phi]$, we have

$$\|u_r - \varphi\|_p \leq \|u_r - v_r\|_p + \|v_r - \phi\|_p + \|\varphi - \phi\|_p.$$

Now $u_r - v_r = P[\varphi - \phi]_r$, hence $\|u_r - v_r\|_p < \varepsilon$. Note also that $\|v_r - \phi\|_p \leq \|v_r - \phi\|_\infty$. Thus, $\|u_r - \varphi\|_p \leq \|v_r - \phi\|_\infty + 2\varepsilon$. Since $\phi \in C(\mathbb{T})$, we have $\|v_r - \phi\|_\infty \rightarrow 0$, as $r \rightarrow 1$. It follows that $\limsup_{r \rightarrow 1} \|u_r - \varphi\|_p \leq 2\varepsilon$. Since ε is arbitrary, we have $\|u_r - \varphi\|_p \rightarrow 0$ as desired. \square

2.3.2 Weak- * Convergence Properties of the Poisson Integral

Proposition 2.3.2 fails when $p = \infty$. In fact, for $\varphi \in L^\infty(\mathbb{T})$ and $u = P[\varphi]$, we have $\|u_r - \varphi\|_\infty \rightarrow 0$, as $r \rightarrow 1$, if and only if $\varphi \in C(\mathbb{T})$, as can be easily verified.

In the case when $\mu \in M(\mathbb{T})$ and $u = P[\mu]$, one might ask whether the L^1 functions u_r always converge to μ in $M(\mathbb{T})$. Here the answer is negative as well. Since $L^1(\mathbb{T})$ is a

closed subspace of $M(\mathbb{T})$, $u_r \rightarrow \mu$ in $\mu(\mathbb{T})$ precisely when μ is absolutely continuous with respect to σ .

The following result is a substitute for Proposition 2.3.2 in the two cases mentioned above.

Proposition 2.3.3 *The Poisson integrals have the following weak- \star convergence properties:*

- (i) if $\mu \in M(\mathbb{T})$ and $u = P[\mu]$, then $u_r \rightarrow \mu$ weak- \star in $M(\mathbb{T})$ as $r \rightarrow 1$.
- (ii) if $\varphi \in L^\infty(\mathbb{T})$ and $u = P[\varphi]$, then $u_r \rightarrow \varphi$ weak- \star in $L^\infty(\mathbb{T})$ as $r \rightarrow 1$.

Proof (i) Recall that $C(\mathbb{T})' = M(\mathbb{T})$. Suppose $\mu \in M(\mathbb{T})$, $u = P[\mu]$, and $v \in C(\mathbb{T})$. To prove (i), we need to show

$$\lim_{r \rightarrow 1} \int_{\mathbb{T}} v u_r d\sigma = \int_{\mathbb{T}} v d\mu.$$

Working with $\int_{\mathbb{T}} v u_r d\sigma$, we have

$$\begin{aligned} \int_{\mathbb{T}} v u_r d\sigma &= \int_{\mathbb{T}} v(\xi) \int_{\mathbb{T}} P_{r\xi}(\eta) d\mu(\eta) d\sigma(\xi) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} P_{r\eta}(\xi) v(\xi) d\sigma(\xi) d\mu(\eta) \\ &= \int_{\mathbb{T}} P[v](r\eta) d\mu(\eta). \end{aligned}$$

Because $v \in C(\mathbb{T})$, $P[v](r\eta) \rightarrow v(\eta)$ uniformly on \mathbb{T} as $r \rightarrow 1$. This proves (i).

(ii) The proof of (ii) is similar. Use the duality $L^1(\mathbb{T})' = L^\infty(\mathbb{T})$. □

2.4 The Maximal Function of a Measure on \mathbb{T}

2.4.1 The Maximal Function of a Measure on \mathbb{T}

The maximal function of a complex measure $\mu \in M(\mathbb{T})$ is the function $\mathcal{M}\mu : \mathbb{T} \mapsto [0, \infty]$ defined by

$$\mathcal{M}\mu(e^{i\theta}) = \sup_{\delta > 0} \frac{|\mu|(Q(e^{i\theta}, \delta))}{2\delta},$$

where $Q(e^{i\theta}, \delta) = \{e^{it} \in \mathbb{T} : |t - \theta| < \delta\}$. Note that $\mathcal{M}\mu = \mathcal{M}|\mu|$.

