# Chapter 2
# Further Study of Sequences and Series

As you would see earlier in Chapter 1, some problems would ask you to add the first ten terms or even evaluate the sum of the first $k$ terms of a sequence or maybe investigate whether the limit of such sum exists. Expressions such as

$$1 + 4 + 7 + 10 + 13 + \ldots \tag{2.1}$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots \tag{2.2}$$

$$1 + 4 + 9 + 16 + 25 + 36 + \ldots \tag{2.3}$$

are called series and in all three cases can be evaluated exactly for the sum of any finite number of terms. Since Eq. 2.1 represents an arithmetic series with first term 1 and common difference 3, we can use the formula for the sum of the first $n$ terms that is derived in the earlier section. We can write the sum as

$$\begin{aligned} S_n &= 1 + 4 + 7 + 10 + \ldots = \frac{2a_1 + (n-1)d}{2} \cdot n = \frac{2 \cdot 1 + (n-1)3}{2} \cdot n \\ &= \frac{(3n-1)n}{2}. \end{aligned} \tag{2.4}$$

Since Eq. 2.2 represents a geometric series with the first term 1/2 and common ratio 1/2, then the formula for the sum of the first $n$ terms is known. We have

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{b_1(1-r^n)}{1-r} = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n. \tag{2.5}$$

The sum of the last series of Eq. 2.3 can be evaluated exactly as well. We prove this formula in Chapter 1 and prove it in a different way in the following subsection,


E. Grigorieva, *Methods of Solving Sequence and Series Problems*,
DOI 10.1007/978-3-319-45686-7_2

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.6}$$

What do these series of Eqs. 2.1–2.3 have in common? Their partial sums can be evaluated exactly for any number of terms $n$. So we could add the first 25, the first 100 or even the first 2011 terms and get an exact answer for the sum using Eqs. 2.1–2.3 by replacing $n$ by 25, 100, or 2011, respectively. However, if the number of terms, $n$, were to become infinitely large, then we would see some differences. For example, if we increase $n$ then the partial sums of Eqs. 2.4 and 2.6 would increase without limit. The result is different for the sum of Eq. 2.5; it will approach its limit of one since the second term will approach zero. This behavior is typical for any infinite geometric series with common ratio less than one as we established earlier.

We say that the series of Eqs. 2.1 and 2.3 diverge and the series of Eq. 2.2 converges. Serious study of convergence and divergence is a subject of mathematical analysis. For now we simply determine whether or not the series are divergent or convergent and why. Many challenging math contest problems are dedicated to finding an exact sum of the first $n$ terms of a series. The determination of the partial and infinite sums is the topic of the first section of this chapter.

## 2.1 Methods of Finding Partial and Infinite Sums

Let us derive again Eq. 2.6 for the sum of squares of the first $n$ natural numbers and Eq. 1.31 for the sum of the cubes of $n$ natural numbers.

**Problem 47** Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

**Proof.** We need to prove that the following relationship is true:

$$N = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + (n-2)^2 + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Arranging sums in ascending and descending order does not help. We need to find a different approach. If you have read Chapter 1 of the book then you probably have an idea of how to start. Let us consider the difference of two consecutive cubes, $\boxed{n^3 - (n-1)^3 = 3n^2 - 3n + 1}$.

$$\begin{aligned}
1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1 = 1 \\
2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1 \\
3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1 \\
&\dots \\
(n-2)^3 - (n-3)^3 &= 3 \cdot (n-2)^2 - 3 \cdot (n-2) + 1 \\
(n-1)^3 - (n-2)^3 &= 3 \cdot (n-1)^2 - 3(n-1) + 1 \\
n^3 - (n-1)^3 &= 3n^2 - 3n + 1
\end{aligned}$$

Adding the left and the right sides, we obtain $n^3 = 3\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right) - 3(1 + 2 + 3 + \ldots + n) + 1 \cdot n$. This can be written using sigma notation as $n^3 = 3\sum\limits_{k=1}^{n} k^2 - 3\sum\limits_{k=1}^{n} k + n$. Solving this for $\sum\limits_{k=1}^{n} k^2$ and assuming that we know the formula for the sum of the first $n$ natural numbers we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{2n^3 - 2n + 3n(n+1)}{6} = \frac{n(2n^2 + 3n + 1)}{6} \\
\sum_{k=1}^{n} k^2 &= \frac{n(2n+1)(n+1)}{6}.
\end{aligned}$$

The statement is proven.

**Problem 48** Prove that $\sum\limits_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

**Solution.** Try to use a similar approach so consider the difference of the fourth powers of two consecutive integers $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$. Write this out for the first few terms and then for the values as we reach $n$,

$$\begin{aligned}
1^4 - 0^4 &= 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1 \\
2^4 - 1^4 &= 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1 \\
3^4 - 2^4 &= 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1 \\
&\dots \\
(n-2)^4 - (n-3)^4 &= 4(n-2)^3 - 6(n-2)^2 + 4 \cdot (n-2) - 1 \\
(n-1)^4 - (n-2)^4 &= 4(n-1)^3 - 6(n-1)^2 + 4 \cdot (n-1) - 1 \\
n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4 \cdot n - 1.
\end{aligned}$$

Next, we add the left and the right sides together as we did in the previous problem using sigma notation and solve the equation for the unknown sum,

$$n^4 = 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4 \cdot \sum_{k=1}^{n} k - n$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + n(n+1)(2n+1) + n - 2n(n+1)}{4}$$

$$\sum_{k=1}^{n} k^3 = \frac{n(n^3+1) + n(n+1)(2n-1)}{4} = \frac{n(n+1)(n^2 - n + 1 + 2n - 1)}{4}$$

$$\sum_{k=1}^{n} k^3 = \frac{n(n+1)(n^2+n)}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

This is a very interesting relationship because we established again that the sum of the first $n$ cubes equals the square of the sum of the first $n$ natural numbers. For example, $1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2 = 100$.

**Remark.** Earlier we proved the same formula using the geometric approaches of ancient Babylonians and Greeks to demonstrate that the sum of the first $n$ cubes equals the sum of the first $m = \frac{n(n+1)}{2}$ odd consecutive numbers.

**Problem 49** Find the sum, $1 + 11 + 111 + 1111 + \ldots + 11\ldots.111$, where the last number consists of $n$ repetitions of the digit 1. Evaluate the sum for $n = 9$.

**Solution.** We solve this problem in three different ways so you can compare the different methods.

**Method 1.** At first glance, we notice that 1, 11, 111, 1111,…. is neither an arithmetic nor a geometric sequence. Hence, we have to rewrite the sum in another form. For example,

$$\begin{aligned} 1 &= 1 \\ 11 &= 1 + 10 \\ 111 &= 1 + 10 + 100 \\ 1111 &= 1 + 10 + 100 + 1000 \\ 111...11 &= 1 + 10 + 10^2 + 10^3 + 10^4 + \ldots + 10^{n-2} + 10^{n-1} \end{aligned}$$

Each number on the left containing digit 1 repeated $n$ times can be written as a sum of the first $n$ terms of a geometric sequence with the first term equals 1 and a common ratio 10. Thus,

$$\begin{aligned}
1 &= S_1 = 1 \\
11 &= S_2 = 1 + 10 \\
111 &= S_3 = 1 + 10 + 100 \\
&\dots \\
111...11 &= S_n = \frac{1 \cdot (10^n - 1)}{10 - 1} = \frac{10^n - 1}{9}
\end{aligned}$$

Adding over the left and right sides, $1 + 11 + 111 + \ldots + 111\ldots11 = S_1 + S_2 + \ldots + S_n$ and using the formula for the sum of $n$ terms of a geometric sequence and properties of $\sum$ - notation we have

$$S = \sum_{k=1}^{n} \frac{10^k - 1}{9} = \sum_{k=1}^{n} \frac{10^k}{9} - \frac{1}{9}\sum_{k=1}^{n} 1 = \frac{1}{9}\left(\sum_{k=1}^{n} 10^k - n\right) \tag{2.7}$$

Let us consider the first term of difference of Eq. 2.7, $\sum_{k=1}^{n} 10^k = 10 + 10^2 + 10^3 + \ldots + 10^n$. The expression on the right is again a geometric sequence with $b_1 = 10$ and $r = 10$ and

$$\sum_{k=1}^{n} 10^k = \frac{10 \cdot (10^n - 1)}{9} = \frac{10^{n+1} - 10}{9} \tag{2.8}$$

Substituting Eq. 2.8 into Eq. 2.7 we obtain a formula for $S$, $S = \frac{10^{n+1} - 10 - 9n}{81}$.

This formula can be used in order to find a sum like $1 + 11 + 111 + \ldots + 111\ldots.11$ for any specific number $n$. Thus, when $n = 9$, $S = 1 + 11 + 111 + \ldots + 111111111 = \frac{10^{10} - 10 - 9 \cdot 9}{81} = 123,456,789$.

**Method 2.** Denote the total sum by $S$ as $S = 1 + 11 + 111 + 1111 + 11111 + \ldots + 11\ldots1$. Multiplying $S$ by 10, we obtain $10S = 10 + 110 + 1110 + 11110 + 111110 + \ldots$. If we subtract the first sum from the second, we obtain (It may help to rewrite $S$ as $S = 1 + (10 + 1) + (110 + 1) + (1110 + 1) + \ldots$).

Then $9S = \overbrace{111...1}^{n \text{ times}}0 - n \cdot 1$ which leads us to the answer, $S = \dfrac{\overbrace{111...1}^{n \text{ times}}0 - n \cdot 1}{9}$.

**Method 3.** We can notice that $9 = 10 - 1$, $99 = 100 - 1$, $999 = 1000 - 1$, etc. If we multiply and divide the given sum by 9 we can easily evaluate it using a formula for geometric series.

$$S = \frac{1}{9}(10 - 1 + 100 - 1 + 1000 - 1 + 10000 - 1 + \ldots + 100...0 - 1)$$
$$= \frac{\left(10 + 10^2 + \ldots + 10^n - n\right)}{9}$$
$$S = \frac{1}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$$

Our series is divergent because $S$ increases without bound as $n$ increases.

As we mentioned above, evaluating an exact sum for a finite series or a partial sum for an infinite series can be a challenging task, and this is why many such problems appear in different contests. Each problem is unique but we are going to share with you some ideas of finding such sums; you may find them helpful and applicable to other or similar problems.

**Problem 50** Find the sum: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{1998\cdot 1999} + \frac{1}{1999\cdot 2000}$

**Solution.** Sometimes it is a good idea to rewrite a sum in a different but equivalent form by noticing something that the terms have in common, some pattern. One thing you might notice is that the denominator of each fraction is a product of two consecutive natural numbers. How can we obtain a product of two such numbers within a denominator? What operation can give us a product? Answer: When we put together (add or subtract) two fractions with different denominators, that have no common factors, the least common denominator is going to be a product of these numbers. In general,

$$\frac{1}{c} + \frac{1}{d} = \frac{d + c}{c \cdot d}$$
$$\frac{1}{\mathrm{c}} - \frac{1}{d} = \frac{d - c}{c \cdot d}$$

Looking at the second formula above, we can find the way of solving the problem. If $c$ and $d$ differ by 1, i.e., $d - c = 1$, then

$$\frac{1}{c} - \frac{1}{d} = \frac{1}{c \cdot d}$$
$$\frac{1}{1} - \frac{1}{2} = \frac{1}{1 \cdot 2}$$
$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2 \cdot 3}$$
$$\frac{1}{3} - \frac{1}{4} = \frac{1}{3 \cdot 4}$$
$$\cdots$$
$$\frac{1}{1999} - \frac{1}{2000} = \frac{1}{1999 \cdot 2000}.$$

Using these, we replace each fraction on the right by the difference on the left obtaining

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{1998 \cdot 1999} + \frac{1}{1999 \cdot 2000} =$$
$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{1998} - \frac{1}{1999} + \frac{1}{1999} - \frac{1}{2000}$$

In this sum all middle terms cancel each other except the first term, 1, and the last term, $-\frac{1}{2000}$. This gives us $S_{1999} = 1 - \frac{1}{2000} = \frac{1999}{2000}$. Evaluating this sum when $n = 1999$ (a big number), we see that $S_{1999} = \frac{1999}{2000}$ is almost 1. On the other hand, $S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = 1 - \frac{1}{5} = \frac{4}{5} = 0.8$. Four is not a "big" number, hence 0.8 is not as close to 1. Using the same technique, we can find the sum to infinity of the series:

$S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots$ so $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ and also have that $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = 1$.

**Remark.** In order to be considered for possible convergence, the series must first pass the necessary condition for the limit of its $n^{\text{th}}$ term, that is, does $\lim_{n \to \infty} u_n = 0$. If we try to look at the $n^{\text{th}}$ term of this sum, $\frac{1}{n(n+1)}$, we can see that $\lim_{n \to \infty} \frac{1}{n(n+1)} = 0$. We also find that the limit of the partial sums exists, $\lim_{\mathrm{n} \to \infty} S_n = S$ where $S$ is a finite number 1. However, in general, satisfying the necessary condition is not sufficient.

Convergence or divergence of series is established with the use of sufficient convergence theorems. We list some of these rules in Chapter 3.

Why didn't we use a calculator approach? A calculator can be used to find a sum like $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}$, i.e., $sum\big(seq(1/(x(x+1)), x, 1, 3\big) = 0.75$ This is an exact answer. A calculator can evaluate this as $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{100 \cdot 101} = 0.990094$, i.e., $sum\big(seq(1/x/(x+1), x, 1, 100\big) = 0.990094$. Even this: $\frac{1}{1 \cdot 2} + \ldots.. + \frac{1}{500 \cdot 501} = 0.99800$. But if we have more than 100 terms in summation, for example, $x = 1999$, such as our original problem, TI83/84 graphing calculators cannot be used. We might have some idea that this number gets closer and closer to 1. But how close? What if we need to find the exact answer or figure out the value of $S_n$, the sum of the first $n$ terms for any $n$? Remember that since $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$, we evaluated its limit analytically as $\lim_{n\to\infty} S_n = 1$.

In the preceding problem numbers within each denominator differed by 1. But the idea of replacing each fraction by a difference is so elegant, we wonder, "What happens if two numbers in each fraction differ by the same number but not by 1? Can we use the same technique here?"

**Problem 51** Evaluate $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \frac{1}{9\cdot13} + \ldots + \frac{1}{197\cdot201}$.

**Solution.** Look at the sequence of the first numbers of each denominator: $1, 5, 9, \ldots, 197$. They are terms of an arithmetic sequence with $a_1 = 1$ and $d = 4$. Let us find the number of the term that is 197.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 197 &= 1 + (n-1)4 \\ n &= 50 \end{aligned}$$

This means that we have to add 50 fractions together. Look at the differences:

$$\begin{aligned} 1 - \frac{1}{5} &= \frac{5-1}{1\cdot5} = \frac{4}{1\cdot5} = 4\cdot\frac{1}{1\cdot5} \\ \frac{1}{5} - \frac{1}{9} &= \frac{9-5}{5\cdot9} = \frac{4}{5\cdot9} = 4\cdot\frac{1}{5\cdot9} \\ &\ldots \\ \frac{1}{197} - \frac{1}{201} &= \frac{4}{197\cdot201} = 4\cdot\frac{1}{197\cdot201} \end{aligned}$$

Now the given sum can be written in the form:

$$S_{50} = \frac{1}{4}\left(1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \ldots. + \frac{1}{193} - \frac{1}{197} + \frac{1}{197} - \frac{1}{201}\right)$$

$$S_{50} = \frac{1}{4}\left(1 - \frac{1}{201}\right) = \frac{50}{201}.$$

Notice that the $n^{\text{th}}$ term of the series can be written as $\frac{1}{(4n-3)(4n+1)}$. We can evaluate the partial sum (the sum of the first $n$ terms) as $S_n = \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) = \frac{n}{4n+1}$. If $n \to \infty$, $S_n \to \frac{1}{4}$. Therefore, the series is convergent.

