Chapter 2
Stochastic Discounting

In this chapter we define a mathematically consistent model for calculating (time) values of cash flows. The key objects are so-called deflators which play the role of stochastic discount factors. Our definition (via deflators) leads to values which are consistent with the classical financial theory that involves risk neutral valuation. Typically, in financial mathematics the pricing formulas are based on equivalent martingale measures (see, for example, Föllmer–Schied [FS11]), economists use the notion of state price densities (see Malamud et al. [MTW08]) and actuaries use the terminology of deflators under the real world probability measure (see Duffie [Du96] and Bühlmann et al. [BDES98]). In this chapter we introduce and describe these terminologies.

Moreover, we emphasize that in financial mathematics one usually works under risk neutral measures (equivalent martingale measures) for pricing financial assets. In actuarial mathematics, however, one should also analyze the processes under the real world probability measure (physical measure). This makes it necessary for us to understand the connection between these measures as well as the measure transformation techniques.

2.1 The Basic Discrete and Finite Time Model

In this chapter we develop the theoretical foundations of market-consistent valuation. We work in a discrete and finite time setting which has the advantage that the mathematical machinery becomes simpler; for continuous and infinite time horizon models we refer to the standard literature in financial mathematics, see for example Jeanblanc et al. [JYC09] or Elliott–Kopp [EK99].

Fix $n \in \mathbb{N}$; this n denotes the final time horizon. Then, w.l.o.g., we measure time in years and we consider cash flows on the yearly time grid $t = 0, 1, \ldots, n$.

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We choose a sufficiently rich probability space $(\Omega, \mathcal{F}, P)$ and an increasing sequence of σ-fields $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,n}$ on $(\Omega, \mathcal{F}, P)$ with

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subseteq \mathcal{F}, \tag{2.1}$$

and for simplicity we assume $\mathcal{F}_n = \mathcal{F}$. We call $(\Omega, \mathcal{F}, P, \mathbb{F})$ a filtered probability space with filtration $\mathbb{F}$. The σ-field $\mathcal{F}_t$ plays the role of the information available/known at time t. This may include demographic information, insurance technical information on insurance contracts, financial and economic information and any other information (weather conditions, legal changes, politics, etc.) that is available at time t.

Then, we consider $\mathbb{F}$-adapted random vectors

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \tag{2.2}$$

on the filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, implying that every component X_k of $\mathbf{X}$ is $\mathcal{F}_t$-measurable for $k \leq t$.

Interpretation and Aim.
$\mathbf{X}$ models (random) cash flows with single payments at times t described by X_t. If we have information $\mathcal{F}_t$, then X_k is known for all $k \leq t$, and otherwise it reflects a random payment in the future. On the one hand, we aim at predicting future payments $X_k, k > t$, based on the information $\mathcal{F}_t$ available at time t. On the other hand, our goal is to determine the (time) value of such cash flows $\mathbf{X}$ at any time point $t = 0, \ldots, n$, see also Fig. 2.1.

We make the following technical assumption.

Assumption 2.1 Assume that every component of the $\mathbb{F}$-adapted cash flow $\mathbf{X}$ on $(\Omega, \mathcal{F}, P, \mathbb{F})$ is square integrable.

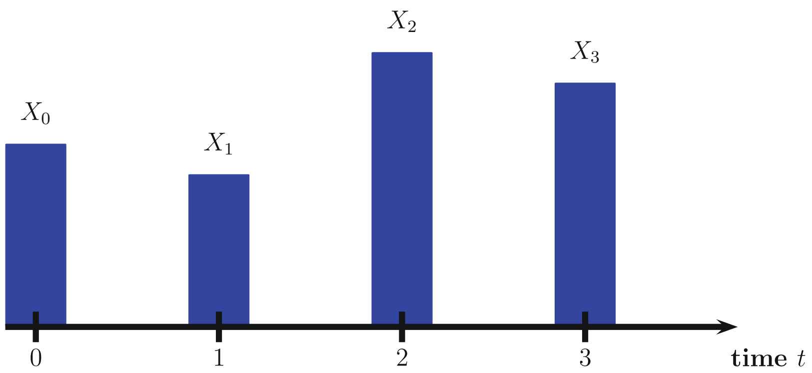

Fig. 2.1 Cash flow $\mathbf{X} = (X_0, X_1, \ldots, X_n)$ for $n = 3$

For the space of all cash flows $\mathbf{X}$ satisfying Assumption 2.1 we write

$$\mathbf{X} = (X_0, X_1, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}). \tag{2.3}$$

We remark that $L^2_{n+1}(P, \mathbb{F})$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$:

$$E\left[\sum_{t=0}^{n} X_t^2\right] < \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P, \mathbb{F}), \tag{2.4}$$

$$\langle \mathbf{X}, \mathbf{Y} \rangle = E\left[\sum_{t=0}^{n} X_t Y_t\right] < \infty \quad \text{for all } \mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P, \mathbb{F}), \tag{2.5}$$

$$\|\mathbf{X}\| = \langle \mathbf{X}, \mathbf{X} \rangle^{1/2} < \infty \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(P, \mathbb{F}). \tag{2.6}$$

Technical Remark. The equality $\|\mathbf{X} - \mathbf{Y}\| = 0$ implies that $\mathbf{X} = \mathbf{Y}$, P-a.s. As usually done, we identify random variables which are equal, P-a.s.

