The Arc Rampant in 1673: Abraham Bosse, François Blondel, Philippe de la Hire, and conic sections

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Abstract An *arc rampant* is an arc of a conic section determined by tangents at two given endpoints and by an additional tangent line. In *Résolution des quatres principaux problèmes d'architecture*, 1673, François Blondel (1618 - 1686) addressed the construction of an arc rampant; he included, perhaps for the first time in print, justification of Pappus's *Book* VIII *Prop*14. Independently, Philippe de la Hire (1640 - 1718) took up the problem in a short work of 1672, and incorporated it into works of projective geometry of 1673 and 1685. For both, the problem was suggested by Abraham Bosse, engraver and collaborator of Girard Desargues. Both showed that the construction involved harmonic conjugates, and both used, for the first time, the term "harmonic."

1 Introduction

The tracing of the *arc rampant* is the second problem of *Résolution des quatres principaux problèmes d'architecture*, 1673, by (Nicolas)-François Blondel (1618 - 1686):

> to find a Conic Section tangent to three given straight lines, in one plane, at a given point on two of these lines: in other words, to describe geometrically the *arcs rampans* of all types of foot segments (*pieds droits*) and heights.

In his *Cours d'Architecture* (p 424, 1683) [also see Gerbino (2005)], Blondel explained that these oblique arcs are used "rather frequently in architecture, as in passage ways, arches of vaults, lunettes ..., and especially in ramps of staircases."

We begin our account with Abraham Bosse (1604? - 1676), a master of the art of etching, who, under the influence of Girard Desargues, in Paris, sought to bring geometry to various technical arts, including stone cutting and architecture. He sought practical rules for finding and tracing an *arc rampant*. We note in Figure 1,

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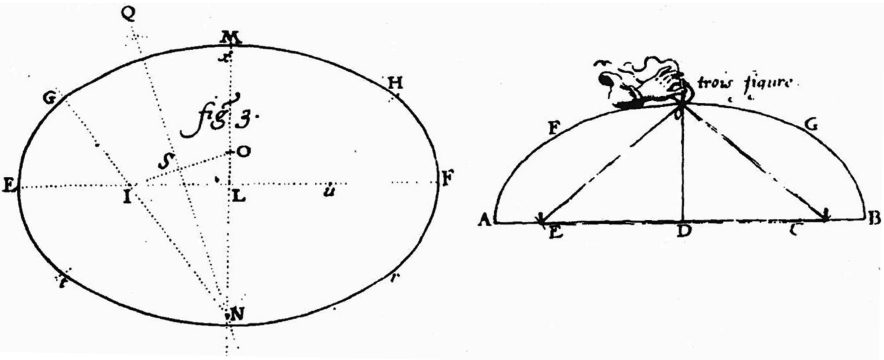


Fig. 1 From Bosse, 1665, *Traité des practiques géométrales et perspectives enseignées dans l'Académie royale de la peinture et sculpture* Left: Plate p 14 *To trace segmental Arcs and Ovals by two Compass openings* Right: Plate p 15 Tracing an ellipse by foci and cord

from Bosse (1665), that Bosse grouped the conic sections with ovals, allowing an approximation of an ellipse by a pair of congruent smaller circular arcs joined smoothly by a pair of arcs from a larger circle. As the title states, Bosse (1665) is based on earlier lectures at the l'Académie royales de la peinture et sculpture, from which Bosse had been excluded in 1661 — an indication of Bosse's affinity for controversy.

Desargues left Paris around 1649. Although he returned in 1657 (Field and Gray 1987, p 35) he no longer worked in mathematics; he died in 1661. So when Bosse later took up the problem of the *arc rampant* he would have needed a mathematical collaborator. He turned to the son, Philippe, of his friend, the painter Laurant de la Hire. The result was LaHire (1672).

Philippe de la Hire was born in 1640, at Paris. He prepared to follow his father's profession, but, even in his teens, geometric aspects of painting, such as perspective, intrigued him. When he traveled to Italy in 1660, for reasons of both his health and his art, he fell in love with Greek geometry. He particularly took to the *Conics* of Apollonius. Bernard de Fontenelle reported, "Geometry began to prevail with him, although dressed in this thorny and frightening form which it took on in the books of the ancients." (Fontenelle 1718, p 77)

The first work by La Hire that comes to us is his short piece of 1672, on *arcs rampants*. Working with a slice of a three-dimensional cone, he found the point of tangency on the given tangent line, the line which La Hire called the *moyenne*, and he found the center of the conic. In the next year, LaHire (1673) appeared, a comprehensive work based primarily on the harmonic relationship associated with conic sections.

