

Chapter 2

Parametric Generalized R -norm Fuzzy Information and Divergence Measures

This chapter deals with two new generalized R -norm fuzzy entropy measures and some of their interesting properties. In addition, we propose two new generalized R -norm fuzzy divergence measures. In order to check the validity of proposed measures, their essential properties are studied.

Entropy is very important for measuring uncertain information first introduced by Shannon [14] to measure the uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P = (p_1, p_2, \dots, p_n)$ in an experiment. The information contained in this experiment is given by

$$H(P) = - \sum_{i=1}^n p_i \log p_i \quad (2.1)$$

which is the well-known Shannon [14] entropy.

Fuzziness, a feature of uncertainty, is found in our verbal communication, in our judgment and in the way we process information. The fuzzy entropy is one of important digital features of fuzzy sets and occupies an important place in system model and system design. It is an important concept for measuring fuzzy information. Fuzziness, an element of uncertainty, results from the lack of sharp difference of being or not being a member of the set. A measure of fuzziness used and cited in literature is called the fuzzy entropy, first mentioned by Zadeh [15]. It has wide applications in the area of image processing, pattern recognition, speech recognition, decision-making, etc.

In Sect. 2.1 we review the definition of fuzzy information measure, R -norm fuzzy measures of information and there generalization existing in the literature. Two new parametric generalizations of R -norm measure of fuzzy information are defined and the essential properties are proved to check their authenticity in Sect. 2.2. In order to make a comparison between the proposed generalized R -norm measures of fuzzy information with one existing R -norm fuzzy information

measure, a numerical example is given in Sect. 2.3. In Sect. 2.4 we review some existing R -norm fuzzy measures of divergence. Section 2.5 introduces and validates two new parametric generalized R -norm fuzzy divergence measures. Some of interesting properties of these two proposed fuzzy divergence measures are established and proved in Sect. 2.6. A relation between the new generalized R -norm measures of fuzzy information and divergence is established in Sect. 2.7. The concluding remarks are drawn in the last section.

2.1 Generalized R -norm Fuzzy Information Measures

We begin by reviewing the definition of fuzzy information measure, R -norm fuzzy measures of information and their generalization existing in the literature. De Luca and Termini [4] introduced the measure of fuzzy entropy corresponding to Shannon [14] entropy given in Eq. 2.1 as

$$H(A) = - \sum_{i=1}^n [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))] \quad (2.2)$$

satisfying the following essential properties:

- (P1) $H(A)$ is minimum if and only if A is a crisp set, i.e.
 $\mu_A(x_i) = 0$ or 1 for all $x_i : i = 1, 2, \dots, n$.
- (P2) $H(A)$ is maximum if and only if A is most fuzzy set, i.e.
 $\mu_A(x_i) = 0.5$ for all $x_i : i = 1, 2, \dots, n$.
- (P3) $H(A) \geq H(A^*)$, where A^* is sharpened version of A .
- (P4) $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement of A .

Later on Bhandari and Pal [2] defined the following exponential fuzzy entropy corresponding to Pal and Pal [13] exponential entropy as

$$E(A) = \frac{1}{n(\sqrt{e} - 1)} \sum_{i=1}^n [\mu_A(x_i)e^{(1-\mu_A(x_i))} + (1 - \mu_A(x_i))e^{\mu_A(x_i)} - 1] \quad (2.3)$$

Boeke and Lubbe [3] defined and studied R -norm information measure, which has been mentioned by Arimoto [1] as an example of a generalized class of information measures of the distribution P for $R \in \mathbf{R}$ as given by

$$H_R(P) = \frac{R}{R-1} \left[1 - \left(\sum_{i=1}^n p_i^R \right)^{\frac{1}{R}} \right]; \quad R > 0, R \neq 1. \quad (2.4)$$

The R -norm information measure (2.4) is a real function $\Delta_n \rightarrow \mathbf{R}$, defined on Δ_n (Set of all n -ary probability distributions which satisfies the condition $p_i \geq 0, i = 1, 2, 3, \dots, n$ and $\sum_{i=1}^n p_i = 1$) where $n \geq 2$ and $\mathbf{R}$ is the set of real numbers.

The most interesting property of this measure is that when $R \rightarrow 1$, R -norm information measure (2.4) approaches to Shannon's entropy (2.1) and in case $R \rightarrow \infty$, $H_R(P) \rightarrow (1 - \max p_i)$, $i = 1, 2, \dots, n$.

Analogous to measure (2.4) Hooda [5] proposed the following R -norm fuzzy measure of information

$$H_R(A) = \frac{R}{R-1} \sum_{i=1}^n \left[1 - (\mu_A^R(x_i) + (1 - \mu_A(x_i))^R)^{\frac{1}{R}} \right]; R(>0) \neq 1. \quad (2.5)$$

Further, from the significant studies it is noted that Hooda and Bajaj [6] and Hooda and Jain [7] provide the generalization of R -norm fuzzy information measure (2.5) corresponding to generalized R -norm information measures proposed by Hooda and Ram [8] and Hooda and Sharma [9] respectively.

Kumar [11] generalized the measure (2.4) and gave R -norm measure of information of order α which is

$$H_R^\alpha(P) = \frac{R}{R-\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} \right]; \quad 0 < \alpha \leq 1, R(>0) \neq 1. \quad (2.6)$$

In addition, Kumar and Choudhary [12] generalized the measure (2.4) and gave the R -norm information measure of degree m as

$$H_R^m(P) = \frac{R-m+1}{R-m} \left[1 - \left(\sum_{i=1}^n p_i^{R-m+1} \right)^{\frac{1}{R-m+1}} \right]; \quad (2.7)$$

$R-m+1 > 0, R \neq m, R, m > 0 (\neq 1).$

2.2 New Parametric Generalized R -norm Fuzzy Information Measures

Let A be the fuzzy set defined in a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ having the membership values $\mu_A(x_i)$, $i = 1, 2, \dots, n$. Corresponding to measures given by Kumar [11] (2.6) and Kumar and Choudhary [12] (2.7), the following generalized R -norm fuzzy information measures are proposed as follows:

$$H_R^\alpha(A) = \frac{R}{R-\alpha} \sum_{i=1}^n \left[1 - \left\{ \mu_A^{\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} \right\}^{\frac{\alpha}{R}} \right]; \quad 0 < \alpha \leq 1, R(>0) \neq 1. \quad (2.8)$$

$$H_R^m(A) = \frac{R-m+1}{R-m} \sum_{i=1}^n \left[1 - \left\{ \mu_A^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} \right]; \quad (2.9)$$

$R-m+1 > 0, R \neq m, R, m > 0 (\neq 1).$

Theorem 2.1 *The generalized R -norm fuzzy information measures $H_R^\alpha(A)$ and $H_R^m(A)$ are valid measures of fuzzy information.*

Proof P1 (Sharpness): The measures (2.8) and (2.9) clearly satisfy the property P1, i.e. $H_R^\alpha(A) = 0$ and $H_R^m(A) = 0$ if and only if A is non-fuzzy set or crisp set.

P2 (Maximality): To verify that the proposed measure (2.8) is concave; the values of $H_R^\alpha(A)$ are computed first for a fixed value of R and different values of α , second for a fixed value of α and different values of R .

Case 1: Let us assume a particular value of $R = 0.5$ and different values of α . The computed values of $H_R^\alpha(A)$ using (2.8) with different values of α are given in Table 2.1 which show the concavity of $H_R^\alpha(A)$ with respect to α .

A glance at Fig. 2.1 more precisely shows the concave behaviour of $H_R^\alpha(A)$ with respect to α .