**Answer.** $\frac{50}{201}$

Now we can make a trivial but very useful conclusion. For any real $c$ and $d$ such that $c \neq d$

$$\boxed{\frac{1}{c \cdot d} = \frac{1}{d - c} \cdot \left[\frac{1}{c} - \frac{1}{d}\right]} \tag{2.9}$$

**Problem 52** Numbers $a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1}$ are terms of an arithmetic sequence. Prove that $\frac{1}{a_1 \cdot a_2} + \frac{1}{a_2 \cdot a_3} + \ldots + \frac{1}{a_n \cdot a_{n+1}} = \frac{n}{a_1 \cdot a_{n+1}}$

**Proof.** $a_1,\ a_2,\ \ldots,\ a_n,\ a_{n+1}$ are terms of an arithmetic sequence, then $a_2 - a_1 = a_3 - a_2 = \ldots = a_{n+1} - a_n = d$, where $d$ is a common difference of the sequence. Using (Eq. 2.9) we can state the following:

$$\begin{aligned} \frac{1}{a_1 a_2} &= \left(\frac{1}{a_1} - \frac{1}{a_2}\right) \cdot \frac{1}{d} \\ \frac{1}{a_2 a_3} &= \left(\frac{1}{a_2} - \frac{1}{a_3}\right) \cdot \frac{1}{d} \\ &\cdots \\ \frac{1}{a_n a_{n+1}} &= \left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right) \cdot \frac{1}{d} \end{aligned}$$

Replacing each term on the left of the given expression by formulas above and factoring out $1/_d$ we obtain

$$\begin{aligned} S &= \frac{1}{d}\left\{\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \ldots + \frac{1}{a_n} - \frac{1}{a_{n+1}}\right\} \\ &= \frac{1}{d} \cdot \frac{(a_{n+1} - a_1)}{a_1 \cdot a_{n+1}} \end{aligned} \tag{2.10}$$

But $a_{n+1} = a_1 + nd$, then

$$a_{n+1} - a_1 = nd \tag{2.11}$$

Replacing Eq. 2.11 into Eq. 2.10 we have the required expression for $S$, $S = \frac{nd}{d(a_1 \cdot a_{n+1})} = \frac{n}{a_1 \cdot a_{n+1}}$.

The proof is complete.

**Problem 53** Prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$.

**Proof.** Denote the given sum by $S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$. In addition, consider another series, made of one that we have already seen and evaluated:

$$\Sigma = 1 + \left( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(n-1)n} \right)$$

Each term of this auxiliary series, starting from the second term, is greater than the corresponding term of the given series, such that

$$\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}, \quad n \geq 2, \ n \in \mathbb{N}$$

Hence, the sum of all terms of the given series is less than the sum of the auxiliary series:

$$S < \Sigma = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n}, \quad n \in \mathbb{N}.$$

Therefore, we can state that $S < 2 - \frac{1}{n} < 2, \ n \in \mathbb{N}$. The statement is proven.

An interesting approach of rewriting a fraction as a difference of two other fractions can be applied to many other math problems. For example, we can use this approach in calculus when evaluating integrals like this: $\int \frac{du}{u^2 - 1}$ or any integral of the form: $\int \frac{du}{u^2 - m^2}$, where $m$ is any integer. Let us do the following problem.

**Problem 54** Evaluate the integral, $\int \frac{du}{u^2 - 1}$.

**Solution.** Consider the rational expression under a symbol of an integral. Because the quantities, $(u - 1)$ and $(u + 1)$ differ by 2, we can use the same technique (Eq. 2.9) of rewriting this as a difference of two fractions multiplied by (1/2):

$$\frac{1}{u^2 - 1} = \frac{1}{(u-1)(u+1)} = \left( \frac{1}{u-1} - \frac{1}{u+1} \right) \cdot \frac{1}{2}$$

and

$$\int \frac{du}{u^2 - 1} = \frac{1}{2} \cdot \left( \int \frac{du}{u-1} - \int \frac{du}{u+1} \right)$$
$$= \frac{1}{2} (\ln|u-1| - \ln|u+1|) + C = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C$$

**Answer.** $\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$.

**Problem 55** Prove that $\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$.

**Proof.** Would it be nice to have the sum of the first $n$ squares or the sum of $n$ fractions with those denominators but unit in each numerator? Yes. We would evaluate such sums without any troubles. These little observations can help us to prove the statement. Denote the unknown sum by $S = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{n^2}{(2n-1)(2n+1)}$ and then rewrite it using sigma notation and by applying the difference of squares formula to the $n^{\text{th}}$ term, $\sum_{n=1}^{n} \frac{n^2}{4n^2-1} = S$. Let us multiply both sides by 4 and put 4 inside the summation:

$$4 \cdot \sum_{n=1}^{n} \frac{n^2}{4n^2 - 1} = 4S$$
$$\sum_{n=1}^{n} \frac{4n^2}{4n^2 - 1} = 4S$$

Would it be nice to add just $n$ units instead? We do not have it but the following operation will make it possible

$$\sum_{n=1}^{n} \frac{4n^2}{4n^2 - 1} - \sum_{n=1}^{n} \frac{1}{4n^2 - 1} = 4S - \sum_{n=1}^{n} \frac{1}{4n^2 - 1}$$
$$\sum_{n=1}^{n} \frac{4n^2 - 1}{4n^2 - 1} = 4S - \sum_{n=1}^{n} \frac{1}{(2n-1)(2n+1)}$$
$$n = 4S - \sum_{n=1}^{n} \frac{1}{(2n-1)(2n+1)}$$

The sum on the right hand side looks familiar to you because denominator of each term consists of a product of two consecutive odd numbers that differ by 2.

$$\begin{aligned}\sum_{n=1}^{n} \frac{1}{(2n-1)(2n+1)} &= \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)} \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \ldots - \frac{1}{2n+1}\right) = \frac{2n+1-1}{2(2n+1)} \\ &= \frac{n}{2n+1}\end{aligned}$$

Finally, we have $n = 4S - \frac{n}{2n+1}$. Solving this equation for $S$, we obtain the requested quantity:

$$4S = n + \frac{n}{2n+1}$$
$$S = \frac{2n^2 + 2n}{4(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

The proof is complete.

Problem 56 demonstrates another approach for finding sums.

**Problem 56** Find the sum $S = \frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \ldots$.

**Solution.** Notice that the $n^{\text{th}}$ term of the series can be represented as $u_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$.

Let us rewrite it as follows:

$$u_n = \frac{A}{2n-1} + \frac{B}{2n+1} + \frac{C}{2n+3} = \frac{1}{(2n-1)(2n+1)(2n+3)} \tag{2.12}$$

where $A$, $B$, and $C$ are some constants to be determined.

If we put expressions on the left side of Eq. 2.12 over the common denominator, and equate both sides, we can find these constants:

$$A(4n^2 + 8n + 3) + B(4n^2 + 4n - 3) + C(4n^2 - 1) = 1$$
$$4n^2(A + B + C) + 4n(2A + B) + (3A - 3B - C) = 1$$

Since $n \neq 0$ we have to solve the system:

$$\begin{cases} A + B + C = 0 \\ 2A + B = 0 \\ 3A - 3B - C = 1 \end{cases} \Leftrightarrow A = C = 1/8, \;\; B = -\frac{1}{4} \tag{2.13}$$

Using Eq. 2.13 the given sum can be written as

$$\begin{aligned} &\frac{1}{8}(1 + 1/3 + 1/5 + 1/7 + \ldots) - \frac{1}{4}(1/3 + 1/5 + 1/7 + 1/9 + \ldots) \\ &+ \frac{1}{8}(1/5 + 1/7 + 1/9 + \ldots) \\ &= \frac{1}{8}(1 + 1/3) - \frac{1}{4}\cdot\frac{1}{3} - \frac{1}{4}(1/5 + 1/7 + \ldots) + \frac{1}{4}(1/5 + 1/7 + \ldots) \\ &= \frac{1}{12} \approx 0.0833 \end{aligned} \tag{2.14}$$

**Answer.** 1/12.

**Remark.** The required sum can be evaluated using properties of sigma notation as

$$\begin{aligned}8S &= \sum_{n=1}^{\infty}\frac{1}{2n-1} - 2\sum_{n=1}^{\infty}\frac{1}{2n+1} + \sum_{n=1}^{\infty}\frac{1}{2n+3}\\ &= \left(\sum_{n=3}^{\infty}\frac{1}{2n-1} - 2\sum_{n=3}^{\infty}\frac{1}{2n-1} + \sum_{n=3}^{\infty}\frac{1}{2n-1}\right) + 1 + \frac{1}{3} - 2\cdot\frac{1}{3} = \frac{2}{3}\\ S &= \frac{1}{12}.\end{aligned}$$

Additionally, notice that 1/12 in Eq. 2.14 is the sum of the infinite series. If the number of terms, $\boldsymbol{k}$, is some counting number we can evaluate the sum exactly as $S_k = \frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \ldots \frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{1}{12} + \frac{1}{8}\left(\frac{1}{2k+3} - \frac{1}{2k+1}\right) = \frac{1}{12} + \frac{1}{2(2k+1)(2k+3)} \to \frac{1}{12}$ as $k \to \infty$. We say that the series is convergent to 1/12. $k \to \infty$. However, for small $k$ and sums up to, for example $\frac{1}{11\cdot13\cdot15}$ ($k = 6$ and we have to add only six terms), we should use the exact formula for the partial sum above, that yields $\frac{1}{12} + \frac{1}{2\cdot13\cdot15} = \frac{201}{2340} \approx \frac{1}{12} + 0.002564 \approx .0859$

**Problem 57** Find the sum $S_n = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} \quad + \ldots + \quad \frac{1}{n(n+1)(n+2)}$.

**Solution.** Let us rewrite the $k^{\text{th}}$ term as

$$\begin{aligned}\frac{1}{k(k+1)(k+2)} &= \frac{1}{k+1}\cdot\frac{1}{k}\cdot\frac{1}{k+2} = \frac{1}{k+1}\cdot\frac{1}{2}\left[\frac{1}{k} - \frac{1}{k+2}\right]\\ &= \frac{1}{2}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right]\end{aligned}$$

Therefore, the partial sum is

$$\begin{aligned}S_n &= \frac{1}{2}\left[\frac{1}{1\cdot2} - \frac{1}{2\cdot3} + \frac{1}{2\cdot3} - \frac{1}{3\cdot4} + \frac{1}{3\cdot4} - \frac{1}{4\cdot5} + \ldots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]\\ &= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{n(n+3)}{4(n+1)(n+2)}\end{aligned}$$

Notice that $\lim_{n\to\infty} S_n = \frac{1}{4}$. The series is convergent.

**Answer.** $S_n = \frac{n(n+3)}{4(n+1)(n+2)}$

Here is another example of how these ideas can be applied in Calculus when taking integrals:

**Problem 58** Evaluate $\int \frac{dx}{x(x+1)(x+2)}$ for all positive $x$.

**Solution.** Noticing that $\frac{2}{(x)(x+1)(x+2)} = \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}$ we can evaluate the integral as $\int \frac{dx}{x(x+1)(x+2)} = \frac{1}{2}\left\{\ln[x(x+2)] - \ln(x+1)^2\right\} + C = \frac{1}{2}\ln\frac{x(x+2)}{(x+1)^2} + C$.

**Answer.** $\ln\frac{\sqrt{x(x+2)}}{x+1} + C$

**Problem 59** Evaluate $\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}$.

**Solution.** Denote $S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}$. Multiplying this by two and regrouping terms, we obtain $2 \cdot S_n = 1 + \frac{3}{2^1} + \frac{5}{2^2} + \ldots + \frac{2n-1}{2^{n-1}}$. Within this sum, we recognize a geometric series and the original sum minus the last term, The first term is 1 and the common ratio is ½.

$$1 + \left(\frac{2}{2} + \frac{1}{2}\right) + \left(\frac{2}{2^2} + \frac{3}{2^2}\right) + \left(\frac{2}{2^3} + \frac{5}{2^3}\right) + \ldots + \left(\frac{2}{2^{n-1}} + \frac{2n-3}{2^{n-1}}\right)$$

$$2S_n = 1 + \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} + S_n - \frac{2n-1}{2^n}$$

Solving this for $S_n$, $S_n = 3 - \frac{2n+3}{2^n}$. This series is convergent because if $\boldsymbol{n}$ increases the second term will approach zero and the limit of partial sums will approach 3, i.e., $\lim_{n\to\infty} S_n = 3$.

**Answer.** $S_n = 3 - \frac{2n+3}{2^n}$.

**Problem 60** Evaluate the sum: $S = \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots \frac{1}{\sqrt{1977}+\sqrt{1978}} + \ldots + \frac{1}{\sqrt{2016}+\sqrt{2017}}$.

**Solution.** This problem was given at the Volgograd District Math Olympiad, with the only difference being that the last term ended in $\frac{1}{\sqrt{1977}+\sqrt{1978}}$, because the current year was 1978. Despite the different last term, the method of solving this problem is the same: we rationalize the denominator of each fraction:

$$\frac{1}{\sqrt{n-1}+\sqrt{n}} = \frac{\sqrt{n-1}-\sqrt{n}}{\left(\sqrt{n-1}+\sqrt{n}\right)\cdot\left(\sqrt{n-1}-\sqrt{n}\right)} = \sqrt{n}-\sqrt{n-1}$$

$S$ can be written as
$S = \sqrt{2}-1+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\ldots+\sqrt{2017}-\sqrt{2016} = \sqrt{2017}-1$. Next, we can easily add the first $n$ terms of the series and find $S_n$:

$$S_n = \frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1}-1$$

This partial sum can be evaluated precisely for any natural $n$. The series is divergent because this sum will increase without bound.

**Answer.** $\sqrt{2017}-1$.

Next, using similar idea, let us solve the following problem.

**Problem 61** Positive numbers $a_1, a_2, \ldots, a_n$ form an arithmetic progression. Prove the following: $S_n = \frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$.

**Proof.** Since this looks similar to the sum we just evaluated in the previous problem, let us try the same idea: we rationalize each denominator,

$$S_n = \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}+\frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

For any arithmetic progression the differences in these denominators are the differences between consecutive terms of the arithmetic sequence and must be the same. We denote it by $d$. Next, after substitution and eliminating opposite terms, this expression will be written as

$$S_n = \frac{\sqrt{a_2}-\sqrt{a_1}}{d}+\frac{\sqrt{a_3}-\sqrt{a_2}}{d}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d} = \frac{\sqrt{a_n}-\sqrt{a_1}}{d}$$

Since the original problem does not have any information about common difference of the progression, then we can find $d$ from the formula that connects the first and the $n^{\text{th}}$ term of any arithmetic progression:

$$a_n = a_1+(n-1)d$$
$$d = \frac{a_n-a_1}{n-1}$$

Therefore, $S_n = \frac{\sqrt{a_n} - \sqrt{a_1}}{d} = \frac{(n-1)\left(\sqrt{a_n} - \sqrt{a_1}\right)}{a_n - a_1} = \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}}$

The proof is complete.