(Nicolas)-François Blondel, born in 1618, rose through the armed forces. In the 1640s he designed fortifications in France and the West Indies, followed over the next thirty years by various important building projects. He served as diplomat on missions for Richelieu and Mazarin. In 1656, he was named professor of

Mathematics and Fortifications at the *Collège Royal*, then professor and the first director of the *Académie Royale d'Architecture* in the year of its founding, 1671, and mathematics tutor to the dauphin, in 1673.

His *Résolution des quatres principaux problèmes d'architecture* brought together his work of several decades. In addition to his treatment of *arcs ramparts*, the book also gave mathematical consideration to the ideal shape, with regard to strength, of columns and of beams. Throughout, his objective was to provide precise guidance, based on mathematical theory, to the workers who would carry out the designs. Specifically, Blondel provided a step-by-step straightedge-compass construction of the foci of the (unique) ellipse or hyperbola which satisfies the given conditions for the *arc rampant*. He made use of propositions of Apollonius to locate the point of tangency, and then a pair of conjugate diameters.

La Hire and Blondel came differently to the problem of *arcs ramparts*. Blondel wished to establish, by mathematics, rules of practice in several arts, including architecture and ballistics as needed for artillery. Architecture, going back to Vitruvius, was a learned discipline, but in practice it was a trade governed by tradition and rule of thumb. In an effort to make building construction a science, the *Académie Royale d'Architecture*, a project of Minister Colbert, was created in 1671, with Blondel as its first director.

The late Renaissance saw mathematics brought to practical arts: perspective drawing; ballistics, from the time of Tartaglia; and strength of beams, from Galileo's *Two New Sciences* of 1638. Blondel was a player in this movement, as were Girard Desargues and his disciple Abraham Bosse.

La Hire, on the other hand, seems to have first fallen in love with the mathematics and only slowly developed interest in applications. Fittingly, after a decade of service in mapping, astronomy, and public works, La Hire was named as a member of that same *Académie Royale d'Architecture*, in 1687, taking the place of Blondel, who had died the year before.

2 Harmonic Division

In 1673, the term *harmonic*, describing the division of a line segment in relation to a conic section, first appeared, independently, in two works, LaHire (1673) and Blondel (1673). La Hire's 1673 work begins

> I call the straight line AD cut in 3 parts harmonically when the rectangle contained by all AD and the middle part BC is equal to the rectangle contained by the two extreme parts AB, CD.

[In Fig 2 Left, AD is cut harmonically at B and C.] We now say that A and C are *harmonic conjugates* of B and D when $AD \cdot BC = AB \cdot CD$ and exactly one of A and C is between B and D. This can be written $H(AC, BD)$. Blondel observed that length AC is, then, the *harmonic mean* of lengths AB and AD. When the pairing of the points is understood, the four points will simply be called a *harmonic set*. Both

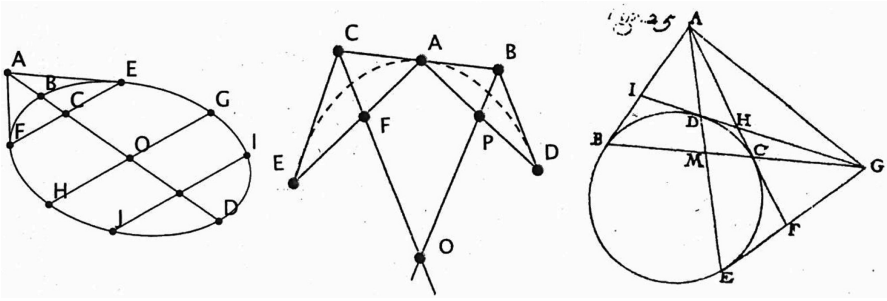


Fig. 2 Left: Conjugate Diameters *HG* and *BD*. Center: Based on Fig 1 of La Hire 1672 Right: La Hire 1673, for Lemma 17.