Case 2: Let us assume a particular value of $\alpha = 0.7$ and different values of R . The computed values of $H_R^\alpha(A)$ for $\alpha = 0.7$ using (2.8) and different values of R are given in Table 2.2.

Table 2.2 and Fig. 2.2 clearly depict the concavity of $H_R^\alpha(A)$ with respect to R using (2.8).

Thus $H_R^\alpha(A)$ is a concave function with respect to α and R and its maximum value exists at $\mu_A(x_i) = 0.5$.

Now to verify that the proposed measure $H_R^m(A)$ in (2.9) is concave; the values of $H_R^m(A)$ first for a fixed value of R and different values of m , second for a fixed value of m and different values of R are computed.

Case 1: Let us assume a particular value of $R = 0.5$ and different values of m . The computed values of $H_R^m(A)$ using (2.9) with different values of m are given in Table 2.3 and Fig. 2.3 which depict the concavity of $H_R^m(A)$ using (2.9) with respect to m .

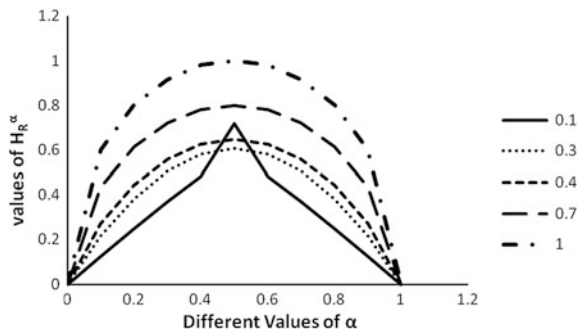
Case 2: Let us assume a particular value of $m = 0.7$ and different values of R . The computed values of $H_R^m(A)$ for $m = 0.7$ using (2.9) and different values of R are given in Table 2.4.

It may be seen from the results presented in Table 2.4 and Fig. 2.4 that behaviour of $H_R^m(A)$ with respect to R is concave.

Table 2.1 Different values of $H_R^\alpha(A)$ for $R = 0.5$ and for different values of α

α	$\mu_A(x_i)$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	$H_{0.5}^{0.1}(A)$	0	0.1250	0.2498	0.3725	0.4812	0.7179	0.4812	0.3725	0.2498	0.1250	0
0.3	$H_{0.5}^{0.3}(A)$	0	0.2155	0.3832	0.5054	0.5802	0.6053	0.5802	0.5054	0.3832	0.2155	0
0.4	$H_{0.5}^{0.4}(A)$	0	0.2705	0.4437	0.5588	0.6254	0.6472	0.6254	0.5588	0.4437	0.2705	0
0.7	$H_{0.5}^{0.7}(A)$	0	0.4319	0.6124	0.7204	0.7797	0.7988	0.7797	0.7204	0.6124	0.4319	0
1.0	$H_{0.5}^{1.0}(A)$	0	0.6	0.8	0.9165	0.9798	1	0.9798	0.9165	0.8	0.6	0

Fig. 2.1 Concave behaviour of $H_R^\alpha(A)$ with respect to α



Thus $H_R^m(A)$ is a concave function with respect to m and R and its maximum value exists at $\mu_A(x_i) = 0.5$.

Hence the measures (2.8) and (2.9) satisfy the property P2.

P3 (Resolution): Since $H_R^\alpha(A)$ and $H_R^m(A)$ are increasing functions of $\mu_A(x_i)$ in the range $[0, 0.5)$ and decreasing function in the range $(0.5, 1]$, therefore

$$\begin{aligned} \mu_{A^*}(x_i) \leq \mu_A(x_i) &\Rightarrow H_R^\alpha(A^*) \leq H_R^\alpha(A) \text{ in } [0, 0.5] \\ \text{and } \mu_{A^*}(x_i) \geq \mu_A(x_i) &\Rightarrow H_R^\alpha(A^*) \geq H_R^\alpha(A) \text{ in } (0.5, 1] \end{aligned}$$

Taking the above equations together, it comes

$$H_R^\alpha(A^*) \leq H_R^\alpha(A).$$

Similarly, $H_R^m(A^*) \leq H_R^m(A)$.

P4 (Symmetry): From the definition of $H_R^\alpha(A)$ and $H_R^m(A)$ and with $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$, it is obvious that $H_R^\alpha(\bar{A}) = H_R^\alpha(A)$ and $H_R^m(\bar{A}) = H_R^m(A)$.

Hence in view of the definition of fuzzy information measure of De Luca and Temini [4] provided in Sect. 2.1, $H_R^\alpha(A)$ and $H_R^m(A)$ are valid measures of fuzzy information. The measure (2.8) can be called as the generalized $\alpha - R$ -norm fuzzy information measure and measure (2.9) as the generalized R -norm fuzzy information measure of type R and degree m .

Limiting and Particular Cases:

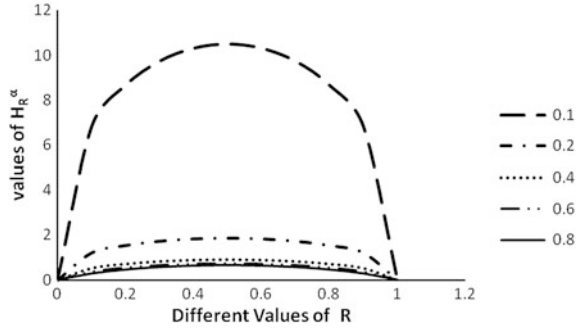
- (i) When $\alpha = 1$ and $m = 1$, (2.8) and (2.9) reduce to $H_R(A)$ in (2.5).
- (ii) When $\alpha = 1$, $m = 1$ and $R \rightarrow 1$, (2.8) and (2.9) reduce to $H(A)$ in (2.2).
- (iii) When $\alpha = 1$, $m = 1$ and $R \rightarrow \infty$, (2.8) and (2.9) reduce to

$$\sum_{i=1}^n [1 - \max\{\mu_A(x_i), 1 - \mu_A(x_i)\}].$$

Table 2.2 Different values of $H_R^\alpha(A)$ for $\alpha = 0.7$ and for different values of R

R	$\mu_A(x_i)$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	$H_{0.1}^{0.7}(A)$	0	6.8071	8.6641	9.7356	10.3152	10.5	10.3152	9.7356	8.6641	6.8071	0
0.2	$H_{0.2}^{0.7}(A)$	0	1.2087	1.5379	1.7276	1.8301	1.8627	1.8301	1.7276	1.5379	1.2087	0
0.4	$H_{0.4}^{0.7}(A)$	0	0.5275	0.7174	0.8289	0.8896	0.9090	0.8896	0.8289	0.7174	0.5275	0
0.6	$H_{0.6}^{0.7}(A)$	0	0.3698	0.5467	0.6552	0.7153	0.7348	0.7153	0.6552	0.5467	0.3698	0
0.8	$H_{0.8}^{0.7}(A)$	0	0.2911	0.4652	0.5783	0.6429	0.6640	0.6429	0.5783	0.4652	0.2911	0

Fig. 2.2 Concave behaviour of $H_R^\alpha(A)$ with respect to R



Theorem 2.2 For $A, B \in FS(X)$, $H_R^\alpha(A \cup B) + H_R^\alpha(A \cap B) = H_R^\alpha(A) + H_R^\alpha(B)$.