For each fixed $\delta > 0$, the above quotient is easily seen to be a lower semicontinuous function of $e^{i\theta}$. Hence $\mathcal{M}\mu$ is lower semicontinuous.

The operator \mathcal{M} which maps μ to $\mathcal{M}\mu$ is called the *Hardy-Littlewood maximal operator*.

Theorem 2.4.1 *If $\mu \in M(\mathbb{T})$ then*

$$m(\{\mathcal{M}\mu > t\}) \leq Ct^{-1}\|\mu\|, \quad \text{for every } t > 0. \quad (2.10)$$

The notation used on the left-hand side of (2.10) replaces the more cumbersome expression $m(\{e^{i\theta} \in \mathbb{T} : (\mathcal{M}\mu)(e^{i\theta}) > t\})$. We shall often simplify the notation in this way.

Proof Fix μ and t . Let K be a compact subset of the open set $\{\mathcal{M}\mu > t\}$. By the definition of $\mathcal{M}\mu$ and the compactness of K there are open arcs Q_j , $j = 1, 2, \dots, m$ such that $K \subset \cup_{j=1}^m Q_j$ and $|\mu(Q_j)| > tm(Q_j)$. Assume that the sequence $m(Q_j)$ is decreasing. Let $J_1 = Q_1$. Let $J_2 = Q_k$, where k is the smallest i for which $Q_i \cap J_1 = \emptyset$. Then let $J_3 = Q_n$ where n is the smallest $i > k$ such that $Q_i \cap (J_1 \cup J_2) = \emptyset$. Continuing this way we find a sequence J_j of pairwise disjoint arcs on \mathbb{T} such that $\cup_i Q_i \subset \cup_j J_j^*$, where, for each j , J_j^* is the arc “concentric” with J_j and $m(J_j^*) = 3m(J_j)$. It follows that

$$m(K) \leq \sum_j m(J_j^*) \leq 3t^{-1}\|\mu\|.$$

Now (2.10) follows by taking the supremum over all compact $K \subset \{\mathcal{M}\mu > t\}$. \square

2.4.2 Lorentz Spaces

Let $L^0(\mathbb{T}, \sigma) = L^0(\mathbb{T})$ be the space of complex-valued Lebesgue measurable functions on \mathbb{T} . For $u \in L^0(\mathbb{T})$ and $s \geq 0$, we write $\lambda_u(s) = \sigma(\{\xi \in \mathbb{T} : |u(\xi)| > s\})$ for the distribution function and $u^*(s) = \inf(\{t \geq 0 : \lambda_u(t) \leq s\})$ for the *decreasing rearrangement* of $|u|$, each taken with respect to σ . The *Lorentz functional* $\|\cdot\|_{p,q}$ is defined for $u \in L^0(\mathbb{T})$ by

$$\|u\|_{p,q} = \left(\int_0^\infty [u^*(s)s^{1/p}]^q \frac{ds}{s} \right)^{1/q} \quad \text{for } 0 < q < \infty$$

and

$$\|u\|_{p,\infty} = \sup_{s \geq 0} [u^*(s)s^{1/p}].$$

The corresponding *Lorentz space* is defined by

$$L^{p,q}(\mathbb{T}, \sigma) = L^{p,q}(\mathbb{T}) = \{u \in L^0(\mathbb{T}) : \|u\|_{p,q} < \infty\}.$$

It is not difficult to show that the space $L^{p,q}(\mathbb{T})$ is separable if and only if $q \neq \infty$. The class of functions $u \in L^0(\mathbb{T})$ satisfying $\lim_{s \rightarrow 0} (u^*(s)s^{1/p}) = 0$ is a separable closed subspace of $L^{p,\infty}(\mathbb{T})$ which is denoted by $L_0^{p,\infty}(\mathbb{T})$.

The cases of main interest are, of course, $p = q$ and $q = \infty$. Indeed, $L^{p,p}(\mathbb{T})$ is nothing but $L^p(\mathbb{T})$ and $L^{p,\infty}(\mathbb{T})$ is the so-called *weak- $L^p(\mathbb{T})$ space*.

2.4.3 The Maximal Theorem

If $f \in L^1(\mathbb{T}, \sigma)$ and $t > 0$, then the inequality $\sigma(\{|f| > t\}) \leq t^{-1} \|f\|_{L^1(\mathbb{T})}$ is obvious (and holds equally well for any positive measure in place of σ).