Blondel and La Hire began their works of 1673 with the construction of the fourth point of a harmonic set when three points are given.

A reminder of fundamentals and vocabulary: Given a conic section and any collection of parallel chords, then the midpoints of those chords are collinear. The set of midpoints is called a *diameter* and the corresponding half-chords are the *ordinates*. A diameter meets the conic itself at two *vertices* in the case of the *central conics*, the ellipse and hyperbola, or in one *vertex* in the case of a parabola, with the tangents at the vertices parallel to the ordinates. When a diameter is an ordinate for another diameter, those are called *conjugate* diameters. Perpendicular conjugate diameters are called *axes*. In the central conics, all diameters meet at one point, the *center*. [In Fig 2 Left, *BD* is a diameter, and *GH* the conjugate diameter.]

Harmonic sets appear in the *Conics* of Apollonius. We have, for example [See Figure 2 Left], Prop 34 of Book I of Apollonius (1998) [Its converse is Prop 37 of Book I or Prop 37 of Book III], which tells us

> Let *BD* be a diameter of an ellipse. From *C* on *BD*, taken closer to *B* than *D*, draw the ordinate meeting the ellipse at *E*. Extend *BD* past *B* to *A* so $AD \cdot BC = AB \cdot CD$, [i.e., $H(AC, BD)$]. Then *EA* is tangent to the ellipse.

When *O* is the midpoint of *BD*, then the harmonic relation $AD \cdot BC = AB \cdot CD$ is equivalent to $OB^2 = OA \cdot OC$.

Both Blondel and La Hire made use of Apollonius's Prop 29 of Book II, that if two tangents from an outside point *A* meet a conic at *E* and *F*, then the line on *A* and the midpoint of *EF* is a diameter of the conic.

3 La Hire's Work of 1672 and 1673

Fig. 2 Center, based on *Figure 1* of LaHire (1672) illustrates the *arc rampant* problem: *EC*, tangent at *E*, *DB*, tangent at *D*, and tangent *CB* are given. One is to find the unique conic section — dotted in the diagram — with the given tangents,

including finding the point of tangency, A. That A is a point in a harmonic set, conjugate with $CB \cap ED$, with respect to C and B, was clear from Apollonius's *Conics* .

LaHire (1672) took a conic section as a slice through a cone, and he considered a cone whose base circle was in a plane perpendicular to the plane of reference and with a vertex, H, not in the plane of reference, for which the three given tangents determine tangent planes to the cone. He would say in (LaHire 1673, p 41) that the demonstrations of 1672 were "according to the method of the ancients." [One can dispute this claim: the overall plan of Apollonius was to derive abscissa-ordinate equations of the conic sections early in his first book, based on their three-dimensional definition as slices through a cone, and thereafter to use the equations and not the three-dimensional cone.]

Bosse published LaHire (1672). He wrote a *Preamble* [See Taton 1953, p 97] and attached his own article, *Regle universelle, pour décrire toutes sortes d'arcs rampans dans toutes les surjections que l'on puisse proposer, sans se servir des Axes, Des Foyers, ny du Cordeau*. In the *Preamble*, Bosse gave a history of the 1672 publication.

> Several years ago I began to work on a second part of the book on stone cutting, on the diagrams of a very capable person. But as we came to examine the *Arcs rampans* in all their positions, we judged it appropriate to confer [with Philippe de la Hire], who, having worked on this subject, more and more involved himself in the deeper properties of the various conic sections which served to form them in all their manifestations. And since he had given me his observations, I believed that they merited publication, with his consent.

Thus, if Bosse is correct, La Hire was studying conic sections, in what would become LaHire (1673), when he composed LaHire (1672). René Taton judged, in Taton (1953, p 97), that the "very capable person" was master mason Monsieur Rouget (the elder).

In LaHire (1673), a conic is the projection, from the cone's vertex, of the base circle of a cone onto a cutting plane. That work incorporates all that is needed to find the point of tangency, A, and the center of the conic section which would serve as the *arc rampant*, but there is no specific mention of the *arc rampant* until the last two pages. The work begins with propositions showing that the harmonic relation among four collinear points is preserved under projection, and that a line cutting a circle yields a harmonic set, as in Lemma 17 [Fig 2 Right]. In the diagram for Lemma 17, treating a circle, we have the harmonic relation $H(AM, DE)$, from which $H(AC, HF)$ follows. [Tangents from A meet the circle at B and C, with M lying on BC.]