Proof Let us consider the sets

$$X_1 = \{x/x \in X, \mu_A(x_i) \geq \mu_B(x_i)\} \quad (2.10)$$

$$X_2 = \{x/x \in X, \mu_A(x_i) < \mu_B(x_i)\} \quad (2.11)$$

where $\mu_A(x_i)$ and $\mu_B(x_i)$ be the fuzzy membership functions of A and B respectively.

$$\begin{aligned} H_R^\alpha(A \cup B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[1 - (\mu_{A \cup B}^{\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cup B}(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} \left[\sum_{X_1} \left[1 - (\mu_A^{\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \right. \\ &\quad \left. + \sum_{X_2} \left[1 - (\mu_B^{\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \right] \end{aligned} \quad (2.12)$$

$$\begin{aligned} H_R^\alpha(A \cap B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[1 - (\mu_{A \cap B}^{\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cap B}(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \\ &= \frac{R}{R-\alpha} \left[\sum_{X_1} \left[1 - (\mu_B^{\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \right. \\ &\quad \left. + \sum_{X_2} \left[1 - (\mu_A^{\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \right] \end{aligned} \quad (2.13)$$

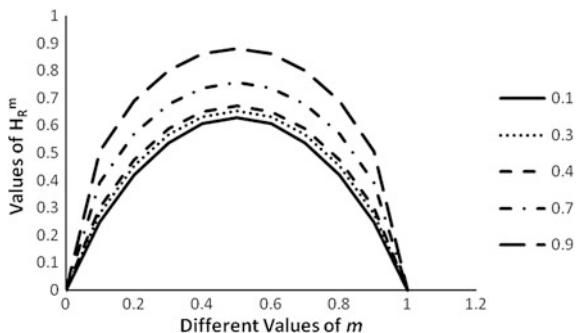
Adding Eqs. 2.12 and 2.13 gives

$$H_R^\alpha(A \cup B) + H_R^\alpha(A \cap B) = H_R^\alpha(A) + H_R^\alpha(B).$$

Table 2.3 Different values of $H_R^m(A)$ for $R = 0.5$ and for different values of m

α	$\mu_A(x_i)$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	$H_{0.5}^{0.1}(A)$	0	0.2469	0.4184	0.5363	0.6058	0.6288	0.6058	0.5363	0.4184	0.2469	0
0.3	$H_{0.5}^{0.3}(A)$	0	0.2797	0.4533	0.5675	0.6331	0.6546	0.6331	0.5675	0.4533	0.2797	0
0.4	$H_{0.5}^{0.4}(A)$	0	0.3004	0.4749	0.5872	0.651	0.6718	0.651	0.5872	0.4749	0.3004	0
0.7	$H_{0.5}^{0.7}(A)$	0	0.3920	0.5700	0.6780	0.7376	0.7568	0.7376	0.6780	0.5700	0.3920	0
0.9	$H_{0.5}^{0.9}(A)$	0	0.5043	0.6915	0.8017	0.8619	0.8811	0.8619	0.8017	0.6915	0.5043	0

Fig. 2.3 Concave behaviour of $H_R^m(A)$ with respect to m



Hence, 2.2 holds.

In particular,

For $A \in FS(X)$, $A \in FS(X)$ where $\bar{A}$ is the complement of fuzzy set A , it gets

$$H_R^z(A) = H_R^z(\bar{A}) = H_R^z(A \cup \bar{A}) = H_R^z(A \cap \bar{A}) \quad (2.14)$$

Theorem 2.3 For $A \in FS(X)$, $H_R^m(A \cup B) + H_R^m(A \cap B) = H_R^m(A) + H_R^m(B)$.

Proof Clearly the result can be proved on similar lines as in Theorem 2.2.

In particular,

For $A \in FS(X)$, $A \in FS(X)$ where $\bar{A}$ is the complement of fuzzy set A , it gets

$$H_R^m(A) = H_R^m(\bar{A}) = H_R^m(A \cup \bar{A}) = H_R^m(A \cap \bar{A}). \quad (2.15)$$

2.3 Numerical Example

Example Let $A = \{(x_i, \mu_A(x_i))/x_i \in X\}$ be a fuzzy set in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$. For any real number n , from the operation of power of a fuzzy set:

$$A^n = \{(x_i, [\mu_A(x_i)]^n)/x_i \in X\}.$$

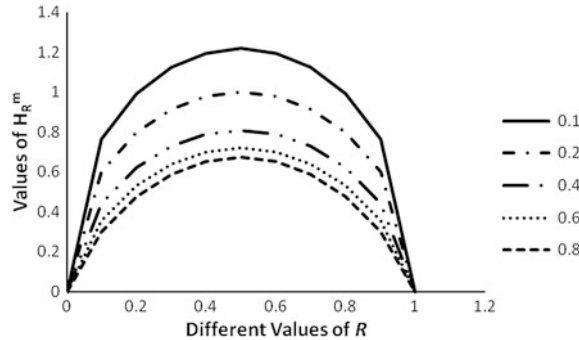
Let us assume a standard fuzzy set A on $X = \{x_1, x_2, \dots, x_n\}$ defined as:

$$A = (0.2, 0.3, 0.2, 0.4, 0.5)$$

Table 2.4 Different values of $H_R^z(A)$ for $m = 0.7$ and for different values of R

R	$\mu_A(x_i)$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	$H_{0,1}^{0.7}(A)$	0	0.7630	0.9920	1.1244	1.1961	1.219	1.1961	1.1244	0.9920	0.7630	0
0.2	$H_{0,2}^{0.7}(A)$	0	0.6000	0.8000	0.9165	0.9798	1	0.9798	0.9165	0.8000	0.6000	0
0.4	$H_{0,4}^{0.7}(A)$	0	0.4396	0.6207	0.7288	0.7881	0.8071	0.7881	0.7288	0.6207	0.4396	0
0.6	$H_{0,6}^{0.7}(A)$	0	0.3549	0.5313	0.6402	0.7009	0.7205	0.7009	0.6402	0.5313	0.3549	0
0.8	$H_{0,8}^{0.7}(A)$	0	0.3004	0.4749	0.5872	0.6510	0.6718	0.6510	0.5872	0.4749	0.3004	0

Fig. 2.4 Concave behaviour of $H_R^m(A)$ with respect to R



By taking into consideration the characterization of linguistic variables, A is considered as “LARGE” on X . Using the above operation:

- $A^{1/2}$ may be treated as “More or less LARGE”;
- A^2 may be treated as “Very LARGE”;
- A^3 may be treated as “Quite very LARGE”;
- A^4 may be treated as “Very very LARGE”

Next these fuzzy sets are used to compare the above proposed fuzzy entropy measures with one of R -norm fuzzy entropy measure provided in Eq. 2.5. From the point of logical consideration, it may be mentioned that the entropies of fuzzy sets are required to follow the following order pattern:

$$H_R(A^{1/2}) > H_R(A) > H_R(A^2) > H_R(A^3) > H_R(A^4). \tag{2.16}$$

The calculated numerical values of existing fuzzy information measure H_R for these cases are given in Table 2.5.

For any particular value of $\alpha = 0.7$ and different values of R , the calculated numerical values of the proposed measure H_R^α are presented in Table 2.6.

Table 2.7 provides the calculated numerical values of the proposed measure H_R^m , for any particular value of $m = 0.7$ and different values of R .