**Problem 62** Find the sum $S_n = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$

**Solution.** The following is true for the $n$th term of the series

$$n \cdot n! = (n+1)! - n! = n!(n+1) - n! = n!n$$

The given sum can be written as

$$\begin{aligned} S_n &= 2! - 1! + 3! - 2! + 4! - 3! + \ldots + n! - (n-1)! + (n+1)! - n! \\ &= (n+1)! - 1. \end{aligned}$$

The series is divergent since the limit of the partial sums does not exist.

**Answer.** $(n+1)! - 1$.

**Problem 63** Evaluate the sum: $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \ldots + 100 \cdot 2^{99}$. Find a general formula for the sum of the first $N$ terms of series $S_N = 1 + 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + 5 \cdot 16 + \ldots + N \cdot 2^{N-1}$.

**Solution. Method 1.**
Denote the required sum as $S$ and multiply it by 2, $2S = 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots + 99 \cdot 2^{99} + 100 \cdot 2^{100}$. Next, we subtract $S$ from $2S$,

$$\begin{aligned} S &= 100 \cdot 2^{100} - \left(1 + 2 + 2^2 + 2^3 + \ldots + 2^{99}\right) \\ &= 100 \cdot 2^{100} - \left(2^{100} - 1\right) \\ &= 99 \cdot 2^{100} + 1 \end{aligned}$$

Clearly, a general formula is $S_N = (N-1) \cdot 2^N + 1$.

**Method 2.**
We can rewrite this series as follows:

$$\begin{aligned}
1+2\cdot 2+3\cdot 2^2+\ldots+N\cdot 2^{N-1} &= \left(1+2+2^2+\ldots+2^{N-1}\right)\\
&\quad+\left(2+2^2+\ldots+2^{N-1}\right)+\left(2^2+2^3+\ldots+2^{N-1}\right)\\
&\quad+\left(2^3+2^4+\ldots+2^{N-1}\right)+\ldots+2^{N-1}\\
&=\left(2^N-1\right)+2\left(2^{N-1}-1\right)+2^2\left(2^{N-2}-1\right)\\
&\quad+2^3\left(2^{N-3}-1\right)+\ldots+2^{N-1}(2-1)\\
&=N\cdot 2^N-\left(1+2+2^2+\ldots+2^{N-1}\right)\\
&=N\cdot 2^N-\left(2^N-1\right)\\
S&=(N-1)2^N+1.
\end{aligned}$$

Therefore, we obtain $\sum_{N=1}^{N} N\cdot 2^{N-1}=1+2^N(N-1)$.

**Method 3.** (Using a derivative).

Consider a polynomial $P(x)=x+x^2+x^3+\ldots+x^N$ and its first derivative $P'(x)=1+2x+3x^2+\ldots+N\cdot x^{N-1}$. We can evaluate the sum of all terms of the polynomial as the sum of the $N$ terms of geometric series, $P(x)=\frac{x(x^N-1)}{x-1}=\frac{x^{N+1}-x}{x-1}$. The derivative of this sum will be

$$\begin{aligned}
P'(x)&=\frac{\left(x^{N+1}-x\right)'\cdot(x-1)-\left(x^{N+1}-x\right)(x-1)'}{(x-1)^2}=\\
&=\frac{N\cdot x^{N+1}-(N+1)x^N+1}{(x-1)^2}
\end{aligned}$$

If we replace $x=2$, we obtain that the given sum is

$$\begin{aligned}
P'(x=2)&=N\cdot 2^{N+1}-(N+1)2^N+1\\
S&=1+2^N\cdot(N-1).
\end{aligned}$$

**Answer.** $S_N=1+2^N(N-1)$.

**Problem 64** Evaluate $S=1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots+n(n+1)^2$.

**Solution.** Notice that $2\cdot 3^2+3^2=3^3,\ 3\cdot 4^2+4^2=4^3,\ \ldots,n(n+1)^2+(n+1)^2=(n+1)^3$. Hence, $a_n=(n+1)^3-(n+1)^2$. We can evaluate the series as follows:

$$\begin{aligned}
&1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots+n(n+1)^2\\
&+\\
&\quad 2^2+\quad 3^2+\quad 4^2+\ldots+\quad (n+1)^2\\
&=2^3+3^3+4^3+\ldots+(n+1)^3
\end{aligned}$$

The sum above can be rewritten as

$$\begin{aligned}
&S+\sum_{n=2}^{n+1} n^2=\sum_{n=2}^{n+1} n^3\\
&S+\frac{(n+1)(n+2)(2(n+1)+1)}{6}-1=\frac{(n+1)^2(n+2)^2}{4}-1\\
&S=\frac{(n+1)(n+2)}{2}\left(\frac{2n+3}{3}-\frac{(n+1)(n+2)}{2}\right)\\
&\ =\frac{n(n+1)(n+2)(3n+5)}{12}
\end{aligned}$$

For example, we can check this formula as $S_4=1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+4\cdot 5^2=170=\frac{4\cdot 5\cdot 6\cdot(4\cdot 3+5)}{12}=170$.

**Answer.** $S=\frac{n(n+1)(n+2)(3n+5)}{12}$.

**Problem 65** Evaluate the sum: $S=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{2015}{2016!}$.

**Solution.** Let us find the formula for the $n$th term. We can see that $a_n=\frac{n}{(n+1)!}$. Notice that $\frac{n}{(n+1)!}+\frac{1}{(n+1)!}=\frac{n+1}{(n+1)!}=\frac{1}{n!}$. Hence $a_n+\frac{1}{(n+1)!}=\frac{1}{n!}$. Since $a_{n-1}=\frac{n-1}{n!}$, then $a_{n-1}+\frac{1}{n!}=\frac{n-1}{n!}+\frac{1}{n!}=\frac{n}{n!}$ and $a_{n-1}+\frac{1}{n!}=\frac{1}{(n-1)!}$, which can be continued until we have the last term $\frac{1}{2!}+\frac{1}{2!}=\frac{2}{2!}=1$. Therefore, if we add $\frac{1}{2016!}$ to the given sum and start adding the terms by pairing them from right to left, we obtain

$$S+\frac{1}{2016!}=1$$

$$\boxed{S=1-\frac{1}{2016!}}$$

In general, we can evaluate the partial sum for any number of terms $n$, $S_n=1-\frac{1}{(n+1)!}$.

It is clear that the series is convergent because the limit of the partial sum equals 1.

**Answer.** $S=1-\frac{1}{2016!}$.

**Problem 66** Prove that
$S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

**Proof.**
**Method 1.** Consider the $n^{\text{th}}$ term of the series and rewrite it as $a_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$. Hence using sigma notation we can rewrite this sum as $\sum_{n=1}^{n} n(n+1)(n+2) = \sum_{n=1}^{n} n^3 + 3 \cdot \sum_{n=1}^{n} n^2 + 2 \cdot \sum_{n=1}^{n} n$. If we substitute Eqs. 1.29–1.31, the right hand side is rewritten as

$$S = \frac{(n+1)^2 n^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} = \frac{n(n+1)(n^2+5n+6)}{4}$$
$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

**Method 2.** On the other hand, the $n^{\text{th}}$ term and the corresponding partial sum can be evaluated as

$$a_n = (n+1)[(n+2)n] = (n+1) \cdot (n^2 + 2n) = (n+1)\left((n+1)^2 - 1\right)$$
$$= (n+1)^3 - (n+1).$$
$$S = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)((n+1)(n+2)-2)}{4}$$
$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

Which allows us to evaluate the requested sum as a difference between the sum of cubes and the sum of all natural numbers from 1 to $(n+1)$. The proof is complete.

**Problem 67** Prove that for any natural $n \geq 2,\ n \in \mathbb{N}$, the sum $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$ is greater than ½ but less than 1.

**Proof.** Consider the chain of true inequalities,

$$\frac{1}{2n} < \frac{1}{n+1} < \frac{1}{n}$$
$$\frac{1}{2n} < \frac{1}{n+2} < \frac{1}{n}$$
$$\frac{1}{2n} < \frac{1}{n+3} < \frac{1}{n}$$
$$\ldots$$
$$\frac{1}{2n} \le \frac{1}{2n} < \frac{1}{n}$$

Adding all these inequalities, we obtain $\frac{n}{2n} = \frac{1}{2} < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \frac{n}{n} = 1$.
The proof is complete.

**Problem 68** Prove the following statements:

a) $\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$

b) $\frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$

c) $\frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots$

**Proof.**

a) The partial and infinite sums for the first infinite series can be rewritten and evaluated as:

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$
$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = 1.$$

b) Consider the second sum: $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$

**Method 1.** Would it be nice to recognize a similar pattern here? Can we rewrite each term as a difference of two other terms? Let us rewrite this sum by factoring out two from each fraction:

$$2\left(\frac{1}{6}+\frac{1}{24}+\frac{1}{60}+\frac{1}{120}+\frac{1}{210}+\dots\right)$$
$$=2\cdot\left(\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{4\cdot 5\cdot 6}+\frac{1}{5\cdot 6\cdot 7}+\dots+\right)$$
$$=2\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$$

This formula must look familiar to you (Prob. 57). The sum above can be found as

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right)$$
$$=\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\dots-\frac{1}{(n+1)(n+2)}$$
$$=\frac{1}{2}-\frac{1}{(n+1)(n+2)}.$$

Therefore, the sum of infinite series is ½.

**Method 2.** One could also notice the following:

$$\frac{1}{3}=\frac{1}{2}-\frac{1}{6}$$
$$\frac{1}{12}=\frac{1}{6}-\frac{1}{12}$$
$$\frac{1}{30}=\frac{1}{12}-\frac{1}{20}$$
$$\frac{1}{60}=\frac{1}{20}-\frac{1}{30}$$
$$\frac{1}{105}=\frac{1}{30}-\frac{1}{42}$$
$$\dots$$

It looks like if we add the left and right sides of the relationships, we can evaluate the corresponding sums of the first two, first three, first four and first five terms of the series as follows:

$$S_2 = \frac{1}{3} + \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{1}{2} - \frac{1}{3 \cdot 4}$$
$$S_3 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} = \frac{1}{2} - \frac{1}{20} = \frac{1}{2} - \frac{1}{4 \cdot 5}$$
$$S_4 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} = \frac{1}{2} - \frac{1}{30} = \frac{1}{2} - \frac{1}{5 \cdot 6}$$
$$S_5 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} = \frac{1}{2} - \frac{1}{42} = \frac{1}{2} - \frac{1}{6 \cdot 7}$$

By induction, the formula for the sum of the first $n$ term is

$$S_n = \frac{1}{2} - \frac{1}{(n+1) \cdot (n+2)} \tag{2.15}$$

Using Eq. 2.15 and subtracting the sum of the first $n$ terms and the sum of the first $(n-1)$ terms we obtain the formula for the $n^{\text{th}}$ term:

$$a_n = S_n - S_{n-1} = \frac{1}{n+1}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{2}{n(n+1)(n+2)} \tag{2.16}$$

By replacing $n$ by 1, 2, 3, 4, and 5, we obtain correct values of the terms. For example,

$$a_3 = \frac{2}{3 \cdot 4 \cdot 5} = \frac{1}{30}$$
$$a_5 = \frac{2}{5 \cdot 6 \cdot 7} = \frac{1}{105}$$

Now we can predict any term of the series, $a_6 = \frac{1}{168}$, $a_7 = \frac{1}{252}$, $a_8 = \frac{1}{360}, \ldots$. Therefore, Eq. 2.15 is correct and then the infinite series sum is ½.

The proof is complete.

The second method of proof can help us to introduce the so-called **Leibniz triangle**.

The Leibniz harmonic triangle is a triangular arrangement of fractions in which each row starts with the reciprocal of the row number and every entry of the triangle is equal to the sum of the two fractions below it. For example, $\frac{1}{42} = \frac{1}{56} + \frac{1}{168}$ or $\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$, etc.. In order to see a connection between Leibniz and Pascal's triangles, we place them together as in Figure 2.2. Instead of showing the fractions as in Figure 2.1, we record only the denominators of the fractions in the Leibniz triangle. Note that the first row for both triangles corresponds to $i = 0$.

Whereas each entry in Pascal's triangle is the sum of the two entries in the above row, each entry in the Leibniz triangle is the sum of the two entries in the row below it. Denote by $P(i,j)$, $L(i,j)$, $z(i,j)$ the entries of Pascal, Leibniz, and modified Leibniz triangles, respectively. For example, in the $5^{\text{th}}$ row of Pascal triangle, the

$$1$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$
$$\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}$$
$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$
$$\frac{1}{8} \quad \frac{1}{56} \quad \frac{1}{168} \quad \frac{1}{280} \quad \frac{1}{280} \quad \frac{1}{168} \quad \frac{1}{56} \quad \frac{1}{8}$$
$$\vdots \qquad \vdots \qquad \vdots$$

**Figure 2.1** Leibniz triangle

entry $P(5,2) = 10$ is the sum of 4 and 6 in the 4th row. On the other hand, in the 5th row of the Leibniz triangle the corresponding entry $L(5,2) = 1/60$ is the sum of 1/105 and 1/140 in the 6th row. Just as Pascal's triangle can be computed by using binomial coefficients, so can Leibniz's triangle. The connection between the entries of three triangles is summarized by Eq. 2.17.

$$\begin{aligned} &L(i,j) = \frac{1}{z(i,j)} \\ &z(i,j) = \left(\frac{1}{z(i+1,j)} + \frac{1}{z(i+1,j+1)}\right)^{-1} \\ &P(i,j) = P(i-1,j-1) + P(i-1,j) \\ &P(i,j) = C_i^j = \frac{i!}{j!(i-j)!} \\ &z(i,j) = (i+1)\cdot P(i,j) \quad i = 0,\ 1,\ 2,\ \ldots \end{aligned} \tag{2.17}$$

Because any Leibniz triangle entry $L(n-1,k-1)$ is the sum of two entires, $L(n,k-1)$ and $L(n,k)$, the following is true:

$$\begin{aligned} \frac{1}{n\cdot C_{n-1}^{k-1}} &= \frac{1}{(n+1)C_n^{k-1}} + \frac{1}{(n+1)C_n^k} \\ L(n-1,k-1) &= L(n,k-1) + L(n,k) \end{aligned} \tag{2.18}$$

Please prove it yourself by using Eq. 2.17 for binomial coefficients and by adding fractions.

Consider $P(6,2) = 15$ in Pascal's triangle, $P(6,2) = \frac{6!}{(6-2)!2!} = 15$. Corresponding to it Leibniz number is $L(6,2) = \frac{1}{(6+1)P(6,2)} = \frac{1}{7\cdot 15} = \frac{1}{105}$ (Please use Figure 2.2 to see that $z(6,2) = 105$).