Example 2.1 (*Life insurance*) We consider a general life insurance policy financed by a regular premium income stream $(\Pi_0, \ldots, \Pi_n)$, where Π_t denotes the premium payment made at time t. Furthermore, cash outflows comprise the expenses and the benefit payments occurring in the time interval $(t-1, t]$. If we map all cash flows occurring in the time interval $(t-1, t]$ to its right end point t, we obtain a discrete time cash flow for $t \in \{0, \ldots, n\}$ given by

$$X_t = -\Pi_t + \text{benefits and expenses paid within } (t-1, t]. \tag{2.7}$$

Henceforth, $\mathbf{X} = (X_0, \ldots, X_n)$ denotes the cash flow generated by this insurance policy. The aim is to find an appropriate stochastic model that is able to describe the key features of $\mathbf{X}$. □

Example 2.2 (*Non-life insurance*) In non-life insurance the insurance company usually receives a (risk) premium at the beginning of a well-defined insurance period (upfront premium). Within this insurance period well-defined potential (random) financial losses are covered. We denote the upfront premium payment by $\Pi = -X_0$. The occurrence of an insured event (covered claim) during the insurance period typically generates a sequence of future cash outflows, the so-called claims payments, until the claim is finally settled. That is, usually the insurance company cannot immediately settle a claim, but it takes time until the ultimate claim amount is known. The delay in the settlement is due to the fact that, for example, it takes time until the total medical expenses are known, until the claim is settled at court, until the damaged building is fixed, until the recovery process is understood, etc. (see also Wüthrich–Merz [WM08, WM15] and Wüthrich [Wü13]).

Since one does not wait with the payments until the ultimate claim amount is known (e.g. medical expenses and salaries are paid when they occur) a claim consists of several single payments X_t which reflect the ongoing recovery process. Hence, the total or ultimate nominal claim amount is given by

$$C_n = \sum_{t=1}^{n} X_t, \tag{2.8}$$

where X_t, $t = 1, \ldots, n-1$, denote the single claims payments in the corresponding accounting years and X_n denotes the final payment when the claim is finally settled. Typically, at time t we have information $\mathcal{F}_t$ and the payments X_k, $k \leq t$, are already made, whereas the future payments X_k, $k > t$, need to be predicted based on the information $\mathcal{F}_t$ available at time t. These predictions of future payments determine the claims reserves in the balance sheet of the insurance company.

The underwriting loss (nominal loss) can then be written as

$$UL = \sum_{t=0}^{n} X_t = -\Pi + C_n. \tag{2.9}$$

Remark. UL does not necessarily need to be negative to run this non-life insurance business successfully. The nominal underwriting loss UL does not consider the financial income during the settlement of the claim. That is, the delay in the claims payments allows us to discount these payments, which in the profit and loss statement is considered similarly to investment income on financial assets at the insurance company (see the next sections). □

2.2 Market-Consistent Valuation in the Basic Discrete Time Model

We now value the stochastic cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. We proceed as in Bühlmann [Bü92, Bü95] using a linear, positive and continuous (valuation) functional on the space of cash flows $L^2_{n+1}(P, \mathbb{F})$. Since we are dealing with random vectors $\mathbf{X}$ the definition of positivity needs some care. This we are going to introduce first.

Definition 2.2 (*Positivity*) Choose $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. We define the following notions of positivity for $\mathbf{X}$.

- $\mathbf{X} \geq 0 \iff X_t \geq 0$, P-a.s., for all $t = 0, \ldots, n$.
- $\mathbf{X} > 0 \iff \mathbf{X} \geq 0$ and there exists a $k \in \{0, \ldots, n\}$ such that $X_k > 0$ with positive probability.
- $\mathbf{X} \gg 0 \iff X_t > 0$, P-a.s., for all $t = 0, \ldots, n$.

Assumption 2.3 (*Valuation functional*) Assume that $Q : L^2_{n+1}(P, \mathbb{F}) \to \mathbb{R}$ is a (1) linear, (2) positive, (3) continuous, and (4) normalized functional on $L^2_{n+1}(P, \mathbb{F})$.