The numerical values given in Table 2.5 reveal that the R -norm fuzzy entropy measure $H_R(A)$ satisfies Eq. 2.16. The results presented in Tables 2.6 and 2.7

Table 2.5 Numerical values of the R -norm fuzzy entropy measure H_R

H_R	$A^{1/2}$	A	A^2	A^3	A^4
$R = 0.1$	328.9041	258.0100	179.0883	118.0176	79.5181
$R = 0.5$	4.8587	4.4963	2.9554	1.8314	1.1392
$R = 0.9$	4.4643	3.1244	1.7781	0.9322	0.4897
$R = 2$	2.7415	2.3220	1.0367	0.4409	0.1940
$R = 5$	2.3658	1.8854	0.7242	0.2900	0.1242

Table 2.6 Numerical values of the R -norm fuzzy entropy measure H_R^z

H_R^z	$A^{1/2}$	A	A^2	A^3	A^4
$R = 0.1$	51.2045	47.8754	33.5312	22.6947	15.6592
$R = 0.5$	3.8610	3.5240	2.1409	1.2058	0.6798
$R = 0.9$	3.0597	3.3325	1.3683	0.5984	0.3062
$R = 2$	2.5640	2.1088	0.8703	0.3548	0.1527
$R = 5$	2.2744	1.7958	0.6744	0.2698	0.1156

Table 2.7 Numerical values of R -norm fuzzy entropy measure H_R^m

H_R^m	$A^{1/2}$	A	A^2	A^3	A^4
$R = 0.1$	5.9352	5.5234	3.7551	2.4330	1.5886
$R = 0.5$	3.6502	3.3124	1.9470	1.0613	0.5777
$R = 0.9$	3.1255	2.7618	1.4353	0.6890	0.3335
$R = 2$	2.6660	2.2309	0.9625	0.4010	0.1743
$R = 5$	2.3492	1.8684	0.7144	0.2859	0.1225

clarify that the proposed new generalized R -norm fuzzy entropy measures $H_R^z(A)$ and $H_R^m(A)$ satisfy the same:

$$H_R^z(A^{1/2}) > H_R^z(A) > H_R^z(A^2) > H_R^z(A^3) > H_R^z(A^4) \quad (2.17)$$

$$\text{and } H_R^m(A^{1/2}) > H_R^m(A) > H_R^m(A^2) > H_R^m(A^3) > H_R^m(A^4). \quad (2.18)$$

Thus, the behaviour of new generalized R -norm fuzzy entropy measures $H_R^z(A)$ and $H_R^m(A)$ is also consistent for the structured linguistic variables. Table 2.8 and Fig. 2.5 display a declining trend in the numerical values of three entropy measures corresponding to the logical order of fuzzy sets.

Figure 2.5 clearly depicts the similarity between the proposed entropy measures and the existing one. An inequality among two proposed measures of fuzzy information and the existing one may also be seen from Fig. 2.5. That is: $H_R \geq H_R^z \geq H_R^m$.

Table 2.8 Calculated numerical values of proposed fuzzy entropy measures and existing one

$R = 0.5$	H_R (Existing)	H_R^z (Proposed)	H_R^m (Proposed)
$A^{1/2}$	4.8587	3.8610	3.6502
A	4.4963	3.5240	3.3124
A^2	2.9554	2.1409	1.9470
A^3	1.8314	1.2058	1.0613
A^4	1.1392	0.6798	0.5777

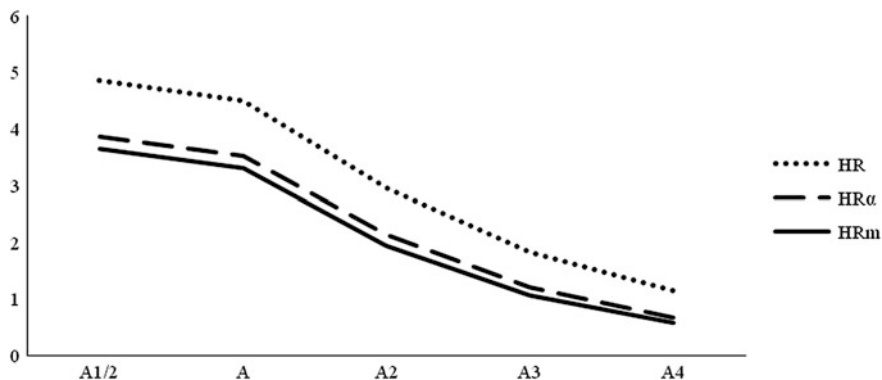


Fig. 2.5 Comparison of numerical values of H_R , H_R^α and H_R^m

2.4 Generalized Measures of R -norm Fuzzy Divergence

As mentioned in Chap. 1, in the recent years, many authors have introduced various measures of divergence for probability distributions and their corresponding divergence measures between fuzzy sets.

Hooda and Sharma [9] introduced the R -norm measure of divergence between the probability distribution $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ given by

$$D_R(P, Q) = \frac{R}{R-1} \left[\left(\sum_{i=1}^n p_i^R q_i^{1-R} \right)^{\frac{1}{R}} - 1 \right]; R > 0, R \neq 1. \quad (2.19)$$

It is noticed that when $R \rightarrow 1$, the measure (2.19) reduces to Kullback and Leibler [10] divergence measure

$$D(P : Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i}$$

Analogous to the measure (2.19), Hooda and Bajaj [6] defined the R -norm fuzzy divergence measure for fuzzy sets A and B as

$$I_R(A, B) = \frac{R}{R-1} \sum_{i=1}^n \left[\{ \mu_A^R(x_i) \mu_B^{1-R}(x_i) + (1 - \mu_A(x_i))^R (1 - \mu_B(x_i))^{1-R} \}^{\frac{1}{R}} - 1 \right] \\ R(>0) \neq 1. \quad (2.20)$$

and the corresponding R -norm fuzzy symmetric divergence measure as

$$J_R(A, B) = I_R(A, B) + I_R(B, A) \quad (2.21)$$

2.5 Two New Generalized R -norm Fuzzy Divergence Measures

We now propose the parametric generalized R -norm fuzzy divergence measures corresponding to the measures (2.8) and (2.9) for fuzzy sets A and B of universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ having the membership values $\mu_A(x_i), \mu_B(x_i), i = 1, 2, \dots, n$, as

$$I_R^\alpha(A, B) = \frac{R}{R - \alpha} \sum_{i=1}^n \left[(\mu_A^\alpha(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right],$$

$$0 < \alpha \leq 1, R(> 0) \neq 1. \quad (2.22)$$

and

$$I_R^m(A, B) = \frac{R - m + 1}{R - m} \sum_{i=1}^n \left[\{ \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \}^{\frac{1}{R-m+1}} - 1 \right],$$

$$R - m + 1 > 0, R \neq m, R, m > 0 (\neq 1). \quad (2.23)$$

Theorem 2.4 *The generalized R -norm fuzzy divergence measures $I_R^\alpha(A, B)$ and $I_R^m(A, B)$ defined in Eqs. 2.22 and 2.23 are valid measures of fuzzy divergence.*

Proof It is clear from the definitions of measures $I_R^\alpha(A, B)$ and $I_R^m(A, B)$ in (2.22) and (2.23), respectively, that

- (i) $I_R^\alpha(A, B) \geq 0$ and $I_R^m(A, B) \geq 0$,
- (ii) the equality holds if $\mu_A(x_i) = \mu_B(x_i), \forall i = 1, 2, \dots, n$.
- (iii) We now prove the convexity of measure $I_R^\alpha(A, B)$.

Let us assume that $\sum_{i=1}^n \mu_A(x_i) = \alpha_1, \sum_{i=1}^n \mu_B(x_i) = \beta_1$.

The measure $I_R^\alpha(A, B)$ in (2.22) can be written as

$$I_R^\alpha(A, B) = t_1 \sum_{i=1}^n \left[(\mu_A^{\delta_1}(x_i) \mu_B^{1-\delta_1}(x_i) + (1 - \mu_A(x_i))^{\delta_1} (1 - \mu_B(x_i))^{1-\delta_1})^{\delta_1} - 1 \right]$$

where $t_1 = \frac{R}{R-\alpha}$ and $(\delta_1 = \frac{R}{\alpha} > 0) \neq 1$.