Theorem 2.4.1 restricted to $L^1(\mathbb{T}, \sigma)$ can be restated by saying that the Hardy-Littlewood maximal operator \mathcal{M} maps $L^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$. Since the operator \mathcal{M} is *subadditive*: $\mathcal{M}(f + g) \leq \mathcal{M}f + \mathcal{M}g$ and since the inequality $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$ is trivial, the following L^p result is a consequence of the Marcinkiewicz interpolation theorem [7, 15].

Theorem 2.4.2 (Maximal theorem) *For $1 < p < \infty$ there are constants $C_p < \infty$ such that*

$$\int_0^{2\pi} [\mathcal{M}u(e^{i\theta})]^p d\theta \leq C_p \int_0^{2\pi} |u(e^{i\theta})|^p d\theta$$

for every $u \in L^p(\mathbb{T})$.

2.5 Nontangential Maximal Function and Fatou's Theorem

2.5.1 Nontangential Maximal Function

For $e^{i\theta} \in \mathbb{T}$ and $\alpha > 1$ we define the *Stolz (nontangential) approach region* of aperture α :

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in \mathbb{D} : |z - e^{i\theta}| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

For any $u : \mathbb{D} \rightarrow \mathbb{C}$, define the *nontangential maximal function* of u by

$$M_\alpha u(e^{i\theta}) = \sup_{z \in \Gamma_\alpha(e^{i\theta})} |u(z)|,$$

and the *radial maximal function* of u is the function $M_{rad}u$ defined on \mathbb{T} by

$$M_{rad}u(e^{i\theta}) = \sup_{0 \leq r < 1} |u(re^{i\theta})|.$$

Lemma 2.5.1 *For each $\alpha > 1$ there is a finite positive constant C_α such that the inequality*

$$M_\alpha P[\mu](e^{i\theta}) \leq C_\alpha \mathcal{M}\mu(e^{i\theta}), \quad e^{i\theta} \in \mathbb{T}, \quad (2.11)$$

holds for every complex measure μ on \mathbb{T} .

Proof Clearly, we can assume that $\theta = 0$. Choose $re^{i\phi} \in \Gamma_\alpha(1)$. Then one can show that $|\phi| \leq 2\alpha(1-r)$. We have

$$\begin{aligned} 2\pi P[\mu](re^{i\phi}) &= \int_{|\psi-\phi| \leq 4\alpha(1-r)} P_{re^{i\phi}}(\psi) d\mu(\psi) + \int_{4\alpha(1-r) < |\psi-\phi| \leq \pi/2} P_{re^{i\phi}}(\psi) d\mu(\psi) \\ &\quad + \int_{\pi/2 < |\psi-\phi|} P_{re^{i\phi}}(\psi) d\mu(\psi). \end{aligned} \quad (2.12)$$

Denote the last three integral terms by I_1 , I_2 and I_3 respectively. They can be estimated separately. Since $|P_{re^{i\phi}}(\psi)| \leq 2/(1-r)$, we obtain

$$|I_1| \leq \frac{2}{1-r} \int_{|\psi-\phi| < 5\alpha(1-r)} d|\mu|(\psi) \leq \frac{2}{1-r} \int_{|\psi| < 7\alpha(1-r)} d|\mu|(\psi) \leq 28\alpha \mathcal{M}\mu(1). \quad (2.13)$$

Next, for $|\psi - \phi| > \pi/2$ we have $P_{re^{i\phi}}(\psi) \leq 1$ and therefore

$$|I_3| \leq 2 \int_{|\psi-\phi| > \pi/2} d|\mu|(\psi) \leq 2|\mu|(\mathbb{T}) \leq 4\pi \mathcal{M}\mu(1). \quad (2.14)$$

In order to estimate I_2 , we set $S_j = \{\psi : 2^j\alpha(1-r) < |\psi - \phi| \leq 2^{j+1}\alpha(1-r)\}$ and let N be the largest integer such that S_N intersects $S = \{\psi : |\psi - \phi| \leq \pi/2\}$. Then we have $I_2 = \sum_{j=2}^N \int_{S_j \cap S} P_{re^{i\phi}}(\psi) d\mu(\psi)$. Since

$$P_{re^{i\phi}}(\psi) = \frac{1-r^2}{(1-r)^2 + 4r \sin^2(\psi - \phi)/2}$$

and $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$ (this is *Jordan's inequality* from elementary calculus) we have, for $\psi \in S_j \cap S$,

$$P_{re^{i\phi}}(\psi) \leq \frac{1-r^2}{(1-r)^2 + 16/\pi^2 |\psi - \phi|^2} \leq \frac{1-r^2}{(1-r)^2 + 2^{2j}\alpha^2(1-r)^2}.$$