This means that the functional Q satisfies the following four properties:

(1) Linearity: For all $\mathbf{X}, \mathbf{Y} \in L^2_{n+1}(P, \mathbb{F})$ and $a, b \in \mathbb{R}$ we have

$$Q\left[a\mathbf{X} + b\mathbf{Y}\right] = aQ\left[\mathbf{X}\right] + bQ\left[\mathbf{Y}\right]. \tag{2.10}$$

(2) Positivity: For any $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ with $\mathbf{X} > 0$ we have $Q[\mathbf{X}] > 0$.
(3) Continuity: For any sequence $(\mathbf{X}^{(k)})_k \subset L^2_{n+1}(P, \mathbb{F})$ with $\mathbf{X}^{(k)} \to \mathbf{X}$ in $L^2_{n+1}(P, \mathbb{F})$ as $k \to \infty$, we have $Q\left[\mathbf{X}^{(k)}\right] \to Q\left[\mathbf{X}\right]$ in $\mathbb{R}$ as $k \to \infty$.
(4) Normalization: For all $\mathbf{X}_0 = (x_0, 0, \ldots, 0) \in L^2_{n+1}(P, \mathbb{F})$ we have $Q\left[\mathbf{X}_0\right] = x_0$.

Interpretation.
The mapping $\mathbf{X} \mapsto Q[\mathbf{X}]$ assigns a monetary value $Q[\mathbf{X}] \in \mathbb{R}$ to every cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$, which can be seen as the price of $\mathbf{X}$ at time 0. As we will see below, this valuation can be extended in a consistent way to future time points which leads to a risk neutral valuation scheme on $L^2_{n+1}(P, \mathbb{F})$. We will call Q satisfying Assumption 2.3 a *valuation functional*.

Remark. Assumptions (1) and (2) ensure that one can develop an arbitrage-free pricing system (see Lemma 2.9 and Remark 2.15, below).

Lemma 2.4 *Assumptions (1) and (2) imply (3) in Assumption 2.3.*

Proof Choose a sequence $(\mathbf{X}^{(k)})_k \subset L^2_{n+1}(P, \mathbb{F})$ with $\mathbf{X}^{(k)} \to \mathbf{X}$ in $L^2_{n+1}(P, \mathbb{F})$ as $k \to \infty$. Define $\mathbf{Y}^{(k)} = \mathbf{X}^{(k)} - \mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. Due to the linearity of Q it suffices to prove that $\mathbf{Y}^{(k)} \to 0$ in $L^2_{n+1}(P, \mathbb{F})$ implies that $Q\left[\mathbf{Y}^{(k)}\right] \to 0$ in $\mathbb{R}$.

In the first step we assume that $\mathbf{Y}^{(k)} \geq 0$ for all k. Then we claim

$$\mathbf{Y}^{(k)} \to 0 \text{ in } L^2_{n+1}(P, \mathbb{F}) \text{ implies } Q\left[\mathbf{Y}^{(k)}\right] \to 0 \text{ as } k \to \infty. \tag{2.11}$$

Assume (2.11) does not hold true, hence (using the positivity of the linear functional Q) there exists $\varepsilon > 0$ and an infinite subsequence k' of k such that for all k'

$$Q\left[\mathbf{Y}^{(k')}\right] \geq \varepsilon. \tag{2.12}$$

Choose an infinite subsequence k'' of k' with

$$\sum_{k''} \left\| \mathbf{Y}^{(k'')} \right\| < \infty. \tag{2.13}$$

We define

$$\mathbf{Y} = \sum_{k''} \mathbf{Y}^{(k'')}. \tag{2.14}$$

Due to the completeness of $L^2_{n+1}(P, \mathbb{F})$ we know that $\mathbf{Y} \in L^2_{n+1}(P, \mathbb{F})$. But linearity and positivity implies

$$Q\left[\mathbf{Y}\right] \geq Q\left[\sum_{k''=1}^{K} \mathbf{Y}^{(k'')}\right] \geq K\,\varepsilon \qquad \text{for every } K. \tag{2.15}$$

This implies that $Q\left[\mathbf{Y}\right] = \infty$ is not finite, which is a contradiction.

Second step: Decompose $\mathbf{Y}^{(k)} = \mathbf{Y}_+^{(k)} - \mathbf{Y}_-^{(k)}$ into a positive and a so-called negative part. Since $\|\mathbf{Y}_+^{(k)}\| \le \|\mathbf{Y}^{(k)}\| \to 0$ and $\|\mathbf{Y}_-^{(k)}\| \le \|\mathbf{Y}^{(k)}\| \to 0$ we see that both $\mathbf{Y}_+^{(k)}$ and $\mathbf{Y}_-^{(k)}$ tend to 0. Because $\mathbf{Y}_+^{(k)} \ge 0$ and $\mathbf{Y}_-^{(k)} \ge 0$ we have – as proved in the first step –

$$Q\left[\mathbf{Y}_+^{(k)}\right] \to 0 \quad \text{and} \quad Q\left[\mathbf{Y}_-^{(k)}\right] \to 0. \tag{2.16}$$