Then two cases arise.

Case 1: $R < \alpha$

In this case we have $t_1 < 0$, $\delta_1 < 1$.

$$\begin{aligned} & \left[\sum_{i=1}^n \left(\frac{\mu_A(x_i)}{\alpha_1} \right)^{\delta_1} \left(\frac{\mu_B(x_i)}{\beta_1} \right)^{1-\delta_1} - 1 \right] \leq 0 \\ \text{or } & \sum_{i=1}^n \mu_A^{\delta_1}(x_i) \mu_B^{1-\delta_1}(x_i) \leq \alpha_1^{\delta_1} \beta_1^{1-\delta_1} \end{aligned} \quad (2.24)$$

Similarly,

$$\sum_{i=1}^n (1 - \mu_A(x_i))^{\delta_1} (1 - \mu_B(x_i))^{1-\delta_1} \leq (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} \quad (2.25)$$

Adding Eqs. 2.24 and 2.25

$$\begin{aligned} & \sum_{i=1}^n \left(\mu_A^{\delta_1}(x_i) \mu_B^{1-\delta_1}(x_i) + (1 - \mu_A(x_i))^{\delta_1} (1 - \mu_B(x_i))^{1-\delta_1} \right) \\ & \leq \left(\alpha_1^{\delta_1} \beta_1^{1-\delta_1} + (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} \right) \\ I_R^\alpha(A, B) & \geq t_1 \left(\alpha_1^{\delta_1} \beta_1^{1-\delta_1} + (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} - n \right) \end{aligned} \quad (2.26)$$

Case 2: $R > \alpha$

In this case we have $t_1 > 0$, $\delta_1 > 1$.

$$\begin{aligned} & \left[\sum_{i=1}^n \left(\frac{\mu_A(x_i)}{\alpha_1} \right)^{\delta_1} \left(\frac{\mu_B(x_i)}{\beta_1} \right)^{1-\delta_1} - 1 \right] \geq 0 \\ \text{or } & \sum_{i=1}^n \mu_A^{\delta_1}(x_i) \mu_B^{1-\delta_1}(x_i) \geq \alpha_1^{\delta_1} \beta_1^{1-\delta_1} \end{aligned} \quad (2.27)$$

Similarly,

$$\sum_{i=1}^n (1 - \mu_A(x_i))^{\delta_1} (1 - \mu_B(x_i))^{1-\delta_1} \geq (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} \quad (2.28)$$

Adding Eqs. 2.27 and 2.28

$$\begin{aligned} & \sum_{i=1}^n \left(\mu_A^{\delta_1}(x_i) \mu_B^{1-\delta_1}(x_i) + (1 - \mu_A(x_i))^{\delta_1} (1 - \mu_B(x_i))^{1-\delta_1} \right) \\ & \geq \left(\alpha_1^{\delta_1} \beta_1^{1-\delta_1} + (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} \right) \\ & I_R^\alpha(A, B) \geq t_1 \left(\alpha_1^{\delta_1} \beta_1^{1-\delta_1} + (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} - n \right) \end{aligned} \quad (2.29)$$

$$\text{Now let } f(\alpha_1) = t \left(\alpha_1^{\delta_1} \beta_1^{1-\delta_1} + (n - \alpha_1)^{\delta_1} (n - \beta_1)^{1-\delta_1} - n \right)$$

So that we have

$$f'(\alpha_1) = t_1 \left[\delta_1 \left(\frac{\alpha_1}{\beta_1} \right)^{\delta_1-1} - \delta_1 \left(\frac{n - \alpha_1}{n - \beta_1} \right)^{\delta_1-1} \right] \quad (2.30)$$

$$f''(\alpha_1) = t_1 \delta_1 (\delta_1 - 1) \left[\frac{1}{\beta_1} \left(\frac{\alpha_1}{\beta_1} \right)^{\delta_1-2} + \frac{1}{n - \beta_1} \left(\frac{n - \alpha_1}{n - \beta_1} \right)^{\delta_1-2} \right] > 0 \quad (2.31)$$

So this shows that $f(\alpha_1)$ is a convex function of α_1 whose minimum value exists when $\frac{\alpha_1}{\beta_1} = \frac{n - \alpha_1}{n - \beta_1}$, i.e. at $\alpha_1 = \beta_1$ and is equal to zero. So that $f(\alpha_1) > 0$ and vanishes only when $\alpha_1 = \beta_1$.

Hence in view of the definition of fuzzy divergence measure of Bhandari and Pal [2] provided in Sect. 1.3 of Chap. 1, $I_R^\alpha(A, B)$ is a valid measure of R -norm fuzzy divergence of order α and consequently it establishes the validity of $J_R^\alpha(A, B) = I_R^\alpha(A, B) + I_R^\alpha(B, A)$ as measure of R -norm symmetric divergence of order α between fuzzy sets A and B .

We now prove the convexity of measure $I_R^m(A, B)$.

Let us assume that $\sum_{i=1}^n \mu_A(x_i) = \alpha_2$, $\sum_{i=1}^n \mu_B(x_i) = \beta_2$.

The measure $I_R^m(A, B)$ in (2.23) can be written as

$$I_R^m(A, B) = t_2 \sum_{i=1}^n \left[(\mu_A^{\delta_2}(x_i) \mu_B^{1-\delta_2}(x_i) + (1 - \mu_A(x_i))^{\delta_2} (1 - \mu_B(x_i))^{1-\delta_2})^{\delta_2} - 1 \right]$$

where $t_2 = \frac{R-m+1}{R-m}$ and $((\delta_2 = R - m + 1) > 0) \neq 1$.

Then two cases arise.

Case 1: $R - m + 1 < 1$

In this case we have $t_2 < 0$, $\delta_2 < 1$.

$$\left[\sum_{i=1}^n \left(\frac{\mu_A(x_i)}{\alpha_2} \right)^{\delta_2} \left(\frac{\mu_B(x_i)}{\beta_2} \right)^{1-\delta_2} - 1 \right] \leq 0 \quad (2.32)$$

or $\sum_{i=1}^n \mu_A^{\delta_2}(x_i) \mu_B^{1-\delta_2}(x_i) \leq \alpha_2^{\delta_2} \beta_2^{1-\delta_2}$

Similarly,

$$\sum_{i=1}^n (1 - \mu_A(x_i))^{\delta_2} (1 - \mu_B(x_i))^{1-\delta_2} \leq (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} \quad (2.33)$$

Adding Eqs. 2.32 and 2.33

$$\begin{aligned} & \sum_{i=1}^n \left(\mu_A^{\delta_2}(x_i) \mu_B^{1-\delta_2}(x_i) + (1 - \mu_A(x_i))^{\delta_2} (1 - \mu_B(x_i))^{1-\delta_2} \right) \\ & \leq \left(\alpha_2^{\delta_2} \beta_2^{1-\delta_2} + (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} \right) \\ & I_R^m(A, B) \geq t_2 \left(\alpha_2^{\delta_2} \beta_2^{1-\delta_2} + (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} - n \right) \end{aligned} \quad (2.34)$$

Case 2: $R - m + 1 > 1$

In this case we have $t_2 > 0$, $\delta_2 > 1$.

$$\left[\sum_{i=1}^n \left(\frac{\mu_A(x_i)}{\alpha_2} \right)^{\delta_2} \left(\frac{\mu_B(x_i)}{\beta_2} \right)^{1-\delta_2} - 1 \right] \geq 0 \quad (2.35)$$

or $\sum_{i=1}^n \mu_A^{\delta_2}(x_i) \mu_B^{1-\delta_2}(x_i) \geq \alpha_2^{\delta_2} \beta_2^{1-\delta_2}$