In the diagram for La Hire's Lemma 17, we recognize the *arc rampant* when it is an arc of a circle. C is the point of tangency on *moyenne HF* for the *arc rampant* with *pieds droits DH* and *EF*. When that configuration is projected to a conic section, on p 39, we still have $H(AC, HF)$, with C the point of tangency sought.

At the very end, p 93 and 94, of LaHire (1673), the argument of p 39 is repeated, with the claim that a harmonic set is constructed:

> This is demonstrated in the circle by Lemma 17, and following this method the same holds in the curves.

> *This property of curves is entirely necessary to trace the arcs ramparts in all classes of given conditions; and this is what I did, and was printed in 1672 by Monsieur Bosse, with the particularity for the practice of these arcs according to the method of Apollonius, after having seen what Monsieur Rouget, the elder, master mason, very intelligent in stone cutting, had done on this practice. I also knew that Monsieur Blondel, master of mathematics of Monseigneur le Dauphin, had worked on this before I had thought about it, but I was not able to see what he had written on it; and I have no doubt that it must have been something beautiful since it comes from so grand a man.* [Italics in the original]

In LaHire (1685), the specific problem of finding the point of tangency and then the center of the conic section appear in the single Proposition 41, the last of Book 2.

4 Blondel's Work of 1673

Blondel's work opened with the construction, with justification, of the geometric mean and the harmonic mean of two lengths. Then he turned to the question of the *arc rampant* in a large number of cases. There are separate cases for the three different conic sections, for whether the *pieds droits* are or are not parallel, with or without a given *moyenne BC* [as in Figure 2 Center], and whether *BC* is or is not parallel to *DE*. He began with the construction of the point of tangency, *A*, when the *moyenne* is given. Then he tells how to construct a pair of conjugate diameters, when the *arc rampant* is an ellipse or hyperbola, and then how to construct axes for the same conic, and finally how to construct the foci. One assumes he would counsel the workers to use a string, tacked down at the foci, to trace the ellipse. The entire process is very lengthy, but it shows a mastery of conic section properties. He proved, in many cases, that the conic produced does, indeed, fit the requirements of the problem.

Let us examine the steps more carefully. For Blondel, we take his *Quatrième Observation*, his "universal method." Although the method is illustrated in a number of cases, we follow his Figure IV of plate VI, in which the conic is an ellipse. This is our Figure 3 Left, in which the points are renamed so they agree with Figure 2 Center. We let *CB* and *ED* meet at *I*, and then construct the harmonic conjugate, *A*, of *I* with respect to *B* and *C*. Let *Z* be the midpoint of *ED* and let *G* be the point at which the tangents at *E* and *D* meet. By Prop 29 of Book 2 of Apollonius (1998), *ZG* is a diameter. Likewise, when *P* is the midpoint of *AD*, then *BP* is also a diameter. *BP* and *ZG* must meet at the center, *O*, of the conic.

Then Blondel found conjugate diameters. Any line joining a point of an ellipse to its center is a diameter, so *AO* is a semi-diameter, and the line on center *O* that is parallel to the tangent at *A* contains the conjugate diameter. How long is the conjugate semi-diameter, *OY*? We apply the harmonic property: Let the tangent *BD* meet the conjugate diameter extended at *X*. Mark *N* between *O* and *X* so *DN* is parallel to *AO*. By the harmonic property, equivalent to Book I Prop 37 of Apollonius (1998), $XO \cdot NO = OY^2$, from which *Y* can be constructed.