Using once more the linearity of Q completes the proof. □

Theorem 2.5 (Riesz' representation) *Assume that the functional $Q : L^2_{n+1}(P, \mathbb{F}) \to \mathbb{R}$ fulfills Assumption 2.3. There exists a $\varphi \in L^2_{n+1}(P, \mathbb{F})$ such that for all $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ we have*

$$Q[\mathbf{X}] \;=\; \langle \varphi, \mathbf{X} \rangle \;=\; E\left[\sum_{t=0}^{n} \varphi_t X_t\right]. \tag{2.17}$$

This $\varphi \in L^2_{n+1}(P, \mathbb{F})$ has the following properties:

- *φ is $\mathbb{F}$-adapted;*
- *φ has square integrable components φ_t for $t = 0, \dots, n$;*
- *φ is unique;*
- *$\varphi \gg 0$; and*
- *$\varphi_0 = 1$.*

Proof of Theorem 2.5. The classical Riesz' representation theorem provides for every linear and continuous functional Q the existence of a $\varphi \in L^2_{n+1}(P, \mathbb{F})$ such that (2.17) holds for every $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. Thus, in view of Assumption 2.3 and Lemma 2.4 we may relax the assumptions to apply the classical Riesz' representation theorem.

There remain the proofs of the properties of φ: $\mathbb{F}$-adaptedness and square integrability are immediately clear since $\varphi \in L^2_{n+1}(P, \mathbb{F})$. Next we prove uniqueness. Assume that there are two random vectors φ and φ^* in $L^2_{n+1}(P, \mathbb{F})$ satisfying for all $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$

$$Q[\mathbf{X}] = \langle \varphi, \mathbf{X} \rangle = \langle \varphi^*, \mathbf{X} \rangle. \tag{2.18}$$

But then we may choose $\mathbf{X} = \varphi - \varphi^* \in L^2_{n+1}(P, \mathbb{F})$. This and (2.18) imply

$$0 = \langle \varphi - \varphi^*, \mathbf{X} \rangle = \|\varphi - \varphi^*\|^2, \tag{2.19}$$

which immediately gives $\varphi = \varphi^*$, P-a.s. This proves uniqueness.

Next we prove $\varphi \gg 0$. Assume that the latter does not hold true. Then there exists a $k \in \{0, \dots, n\}$ such that $P[\varphi_k \le 0] > 0$. We define the cash flow

$$\mathbf{X} = (0, \dots, 0, 1_{\{\varphi_k \le 0\}}, 0, \dots, 0) \in L^2_{n+1}(P, \mathbb{F}), \tag{2.20}$$

with the non-zero entry being in the $(k+1)$-st component of $\mathbf{X}$. Note that this cash flow satisfies $\mathbf{X} > 0$. Positivity of Q then implies

$$0 < Q[\mathbf{X}] = \langle \varphi, \mathbf{X} \rangle = E\left[\varphi_k 1_{\{\varphi_k \le 0\}}\right] \le 0, \tag{2.21}$$

which is the desired contradiction. Finally, normalization follows from the assumption $\mathcal{F}_0 = \{\emptyset, \Omega\}$ which implies for every non-zero cash flow $\mathbf{X}_0 = (x_0, 0, \ldots, 0) \in L^2_{n+1}(P, \mathbb{F})$

$$x_0 = Q[\mathbf{X}_0] = \langle \varphi, \mathbf{X}_0 \rangle = E[\varphi_0 x_0] = \varphi_0 x_0. \tag{2.22}$$

This proves the claim. □

Definition 2.6 The vector φ (and its single components φ_t) satisfying the properties in Theorem 2.5 is called a (state price) deflator.

The terminology (state price) deflator was introduced by Duffie [Du96] and Bühlmann et al. [BDES98]. In economic theory deflators are called "state price densities" and in financial mathematics "financial pricing kernels", "stochastic interest rates" or "stochastic discount factors".

Remarks.

- We have assumed that $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ in order to find the state price deflator φ. This can be generalized to cash flows $\mathbf{X} \in L^p_{n+1}(P, \mathbb{F})$, $1 \le p \le \infty$, and then the deflator φ would be in $L^q_{n+1}(P, \mathbb{F})$ with $1/p + 1/q = 1$. Or, even more generally, we can take a fixed deflator $\varphi \in L^1_{n+1}(P, \mathbb{F})$ with $\varphi \gg 0$ and then define the set of cash flows that can be priced by

$$\mathcal{L}_\varphi = \left\{ \mathbf{X} \in L^1_{n+1}(P, \mathbb{F}) : \; E\left[\sum_{t=0}^{n} \varphi_t |X_t|\right] < \infty \right\}. \tag{2.23}$$

  For these cash flows we then define the valuation functional Q on $\mathcal{L}_\varphi$ by $Q[\mathbf{X}] = \langle \varphi, \mathbf{X} \rangle$. For more details we refer to Wüthrich–Merz [WM13].
- There is a one-to-one correspondence between the valuation functional Q of Assumption 2.3 and the state price deflator φ according to Definition 2.6. Theorem 2.5 proves one direction and the other direction is immediate.