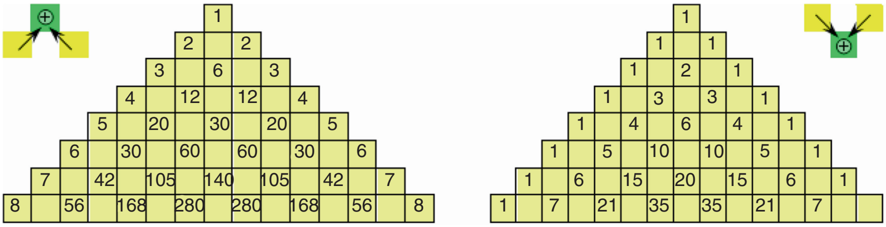


**Figure 2.2** Modified Liebnitz (*left*) and Pascal (*right*) triangles

Moreover, each diagonal of Leibniz triangle does not only relate to the corresponding Pascal's triangle diagonals but also relates to a certain modification of the figurate numbers. Consider a sequence of the numbers in the second diagonal, just $z$ numbers presented by the left diagram in Figure 2.2: 2, 6, 12, 20, 42, 56, … Each term of this sequence is 2 times the corresponding triangular number 1, 3, 6, 10, 15, 21, 28, … and can be written as $b_n = 2T_n = 2 \cdot \frac{n(n+1)}{2}$. Hence, an $n^{\text{th}}$ entry of the second Leibniz diagonal is its reciprocal, $a_n = \frac{1}{b_n} = \frac{1}{n(n+1)}$.

Consider a sequence of the numbers in the third diagonal, just $z$ numbers presented by the left diagram in Figure 2.2. Each term of this sequence, 3, 12, 30, 60, 105, 168, …, is 3 times the corresponding tethrahedron numbers and can be written as $b_n = 3TH_n = 3 \cdot \frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(n+2)}{2}$. Hence, the corresponding $n^{\text{th}}$ term of the third diagonal of Leibniz triangle (Figure 2.1) is its reciprocal (Eq. 2.16), $a_n = \frac{1}{b_b} = \frac{2}{n(n+1)(n+2)}$. Therefore, we can also state that the infinite series of the reciprocals of tetrahedral numbers is convergent and its sum is 3/2, $\sum\limits_{n=1}^{\infty} \dfrac{6}{n(n+1)(n+2)} = \dfrac{3}{2}$. The proof is complete.

Further, the first Leibniz diagonal consists of reciprocals of natural numbers, $z = 1, 2, 3, 4, 5, 6, \ldots$ The second diagonal consists of 1/(2x triangular numbers), $z = 2 \cdot 1,\ 2 \cdot 3,\ 2 \cdot 6,\ 2 \cdot 10,\ 2 \cdot 15,\ 2 \cdot 21, \ldots$ (Here 1,3,6,10,15,21, … are triangular numbers). The third diagonal consists of 1/(3x tetrahedral numbers) and so on.

**Method 3.** Consider again the sum $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$
We can see that this infinite series represent the sum of all fractions in the third diagonal of Leibniz triangle. Hence, each fraction can be replaced by the difference of two others using Eq. 2.18,

$$L(n, k-1) = L(n-1, k-1) - L(n, k) \tag{2.19}$$

For example, $z(3,1) = 12,\ P(3,1) = 3,\ L(3,1) = \frac{1}{12} = \frac{1}{(3+1)P(3,1)} = \frac{1}{3 \cdot 4}$

For some terms of the series b) we obtain the following:

$$\begin{aligned}
&\boxed{\frac{1}{3}=\frac{1}{2}-\frac{1}{6}}\\
a_1&=L(2,0)=L(1,0)-L(2,1)\\
&\boxed{\frac{1}{12}=\frac{1}{6}-\frac{1}{12}}\\
a_2&=L(3,1)=L(2,1)-L(3,2)\\
&\boxed{\frac{1}{30}=\frac{1}{12}-\frac{1}{20}}\\
a_3&=L(4,2)=L(3,2)-L(4,3)\\
&\boxed{\frac{1}{60}=\frac{1}{20}-\frac{1}{30}}\\
a_4&=L(5,3)=L(4,3)-L(5,4)\\
&\dots
\end{aligned} \tag{2.20}$$

Additionally, for this chain of equations, by induction, we can find the formula of the $n^{\text{th}}$ term of this series, $\boxed{a_n = L(n+1,n-1) = L(n,n-1) - L(n+1,n)}$. We can see that if we add the left and right sides of Eq. 2.20, then on the left we have the given series and on the right, all the terms except the first one and the last one are cancelled and that the partial sum is

$$\begin{aligned}
S_n &= \frac{1}{2} - L(n+1,n) = \frac{1}{2} - \frac{1}{(n+2)C_{n+1}^{n}}\\
&= \frac{1}{2} - \frac{1}{(n+2)(n+1)}.\\
S_\infty &= \frac{1}{2}.
\end{aligned}$$

This matches with our other formula found earlier and proves the statement.

c) Let us now prove that $\frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \dots$.

**Method 1.** Denote the sum by $S = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \dots$, and multiply and divide the right side by 6,

$$S = 6 \cdot \left(\frac{1}{4\cdot 6} + \frac{1}{20\cdot 6} + \frac{1}{60\cdot 6} + \frac{1}{140\cdot 6} + \dots\right)$$
$$S = 6\left(\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \frac{1}{3\cdot 4\cdot 5\cdot 6} + \dots\right)$$

Hence the given sum is six times the sum inside the parentheses. We have seen such a series earlier in this chapter. It can be evaluated as $S = 6 \cdot \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$. In order to evaluate an infinite sum of this series, we rewrite the $n^{\text{th}}$ term in a different form and firstly, we multiply two inner and two outer factors of the denominator, $6 \cdot \left(\frac{1}{(n^2+3n)\cdot(n^2+3n+2)}\right)$.

We obtained a familiar structure: two quantities in the denominator differ by two, and we can rewrite the fraction again and again decompose it into two new fractions as follows:

$$\begin{aligned} a_n &= \frac{6}{2} \cdot \left(\frac{1}{n(n+3)} - \frac{1}{(n+1)(n+2)}\right) = \frac{3}{3} \cdot \left(\frac{1}{n} - \frac{1}{n+3}\right) - 3\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{n} - \frac{1}{n+3} - \frac{3}{n+1} + \frac{3}{n+2}. \end{aligned}$$

This $n^{\text{th}}$ term can be rewritten in a little different form so we can calculate the partial sum of the series easily:

$$a_n = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - 2 \cdot \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - 2 \cdot \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right)$$

Now, the sum of each quantity can be evaluated separately and the final answer will be the sum of these three answers:

$$\sum_{n=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{n} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{n+3}$$

$$-2\sum_{n=1}^{n} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = -\frac{2}{2} + \frac{2}{n+2}$$

$$\boxed{S_n = \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+3} + \frac{2}{n+2}}$$

Obviously, as $n$ goes to infinity, the partial sum will go to 1/3. Therefore, $S_\infty = \frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots$.

**Method 2.** Please notice that the series is the sum of all fractions in the fourth diagonal of Leibniz triangle. Looking at that diagonal of the Leibniz triangle in Figure 2.2, and using Eq. 2.19 we have the following chain of the true relationships:

$$\begin{aligned}
&a_1 = \frac{1}{4} = \frac{1}{3} - \frac{1}{12} \;\Leftrightarrow\; a_1 = L(3,0) = L(2,0) - L(3,1)\\
&a_2 = \frac{1}{20} = \frac{1}{12} - \frac{1}{30} \;\Leftrightarrow\; a_2 = L(4,1) = L(3,1) - L(4,2)\\
&a_3 = \frac{1}{60} = \frac{1}{30} - \frac{1}{60} \;\Leftrightarrow\; a_3 = L(5,2) = L(4,2) - L(5,3)\\
&a_4 = \frac{1}{140} = \frac{1}{60} - \frac{1}{105} \;\Leftrightarrow\; a_4 = L(6,3) = L(5,3) - L(6,4)\\
&a_5 = \frac{1}{280} = \frac{1}{105} - \frac{1}{168} \;\Leftrightarrow a_5 = L(7,4) = L(6,4) - L(7,5)\\
&\ldots
\end{aligned}$$

From these relationships, by induction, we can recognize the formula for $n^{\text{th}}$ term of the series and evaluate its $n^{\text{th}}$ partial sum,

$$\begin{aligned}
a_n &= L(n+2, n-1) = L(n+1, n-1) - L(n+2, n)\\
S_n &= \sum_{i=1}^{n} a_i = \frac{1}{3} - L(n+2, n)
\end{aligned} \tag{2.21}$$

It follows from Eq. 2.21 that the $n^{\text{th}}$ partial sum of the series is 1/3 minus the Leibniz entry $L(n+2,\ n)$. Additionally, we can evaluate the $n^{\text{th}}$ term of the series by using Eq. 2.17 for $L(i,j)$,

$$\begin{aligned}
L(n+1, n-1) &= \frac{1}{(n+2)C_{n+1}^{n-1}} = \frac{1}{(n+2)\dfrac{(n+1)!}{(n-1)!\cdot 2!}}\\
&= \frac{2}{(n+2)(n+1)n}\\
L(n+2, n) &= \frac{1}{(n+3)C_{n+2}^{n}} = \frac{1}{(n+3)\dfrac{(n+2)!}{(n)!\cdot 2!}}\\
&= \frac{2}{(n+3)(n+2)(n+1)}
\end{aligned} \tag{2.22}$$

Subtracting the left and right sides, we obtain

$$\begin{aligned}
a_n &= \frac{2}{(n+2)(n+1)n} - \frac{2}{(n+3)(n+2)(n+1)} \\
&= 2 \cdot \left( \frac{1}{(n+2)(n+1)n} - \frac{1}{(n+3)(n+2)(n+1)} \right) a_n \\
&= \frac{2}{(n+2)(n+1)} \cdot \left( \frac{1}{n} - \frac{1}{n+3} \right)
\end{aligned}$$

$$\boxed{a_n = \frac{6}{n(n+1)(n+2)(n+3)}}$$

Making substitutions of the Leibniz entry $L(n+2, n)$ from Eq. 2.22 into Eq. 2.21, we have

$$S_n = \frac{1}{3} - \frac{2}{(n+3)(n+2)(n+1)} \tag{2.23}$$

If $n$ increases without bound, then the partial sum above will get closer and closer to 1/3. Therefore, the sum in part (c) is 1/3. The statement is proven.

Consider again Figure 2.2. Start counting the rows from the top $i = 1$. Take the numbers of the $n^{\text{th}}$ row and add them. For example, for the $4^{\text{th}}$ row, we have $5 + 20 + 30 + 20 + 5 = 80 = 5 \cdot 2^4$. The following statement is true.

**Lemma 2.1** The sum of the numbers in the $n^{\text{th}}$ row of a triangle made of the denominators of Leibniz triangle equals $n \cdot 2^{n-1}$.

**Proof.** The sum of all numbers in the $n^{\text{th}}$ row is the sum of the z-numbers and hence, it can be written using a definition of a $z$ number as $\sum_{k=0}^{n-1} n \cdot C_{n-1}^{k} = n \sum_{k=0}^{n-1} C_{n-1}^{k} = n \cdot 2^{n-1}$.

## 2.2 Trigonometric Series

The following problems are very different from anything above. They are trigonometric series. In order to evaluate trigonometric series we need to know trigonometric identities and de Moivre's Formula. Some formulas are given by,

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$
$$\sin x \sin y = \frac{1}{2} \cdot (\cos(x-y) - \cos(x+y)) \tag{2.24}$$
$$\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{y-x}{2}$$

de Moivre's Formula (Abraham de Moivre, French mathematician, 1667-1754)

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n. \tag{2.25}$$

We do not give a proof for the first three formulas because students study them in high school. de Moivre's Formula is not in the regular high school curriculum so we need to discuss it a little more. Let us see how easily it can be derived under assumption that the Euler's relationship below is true.

Euler's Formula

$$e^{ix} = \cos x + i \sin x \tag{2.26}$$

Let us raise the left and the right side of Eq. 2.26 to the second power, then the third, fourth, and so on and apply Eq. 2.25 again each time. We obtain the following chain of the correct equations:

$$\begin{aligned}
(e^{ix})^2 &= (\cos x + i \sin x)^2 \\
e^{i2x} &= \cos 2x + i \sin 2x \\
(e^{ix})^3 &= (\cos x + i \sin x)^3 \\
e^{i3x} &= \cos 3x + i \sin 3x \\
(e^{ix})^4 &= (\cos x + i \sin x)^4 \\
e^{i4x} &= \cos 4x + i \sin 4x \\
&\dots \\
e^{inx} &= \cos nx + i \sin nx
\end{aligned}$$

**Problem 69** Evaluate $S_n = \cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{3\pi}{n} + \ldots + \cos\frac{(n-1)\pi}{n}$

**Solution.** Let us multiply the given sum by $\sin\frac{\pi}{2n}$. Using the first formula of trigonometric identities of Eq. 2.24 and the fact that sine is an odd function $(\sin(-y) = -\sin(y))$, we obtain,

$$2S_n\sin\frac{\pi}{2n} = 2\sin\frac{\pi}{2n}\cos\frac{\pi}{n} + 2\sin\frac{\pi}{2n}\cos\frac{2\pi}{n} + \ldots + 2\sin\frac{\pi}{2n}\cos\frac{(n-2)\pi}{n} +$$
$$2\sin\frac{\pi}{2n}\cos\frac{(n-1)\pi}{n} = \sin\frac{3\pi}{2n} - \sin\frac{\pi}{2n} + \sin\frac{5\pi}{2n} - \sin\frac{3\pi}{2n} +$$
$$\ldots + \sin\frac{(2n-3)\pi}{2n} - \sin\frac{(2n-5)\pi}{2n} + \sin\frac{(2n-1)\pi}{2n} - \sin\frac{(2n-3)\pi}{2n}.$$

After simplification and canceling opposite terms we obtain $2S_n\sin\frac{\pi}{2n} = -\sin\frac{\pi}{2n} + \sin\frac{(2n-1)\pi}{2n} = -\sin\frac{\pi}{2n} + \sin\left(\pi - \frac{\pi}{2n}\right) = 0$. Considering the expression above we notice that the second factor on the left hand side is never zero for any natural $n$, therefore the given sum must be zero.

**Answer.** $S_n = 0$.

**Problem 70** Prove that

$$S_n = \sin x + \sin 2x + \sin 3x + \ldots + \sin nx = \frac{\sin\frac{nx}{2}\cdot\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}.}$$

**Proof.** This proof will involve only knowledge at a high school curriculum level, and trigonometric identities. Multiplying the sum by $2\sin(x/2)$ we obtain:

$$2\sin\frac{x}{2}(\sin x + \sin 2x + \sin 3x + \ldots + \sin nx)$$
$$= 2\sin\frac{x}{2}\sin x + 2\sin\frac{x}{2}\sin 2x + 2\sin\frac{x}{2}\sin 3x + \ldots + 2\sin\frac{x}{2}\sin nx$$
$$= \cos\left(\frac{x}{2} - x\right) - \cos\left(\frac{x}{2} + x\right) + \cos\left(\frac{x}{2} - 2x\right) - \cos\left(\frac{x}{2} + 2x\right)$$
$$+ \cos\left(\frac{x}{2} - 3x\right) - \cos\left(\frac{x}{2} + 3x\right) + \ldots + \cos\left(\frac{x}{2} - nx\right) - \cos\left(\frac{x}{2} + nx\right)$$

Since cosine is an even function, then $\cos(-y) = \cos(y)$ and all terms in the middle of the last formula will be eliminated as $\cos\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right) + \cos\left(\frac{3x}{2}\right) - \cos\left(\frac{5x}{2}\right) + \cos\left(\frac{5x}{2}\right) + \ldots - \cos\left(\frac{x(2n+1)}{2}\right)$. Now we obtain that $S_n\cdot 2\sin\frac{x}{2} = \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}$. Apply the difference of cosines formula ($3^{\text{rd}}$ formula of Eq. 2.24):

$$S_n \cdot 2\sin\frac{x}{2} = 2\sin\frac{\left(\frac{1}{2}+\frac{2n+1}{2}\right)x}{2} \cdot \sin\frac{\left(-\frac{1}{2}+\frac{2n+1}{2}\right)x}{2}$$
$$= 2\sin\frac{(n+1)x}{2} \cdot \sin\frac{nx}{2}$$

Dividing the last row by 2sin($x$/2) we prove the formula:

$$S_n = \sin x + \sin 2x + \sin 3x + \ldots + \sin nx$$
$$= \frac{\sin\frac{nx}{2} \cdot \sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}$$

You will have a chance to prove this formula a second way in a homework problem using de Moivre's Formula. You can use the next problem as an example.