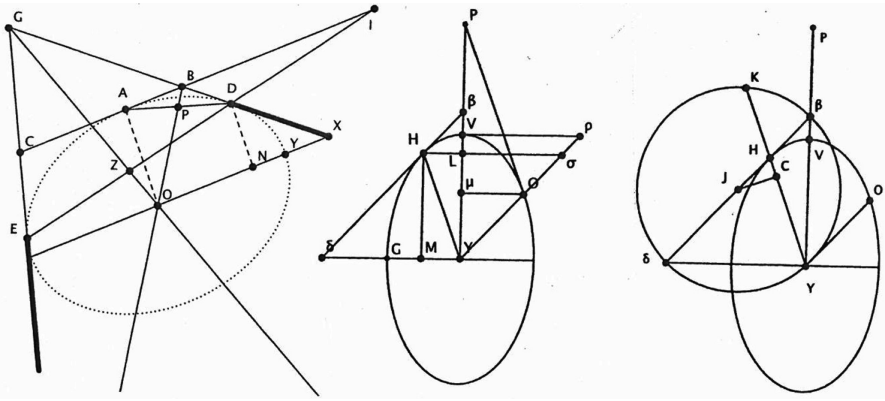


Fig. 3 Left: Based on Blondel 1673, Fig IV of Plate VI. Center: Blondel's Lemma: $YO^2 = \delta H \cdot H\beta$. Right: Constructing axes, based on Pappus Bk VIII Prop 14.

The next-to-last step is most intriguing, finding axes after conjugate diameters have been constructed. Blondel followed the construction of Prop 14 of Book VIII of Pappus's *Mathematical Collection*. See Figure 3 Right. He noted that there is no demonstration in Pappus's Book VIII that the construction is correct. Further, Blondel commented that Commandino, in his 1588 translation and commentary on Pappus's *Mathematical Collection*, expressed surprise that the demonstration was lacking:

> (Pappus 1588, p 470) It is surprising that Pappus, otherwise most diligent, did not address the demonstration of this problem.

Commandino did not supply a demonstration himself and, further, (Blondel 1673, p 51) " I am surprised that Commandino did not note that all the force of the construction of Pappus depends on that the rectangle *YHK*, that is, $\delta H\beta$ equals the square of *OY*." Blondel is correct in the importance of $YH \cdot HK = OY^2$, and he supplies a page-long proof, making use of the harmonic property and similar triangles. [See below.]

Locating the foci, according to well-known principles, is the final step of the construction.

5 Blondel's Lemma

Blondel's Lemma (Blondel 1673, p 52): Given ellipse *GHVO* with center *Y*, axes *VY* and *GY*, and conjugate diameters *HY* and *OY*. Let the tangent at *H* meet *YV* (extended) at β and meet *YG* at δ. [Figure 3 Center] Then

$$\beta H \cdot \delta H = OY^2.$$

Proof. PY is cut harmonically since it is a secant to the ellipse, so $VY^2 = PY \cdot \mu Y$ and, likewise, $VY^2 = \beta Y \cdot YL$. Further, with similar triangles, $\frac{\beta Y}{PY} = \frac{\beta H}{YO}$ and $\frac{\beta L}{\beta H} = \frac{\mu Y}{YO}$. Thus,

$$\frac{VY^2}{\mu Y^2} = \frac{PY \cdot \mu Y}{\mu Y^2} = \frac{PY}{\mu Y} = \frac{YO \cdot \beta Y}{\mu Y \cdot \beta H} = \frac{YO \cdot \beta Y}{\beta L \cdot YO} = \frac{\beta Y}{\beta L}.$$

Again by similar triangles,

$$\frac{\rho Y}{OY} = \frac{VY}{\mu Y} \text{ so } \frac{\rho Y^2}{OY^2} = \frac{VY^2}{\mu Y^2} \text{ and, further, } \frac{\beta Y}{\beta L} = \frac{\beta \delta}{\beta H} = \frac{\beta \delta \cdot \delta H}{\beta H \cdot \delta H}.$$

Since $VY^2 = \beta Y \cdot YL$, then multiplying all lengths by $\frac{\rho Y}{VY}$ we get $\rho Y^2 = \beta \delta \cdot \delta H$. Putting these equations together, we get

$$\frac{\beta \delta \cdot \delta H}{\beta H \cdot \delta H} = \frac{\beta Y}{\beta L} = \frac{VY^2}{\mu Y^2} = \frac{\rho Y^2}{OY^2} = \frac{\beta \delta \cdot \delta H}{OY^2}.$$

Therefore, we have $\beta H \cdot \delta H = OY^2$. □

The Pappus Construction. [Figure 3 Right] On ray YH, we construct point K so $KH \cdot HY = YO^2$. By Blondel's Lemma, $\beta H \cdot \delta H = OY^2$, so $KH \cdot HY = \beta H \cdot \delta H$. This means that K, β, Y, and δ lie on one circle. Therefore, given an ellipse with conjugate diameters HY and YO, we can construct the axes by marking midpoint C of YH, then the perpendicular on C to HY, meeting at J the tangent at H. β and δ lie on the circle on Y with center J.