2.2.1 The Task of Modelling

Find the appropriate valuation functional Q or equivalently find the appropriate $\mathbb{F}$-adapted state price deflator φ!

In the more general setup, one would define/choose $\varphi \in L^1_{n+1}(P, \mathbb{F})$ and then value the cash flows $\mathbf{X} \in \mathcal{L}_\varphi$, see (2.23). The choice of φ will include market risk aversion as well as individual risk aversion, this will be described in the following chapters, and we will also describe the connection between the state price deflators and the risk neutral martingale measures.

The $\mathbb{F}$-adaptedness will be crucial in the sequel. It essentially means that the deflator φ_t (stochastic discount factor) is known at time t and, hence, allows us to make a direct connection between the $\mathcal{F}_t$-measurable cash flow X_t and the behaviour of the financial market at time t described by φ_t. In particular, this implies that φ_t will allow us to model embedded options and guarantees in X_t that depend on economic and financial scenarios.

Examples of state price deflators can be found in Bühlmann [Bü95], for example the Ehrenfest Urn with the limit Ornstein–Uhlenbeck model, in Filipović–Zabczyk [FZ02] or one can easily discretize, for example, the Vasiček model, see Brigo–Mercurio [BM06] and Exercise 2.3. For more discrete time examples we refer to Wüthrich–Merz [WM13].

Exercise 2.3 (*Discrete time Vasiček* [Va77] *model*) Choose a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and assume that $(\varepsilon_t)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted, that ε_t is independent of $\mathcal{F}_{t-1}$ for all $t = 1, \ldots, n$ and standard Gaussian distributed w.r.t. P. Then, we define the stochastic process $(r_t)_{t=0,\ldots,n}$ by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho \varepsilon_t, \tag{2.24}$$

for given parameters b, β, $\rho > 0$. This $(r_t)_{t=0,\ldots,n}$ describes the spot rate dynamics of the Vasiček model under the (real world) probability measure P, see Wüthrich–Merz [WM13], Sect. 3.3.

Next, we choose $\lambda \in \mathbb{R}$ (market price of risk) and define the deflator in the Vasiček model by

$$\varphi_t = \exp\left\{ -\sum_{k=1}^{t} \left[r_{k-1} + \frac{\lambda^2}{2}\, r_{k-1}^2 \right] - \sum_{k=1}^{t} \lambda\, r_{k-1}\, \varepsilon_k \right\}, \tag{2.25}$$

for $t = 1, \ldots, n$ and $\varphi_0 = 1$. Prove that $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n) \in L^1_{n+1}(P, \mathbb{F})$ is a deflator. Moreover, prove that the cash flow $\mathbf{X} = (0, \ldots, 0, 1, 0, \ldots, 0)$ is in $\mathcal{L}_{\varphi}$, see (2.23). □

2.2.2 Understanding Deflators and Zero Coupon Bonds

A *deflator* φ_t transports the cash amount at time t to its value at time 0, see Fig. 2.2. This transportation is a stochastic transportation (stochastic discounting). A cash flow $\mathbf{X}_t = (0, \ldots, 0, X_t, 0, \ldots, 0)$ does not necessarily need to be independent of (or uncorrelated with) φ_t, which, in general, gives

$$Q\left[\mathbf{X}_t\right] = E\left[X_t\, \varphi_t\right] \neq E\left[X_t\right] E\left[\varphi_t\right]. \tag{2.26}$$

$Q\left[\mathbf{X}_t\right]$ describes the value/price of $\mathbf{X}_t$ at time 0, where X_t is stochastically discounted with the $\mathcal{F}_t$-measurable deflator φ_t.