**Problem 71** Evaluate $A = \frac{\cos\frac{\pi}{4}}{2} + \frac{\cos\frac{2\pi}{4}}{2^2} + \ldots + \frac{\cos\frac{\pi n}{4}}{2^n}$.

**Solution.** Denote

$$B = \frac{\sin\frac{\pi}{4}}{2} + \frac{\sin\frac{2\pi}{4}}{2^2} + \ldots + \frac{\sin\frac{\pi n}{4}}{2^n} \tag{2.27}$$

Assuming that $B$ is imaginary part of a complex number $A + \mathrm{i}B$, we multiply Eq. 2.27 by $i$ and add the corresponding $A$:

$$A + iB = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \frac{1}{2^2}\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) + \ldots + \frac{1}{2^n}\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right)$$

Applying de Moivre's Formula (Eq. 2.25) to the previous expression, we have

$$A + iB = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \frac{1}{2^2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2 + \ldots + \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n \tag{2.28}$$

We can notice that Eq. 2.28 is a geometric series with both, first term and the ratio equal to $\frac{1}{2} \cdot \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

Therefore, using the sum of geometric series, Eq. 2.28 can be rewritten in a compact form as follows:

$$A + iB = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{\left(1 - \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right)}{\left(1 - \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)} \tag{2.29}$$

Applying de Moivre's Formula (Eq. 2.25) to Eq. 2.29 again and using the fact that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ we have,

$$A + iB = \frac{1}{2\sqrt{2}}(1+i)\cdot\frac{\left(1 - \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right)}{\left(1 - \frac{1}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right)} \tag{2.30}$$

Rationalizing the denominator and extracting the real part of $A + iB$ in Eq. 2.30, we obtain

$$A = \frac{\left(\sqrt{2}-1\right)\left(2^n - \cos\frac{\pi n}{4}\right) + \sqrt{2}\sin\frac{\pi n}{4}}{2^n\left(5 - 2\sqrt{2}\right)}.$$

**Answer.** $A = \dfrac{\left(\sqrt{2}-1\right)\left(2^n - \cos\frac{\pi n}{4}\right) + \sqrt{2}\sin\frac{\pi n}{4}}{2^n\left(5 - 2\sqrt{2}\right)}$

## 2.3 Using Mathematical Induction for Sequences and Series

The principle of mathematical induction is very helpful in proving many statements about positive integers. According to this principle, a mathematical statement involving the variable $n$ can be shown to be true for any positive integer $n$ by proving the following two statements:

- The statement is true for $n = 1$
- If the statement is true for any positive integer $k$, then it is also true for $n = k + 1$.

Let us show how mathematical induction can help us to prove and solve some problems involving sequences and series.

**Problem 72** Use mathematical induction to prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is true for any positive number $n$.

**Proof. Step 1.** Replacing $n$ by 1 in the above equality gives $2 \cdot (1) - 1 = 1$ which is true, so $n = 1$ satisfies the equation.

**Step 2.** Assume that the equality is true at $n = k$. And let us show that it will be true at $n = k + 1$. If $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ is true, then let us show that for $n = k + 1$ the left side of the equality equals $(k + 1)^2$.

$$1 + 3 + 5 + \cdots + (2k - 1) + \big(2(k + 1) - 1\big).$$

Start with the left hand side, and notice that (because of our assumption) it is equal to $k^2$, plus an additional term.

$1 + 3 + 5 + \ldots + (2k - 1) + (2(k + 1) - 1) = k^2 + 2k + 1 = (k + 1)^2.$

The final equality proves that the equation is true for $n = k + 1$, given that it is true for $n = k$. By the principle of mathematical induction, we have proven the statement.

**Problem 73** Prove that: $\sum\limits_{k=1}^{n} k^3 = \left(\sum\limits_{k=1}^{n} k\right)^2.$

**Proof.** Step 1. Replacing $n$ by 1 in the above equality gives

$1^3 = 1^2$ which is true, so $n = 1$ satisfies the equation.

Step 2. Assume that the equality is true at $n = k$ and let us show that it will be true at $n = k + 1$:

If $1^3 + 2^3 + 3^3 + \ldots + k^3 = (1 + 2 + 3 + \ldots + k)^2 = \frac{k^2(k+1)^2}{4}$ is true, then let us show that for $n = k + 1$ the left side of the given equality equals $(1 + 2 + 3 + \ldots + k + 1)^2 = \frac{(k+1)^2(k+2)^2}{4}$.

We can state that $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (1 + 2 + 3 + \ldots + k)^2 + (k + 1)^3$. Replacing the right hand side, putting fractions over the common denominator and factoring, we obtain the required formula:

$$\begin{aligned}\frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\ &= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}\\ &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

The final equality proves that the equation is true for $n = k + 1$, assuming that it is true for $n = k$. Using the principle of mathematical induction, we have completed our proof.

In Prob. 74, we use mathematical induction for recurrent sequences.

**Problem 74** Given a sequence $\{x_n\}$ such that $x_0 = 2,\ x_1 = \frac{3}{2},\ x_{n+1} = \frac{3}{2}x_n - \frac{1}{2}x_{n-1}$. a) Find the exact formula for $x_n$. b) Evaluate $\lim\limits_{n\to\infty} x_n$, if the limit exists. c) Is series $S_\infty = \sum\limits_{k=0}^{\infty} x_k$ convergent or divergent?

**Solution.**

a) Using the given recursive formula we can calculate a few terms of the sequence:

$$\begin{aligned}
x_0 &= 2 = 1 + 2^0\\
x_1 &= \frac{3}{2} = 1 + \frac{1}{2} = 1 + 2^{-1}\\
x_2 &= \frac{3}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot 2 = 1 + \frac{1}{4} = 1 + 2^{-2}\\
x_3 &= \frac{3}{2}\cdot\frac{5}{4} - \frac{1}{2}\cdot\frac{3}{2} = \frac{9}{8} = 1 + \frac{1}{8} = 1 + 2^{-3}\\
x_4 &= \frac{3}{2}\cdot\frac{9}{8} - \frac{1}{2}\cdot\frac{5}{4} = \frac{17}{16} = 1 + 2^{-4}
\end{aligned}$$

Notice that every $n^{\text{th}}$ term of the sequence is obtained as a sum of 1 and 2 raised to a negative power that is equal to the number of the term. We can assume that

$$x_n = 1 + 2^{-n} \tag{2.31}$$

Using mathematical induction let us prove that Eq. 2.31 is the exact formula for the $n^{\text{th}}$ term of the sequence. Denote by $A(m)$ our statement for $n = m$.
Step 1. $A(1)$ is true because $x_1 = \frac{3}{2} = 1 + 2^{-1}$
Step 2. Assume that $A(k)$ and $A(k-1)$ are true (i.e., Eq. 2.31 holds for $n = k$ and for $n = k - 1$), i.e., $x_k = 1 + 2^{-k}$ and $x_{k-1} = 1 + \frac{1}{2^{k-1}}$.
Step 3. Let us prove that $A(k+1)$ is also true. That is, we need to show that $x_{k+1} = 1 + 2^{-(k+1)} = 1 + 2^{-k-1}$. Indeed, $x_{k+1} = \frac{3}{2}x_k - \frac{1}{2}x_{k-1}$ by the condition of the problem, then

$$\begin{aligned}
x_{k+1} &= \frac{3}{2}\cdot\left(1+\frac{1}{2^k}\right)-\frac{1}{2}\cdot\left(1+\frac{1}{2^{k-1}}\right)\\
&= \frac{1}{2}\cdot\frac{\left(3\cdot 2^k+3-2^k-2\right)}{2^k}\\
&= \frac{1}{2}\cdot\frac{\left(2\cdot 2^k+1\right)}{2^k}\\
&= \frac{1}{2}\cdot\left(2+\frac{1}{2^k}\right)\\
&= 1+\frac{1}{2^{k+1}}\\
&= 1+2^{-(k+1)}
\end{aligned}$$

We proved that Eq. 2.31 is the exact formula for the $n^{\text{th}}$ term of the sequence. The proof is complete.

b) Now, knowing the $n^{\text{th}}$ term of the sequence explicitly, $x_n = 1 + 2^{-n}$, let us find the limit of $\{x_n\}$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\left(1+\frac{1}{2^n}\right) = 1$.

c) The series is divergent because, as we established above, the limit of the $n$th term as $n$ goes to infinity is not zero. Therefore, the series does not pass the Necessary Condition (See Chapter 3 for clarification). We can also evaluate the partial sum for the series exactly as $S_n = 2 + 1 + \frac{1}{2^1} + 1 + \frac{1}{2^2} + 1 + \frac{1}{2^3} + \ldots + 1 + \frac{1}{2^n}$. You can see that this is the sum of $(2 + n\cdot 1)$ and the first $n$ terms of geometric series with $b_1 = \frac{1}{2}$, $r = \frac{1}{2}$. Thus, $S_n = 2 + n + \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 2 + n + 1 - \left(\frac{1}{2}\right)^n$ so $S_n = 3 + n - \left(\frac{1}{2}\right)^n$.

We can see that this partial sum depends on $n$ and increases without bound as $n$ increases.

**Answer.** a) $X_n = 1 + 2^{-n}$; b) $S_n = 3 + n - \left(\frac{1}{2}\right)^n$. c) The series is divergent.

In the homework chapter, you will be asked to find the formula for the $n^{\text{th}}$ term using the knowledge of recursion.

**Problem 75** Given a sequence $\{a_n\}$, $a_n = n(3n+1)$, $n \in \mathbb{N}$. Prove that its $n^{\text{th}}$ partial sum can be evaluated by formula $S_n = n(n+1)^2$.

**Proof.** We prove this by induction. It is easy to see that the formula is true for $n = 1$. Indeed, $S_1 = 1\cdot(1+1)^2 = a_1 = 4$. If we evaluate the sums of two, three, four of even five terms of the sequence, we see that the formula works. However, it does not prove the statement. Assume that this formula is correct for $n = k$, i.e., the sum of the first k terms of the given sequence equals $S_k = k\cdot(k+1)^2$. Let us demonstrate that it will be also true for $n = k+1$, i.e., $S_{k+1} = (k+1)\cdot(k+2)^2$.

We know that the sum of $(k+1)$ terms of the sequence equals the sum of its first $k$ terms plus the $(k+1)^{\text{st}}$ term, $S_{k+1} = S_k + a_{k+1}$. Substituting here the $k^{\text{th}}$ sum and the value of the $(k+1)^{st}$ term of the sequence we obtain

$$\begin{aligned} S_{k+1} &= k(k+1)^2 + (k+1)(3 \cdot (k+1) + 1) \\ &= k(k+1)^2 + (k+1)(3k+4) \\ &= (k+1)(k(k+1) + 3k + 4) \\ &= (k+1)(k+2)^2. \end{aligned}$$

Therefore, the formula is correct for $n = k+1$, hence it is true any natural $n$. The statement is proven.

**Problem 76** Prove that any term of the sequence $a_n = 4 \cdot 6^n + 5n - 4$ is divisible by 25.

**Proof.** We can substitute $n = 1$ and obtain that $a_1 = 25$. Yes, it is divisible by 25.

Assume that the statement is true for $n = k$ and that $a_k = 4 \cdot 6^k + 5k - 4$ is divisible by 25, then it can be written as $4 \cdot 6^k + 5k - 4 = 25m \Rightarrow 4 \cdot 6^k = 25m + 4 - 5k$. Let us prove that the next term, $k+1$, $a_{k+1} = 4 \cdot 6^{k+1} + 5(k+1) - 4$ is also is also divisible by 25. Next, because the $k^{\text{th}}$ term is divisible by 25, we extract the $k$th term of the sequence in the expression of the $(k+1)$ term,

$$a_{k+1} = 6 \cdot 4 \cdot 6^k + 5k + 1 = 6 \cdot \left(4 \cdot 6^k + 5k - 4\right) - 25k + 25.$$

Each term of the sum is a multiple of 25, then the total sum or $(k+1)^{st}$ term is divisible by 25. You could do this proof a little bit differently by replacing the $k^{\text{th}}$ term by $25 \cdot m$:

$$\begin{aligned} a_{k+1} &= 6 \cdot 4 \cdot 6^k + 5k - 4 + 5 \\ &= 6(25m + 4 - 5k) + 5k + 1 \\ &= 150m - 25k + 25 \\ &= 25 \cdot (6m - k + 1) \end{aligned}$$

It is clear that it is divisible by 25. The statement is proven.

**Problem 77** Given a Fibonacci sequence $\{a_n\} : a_1 = a_2 = 1, \; a_n = a_{n-1} + a_{n-2}, \; n > 2$. Prove that the terms of the sequence satisfy the equation: $a_{n+1}^2 - a_n \cdot a_{n+2} = (-1)^n, \; \forall n \in \mathbb{N}$.

**Proof.** We prove this by induction.

1. Notice that the equality is true for $n = 1$ because $a_2^2 - a_1 \cdot a_3 = 1 - 1 \cdot 2 = (-1)^1$.
2. Assume that the statement is true for $n = k$, $a_{k+1}^2 - a_k \cdot a_{k+2} = (-1)^k$. From this it follows that $a_{k+1}^2 = (-1)^k + a_k \cdot a_{k+2}$
3. Let us demonstrate that it is also true for $n = k + 1$, i.e., $a_{k+2}^2 - a_{k+1} \cdot a_{k+3} = (-1)^{k+1}$.

Let us substitute the expression for the $(k+3)^{\text{th}}$ term of Fibonacci sequence, $a_{k+2}^2 - a_{k+1} \cdot (a_{k+1} + a_{k+2}) = a_{k+2}^2 - a_{k+1} \cdot \ a_{k+2} - \boxed{a_{k+1}^2}$. Substituting in this formula the value of the term in the box, we obtain

$$\begin{aligned} a_{k+2}^2 - a_{k+1} \cdot a_{k+3} &= a_{k+2}^2 - a_{k+1} \cdot a_{k+2} - (-1)^k - a_k \cdot a_{k+2} \\ &= a_{k+2}^2 - a_{k+2}(a_k + a_{k+1}) + (-1)^{k+1} \\ &= \underbrace{a_{k+2}^2 - a_{k+2} \cdot a_{k+2}}_{0} + (-1)^{k+1}. \end{aligned}$$

The proof is complete.