Hence

$$\begin{aligned} |I_2| &\leq \sum_{j=2}^N \frac{2}{1-r} \int_{S_j \cap S} \frac{d|\mu|(\psi)}{1 + 2^{2j}\alpha^2} \\ &\leq \sum_{j=2}^N \frac{2}{(1-r)(1 + 2^{2j}\alpha^2)} \int_{|\psi| \leq \alpha(1-r)(2^{j+1}+2)} d|\mu|(\psi) \end{aligned}$$

$$\leq \mathcal{M}\mu(1) \sum_{j=2}^N \frac{4\alpha(2^{j+2} + 2)}{1 + 2^{2j}\alpha^2} \quad (2.15)$$

and since the corresponding series converges, we get $|I_2| \leq C_\alpha \mathcal{M}\mu(1)$. Now (2.11) follows from (2.12), (2.13), (2.14) and (2.15). \square

Theorem 2.5.2 *Let $u \in h(\mathbb{D})$ and $1 < p < \infty$. Then the following statements are equivalent:*

- (i) $u \in h^p$,
- (ii) $M_{rad}u \in L^p(\mathbb{T})$,
- (iii) $M_\alpha u \in L^p(\mathbb{T})$, for some $\alpha > 1$,
- (iv) $M_\alpha u \in L^p(\mathbb{T})$, for any $\alpha > 1$.

Moreover, the norms $\|u\|_p$, $\|M_{rad}u\|_p$ and $\|M_\alpha u\|_p$ are equivalent.

Proof From the above Lemma 2.5.1, Theorems 2.3.1 and 2.4.2 it follows that (i) \implies (iv). The implications (iv) \implies (iii) \implies (ii) \implies (i) are trivial since

$$|u(re^{i\theta})| \leq M_{rad}u(e^{i\theta}) \leq M_\alpha u(e^{i\theta}), \quad 0 \leq r < 1, 0 \leq \theta < 2\pi, \alpha > 1.$$

\square

The inequality (2.11), Theorems 2.2.2 and 2.4.1 together yield

Theorem 2.5.3 *The operator M_α , $\alpha > 1$, maps h^1 continuously into $L^{1,\infty}(\mathbb{T})$.*

2.5.2 Nontangential Limits

A function u on \mathbb{D} is said to have *nontangential limit* L at $e^{i\theta} \in \mathbb{T}$ if, for each $\alpha > 1$, $u(z) \rightarrow L$ as $z \rightarrow e^{i\theta}$ within the set $\Gamma_\alpha(e^{i\theta})$. We write

$$(N_t - \lim u)(e^{i\theta}) = u^*(e^{i\theta}) = L.$$

Theorem 2.5.4 (Fatou's theorem on nontangential limits) *Let $1 \leq p \leq \infty$ and $u \in h^p$. Then the nontangential limit $(N_t - \lim u)(e^{i\theta}) = u^*(e^{i\theta})$ exists for almost all θ .*

If $1 < p \leq \infty$, then $u(z) = P[u^](z)$, $z \in \mathbb{D}$. If $p = 1$, then $u(z)$ is the Poisson integral of a measure $\mu \in M(\mathbb{T})$, and μ is related to the boundary value $u^*(e^{i\theta})$ by $d\mu = u^*d\sigma + d\mu_s$, where $d\mu_s$ is singular to Lebesgue measure.*

Proof In view of Theorems 2.2.2 and 2.3.1, it suffices to prove the following two statements. The first one deals with absolutely continuous measures: if $u(z) = P[\phi](z)$, where $\phi \in L^1(\mathbb{T})$, then $(N_t - \lim u)(e^{i\theta}) = \phi(e^{i\theta})$ for almost all $e^{i\theta} \in \mathbb{T}$. The second one deals with singular measures: if $d\mu$ is a complex measure on

\mathbb{T} singular with respect to the Lebesgue measure and if $u(z) = P[\mu](z)$, then $(N_t - \lim u)(e^{i\theta}) = 0$ a.e on \mathbb{T} .