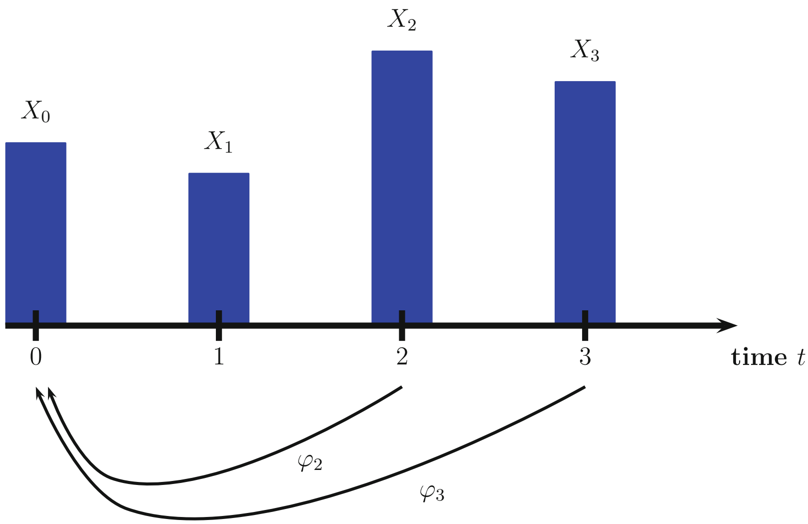

Fig. 2.2 Deflator φ and cash flow $\mathbf{X}$

We decompose the deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$ into its *span-deflators*. Since $\varphi \gg 0$ we can build the following ratios for all $t > 0$, P-a.s.:

$$Y_t = \frac{\varphi_t}{\varphi_{t-1}}. \tag{2.27}$$

Moreover, we define $Y_0 = 1$. Thus, $\mathbf{Y} = (Y_t)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted and satisfies

$$\varphi_t = Y_0\, Y_1\, \cdots\, Y_t = \prod_{k=0}^{t} Y_k. \tag{2.28}$$

$\mathbf{Y} = (Y_t)_{t=0,\ldots,n}$ is called a span-deflator. Span-deflators Y_t, $t \geq 1$, transport the cash amount at time t to its value at time $t-1$ (one-year deflating), see Fig. 2.3. For more information we refer to p. 28.

Question. How is the deflator φ related to zero coupon bonds and classical financial discounting?

Definition 2.7 A (default-free) zero coupon bond is a financial instrument that pays one unit of currency at a fixed maturity date $t \in \{0, \ldots, n\}$. Its cash flow is denoted by $\mathbf{Z}^{(t)} = (0, \ldots, 0, 1, 0, \ldots, 0) \in L^2_{n+1}(P, \mathbb{F})$.

For a given state price deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$, the value at time 0 of this zero coupon bond is given by

$$d_{0,t} = Q\left[\mathbf{Z}^{(t)}\right] = E\left[\varphi_t\right]. \tag{2.29}$$

In the financial literature $d_{0,t}$ is often denoted by $P(0, t)$. We come back to this in (2.50).

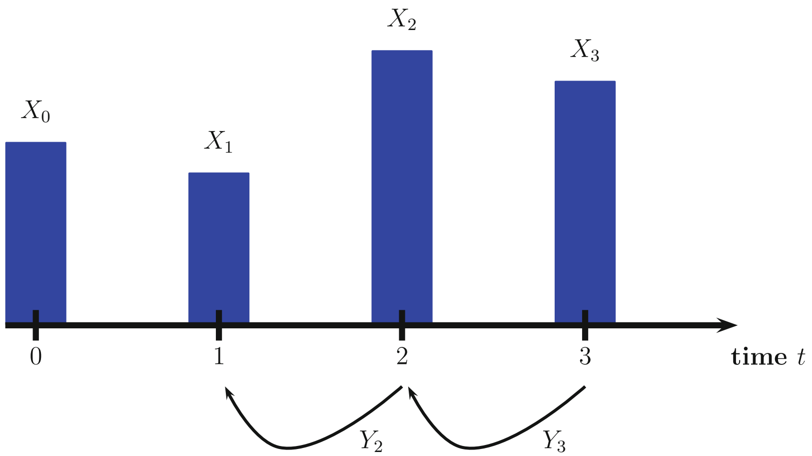

Fig. 2.3 Span-deflators Y_t and cash flow $\mathbf{X}$

Henceforth, $d_{0,t}$ transports the cash amounts at time t to their value at 0. But $d_{0,t}$ is $\mathcal{F}_0$-measurable, whereas φ_t is a $\mathcal{F}_t$-measurable random variable. This means that the deterministic discount factor $d_{0,t}$ is known at the beginning of the time period $(0, t]$, whereas φ_t is only known at the end of the time period $(0, t]$. As long as we are dealing with deterministic cash flows $\mathbf{X}$, we can either work with zero coupon bond prices $d_{0,t}$ or with deflators φ_t to determine the value of $\mathbf{X}$ at time 0. But as soon as the cash flows $\mathbf{X}$ are stochastic we need to work with deflators (see (2.26)) since X_t and φ_t may be influenced by the same risk factors (are dependent). An easy example of this dependence is constructed by choosing X_t as an option that depends on the actual realization of φ_t. Various life insurance companies hold such embedded options and financial guarantees in their insurance portfolios, and henceforth need to use the deflator framework for valuation.

Classical actuarial discounting is taking a constant interest rate $i > 0$. That is, in classical actuarial models φ_t has the following form

$$\varphi_t = (1 + i)^{-t}. \tag{2.30}$$

This deflator gives a consistent theory but its behaviour is far from the economic observations found in practice. This indicates that we have to be very careful with this deterministic modelling choice in a total balance sheet approach, since it implies that we will obtain values which are far away from financial market observations.