Similarly,

$$\sum_{i=1}^n (1 - \mu_A(x_i))^{\delta_2} (1 - \mu_B(x_i))^{1-\delta_2} \geq (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} \quad (2.36)$$

Adding Eqs. 2.35 and 2.36

$$\begin{aligned}
 & \sum_{i=1}^n \left(\mu_A^{\delta_2}(x_i) \mu_B^{1-\delta_2}(x_i) + (1 - \mu_A(x_i))^{\delta_2} (1 - \mu_B(x_i))^{1-\delta_2} \right) \\
 & \geq \left(\alpha_2 \beta_2^{1-\delta_2} + (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} \right) \\
 & I_R^m(A, B) \geq t_2 \left(\alpha_2^{\delta_2} \beta_2^{1-\delta_2} + (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} - n \right)
 \end{aligned} \tag{2.37}$$

Now let $g(\alpha_2) = t_2 \left(\alpha_2^{\delta_2} \beta_2^{1-\delta_2} + (n - \alpha_2)^{\delta_2} (n - \beta_2)^{1-\delta_2} - n \right)$

So that we have

$$g'(\alpha_2) = t_2 \left[\delta_2 \left(\frac{\alpha_2}{\beta_2} \right)^{\delta_2-1} - \delta_2 \left(\frac{n - \alpha_2}{n - \beta_2} \right)^{\delta_2-1} \right] \tag{2.38}$$

$$g''(\alpha_2) = t_2 \delta_2 (\delta_2 - 1) \left[\frac{1}{\beta_2} \left(\frac{\alpha_2}{\beta_2} \right)^{\delta_2-2} + \frac{1}{n - \beta_2} \left(\frac{n - \alpha_2}{n - \beta_2} \right)^{\delta_2-2} \right] > 0 \tag{2.39}$$

So this shows that $g(\alpha_2)$ is a convex function of α_2 whose minimum value exists when $\frac{\alpha_2}{\beta_2} = \frac{n-\alpha_2}{n-\beta_2}$, i.e. at $\alpha_2 = \beta_2$ and is equal to zero. So that $g(\alpha_2) > 0$ and vanishes only when $\alpha_2 = \beta_2$.

Hence in view of the definition of fuzzy divergence measure of Bhandari and Pal [2] provided in Sect. 1.3 of Chap. 1, $I_R^m(A, B)$ is a valid measure of R -norm fuzzy divergence of type R and degree m and consequently it establishes the validity of $J_R^m(A, B) = I_R^m(A, B) + I_R^m(B, A)$ as measure of R -norm symmetric divergence of type R and degree m between fuzzy sets A and B .

Limiting and Particular Cases:

- (i) when $\alpha \rightarrow 1$, $m \rightarrow 1$, measures (2.22) and (2.23) reduce to $I_R(A, B)$.
- (ii) when $\alpha \rightarrow 1$, $m \rightarrow 1$ with $R = 1$, measures (2.22) and (2.23) reduce to Bhandari and Pal [2] fuzzy divergence measure $I(A, B)$.

2.6 Properties of New Generalized R -norm Fuzzy Divergence Measures

The proposed generalized R -norm fuzzy divergence measures (2.22) and (2.23) satisfy the following properties.

Theorem 2.5 For fuzzy sets A and B

- (a) $I_R^\alpha(A \cup B, C) \leq I_R^\alpha(A, C) + I_R^\alpha(B, C)$ and $I_R^m(A \cup B, C) \leq I_R^m(A, C) + I_R^m(B, C)$.
- (b) $I_R^\alpha(A \cap B, C) \leq I_R^\alpha(A, C) + I_R^\alpha(B, C)$ and $I_R^m(A \cap B, C) \leq I_R^m(A, C) + I_R^m(B, C)$.

Proof 2.5(a) Applying Eqs. 2.10 and 2.11

Let us consider the expression

$$\begin{aligned}
 & I_R^\alpha(A, C) + I_R^\alpha(B, C) - I_R^\alpha(A \cup B, C) \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_A^\alpha(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &+ \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_B^\alpha(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &- \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_{A \cup B}^\alpha(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cup B}(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{x_1} \left[\left(\mu_B^\alpha(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &+ \frac{R}{R-\alpha} \sum_{x_2} \left[\left(\mu_A^\alpha(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \geq 0.
 \end{aligned}$$

Now, let us consider the expression

$$\begin{aligned}
 & I_R^m(A, C) + I_R^m(B, C) - I_R^m(A \cup B, C) \\
 &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &+ \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &- \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_{A \cup B}^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_{A \cup B}(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= \frac{R-m+1}{R-m} \sum_{x_1} \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &+ \frac{R-m+1}{R-m} \sum_{x_2} \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \geq 0.
 \end{aligned}$$

Hence, 2.5(a) holds.

Similarly, 2.5(b) can also be proved.

Theorem 2.6 For fuzzy sets A and B

- (a) $I_R^\alpha(A \cup B, C) + I_R^\alpha(A \cap B, C) = I_R^\alpha(A, C) + I_R^\alpha(B, C).$
- (b) $I_R^m(A \cup B, C) + I_R^m(A \cap B, C) = I_R^m(A, C) + I_R^m(B, C).$

Proof 2.6(a) Applying Eqs. 2.10 and 2.11 we get

$$\begin{aligned}
 I_R^\alpha(A \cup B, C) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_{A \cup B}^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cup B}(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \sum_{x_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &\quad + \sum_{x_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.40}$$

$$\begin{aligned}
 I_R^\alpha(A \cap B, C) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_{A \cap B}^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cap B}(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \sum_{x_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &\quad + \sum_{x_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_C^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_C(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.41}$$

On adding Eqs. 2.40 and 2.41 we get the result.

Hence, 2.6(a) holds.

2.6(b)

$$\begin{aligned}
 I_R^m(A \cup B, C) &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_{A \cup B}^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_{A \cup B}(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= \frac{R-m+1}{R-m} \sum_{x_1} \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &\quad + \frac{R-m+1}{R-m} \sum_{x_2} \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right]
 \end{aligned} \tag{2.42}$$

$$\begin{aligned}
I_R^m(A \cap B, C) &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\frac{\{\mu_{A \cap B}^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_{A \cap B}(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1}\}^{\frac{1}{R-m+1}} - 1}{\frac{1}{R-m+1} - 1} \right] \\
&= \frac{R-m+1}{R-m} \sum_{x_1} \left[\frac{\{\mu_B^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1}\}^{\frac{1}{R-m+1}} - 1}{\frac{1}{R-m+1} - 1} \right] \\
&\quad + \frac{R-m+1}{R-m} \sum_{x_2} \left[\frac{\{\mu_A^{R-m+1}(x_i) \mu_C^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_C(x_i))^{R-m+1}\}^{\frac{1}{R-m+1}} - 1}{\frac{1}{R-m+1} - 1} \right] \quad (2.43)
\end{aligned}$$

On adding Eqs. 2.42 and 2.43 we get the result.

Hence, 2.6(b) holds.

Theorem 2.7 For fuzzy sets A and B

- (a) $I_R^z(A \cup B, A \cap B) + I_R^z(A \cap B, A \cup B) = I_R^z(A, B) + I_R^z(B, A)$.
- (b) $I_R^m(A \cup B, A \cap B) + I_R^m(A \cap B, A \cup B) = I_R^m(A, B) + I_R^m(B, A)$.