Let us prove the first statement. Clearly, we can assume ϕ is real valued, and all functions considered in the rest of the proof are real valued. For an integrable ϕ we introduce

$$(\Omega\phi)(e^{i\theta}) = (N_t - \limsup u)(e^{i\theta}) - (N_t - \liminf u)(e^{i\theta}),$$

where these upper and lower limits are taken with respect to Stolz regions $\Gamma_\alpha(e^{i\theta})$ with fixed aperture α . Note that for continuous φ the statement is true by Theorem 2.1.2. Therefore, for any $\varphi \in C(\mathbb{T})$ we have $\Omega\phi = \Omega(\phi - \varphi)$. We also have

$$(\Omega\phi)(e^{i\theta}) \leq 2M_\alpha u(e^{i\theta}) \leq 2C_\alpha \mathcal{M}\phi(e^{i\theta}).$$

Now, let us fix an $\varepsilon > 0$ and consider $U_\varepsilon = \{e^{i\theta} : (\Omega\phi)(e^{i\theta}) > \varepsilon\}$. For any $\eta > 0$ there is a continuous function φ such that $\|\phi - \varphi\|_1 < \eta$. Thus, $U_\varepsilon = \{e^{i\theta} : \Omega(\phi - \varphi)(e^{i\theta}) > \varepsilon\}$ and by the previous estimates the last set is contained in $V_\varepsilon = \{e^{i\theta} : \mathcal{M}(\phi - \varphi)(e^{i\theta}) > \varepsilon/(2C_\alpha)\}$. Now Theorem 2.4.1 implies that $m(U_\varepsilon) \leq m(V_\varepsilon) \leq 2C_\alpha \eta/\varepsilon$. Since $\eta > 0$ is arbitrary, it follows that $m(U_\varepsilon) = 0$, since $\varepsilon > 0$ is arbitrary, it follows that N_t limits exist almost everywhere. It is easily seen that $(N_t - \lim u)(e^{i\theta}) = \phi(e^{i\theta})$ a.e. Indeed, we have $\lim_{r \rightarrow 1} u_r = \phi$ in L^1 norm, and this gives the last assertion.

For the second statement we only sketch the proof. The role of continuous functions is taken by measures supported on compact sets of Lebesgue measure zero; regularity properties of measures allow us to approximate general singular measures by compactly supported ones. After realizing that the statement is true for compactly supported singular measures, the proof continues along the above lines. \square

2.5.3 The Space h_{\max}^p . Atomic Decomposition

The space h_{\max}^p , where $0 < p < \infty$, is defined as the subspace of h^p consisting of all harmonic functions $u \in h(\mathbb{D})$ for which $\|u\|_{p, \max} = \|M_{rad} u\|_{L^p(\mathbb{T})} < \infty$.

From Theorem 2.5.2 it follows that if $1 < p < \infty$ then $h^p = h_{\max}^p$. Note that if $0 < p \leq 1$, then the inclusion $h_{\max}^p \subset h^p$ is proper.

By Theorem 2.3.1, the space $L^p(\mathbb{T})$ is isomorphic to the space $h^p = h_{\max}^p$ whenever $1 < p \leq \infty$. If $0 < p \leq 1$, the space h_{\max}^p turns out to be isomorphic to the space $H_{at}^p(\mathbb{T})$ which we define below.

Let $0 < p \leq 1$. A p -atom is a function $a \in L^\infty(\mathbb{T})$ supported on an interval $I \subset \mathbb{T}$ such that

$$(i) \quad \|a\|_\infty \leq |I|^{-1/p};$$

(ii) $\int_{-\pi}^{\pi} a(e^{it})T(e^{it})dt = 0$ for every trigonometric polynomial $T(e^{it})$ of degree at most $N_p - 1$, where $N_p = \lfloor \frac{1}{p} \rfloor$.

A function φ is said to belong to the space $H_{at}^p(\mathbb{T})$, $0 < p \leq 1$, if there exist a sequence $\{\lambda_k\} \in \ell^p$ and a sequence $\{a_k\}$ of p -atoms such that

$$\varphi(e^{it}) = \sum_{k=1}^{\infty} \lambda_k a_k(e^{it}). \quad (2.16)$$

We write

$$\|\varphi\|_{H_{at}^p(\mathbb{T})} = \inf \left(\sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all sequences $\{\lambda_k\}$ for which (2.16) holds.