Exercise 2.4 (*Zero coupon bond price in the Vasiček model*) We revisit the discrete time Vasiček model presented in Exercise 2.3. Calculate for this model the zero coupon bond prices $d_{0,t}$. We claim that these prices are given by

$$d_{0,t} = \exp\{A(0,t) - r_0\, B(0,t)\}, \tag{2.31}$$

for appropriate functions $A(0,\cdot)$ and $B(0,\cdot)$.

Hint: the claim is proved by induction using properties of log-normal distributions.

Give an interpretation of r_0 in terms of $d_{0,1}$. For more background information we refer to Sect. 3.3 of Wüthrich–Merz [WM13]. □

Before we define prices at arbitrary time points $t = 0, \ldots, n$ we give a finite probability space example which shows the relation between deflators and replication. This is done in the next subsection.

2.2.3 A Toy Example for Deflators

In this subsection we give a toy example of a deflator construction that is based on a finite probability space. In a first step we introduce a statistical model that is calibrated. In a second step we construct the deflator. Our example is borrowed from Jarvis et al. [JSV01].

We consider a one-period model and we assume that there are two possible states at time 1 called ω_1 and ω_2. This setup can be described by the measurable space $(\Omega, \mathcal{F})$ with $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{F} = 2^\Omega$ being the corresponding power set. Finding deflators on finite probability spaces is essentially an exercise in linear algebra. We should also mention that models on finite probability spaces often have the advantage that the crucial mathematical and economic structures are easier to detect (see Malamud et al. [MTW08]).

Step 1. In a first step we construct the *state space securities* SS_1 and SS_2 for the two states ω_1 and ω_2, respectively. A state space security SS_i for state ω_i pays one unit of currency if state ω_i occurs at time 1. These state space securities are used to construct a consistent pricing model. That is, we aim at calibrating the following table.

	SS_1	SS_2
Market price $Q[\cdot]$ at time 0	?	?
Payout if state ω_1 occurs at time 1	1	0
Payout if state ω_2 occurs at time 1	0	1

Since we have two possible states ω_1 and ω_2 we need two linearly independent assets A and B to calibrate the model. Assume that assets A and B have the following prices and payout structures:

	Asset A	Asset B
Market price $Q[\cdot]$ at time 0	1.65	1
Payout if state ω_1 occurs at time 1	3	2
Payout if state ω_2 occurs at time 1	1	0.5

With this information we can construct the two state space securities SS_1 and SS_2 and calculate their (consistent) prices at time 0. For this purpose, say for SS_1, we construct a portfolio that consists of x_1 units of asset A and y_1 units of asset B. The goal is to determine x_1 and y_1 such that the resulting portfolio pays 1 if state ω_1 occurs at time 1 and 0 otherwise. That is, this portfolio exactly replicates the state space security SS_1. Mathematically speaking we need to solve the equation $SS_1 = x_1 A + y_1 B$ on Ω for SS_1, and a similar equation for SS_2. The solution to these two equations provides the following table:

	Units x_i of A	Units y_i of B	Market price Q
State space security SS_1	-1	2	0.35
State space security SS_2	4	-6	0.60

The last column of this table is obtained by requiring that asset prices are linear, in the sense that

$$\text{price of } SS_i = x_i \cdot \text{price of asset } A + y_i \cdot \text{price of asset } B. \tag{2.32}$$

Note that this is similar to the derivation of the Arbitrage Pricing Theory framework (see Ingersoll [Ing87], Chap. 7). Basically, we need that asset A and asset B are linearly independent and that the valuation functional Q is linear. Based on these assumptions the consistent market prices of all other assets and cash flows are calculated using replication arguments. Thus, if we have another asset $\mathbf{X}$ which pays 2 in state ω_1 and 1 in state ω_2, its consistent price at time 0 is given by

$$Q[\mathbf{X}] = 2 \cdot 0.35 + 1 \cdot 0.60 = 1.3. \tag{2.33}$$

Consider the zero coupon bond $\mathbf{Z}^{(1)}$ that pays in both states ω_1 and ω_2 the amount 1:

$$d_{0,1} = Q\left[\mathbf{Z}^{(1)}\right] = 1 \cdot 0.35 + 1 \cdot 0.60 = 0.95, \tag{2.34}$$

which leads to a risk-free return of $(0.95)^{-1} - 1 = 5.26\,\%$. The pricing model is now calibrated so that we have consistent market prices.