Proof 2.7(a) Applying Eqs. 2.10 and 2.11 we get

$$\begin{aligned}
I_R^z(A \cup B, A \cap B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_{A \cup B}^{\frac{R}{\alpha}}(x_i) \mu_{A \cap B}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cup B}(x_i))^{\frac{R}{\alpha}} (1 - \mu_{A \cap B}(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
&= \frac{R}{R-\alpha} \sum_{x_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
&\quad + \frac{R}{R-\alpha} \sum_{x_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \quad (2.44)
\end{aligned}$$

Now

$$\begin{aligned}
I_R^z(A \cap B, A \cup B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_{A \cap B}^{\frac{R}{\alpha}}(x_i) \mu_{A \cup B}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A \cap B}(x_i))^{\frac{R}{\alpha}} (1 - \mu_{A \cup B}(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
&= \frac{R}{R-\alpha} \sum_{x_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
&\quad + \frac{R}{R-\alpha} \sum_{x_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \quad (2.45)
\end{aligned}$$

Adding Eqs. 2.44 and 2.45 we get

$$\begin{aligned}
 & I_R^\alpha(A \cup B, A \cap B) + I_R^\alpha(A \cap B, A \cup B) \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &+ \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= I_R^\alpha(A, B) + I_R^\alpha(B, A)
 \end{aligned}$$

Hence, 2.7(a) holds.

2.7(b)

$$\begin{aligned}
 & I_R^m(A \cup B, A \cap B) \\
 &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_{A \cup B}^{R-m+1}(x_i) \mu_{A \cap B}^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_{A \cup B}(x_i))^{R-m+1} (1 - \mu_{A \cap B}(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= \frac{R-m+1}{R-m} \sum_{x_1} \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &+ \frac{R-m+1}{R-m} \sum_{x_2} \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_A^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_A(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right]
 \end{aligned} \tag{2.46}$$

Now

$$\begin{aligned}
 & I_R^m(A \cap B, A \cup B) \\
 &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_{A \cap B}^{R-m+1}(x_i) \mu_{A \cup B}^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_{A \cap B}(x_i))^{R-m+1} (1 - \mu_{A \cup B}(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= \frac{R-m+1}{R-m} \sum_{x_1} \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_A^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_A(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &+ \frac{R-m+1}{R-m} \sum_{x_2} \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right]
 \end{aligned} \tag{2.47}$$

Adding Eqs. 2.46 and 2.47 we get

$$\begin{aligned}
 & I_R^m(A \cup B, A \cap B) + I_R^m(A \cap B, A \cup B) \\
 &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &+ \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_B^{R-m+1}(x_i) \mu_A^{R-m+1}(x_i) + (1 - \mu_B(x_i))^{R-m+1} (1 - \mu_A(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= I_R^m(A, B) + I_R^m(B, A)
 \end{aligned}$$

Hence, 2.7(b) holds.

Theorem 2.8 For fuzzy sets A and B

- (a) $I_R^\alpha(A, A \cup B) + I_R^\alpha(A, A \cap B) = I_R^\alpha(A, B);$
 $I_R^m(A, A \cup B) + I_R^m(A, A \cap B) = I_R^m(A, B).$
- (b) $I_R^\alpha(B, A \cup B) + I_R^\alpha(B, A \cap B) = I_R^\alpha(B, A);$
 $I_R^m(B, A \cup B) + I_R^m(B, A \cap B) = I_R^m(B, A).$

Proof 2.8(a) Applying Eqs. 2.10 and 2.11 we get

$$\begin{aligned}
 & I_R^\alpha(A, A \cup B) \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_{A \cup B}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_{A \cup B}(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &+ \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.48}$$

Now

$$\begin{aligned}
 & I_R^\alpha(A, A \cap B) \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &+ \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.49}$$

Adding Eqs. 2.48 and 2.49 we get

$$\begin{aligned}
 & I_R^\alpha(A, A \cup B) + I_R^\alpha(A, A \cap B) \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= I_R^\alpha(A, B)
 \end{aligned}$$

Hence, 2.8(a) holds for $I_R^\alpha(A, B)$ and similar result holds for $I_R^m(A, B)$.
2.8(b)

$$\begin{aligned}
 & I_R^\alpha(B, A \cup B) \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_{A \cup B}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_{A \cup B}(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &\quad + \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.50}$$

Now

$$\begin{aligned}
 & I_R^\alpha(B, A \cap B) \\
 &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_{A \cap B}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_{A \cap B}(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &\quad + \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R-\alpha} \sum_{X_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1-\frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right]
 \end{aligned} \tag{2.51}$$

Adding Eqs. 2.50 and 2.51 we get

$$\begin{aligned}
 & I_R^\alpha(B, A \cup B) + I_R^\alpha(B, A \cap B) \\
 &= \frac{R}{R - \alpha} \sum_{x_1} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_1) \mu_A^{1 - \frac{R}{\alpha}}(x_1) + (1 - \mu_B(x_1))^{\frac{R}{\alpha}} (1 - \mu_A(x_1))^{1 - \frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R - \alpha} \sum_{x_2} \left[\left(\mu_B^{\frac{R}{\alpha}}(x_2) \mu_A^{1 - \frac{R}{\alpha}}(x_2) + (1 - \mu_B(x_2))^{\frac{R}{\alpha}} (1 - \mu_A(x_2))^{1 - \frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R - \alpha} \sum_{i=1}^n \left[\left(\mu_B^{\frac{R}{\alpha}}(x_i) \mu_A^{1 - \frac{R}{\alpha}}(x_i) + (1 - \mu_B(x_i))^{\frac{R}{\alpha}} (1 - \mu_A(x_i))^{1 - \frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= I_R^\alpha(B, A)
 \end{aligned}$$

Hence, 2.8(b) holds for $I_R^\alpha(A, B)$ and similar result holds for $I_R^m(A, B)$.

Theorem 2.9 For fuzzy sets A and B

- (a) $I_R^\alpha(A, B) = I_R^\alpha(A^c, B^c)$; $I_R^m(A, B) = I_R^m(A^c, B^c)$.
- (b) $I_R^\alpha(A^c, B) = I_R^\alpha(A, B^c)$; $I_R^m(A^c, B) = I_R^m(A, B^c)$.
- (c) $I_R^\alpha(A, B) + I_R^\alpha(A^c, B) = I_R^\alpha(A^c, B^c) + I_R^\alpha(A^c, B)$;

$$I_R^m(A, B) + I_R^m(A^c, B) = I_R^m(A^c, B^c) + I_R^m(A^c, B).$$

Proof 2.9(a)

$$\begin{aligned}
 & I_R^\alpha(A^c, B^c) \\
 &= \frac{R}{R - \alpha} \sum_{i=1}^n \left[\left(\mu_{A^c}^{\frac{R}{\alpha}}(x_i) \mu_{B^c}^{1 - \frac{R}{\alpha}}(x_i) + (1 - \mu_{A^c}(x_i))^{\frac{R}{\alpha}} (1 - \mu_{B^c}(x_i))^{1 - \frac{R}{\alpha}} \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= \frac{R}{R - \alpha} \sum_{i=1}^n \left[\left((1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1 - \frac{R}{\alpha}} + \mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1 - \frac{R}{\alpha}}(x_i) \right)^{\frac{\alpha}{R}} - 1 \right] \\
 &= I_R^\alpha(A, B)
 \end{aligned}$$

Now

$$\begin{aligned}
 & I_R^m(A^c, B^c) \\
 &= \frac{R - m + 1}{R - m} \sum_{i=1}^n \left[\left\{ \mu_{A^c}^{R-m+1}(x_i) \mu_{B^c}^{R-m+1}(x_i) \right. \right. \\
 &\quad \left. \left. + (1 - \mu_{A^c}(x_i))^{R-m+1} (1 - \mu_{B^c}(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
 &= \frac{R - m + 1}{R - m} \sum_{i=1}^n \left[\left\{ (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \right. \right. \\
 &\quad \left. \left. + \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) \right\}^{\frac{1}{R-m+1}} - 1 \right] = I_R^m(A, B)
 \end{aligned}$$

Hence, 2.9(a) holds.