6 From Theory to Practice

How was the conic section to be traced? With Bosse and Blondel, the enterprise of the *arc rampant* is presented as an aid to the architects and artisans who would need to trace the *arc rampant*. Blondel wrote that in developing, in his own way, the construction of the axes of an ellipse when conjugate diameters are known, he was simplifying the process for the laborers. " What I have added is only to facilitate the execution by the laborers ... where he [Pappus] says it is necessary to make the two squares ... and ... equal to the rectangle ..., and the two squares ... and ... equal to rectangle ...; I made the entire operations by means of the semicircles ... and ...." [p 51]

In Bosse (1672), we read that the work is not just for scholars, but "one needs to give to laborers some method which would be very easy in practice and very little complicated by lines, ... at the same time without making use of foci and cord to trace the ellipse, because to trace it according to the given conditions would require

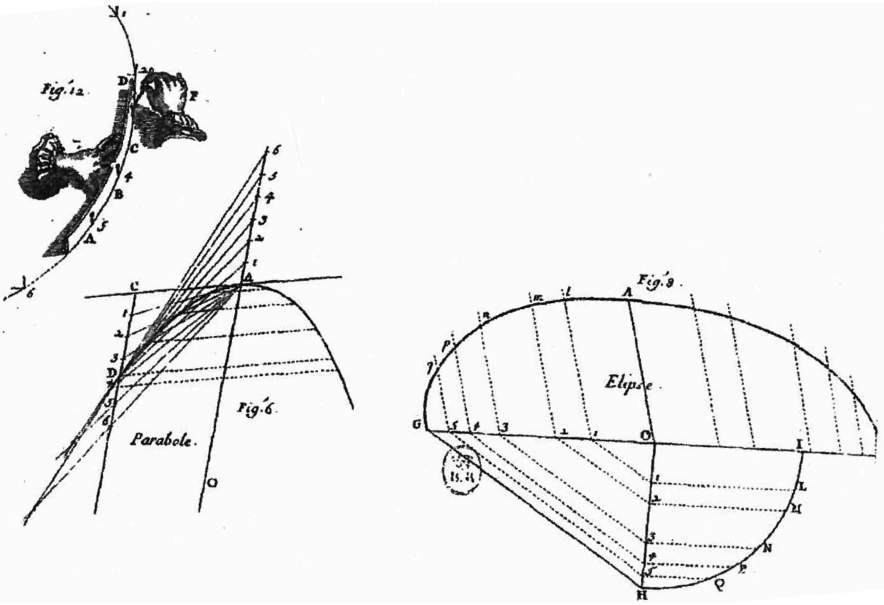


Fig. 4 Bosse 1672. The mechanical spline, points on a parabola, ellipse by a transformation of circle.

too many line operations." Blondel would have agreed with the first part of the statement, but it is likely he is the specific target of the second part, the criticism of procedures that "require too many line operations." Blondel returned the favor. In Blondel (1673, p 33), he criticized "*l'Auteur*" – Bosse? – who "explicated this problem only mechanically." Their squabble likely predates their works of 1672 and 1673, but even in 1672 Bosse defended his approximate methods, still suggesting that an approximation to an ellipse by two congruent circles joined smoothly by a pair of arcs from a larger circle was often acceptable, or that one could find by various methods as many points as needed on the desired curve, and then trace the curve by a mechanical spline held to fit those points. [See Figure 4.] Blondel, on the other hand, criticized approximate methods as "false," in treating Problem 3 of Blondel (1673).

The conclusions of Dominique Raynaud in Raynaud (2016) are worth noting. He finds that the mathematics of Blondel is correct but adds nothing of significance to the theory. As a practical matter, his project is ill advised. The multiplication of cases, many never to appear in practice, and the long lists of steps in the construction, ostensibly to aid the laborers, just make the execution more difficult. His disdain for practical approximation is misplaced. Rather than serving as a helpful corrective to current practice, it is a sterile academic exercise.

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