Theorem 2.5.5 *Suppose $0 < p \leq 1$. If $u \in h(\mathbb{D})$, then $u \in h_{\max}^p$ if and only if u is the Poisson integral of an $\varphi \in H_{at}^p(\mathbb{T})$, i.e.,*

$$u(z) = P[\varphi](z), \quad z \in \mathbb{D}, \quad \text{for some } \varphi \in H_{at}^p(\mathbb{T}).$$

Moreover, $\|M_{rad}u\|_{L^p(\mathbb{T})} \approx \|\varphi\|_{H_{at}^p(\mathbb{T})}$.

2.6 Some Useful Practical Facts

- It can be shown that the closure in h^1 of the set of all harmonic polynomials is the set $\{P[\varphi] : \varphi \in L^1(\mathbb{T})\}$.
- If u is the Poisson integral of a singular measure, then

$$\lim_{r \rightarrow 0} \int_0^{2\pi} |u(re^{it})|^p dt = 0 \quad \text{for } 0 < p < 1.$$

This was shown by J.H. Shapiro [16], Proposition 2.5.

- In [17], D. Kalaj and M. Vuorinen proved the following analogue of the classical Schwarz lemma:

If u is a real-valued function harmonic on \mathbb{D} and $|u(z)| \leq 1$, for $z \in \mathbb{D}$, then

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - u(z)^2}{1 - |z|^2}.$$

The constant $4/\pi$ is optimal.

2.7 Historical and Bibliographical Notes

Fatou's theorem was first proved in [18]. For further details, we refer the reader to [2].

Theorem 2.5.5 is from [8].

Hoffman's classical (although somewhat outdated) book [1] contains a treatment of Fourier series, integral kernels, and Hardy spaces and a discussion of weak-* topology.

2.8 Exercises

1. Show that the inclusion $h_{\max}^p \subset h^p$ is proper, whenever $0 < p \leq 1$.
2. Prove that a real harmonic function in \mathbb{D} belongs to h^1 if and only if it can be written as the difference of two positive harmonic functions in \mathbb{D} .
3. Prove that if a function $u \in h(\mathbb{D})$ is real-valued and positive, then u is the Poisson integral of a finite positive measure.
4. Prove: Let $u \in h(\mathbb{D})$ be a real-valued function such that $|u(z)| \leq 1, z \in \mathbb{D}$. Then

$$|\nabla u(z)| \leq 2 \frac{1 - |u(z)|}{1 - |z|}, \quad z \in \mathbb{D}.$$

The assertion does not hold for complex-valued harmonic functions even if we replace the number 2 by any other constant.

5. Let $\gamma \in BV[0, 2\pi]$. Prove that

$$\begin{aligned} PS[\gamma](re^{i\theta}) &= kP(r, \theta + \pi) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial P}{\partial t}(r, \theta - t) \gamma(t) dt \\ &= kP(r, \theta + \pi) + \frac{\partial}{\partial \theta} P[\gamma](re^{i\theta}), \end{aligned}$$

where

$$k = \frac{\gamma(\pi) - \gamma(-\pi)}{2\pi} \quad \text{and} \quad \frac{\partial P}{\partial t}(r, t) = \frac{-2r \sin t}{1 - 2r \cos t + r^2} P(r, t).$$

6. Prove that the Poisson kernel $P(z) = P_z(1) = \frac{1-|z|^2}{|1-z|^2}$ has the following properties [19]:

$$\begin{aligned} (i) \quad & M_p^p(r, P) = M_{1-p}^{1-p}(r, P), \quad 0 < r < 1, \quad 0 < p < 1 \text{ and} \\ (ii) \quad & M_p(r, P) \approx \begin{cases} 1 - r, & \text{for } 0 < p < 1/2 \\ (1 - r) \left(\log \frac{e}{1-r} \right)^2, & \text{for } p = 1/2 \\ (1 - r)^{1/p-1}, & \text{for } p > 1/2. \end{cases} \end{aligned}$$

7. Let $n \in \mathbb{N}$. Prove the inequality

$$\left| \frac{\partial^n P}{\partial t^n}(r, t) \right| \leq C \frac{1 - r^2}{|1 - re^{it}|^{n+2}}$$

for some fixed constant $C > 0$ and all $r \in [0, 1)$ and $t \in [0, 2\pi]$.

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and Bergman-Type Spaces

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2016, XVI, 323 p., Hardcover

ISBN: 978-3-319-45643-0