2.9(b)

$$\begin{aligned}
I_R^\alpha(A^c, B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_{A^c}^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_{A^c}(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right] \\
&= \frac{R}{R-\alpha} \sum_{i=1}^n \left[((1 - \mu_A(x_i))^{\frac{R}{\alpha}} \mu_B^{1-\frac{R}{\alpha}}(x_i) + \mu_A^{\frac{R}{\alpha}}(x_i) (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right]
\end{aligned} \quad (2.52)$$

$$\begin{aligned}
I_R^\alpha(A, B^c) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_{B^c}^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_{B^c}(x_i))^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right] \\
&= \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_A^{\frac{R}{\alpha}}(x_i) (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}} + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} \mu_B^{1-\frac{R}{\alpha}}(x_i))^{\frac{\alpha}{R}} - 1 \right]
\end{aligned} \quad (2.53)$$

From Eqs. 2.52 and 2.53 we get $I_R^\alpha(A^c, B) = I_R^\alpha(A, B^c)$.

Hence, 2.9(b) holds for $I_R^\alpha(A, B)$ and similar result holds for $I_R^m(A, B)$.

2.9(c) It obviously follows from 2.9(a) and 2.9(b).

2.7 Relation Between Proposed Generalized R -norm Fuzzy Entropy and Divergence Measures

Theorem 2.10 For fuzzy sets A and B , the relation between $H_R^\alpha(A)$ and $I_R^\alpha(A, B)$ is given by

$$I_R^\alpha\left(A, \left[\frac{1}{2}\right]\right) = \left[\left(\frac{1}{2}\right)^{\frac{\alpha-R}{R}} - 1\right] \frac{nR}{R-\alpha} - \left(\frac{1}{2}\right)^{\frac{\alpha-R}{R}} H_R^\alpha(A).$$

Proof We know from Eq. 2.22 that

$$\begin{aligned}
I_R^\alpha(A, B) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_A^{\frac{R}{\alpha}}(x_i) \mu_B^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} (1 - \mu_B(x_i))^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right] \\
I_R^\alpha\left(A, \left[\frac{1}{2}\right]\right) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_A^{\frac{R}{\alpha}}(x_i) \left(\frac{1}{2}\right)^{1-\frac{R}{\alpha}}(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}} \left(\frac{1}{2}\right)^{1-\frac{R}{\alpha}})^{\frac{\alpha}{R}} - 1 \right] \\
&= \frac{R}{R-\alpha} \left(\frac{1}{2}\right)^{\left(1-\frac{R}{\alpha}\right)\frac{\alpha}{R}} \sum_{i=1}^n \left[(\mu_A^{\frac{R}{\alpha}}(x_i) (x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] - \frac{nR}{R-\alpha}
\end{aligned} \quad (2.54)$$

From Eq. 2.8,

$$\begin{aligned}
H_R^\alpha(A) &= \frac{R}{R-\alpha} \sum_{i=1}^n \left[1 - (\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right] \\
\Rightarrow H_R^\alpha(A) &= \frac{Rn}{R-\alpha} - \frac{R}{R-\alpha} \sum_{i=1}^n \left[(\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}} \right]
\end{aligned} \tag{2.55}$$

Using Eq. 2.55 in Eq. 2.54 we get

$$\begin{aligned}
I_R^\alpha\left(A, \left[\frac{1}{2}\right]\right) &= \left(\frac{1}{2}\right)^{\frac{\alpha-R}{R}} \left[\frac{Rn}{R-\alpha} - H_R^\alpha(A) \right] - \frac{nR}{R-\alpha} \\
\Rightarrow I_R^\alpha\left(A, \left[\frac{1}{2}\right]\right) &= \left[\left(\frac{1}{2}\right)^{\frac{\alpha-R}{R}} - 1 \right] \frac{nR}{R-\alpha} - \left(\frac{1}{2}\right)^{\frac{\alpha-R}{R}} H_R^\alpha(A).
\end{aligned}$$

Hence, 2.10 holds.

Theorem 2.11 For fuzzy sets A and B , the relation between $H_R^m(A)$ and $I_R^m(A, B)$ is given by

$$I_R^m\left(A, \left[\frac{1}{2}\right]\right) = \frac{1}{2} \left[\frac{-n(R-m+1)}{R-m} - H_R^m(A) \right].$$

Proof We know from Eq. 2.23 that

$$\begin{aligned}
I_R^m(A, B) &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_A^{R-m+1}(x_i) \mu_B^{R-m+1}(x_i) \right. \right. \\
&\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} (1 - \mu_B(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
I_R^m\left(A, \left[\frac{1}{2}\right]\right) &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[\left\{ \mu_A^{R-m+1}(x_i) \left(\frac{1}{2}\right)^{R-m+1} \right. \right. \\
&\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} \left(\frac{1}{2}\right)^{R-m+1} \right\}^{\frac{1}{R-m+1}} - 1 \right] \\
&= \frac{R-m+1}{R-m} \left(\frac{1}{2}\right) \sum_{i=1}^n \left[\left\{ \mu_A^{R-m+1}(x_i) \right. \right. \\
&\quad \left. \left. + (1 - \mu_A(x_i))^{R-m+1} \right\}^{\frac{1}{R-m+1}} \right] \\
&\quad - \frac{n(R-m+1)}{R-m}
\end{aligned} \tag{2.56}$$

From Eq. 2.9

$$\begin{aligned}
H_R^m(A) &= \frac{R-m+1}{R-m} \sum_{i=1}^n \left[1 - (\mu_A^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1})^{\frac{1}{R-m+1}} \right] \\
&\Rightarrow H_R^m(A) \\
&= \frac{n(R-m+1)}{R-m} - \frac{R-m+1}{R-m} \sum_{i=1}^n \left[(\mu_A^{R-m+1}(x_i) + (1 - \mu_A(x_i))^{R-m+1})^{\frac{1}{R-m+1}} \right]
\end{aligned} \tag{2.57}$$

Using Eq. 2.57 in Eq. 2.56 we get

$$\begin{aligned}
I_R^m\left(A, \left[\frac{1}{2}\right]\right) &= \left(\frac{1}{2}\right) \left[\frac{n(R-m+1)}{R-m} - H_R^m(A) \right] - \frac{n(R-m+1)}{R-m} \\
&\Rightarrow I_R^m\left(A, \left[\frac{1}{2}\right]\right) = \frac{1}{2} \left[\frac{-n(R-m+1)}{R-m} - H_R^m(A) \right].
\end{aligned}$$

Hence, 2.11 holds.

2.8 Concluding Remarks

In this chapter, we have proposed two new parametric generalizations of one of the existing R -norm fuzzy information measures along with proof of their validity. It is noted that the proposed generalized fuzzy measures of information are valid measures which reduce to the known measure on substituting the particular values of parameters. Some of the interesting properties of these measures have also been studied. The similarity of proposed generalized measures of fuzzy information with $H_R(A)$ is proved using a numerical example. In addition, two new parametric generalized R -norm fuzzy divergence measures are introduced along with the proof of their validity. The interesting properties of these proposed generalized R -norm fuzzy divergence measures are also established. Finally, the relation between the proposed generalized R -norm fuzzy entropy and divergence measures are introduced. In view of application, the proposed measures are more flexible.

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