## 2.4 Problems on the Properties of Arithmetic and Geometric Sequences

If three numbers form an arithmetic sequence, the middle term is called the arithmetic mean of the other two. Thus,

$$\begin{aligned} a_3 - a_2 &= a_2 - a_1 \\ 2a_2 &= a_1 + a_3 \\ a_2 &= \frac{a_1 + a_3}{2} \end{aligned}$$

By analogy the arithmetic mean of two numbers is half of their sum. Therefore, $\frac{(a+b)}{2}$ is the arithmetic mean of numbers $a$ and $b$, or the average of two numbers.

Similarly the average or mean value of three numbers $a$, $b$, and $c$ is $\frac{a+b+c}{3}$. In general, the mean value of $n$ positive numbers, $a_1, a_2, a_3, \ldots..a_n$ is $\frac{a_1+a_2+\ldots+a_n}{n}$, that is the average of the sum. Besides the arithmetic mean defined above there is another form of mean value that is defined by the formula: $b = \sqrt{ac}$. Value $b$ is called a geometric mean of numbers $a$ and $c$. Recalling a geometric progression with positive terms $b_1, b_2, \ldots\ldots, b_{n-1}, b_n, b_{n+1}\ldots.$ and common ratio $r$ such that

$\frac{b_n}{b_{n-1}} = \frac{b_{n+1}}{b_n}$, $b_n = \sqrt{b_{n-1}b_{n+1}}$ or ${b_n}^2 = b_{n-1} \cdot b_{n+1}$. Every term of a geometric progression is a geometric mean of the preceding and consequent terms.

Now consider the following problems.

**Problem 78** Peter lives near a bus stop A. The bus stops A, B, C, and D are on the same street. Peter walks for exercise every weekend. He starts at A with a speed of 5 km per hour and goes to D. Reaching D he turns back and goes to B. Walking this rout (A-D-B) requires 5 h. At B Peter takes a bus and goes home. It is known that he can cover the distance between A and C in 3 h. The distances between A and B, B and C, C and D form a geometric sequence in the given order. Find the distance between B and C.

**Solution.** Usually it is a good idea to draw a picture of the problem. A, B, C, and D are on the same street (Figure 2.3). It means that we can draw them as points on the same line, A and D will be the end points of the segment and B and C between them in the order A-B-C-D.

Because our unknown is the distance between B and C it seems obvious to introduce 3 variables $x$, $y$, and $z$ as distances between A and B, B and C, and C and D respectively. Using the condition of the problem and distance $=$ speed $\cdot$ time we write, $x + y + z + z + y = 5 \cdot 5 = 25$ and $x + y = 3 \cdot 5 = 15$.

Now we are going to write the last equation of the system. Because $x$, $y$, and $z$ are consecutive terms of a geometric sequence, then $y^2 = xz$, and we can complete a system:

$$\begin{cases} x + 2y + 2z = 25 \\ x + y = 15 \\ y^2 = xz \end{cases}$$

$$\begin{cases} y + 2z = 10 \\ x = 15 - y \\ y^2 = xz \end{cases}$$

$$\begin{cases} z = \dfrac{10 - y}{2} \\ x = 15 - y \\ y^2 = \dfrac{(15 - y)(10 - y)}{2} \end{cases}$$


**Figure 2.3** Problem 78

Subtracting the second equation from the first of the first system, we can eliminate variable $x$ in the second system. Then we express $z$ and $x$ in terms of $y$ and put them into the third equation of the last system. Let us solve the last equation for $y$. Multiplying both sides by 2, we have

$$2y^2 = 150 - 25y + y^2$$
$$y^2 + 25y - 150 = 0$$
$$y_1 = 5, \quad y_2 = -30$$

Because $y$ is a distance, it has to be a positive, so we choose $y = 5$.

**Answer.** The distance between $B$ and $C$ is 5 km.

**Problem 79** The four numbers $a$, $b$, $c$, and $z$ are given. It is known that the first three numbers form an arithmetic sequence, and the last three numbers form a geometric sequence. A sum of the outer terms is 4 and the sum of the inner terms is 2. Find the numbers.

**Solution.** Let us write the numbers in a row: $a\ b\ c\ z$. If variables $a$, $b$ and $c$ are terms of an arithmetic sequence, then

$$b = \frac{a + c}{2} \tag{2.32}$$

On the other hand,

$$\begin{array}{l} +\begin{cases} a + z = 4 \\ b + c = 2 \end{cases} \\ \hline a + b + c + z = 6 \end{array} \tag{2.33}$$

Replacing $(a + c)$ by $2b$ from Eq. 2.32 into Eq. 2.33, we have

$$3b + z = 6 \tag{2.34}$$

Our purpose now is to eliminate some variables. It would be nice to obtain a system of two equations in just two variables. (for example, $z$ and $b$). Let us use the second part of the condition. If $b$, $c$, and $z$ form a geometric sequence, then $c$ is a geometric mean of $b$ and $z$ or

$$c^2 = b \cdot z \tag{2.35}$$

Expressing $c$ as $(2 - b)$ from system (Eq. 2.33) and substituting into (Eq. 2.35) we derive

$$(2-b)^2 = b \cdot z \tag{2.36}$$

Let us combine Eqs. 2.34 and 2.36 as

$$\begin{aligned}
&(2-b)^2 = b(6-3b)\\
&4 - 4b + b^2 = 6b - 3b^2\\
&2b^2 - 5b + 2 = 0\\
&b_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{5 \pm 3}{4}\\
&b_1 = 2 \qquad b_2 = 0.5
\end{aligned}$$

Two different values of $b$ will give us two sets of $a$, $b$, $c$, and $z$.

1. $b = 2$
   $z = 6 - 3b = 0$
   $a = 3b - 2 = 4$
   $c = 2 - b = 0$
2. $b = 0.5$
   $z = 4.5$
   $a = -0.5$
   $c = 1.5$

**Answer.** $(a, b, c, z) = \{(4,\ 2,\ 0, 0),\ (-0.5,\ 0.5,\ 1.5,\ 4.5)\}$

**Problem 80** The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 19$, $a_9 = 99$, and for any $n \geq 3$, $a_n$ is the arithmetic mean of the first $(n-1)$ terms. Find $a_2$.

**Solution.** Let us write down the formula for the $n^{\text{th}}$ and $(n-1)^{st}$ terms of the sequence:

$$a_n = \frac{a_1 + a_2 + \ldots + a_{n-2} + a_{n-1}}{n-1} \tag{2.37}$$

$$a_{n-1} = \frac{a_1 + a_2 + \ldots + a_{n-2}}{n-2} \tag{2.38}$$

Using Eq. 2.38 we can find that

$$a_1 + a_2 + \ldots + a_{n-2} = (n-2)a_{n-1} \tag{2.39}$$

Plugging Eq. 2.39 into Eq. 2.37 we obtain $a_n \cdot (n-1) = a_{n-1} \cdot (n-2) + a_{n-1} = a_{n-1} \cdot (n-1)$. Therefore,

$$a_n = a_{n-1} \text{ for any } n \geq 3 \tag{2.40}$$

1. Since $a_9 = 99$ we can rewrite Eq. 2.40 as $a_3 = a_4 = \ldots = a_9 = 99$
2. Now we can evaluate $a_2$, $a_3 = \frac{a_1+a_2}{2} \Rightarrow 99 = \frac{19+a_2}{2} \Leftrightarrow a_2 = 2 \cdot 99 - 19 = 179$.

**Answer.** 179

**Problem 81** (MGU Entrance exam 2008). Integers $x$, $y$, $z$ are members of a geometric progression but numbers $7x-3$, $y^2$, $5z-6$ are members of an arithmetic progression. Find $x$, $y$ and $z$.

**Solution.** From the condition of the problem and with the use of geometric and arithmetic means, we have the following two equations,

$$y^2 = xz$$
$$y^2 = \frac{7x - 3 + 5z - 6}{2}$$

from which

$$zx = \frac{7x + 5z - 9}{2}$$
$$2zx = 7x + 5z - 9$$
$$x(2z - 7) = 5z - 9$$
$$x = \frac{5z - 9}{2z - 7}$$

Multiplying both sides of the last equation by 2 we obtain $2x = \frac{2 \cdot 5z - 2 \cdot 9}{2z-7}$. Extracting the largest integer from the numerator of the last fraction we obtain $2x = \frac{5(2z-7)+17}{2z-7} = 5 + \frac{17}{2z-7}$. Since 17 is prime, then in order for $2x$ to be an integer, $(2z-7)$ can take only the following values: $\pm 1; \pm 17$.

Consider the following cases:

a) If $2z - 7 = 1$, then $z = 4$, $x = 11$, $y = \sqrt{xz} = \sqrt{44}$, not a solution
b) If $2z - 7 = -1$, then $z = 3$, $x = -6$, $xz < 0$, not a solution
c) $2z - 7 = 17$, then $z = 12$, $x = 3$, $y = 6$ or $y = -6$
d) $2z - 7 = -17$, $z = -5$, $x = 2$, $zx < 0$, not a solution

**Answer.** $(x,y,z) = \{(3,6,12),(3,-6,12)\}$.

**Problem 82** (Lidsky). Prove that there exists an infinite convergent geometric series $1, r, r^2, \ldots, r^n, \ldots$ each member of which divided by the sum of all terms following it, equals given number $k$. For what value of $k$ does the problem have a solution?

**Solution.** By the condition of the problem $|r| < 1$ and we have $r^n = k(r^{n+1} + r^{n+2} + \ldots) = kr^{n+1} \cdot \frac{1}{1-r}$. From this expression $1 - r = kr$ or solving for $r$, $r = 1/(k+1)$. Since $|r| < 1$, then

$$\left|\frac{1}{k+1}\right| < 1$$
$$k > 0 \text{ or } k < -2$$

**Answer.** $k \in (-\infty, -2) \cup (0, \infty)$.

**Problem 83** (AIME 2000). A sequence of numbers $x_1, x_2, \ldots, x_{100}$ has the property that, for every integer $k$ between 1 and 100, inclusive, the number x $_k$ is $k$ less than the sum of the other 99 numbers. Given that $x_{50} = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, find $(m+n)$.

**Solution.** Because by the condition of the problem, $x_k + k = x_1 + x_2 + \ldots + x_{k-1} + x_{k+1} + \ldots + x_{100}$, then

$$\begin{aligned}
&x_1 + 1 = x_2 + x_3 + \ldots + x_{100}\\
&x_2 + 2 = x_1 + x_3 + \ldots + x_{100}\\
&\ldots\\
&x_{50} + 50 = x_1 + x_2 + \ldots + x_{49} + x_{51} + \ldots + x_{100}\\
&x_{51} + 51 = x_1 + x_2 + \ldots + x_{50} + x_{52} + \ldots + x_{100}\\
&\ldots\\
&x_{99} + 99 = x_1 + x_2 + \ldots + x_{98} + x_{100}\\
&x_{100} + 100 = x_1 + x_2 + \ldots + x_{99}
\end{aligned} \tag{2.41}$$

Let us add $x_k$ to both sides of each equation, where $k$ is the number of the equation:

$$2x_k + k = \sum_{i=1}^{100} x_i, \quad k = 1, 2, \ldots, 100. \tag{2.42}$$

Now the right side of each equation will be the same, $\sum_{i=1}^{100} x_i$. For example, for the 50th equation we have

$$2x_{50} + 50 = \sum_{i=1}^{100} x_i \tag{2.43}$$

**Method 1.** (Using properties of sigma notation)
Adding the left and right sides of Eq. 2.42 for $k = 1,2, ..., 100$ and after simplification, we obtain that $2\sum_{i=1}^{100} x_i + \sum_{i=1}^{100} i = 100\sum_{i=1}^{100} x_i$, which can be simplified as follows: $\frac{100 \cdot 101}{2} = 98\sum_{i=1}^{100} x_i$

On the other hand, replacing the sum here by the left side of Eq. 2.43 for the 50th equation, we obtain $98(2x_{50} + 50) = \frac{100 \cdot 101}{2}$. After simplification and replacement the 50th term by the formula given in the condition of the problem yields

$$50 \cdot 101 = 98\left(2 \cdot \frac{m}{n} + 50\right)$$

Solving for $m/n$ we obtain that

$$\frac{m}{n} = \frac{75}{98}$$

Therefore, $m + n = 75 + 98 = 173$.

**Method 2.** (Using properties of an arithmetic sequence)
Subtracting the left and the right sides of two consecutive equations of Eq. 2.41 and then dividing both sides by 2 we obtain that $x_k - x_{k-1} = -0.5$, where $k = 2$, 3, . . ..100. This means that the sequence $\{x_k\}$ is an arithmetic progression with the common difference $d = -0.5$. Now the 50th term of the sequence can be written as

$$\begin{aligned} x_{50} &= x_1 + 49 \cdot d \\ x_{50} &= x_1 - 0.5 \cdot 49 \end{aligned} \tag{2.44}$$

Using Eq. 1.8 for the sum of an arithmetic sequence, we rewrite Eq. 2.42 for $k = 1$ as follows:

$$\begin{aligned} &2x_1 + 1 = \frac{2x_1 + 99d}{2} \cdot 100; \\ &2x_1 + 1 = (2x_1 + 99 \cdot (-0.5))50; \\ &x_1 = \frac{1238}{49}. \end{aligned}$$

Substituting the value into Eq. 2.44 we obtain $x_{50} = \frac{75}{98} = \frac{m}{n}$ or $m + n = 75 + 98 = 173$.

**Answer.** $m/n = 75/98$ and $m+n = 173$.

Let us compare geometric and arithmetic means of two positive numbers $a$ and $b$. What is greater $\frac{a+b}{2}$ or $\sqrt{ab}$? Because both, $a$ and $b$ are positive, we can raise both sides to the second power: $\frac{(a+b)^2}{4} \quad \wedge \quad ab$. The symbol $\wedge$ will mean "compare" for us. If an arithmetic mean is greater than a geometric mean, then $\frac{(a+b)^2}{4} - ab > 0$, Thus, $(a+b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2 \wedge 0$. Because $(a-b)^2 \geq 0$, then we conclude that the arithmetic mean of two positive numbers is always greater their geometric mean, and is equal to their geometric mean if and only if $a = b$,

$$\frac{a+b}{2} \geq \sqrt{ab}$$
$$a + b \geq 2\sqrt{ab}.$$

Let us solve the following problem:

**Problem 84** For how many ordered pairs $(x, y)$ of integers is it true that the arithmetic mean of $x$ and $y$ is exactly 2 more that the geometric mean of $x$ and $y$?

**Solution.**

$$\frac{x+y}{2} = 2 + \sqrt{xy}$$
$$x + y = 4 + 2\sqrt{xy}$$
$$(x + y - 4)^2 = 4xy$$
$$x^2 + y^2 + 2xy - 8(x + y) + 16 = 4xy$$
$$x^2 - 2xy + y^2 = 8(x + y - 2)$$
$$(x - y)^2 = 8(x + y - 2)$$
$$(x - y)(x - y) = 2 \cdot 2 \cdot 2 \cdot (x + y - 2)$$

Because $x$ and $y$ are integers from last equation above, we can write only three possible systems:

1. $\begin{cases} x - y = 8 \\ x - y = x + y - 2 \end{cases} \qquad \left[\begin{array}{l} x = 9 \\ y = 1 \end{array}\right.$
2. $\begin{cases} x - y = 4 \\ x - y = (x + y - 2) \cdot 2 \end{cases} \qquad \left[\begin{array}{l} x = 4 \\ y = 0 \end{array}\right.$

3. $\begin{cases} x-y=2 \\ x-y=4(x+y-2) \end{cases}$ $\left[\begin{array}{l} x=\dfrac{9}{4}=2.25 \\ y=0.25 \end{array}\right.$

4. $\begin{cases} x-y=1 \\ x-y=8(x+y-2) \end{cases}$

Systems (3) & (4) do not give us integers. So we have two possible ordered pairs:

**Answer.** (9, 1) and (4, 0).

## 2.5 Miscellaneous Problems on Sequences and Series

**Problem 85** (Kaganov) Prove that $(a_1+a_2+\ldots+a_m)^2 \le m\left(a_1^2+\ldots+a_m{}^2\right)$ for any real numbers $a_i$

**Solution.** Let us prove it by mathematical induction.

1. $m=1$. The statement is true for $m=1$.
2. Assume this statement is true for $(a_1+a_2+\ldots+a_{m-1})^2 \le (m-1)\left(a_1^2+\ldots+a_{m-1}{}^2\right)$. Denote the left side by $\alpha$ and the right side by $\beta$, $\alpha \le \beta$.
3. Consider that

$$(a_1+a_2+\ldots+a_m)^2 = \alpha + a_m{}^2 + 2a_m \cdot (a_1+a_2+\ldots+a_m);$$
$$m(a_1{}^2+\ldots+a_{m-1}{}^2+a_m{}^2) = (m-1)(a_1{}^2+\ldots+a_{m-1}{}^2)$$
$$+a_1{}^2+\ldots+a_{m-1}{}^2+ma_m{}^2$$

Since $(a_1+a_2+\ldots+a_{m-1})^2 \le (m-1)\left(a_1^2+\ldots+a_{m-1}{}^2\right)$ is true, then

$$(a_1+a_2+\ldots+a_{m-1}+a_m)^2$$
$$=\alpha + a_m{}^2 + 2a_m(a_1+\ldots+a_{m-1}) \le \beta + \sum_{i=1}^{m-1}\left(a_i{}^2+a_m{}^2\right)+a_m{}^2;$$

From which it follows that

$$\alpha \le (a_1{}^2-2a_1\cdot a_m+a_m{}^2)+(a_2{}^2-2a_2\cdot a_m+a_m{}^2)+\ldots+(a_{m-1}{}^2-2a_{m-1}\cdot a_m+a_m{}^2)$$
$$+\beta; \alpha \le \beta+(a_1-a_m)^2+(a_2-a_m)^2+\ldots+(a_{m-1}-a_m)^2.$$

The proof is complete.

The following problem will demonstrate how the knowledge of sequences helps us to do calculus problems.

**Problem 86** Evaluate $\lim_{x \to 1} \frac{x^{n+1}-1}{x^n-1}$.

**Solution.** This limit cannot be found directly, because when x = 1 the denominator becomes 0. Using formulas for geometric series and applying them for the numerator and denominator:

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}$$
$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

We remove discontinuity and evaluate the limit.

$$\lim_{x \to 1} \frac{1 + x + x^2 + \ldots + x^n}{1 + x + x^2 + \ldots + x^{n-1}} = \frac{(n+1) \cdot 1}{n \cdot 1} = \frac{n+1}{n}, \forall n \in N.$$

**Answer.** $\lim_{x \to 1} \frac{x^{n+1}-1}{x^n-1} = \frac{n+1}{n}$.

**Problem 87** (Rivkin) Given

$1 + a + a^2 + \ldots + a^n = (1+a)(1+a^2)(1+a^4)\cdots\left(1+a^{2^k}\right)$.

Find relationship between $n$ and $k$.

**Solution.** The left side of the formula can be rewritten as $\frac{a^{n+1}-1}{a-1}$. Multiplying the both sides by $(a-1) \neq 0$ and because $a \neq 1$, the given relation can be rewritten as $a^{n+1} - 1 = (a-1)(1+a)(1+a^2)\ldots\left(1+a^{2^n}\right)$. Next, using a difference of squares formula applied several ($k$) times, the right side can be simplified as $(a^2-1)(1+a^2)\cdots\left(1+a^{2^k}\right) = (a^4-1)\ldots\left(1+a^{2^k}\right) = a^{2^{k+1}} - 1$. Therefore, $a^{n+1} = a^{2^{k+1}}$. Because by the condition of the problem $a \neq 0, \pm 1$, then the necessary relationship between $n$ and $k$ is $n + 1 = 2^{k+1}$.

**Answer.** $n = 2^{k+1} - 1$.

**Problem 88** Given a function $g(n) = \frac{f(n+1)}{2-(f(0)+f(1)+f(2)+\ldots+f(n))}$, where $f(x) = \frac{x}{2^x}$. Evaluate $A(m) = \frac{g(m+1)}{g(m)}$ and $B(m) = g(m+1) - g(m)$.

**Solution.** Let us evaluate several values of function,

$$f(x):\ f(0) = 0, f(1) = \frac{1}{2^1},\ f(2) = \frac{2}{2^2},\ f(3) = \frac{3}{2^3},\ \ldots\ ,$$
$$f(n) = \frac{n}{2^n},\ f(n+1) = \frac{n+1}{2^{n+1}}.$$

Substituting this into formula for $g(n)$ we obtain the following

$$g(n) = \frac{\frac{n+1}{2^{n+1}}}{2-\left(\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\ldots+\frac{n-1}{2^{n-1}}+\frac{n}{2^n}\right)}. \tag{2.45}$$

Next, we simplify the sum inside parentheses, by denoting it $S_n = \frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\ldots+\frac{n-1}{2^{n-1}}+\frac{n}{2^n}$. Multiplying both sides of the equality by 2 we get $2\cdot S_n = 1+\frac{2}{2^1}+\frac{3}{2^2}+\frac{4}{2^3}+\frac{5}{2^4}+\ldots+\frac{n-1}{2^{n-2}}+\frac{n}{2^{n-1}}$. Subtracting the left and the right sides of two equations and canceling the same terms, we have

$$\begin{aligned} S_n &= 1-\frac{n}{2^n}+\frac{1}{2}\cdot\frac{\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{\left(1-\frac{1}{2}\right)} \\ &= 2-\frac{n}{2^n}-\frac{1}{2^n} \end{aligned} \tag{2.46}$$

Substituting Eq. 2.46 into Eq. 2.45, we have

$$\begin{aligned} g(n) &= \frac{\frac{n+1}{2^{n+1}}}{2-\left(2-\frac{n}{2^n}-\frac{1}{2^{n-1}}\right)} = \frac{(n+1)\cdot\cancel{2^n}}{2\cdot\cancel{2^n}(n+2)} \\ &= \frac{n+1}{2n+4} \end{aligned}$$

Finally,

$$\frac{g(m+1)}{g(m)} = \frac{(m+2)^2}{(m+1)(m+3)}$$
$$g(m+1)-g(m) = \frac{m+2}{2(m+3)}-\frac{m+1}{2(m+2)} = \frac{1}{2(m+2)(m+3)}.$$

**Answer.** $A(m) = \frac{(m+2)^2}{(m+1)(m+3)}$; $B(m) = \frac{1}{2(m+2)(m+3)}$.

**Problem 89** Find all value of $r$ such that all partial sums of the series $\frac{1}{2} + r\cos x + r^2\cos 2x + r^3\cos 4x + r^4\cos 8x + \dots$ are nonnegative for all real $x$.

**Solution.** Consider the second partial sum, $S_2 = \frac{1}{2} + r\cos x \geq 0 \Rightarrow |r| \leq \frac{1}{2}$. Denote

$$\boxed{\psi(y) = r\cos y + r^2\cos 2y} \Rightarrow$$
$$\psi(2y) = r\cos 2y + r^2\cos 4y \Rightarrow$$
$$\boxed{\psi(4y) = r\cos 4y + r^2\cos 8y}$$
$$\psi(8y) = r\cos 8y + r^2\cos 16y$$
$$\boxed{\psi(16y) = r\cos 16y + r^2\cos 32y}$$

We can see that the given series can be rewritten as

$$\begin{aligned}&\frac{1}{2} + \left(r\cos x + r^2\cos 2x\right) + r^2\left(r\cos 4x + r^2\cos 8x\right)\\&+r^4(r\cos 16x + r^2\cos 32x) + \dots\\&= \frac{1}{2} + \psi(x) + r^2\psi(4x) + r^4\psi(8x) + \dots\end{aligned} \tag{2.47}$$

Let us investigate the behavior of $\psi(y)$. Taking the first derivative of it, we obtain that

$$\begin{aligned}\frac{d\psi}{dy} &= -r\sin y - 4r^2\sin y\cos y\\&= -r\sin y(1 + 4r\cos y) = 0\\\frac{d\psi}{dy} &= 0 \Leftrightarrow y = \pi n \text{ or } \cos y = -\frac{1}{4r}\end{aligned}$$

Case 1. $y = \pi n \Rightarrow \psi(y) = \psi(\pi n) = r\cos \pi n + r^2 2\pi n = (-1)^n r + r^2 \geq -\frac{1}{4}$.

Case 2. $\psi(y) = -\frac{1}{4} + r^2\left(\frac{1}{8r^2} - 1\right) \geq -\frac{3}{8}$.

Hence we can state that $\psi(y) \geq -\frac{3}{8}$ $\forall y \in \mathbb{R}$. Series (Eq. 2.47) are bounded as follows:

$$\begin{aligned}
S_{2n+1} &= \frac{1}{2} + \psi(x) + r^2\psi(4x) + r^4\psi(8x) + \ldots + r^{2(n-1)}\psi\left(4^{n-1}x\right)\\
&\geq \frac{1}{2} - \frac{3}{8}\left(1 + r^2 + r^4 + \ldots + r^{2(n-1)}\right)\\
&\geq \frac{1}{2} - \frac{3}{8}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^{n-1}\right)\\
&= \frac{1}{2} - \frac{3}{8}\left(\frac{1-\left(\frac{1}{4}\right)^{n-1}}{1-\frac{1}{4}}\right) = \frac{1}{2\cdot 4^n} = \frac{1}{2^{2n+1}}
\end{aligned}$$

Let us find the next partial sum, $S_{2n+2} = S_{2n+1} + r^{2n+1} \cdot \cos\left(2^{2n}x\right) \geq S_{2n+1} - \frac{1}{2^{2n+1}} \geq 0$. Finally, we can conclude that if $|r| \leq \frac{1}{2}$, then all partial sums of the series of Eq. 2.47 are nonnegative.

**Problem 90** Given a sequence $S_1 = \sqrt{2},\ S_{n+1} = \sqrt{2+S_n}$, prove that this sequence has a limit. Evaluate it.

**Proof.** Assume that the sequence has a limit $S$, then $S = \sqrt{S+2} \Rightarrow S^2 = S+2 \Rightarrow S = -1$ or $S = 2$

**Answer.** 2.

The following problems will make connection between sequences, number theory and geometry.

**Problem 91** A side of a square is $a$. The midpoints of its sides are joined to form an inscribed square. This process is continued as shown in the diagram. Find the sum of the perimeters of the squares if the process is continued without end.

**Solution.** From the diagram (Figure 2.4), we can see that the sides of the black squares form a geometric progression with the first term of $a$ and common ratio ½: $a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \ldots \frac{a}{2^{n-1}}$. All red squares, in turn, form a geometric progression with the same common ratio but the first term $\frac{a\sqrt{2}}{2}$ (half of the diagonal of the original square): $\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}, \frac{a\sqrt{2}}{8}, \ldots$. Because the perimeter of a square with side $b$ is $4b$, we obtain the following expression for the sum of the perimeters of black and red squares:

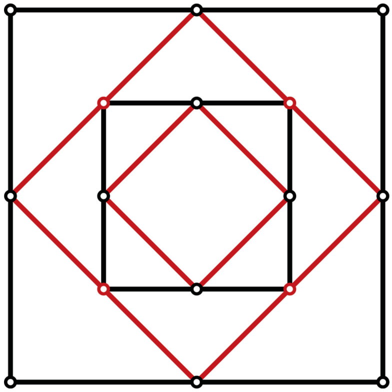

**Figure 2.4** Sketch for Prob. 91

$$P = 4\left(a + \frac{a\sqrt{2}}{2} + a/2 + \frac{a\sqrt{2}}{4} + a/4 + \frac{a\sqrt{2}}{8} + a/8 + \ldots + \frac{a}{2^{n-1}} + \frac{a\sqrt{2}}{2^{n-1}} + \ldots\right)$$
$$= 4a(1 + 1/2 + 1/4 + \ldots.) + 4a \cdot \frac{\sqrt{2}}{2}(1 + 1/2 + 1/4 + \ldots)$$
$$= 4a\left(2 + \sqrt{2}\right)\left(1 - \frac{1}{2^n}\right) = 4a\left(2 + \sqrt{2}\right)$$

**Answer.** $P = 4a\left(2 + \sqrt{2}\right)$

**Problem 92** Given a sequence $a_0 = 2,\ a_1 = 5,\ a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Show that $a_n \cdot a_{n+2} - a_{n+1}^2$ is a perfect square for every $n \geq 0$.

**Proof.** The characteristic polynomial for this recurrent sequence is $r^2 - 5r + 4 = (r-1)(r-4)$, then the general term of the sequence is $a_n = A \cdot 4^n + B \cdot 1^n$. Using the values of the first two terms, the nth term can be written as $a_n = 4^n + 1$. Evaluate

$$a_n \cdot a_{n+2} - a_{n+1}^2 = (4^n + 1)\left(4^{n+2} + 1\right) - \left(4^{n+1} + 1\right)^2$$
$$= 4^n \cdot 9 = (3 \cdot 2^n)^2 = k^2.$$

The proof is complete.

**Problem 93** Prove that there is no infinite arithmetic progression of only prime numbers.

**Proof.** Consider an arithmetic progression with the first term $a \neq 1$ and common difference $d$. Then the $n$th term of this progression can be written as $a_n = a + (n-1)d$. Clearly, if $n = a + 1 \ \Rightarrow a_n = a + a \cdot d = a(d+1)$. Thus, the first and $(a+1)$ st term , $a_{a+1}$, of such arithmetic progression are not relatively primes, and this fact does not depend on the value of the common difference. Moreover, in such infinite progression all terms sitting in the positions of $n = a+1,\ 2a+1,\ 3a+1,\ 4a+1, \ldots$ will be multiples of the first term, $a$.

For example, in the progression $\underline{3}$, 7, 11, $\underline{15}$, 19, 23, $\underline{27}$, 31, 35, $\underline{39}$, ... there are infinitely many members divisible by 3, we underlined some of them . All of them are in the positions $4,\ 7,\ 10,\ \ldots(3k+1), \ldots$. Because our proof was based on an assumption that the first term of a progression is not unit, a reasonable question is what if the first term of an infinite progression equals 1? Can such progression consist of only primes?

The answer is also "no" and the proof of this fact is very similar to the proof above. We just for any given progression start our arguments from the second term. Thus, infinite arithmetic progression $\{a_n\} : 1,\ 1+d,\ 1+2d,\ 1+3d, \ldots$ contains progression $\{b_n\} : 1+d,\ 1+2d,\ 1+3d, \ldots$ the first term of which equals the second term of the first progression, and then again prove that there are infinitely many terms divisible by $(1+d)$.

**Remark.** Any infinite arithmetic progression with natural members will have infinitely many multiples of the first, second, third, or any other term and the location of such multiples will depend only on the value of the selected term of a progression. Suppose a number $b \in \mathbb{N}$ is a term of an infinite arithmetic progression, then there are infinitely numbers of terms divisible by $b$ in the relative location $n = b+1,\ 2b+1,\ 3b+1, \ldots$. Thus if $b$ is the $k^{\text{th}}$ term of the given progression, then all terms divisible by it will have positions of $k,\ k+b,\ k+2b,\ k+3b,\ \ldots$.

For example, since 11 is the third term of the given infinite progression, 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, ..., the terms divisible by 11 will appear at the positions $3,\ 3+11 = 14,\ 3+2\cdot 11 = 25,\ 3+3\cdot 11 = 36,\ \ldots,\ 3+(m-1)\cdot 11,\ \ldots$, where $m$ represent the $m^{\text{th}}$ consecutive multiple of 11. You can see it yourself, 11 is the first multiple of 11, the second is 55, which is $14^{\text{th}}$ term of the given progression, then the third consecutive multiple of 11 in the progression will correspond to the index $n = 25$ and will be evaluated as $3 + (25-1)\cdot 4 = 99$, etc.

We just proved that there is no infinite arithmetic progression that consists of only primes. Is this statement also true for a finite arithmetic progression? The shortest sequence of primes must contain three terms. We can see that the first three terms of the infinite progression discussed above, {3, 7, and 11} are in arithmetic progression given by formula $a_n = 4n-1,\ n = 1,2,3$.

Are there arithmetic progressions with precisely 5, 10 or $N$ prime numbers? The answer is yes, such progressions exist but it is hard to find them.

The previous problem probably gives you some ideas of how to look for such progressions. First, we must select only even numbers as common difference, $d$. Otherwise, even and odd terms would alternate, which would never be a finite arithmetic progression with only prime terms. Obviously, the first term must be an odd prime. The following theorem formulated and proven by Cantor will help to get us started.

**Cantor's Theorem**. If $N$ terms of an arithmetic progression are odd primes, then the difference of the progression is divisible by every prime less than $N$.

The rigorous proof of the existence of an arithmetic sequence with exactly $N$ prime terms was given in 2004 by B. Green and T. Tao. However, their proof does not propose any algorithm of finding such progressions or makes the job of finding it any easy. It is worth to mention that the last longest arithmetic progression of 26 prime numbers was discovered only in 2010.

Here we try to find an arithmetic progression of ten prime terms by solving the following problem.

**Problem 94** Propose a finite arithmetic progression formed by ten prime numbers.

**Solution.** Regarding Cantor's Theorem, the common difference of such progression must be divisible by 2, 3, 5, and 7 (all prime numbers less than $n=10$). The minimal common difference satisfying this conditions is $d = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. Next, we need to find the starting prime, the first term of the progression. It cannot be 11, because $11 + 210 = 221 = 13 \cdot 17$ is not prime.

Can it be 13? The answer is no because 210 divided by 11 leaves a remainder of 1, $210 = 11 \cdot 19 + 1$. Then the remainder of a term when divided by 11 will increase by one each time as $n$ increases. For example, if the starting prime is 13 which give a remainder of 2 divided by 11, then the $n^{\text{th}}$ term has the following form,

$$\begin{aligned} a_n &= 13 + 210 \cdot (n-1) = 11 + 2 + 11 \cdot 19 \cdot (n-1) + n - 1 \\ &= 11k + n + 1 = 11m \end{aligned}$$

We can see that if $n = 10$, then the tenth term will be 1903 that divisible by 11 and not prime.

If a starting prime divided by 11 will leave a different remainder, for example, 3,4,5, etc. then a multiple of 11 will be obtained faster each time. Try yourself to select the first term as next prime, 17. Because $17 = 11 + 6$, then the $5^{\text{th}}$ term of the proposed progression will be a multiple of 11...

$$a_n = 17 + 210 \cdot (n-1) = 11 + 11 \cdot 19 \cdot (n-1) + (6 + n - 1), \ \ a_5 = 858 = 11 \cdot 78.$$

Therefore, the first term must be odd and leave a remainder of one when divided by 11. Let us try $a_1 = 22m + 1$. Consecutive candidates are 23, 67, 89, 199, ... If we try with the first term 23, 67 and 89, we obtain that such progression would have a composite number for the sixth, fourth and second term , respectively. If we set the first term equals 199, then we obtain an arithmetic progression of ten prime numbers, 199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, and 2089.

**Answer.** $a_n = 199 + (n-1)210 = 210n - 11, \ 1 \le n \le 10.$

Let us find out other properties of finite arithmetic progressions in integers by solving the following problems.

**Problem 95** Is there any arithmetic progression of 50 terms such that any two selected terms are relatively primes? If such progression exists, find it.

**Solution.** Let us consider an arithmetic progression with the first term of $a_1 = 1 + 49!$ and common difference $d = 49!$ Its $n$th term can be written as $a_n = 1 + 49! + (n-1)49! = 1 + n \cdot 49!, \ 1 \le n \le 50.$

We can see that any term of the progression is not divisible by any natural number from 1 to 50 because when divided by any such number it gives a remainder of 1. Next, the difference between its $k$th and $m$th terms will be $a_k - a_m = (k-m) \cdot 49!$, which means that the difference of any two terms is not divisible by any prime greater than 49. For example, $a_7 - a_4 = 3 \cdot 49! = 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 49$. This proves that any two selected terms of the arithmetic progression are relatively prime, because the only common factor they have is one.

**Problem 96** Is there an arithmetic progression formed of positive integers such that no term of the progression can be represented as a sum or difference of two primes? If such progression exists, then give an example.

**Solution.** Consider several arithmetic progressions with the corresponding $n^{\text{th}}$ terms $(n \in N)$:

a. 6, 10, 14, 18, $\ldots, 4n+2, \ldots$
b. 11, 19, 27, 35, $\ldots$, $8n + 3, \ldots$
c. 47, 89, 131, 173, $\ldots$, $42n + 5$, $\ldots$
d. 37, 67, 97, 127, $\ldots$, $30n + 7, \ldots$

The proression of "a" is represented by only even numbers, and clearly many its terms can be written as the sum or difference of two primes, for example,

$10 = 3 + 7$, $14 = 17 - 3$. Any even number is either the sum or difference of two even or two odd numbers; in general prime numbers are odd. One can prove that there exist infinitely many even numbers that can be written both as sums and as difference of two primes. Hence, we further consider only sequences of odd numbers.

Additionally, let us eliminate a sequence of odd numbers b) because its terms $19 = 2 + 17$ and $27 = 29 - 2$. If such arithmetic progression exists, then its terms must be odd numbers. Next, if some of its terms can be represented by a sum or difference of two primes, then one of the primes must be even (2).

Consider progression "c" and assume that one of its terms can be written as sum of two primes:

$$\begin{aligned} 42n + 5 &= p_1 + p_2 \\ 42n + 5 &= 2 + p_2 \\ 42n + 3 &= p_2 \\ p_2 &= 3 \cdot (14n + 1). \end{aligned}$$

We can see that $p_2$ is not prime.

Assume that some term of the progression above can be written as a difference of two primes, i.e., $42n + 5 = p_1 - p_2$. Because each term of the progression is odd, then the second prime must be 2.

$$\begin{aligned} 42n + 5 &= p_1 - 2 \\ 42n + 7 &= p_1 \\ p_1 &= 7 \cdot (6n + 1) \end{aligned}$$

Therefore, no terms of an arithmetic progression $a_n = 42n + 5$ can be represented as sum or difference of two primes. A similar conclusion can be made for progression "$d$". We leave it for you as a homework (exercise 89).

**Problem 97** Find all right triangles with integer sides forming consecutive terms of an arithmetic progression.

**Solution.** Assume that such triangle exists and that its sides are $a = a$, $b = a + d$, $c = a + 2d$. then they must satisfy Pythagorean Theorem:

$$\begin{aligned} a^2 + (a + d)^2 &= (a + 2d)^2 \\ a^2 + a^2 + 2ad + d^2 &= a^2 + 4ad + 4d^2 \\ a^2 &= 3d^2 + 2ad \\ (a - d)^2 &= (2d)^2 \\ a - d &= 2d \\ a = 3d,\ b &= 4d,\ c = 5d,\ d \in \mathbb{N}. \end{aligned}$$

Therefore, there are infinitely many such right triangles. For example, the sides of the following right triangles form an arithmetic progression and are Pythagorean triples: (3, 4, 5), (6, 8, 10), (9, 12, 15), (12, 16, 20)...

**Answer.** $(a, b, c) = (3d, 4d, 5d),\ d \in \mathbb{N}$.

**Problem 98** It is known that the numbers $x(x+1),\ y(y+1),\ z(z+1)$ are in increasing arithmetic progression. Find integer numbers $x$, $y$ and $z$.

**Solution.** Assume that such numbers exist and that
$\begin{cases} x = x \\ y = ax + b \\ z = cx + d \end{cases}$, where integer coefficients $a, b, c, d$ are to be determined.

Because $x(x+1),\ y(y+1),\ z(z+1)$ form an arithmetic progression, then $y(y+1) - x(x+1) = z(z+1) - y(y+1)$. Substituting here the expressions from the system above, we obtain the following chain of true equalities:

$$(ax+b)(ax+b+1) - x(x+1) = (cx+d)(cx+d+1) - (ax+b)(ax+b+1)$$
$$2\left(a^2x^2 + 2abx + b^2\right) + 2ax + 3b - x^2 - x = c^2x^2 + 2cdx + d^2 + cx + d$$

By equating the constant terms, the coefficients of linear and quadratic terms, respectively, we obtain the system of three equations in four undetermined integer parameters:

$$\begin{cases} 2b(b+1) = d(d+1) \\ 2a + 4ab - 1 = 2cd + c \\ 2a^2 - 1 = c^2 \end{cases}$$

Consider the last equation of the system, $1 + c^2 = 2a^2$. In order to have any solutions in integers, we know that parameter $c$ must be an odd number, then $c = 2n + 1$. Substituting this back into the equation we have

$$\begin{aligned} &1 + (2n+1)^2 = 2a^2 \\ &1 + 4a^2 + 4a + 1 = 2a^2 \\ &2n^2 + 2n + 1 = a^2 \\ &2n^2 + 2n = a^2 - 1 \\ &2n(n+1) = (a-1)(a+1) \end{aligned}$$

The right hand side is represented by the product of two numbers that differ by 2, hence they either both odd or both even. Because the left side is even then $(a-1)$ and $(a+1)$ must be even, for example, $a - 1 = 2m$, $a + 1 = 2m + 2$. Substituting this into the discussed equation, we obtain

$$2n(n+1) = 2m(2m+2)$$
$$n(n+1) = 2m(m+1).$$

The last equation has solution only if its variables satisfy the system

$$\begin{cases} n = m+1 \\ n+1 = 2m \end{cases} \Rightarrow m+2 = 2m \Rightarrow a = 2m+1 = 2\cdot 2+1 = 5.$$

Knowing $a$, we can evaluate the corresponding positive $c$, $1 + c^2 = 2\cdot 5^2 = 50$ $\Rightarrow c^2 = 49, \ c = 7$. Similarly to the solution of the underlined equation above, we can find positive solution to the first equation of the system.

$$d(d+1) = 2b(b+1)$$
$$\begin{cases} d = b+1 \\ 2b = d+1 \end{cases} \Rightarrow b = 2, \ d = 3.$$

Note that we found all four parameters using only solutions of the first and the last equations. This is very typical when solving equations in integers. The second equation can be used for checking. Thus, $1 + 2\cdot 7\cdot 3 + 7 = 4\cdot 5\cdot 2 + 2\cdot 5 = 50$. Finally, we found that if $x = x, \ y = 5x+2, \ z = 7x+3$, then $x(x+1) = x^2 + x$, $y(y+1) = 25x^2 + 25x + 6$, $z(z+1) = 49x^2 + 49x + 12$ are in the increasing arithmetic progression with common difference $24x^2 + 24x + 6$.

**Answer.** $x = x, \ y = 5x+2, \ z = 7x+3, \ x \in \mathbb{N}$.

**Problem 99** A sequence is defined by $a_n = \frac{1}{n+n^2}$, $n \geq 1$. Given $a_m + a_{m+1} + \ldots + a_{n-1} = \frac{1}{17}$, $m < n$, evaluate $n - m$.

**Solution.** Factoring the denominator of the $n^{\text{th}}$ term, we notice that it can be written as $a_n = \frac{1}{n+n^2} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Replacing each term on the left of the given condition, we have $\frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} + \ldots + \frac{1}{n} - \frac{1}{n+1} = \frac{1}{17}$. After cancelation of the opposite terms we obtain

$$\frac{1}{m} - \frac{1}{n} = \frac{n-m}{mn} = \frac{1}{17}$$
$$17(n-m) = mn$$

The last equation must be solved in integers and can be written as $17m = n(17-m)$. Because from the condition of the problem $17 - m < 17$, and 17 is prime, we know that $n$ must be a multiple of 17. Let $n = 17k$. After substitution we have

$$17m = 17k(17 - m)$$
$$m = k(17 - m).$$

This equation has integer solutions if and only if the second factor on the right hand side equals one, i.e.

$$17 - m = 1$$
$$m = 16,\ k = 17,\ n = 16 \cdot 17.$$
$$n - m = 16 \cdot 17 - 16 = 16^2 = 256.$$

**Answer.** $n - m = 256$.

**Problem 100** Given a sequence $u_1 = 2,\ u_2 = 8,\ \ldots,\ u_n = 4u_{n-1} - u_{n-2},\ n = 3, 4, 5,\ \ldots,$ Prove that $u_n^2 - u_{n+1} \cdot u_{n-1} = 4$.

**Proof.** We can evaluate some terms of the recurrence as $u_3 = 4u_2 - u_1 = 4 \cdot 8 - 2 = 30$, $u_4 = 4u_3 - u_2 = 4 \cdot 30 - 8 = 112$. It is clear that $u_4^2 - u_3 u_2 = 30^2 - 112 \cdot 8 = 4$. Because $ab = ba$, $u_n \cdot 4u_{n-1} = u_{n-1} \cdot 4u_n$. Using the recurrent relationship for the left and right hand sides, we obtain the following chain of true equations:

$$u_n \cdot (u_n + u_{n-2}) = u_{n-1}(u_{n+1} + u_{n-1})$$
$$u_n^2 - u_{n+1}u_{n-1} = u_{n-1}^2 - u_n u_{n-2} = u_{n-2}^2 - u_{n-1}u_{n-3} = \ldots$$
$$= u_2^2 - u_3 u_1 = 8^2 - 30 \cdot 2 = 4.$$

The proof is complete.

Springer

http://www.springer.com/978-3-319-45685-0

Methods of Solving Sequence and Series Problems
Grigorieva, E.
2016, XX, 281 p. 46 illus., 25 illus. in color., Hardcover
ISBN: 978-3-319-45685-0
A product of Birkhäuser Basel