Step 2. Next we construct the deflators. By a slight abuse of notation, we denote by $Q(\omega_i)$ the market price of the state space security SS_i at time 0, i.e. $Q(\omega_1) = 0.35$ and $Q(\omega_2) = 0.60$. Moreover, let $X_1(\omega_i)$ denote the payout of the asset $\mathbf{X} = (0, X_1)$ if state ω_i occurs at time 1. The consistent market price of $\mathbf{X}$ at time 0 is given by, see (2.33),

$$Q[\mathbf{X}] = \sum_{i=1}^{2} X_1(\omega_i)\; Q(\omega_i). \tag{2.35}$$

Note: So far we have not used any probabilities! All arguments are based on linearity and replication only!

Now we complete $(\Omega, \mathcal{F})$ to a probability space by assuming that state ω_1 occurs with probability $p(\omega_1) \in (0, 1)$ and state ω_2 with probability $p(\omega_2) = 1 - p(\omega_1) \in (0, 1)$. Identity (2.35) can be rewritten as follows

$$Q[\mathbf{X}] = \sum_{i=1}^{2} X_1(\omega_i)\ Q(\omega_i) = \sum_{i=1}^{2} p(\omega_i)\ \frac{Q(\omega_i)}{p(\omega_i)}\ X_1(\omega_i) = E\left[\frac{Q}{p}\ X_1\right], \tag{2.36}$$

where E denotes the expected value induced by the probabilities $p(\cdot)$. Henceforth, we may define the random variable

$$\varphi_1 = \frac{Q}{p}, \tag{2.37}$$

which immediately implies the pricing formula

$$Q[\mathbf{X}] = E[\varphi_1 X_1]. \tag{2.38}$$

For an explicit choice of probabilities $p(\omega_i)$, the deflator φ_1 takes the following values:

	Value of deflator φ_1	Probability $p(\omega_i)$
If state ω_1 Occurs at time 1	0.7	0.5
If state ω_2 Occurs at time 1	1.2	0.5

Alternatively to (2.34) this provides the value of the zero coupon bond

$$Q\left[\mathbf{Z}^{(1)}\right] = E[\varphi_1] = \sum_{i=1}^{2} \varphi_1(\omega_i)\ p(\omega_i) = 0.7 \cdot 0.5 + 1.2 \cdot 0.5 = 0.95. \tag{2.39}$$

Note that in our example the deflator φ_1 is not necessarily smaller than 1. With probability 1/2 we will observe that the deflator has a value of 1.2. This may be counter-intuitive from an economic point of view but it makes perfect sense in our model world. Henceforth, the model and parameters need to be specified carefully in order to get economically meaningful models. Moreover, for strict positivity of the deflator, we need that the state space securities have strictly positive prices (i.e. that Q is positive) which is a natural property in consistent pricing systems.

2.3 Valuation at Time $t > 0$

Postulate: Correct prices should eliminate the possibility of "playing games" with cash flows. This should be interpreted as the non-existence of strategies that provide expected gains without downside risks (see also Remark 2.15).

Assume a fixed deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$ is given. We define the price process $(Q_t\,[\mathbf{X}])_{t=0,\ldots,n}$ of the random vector $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ w.r.t. φ as follows: for $t = 0, \ldots, n$ we set

$$Q_t\,[\mathbf{X}] = \frac{1}{\varphi_t}\, E\left[\sum_{k=0}^{n} \varphi_k\, X_k \middle| \mathcal{F}_t\right]. \tag{2.40}$$

Observe that $\varphi \gg 0$ and $\mathbf{X}, \varphi \in L^2_{n+1}(P, \mathbb{F})$ imply that $Q_t\,[\mathbf{X}]$ is well-defined. From definition (2.40) it is also obvious that price $Q_t\,[\mathbf{X}]$ is $\mathcal{F}_t$-measurable for all t, and henceforth $(Q_t\,[\mathbf{X}])_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted. Note that the right-hand side of (2.40) can be decoupled because the payments X_k (and the deflators φ_k) are $\mathcal{F}_t$-measurable for $k \le t$.

Interpretation.
The mapping $\mathbf{X} \mapsto Q_t[\mathbf{X}]$ assigns a monetary value $Q_t[\mathbf{X}]$ at time t to the cash flow $\mathbf{X}$, i.e. it attaches an $\mathcal{F}_t$-measurable price to the cash flow $\mathbf{X}$. Of course, in general, this price is stochastic seen from time 0 and depends on $\mathcal{F}_t$. We will see below that these price processes $(Q_t[\mathbf{X}])_{t=0,\ldots,n}$ lead to a consistent pricing system for a given state price deflator φ. Moreover, this consistent pricing system is equivalent to a risk neutral (and arbitrage-free) valuation scheme (see Lemma 2.9 and Remark 2.15).

By our assumptions we have $Q[\mathbf{X}] = Q_0[\mathbf{X}]$. This is implied by normalization $\varphi_0 = 1$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

The justification of our price process definition $(Q_t[\mathbf{X}])_{t=0,\ldots,n}$ may use an equilibrium argument or alternatively a no-arbitrage argument. The latter (more technical) argument will be provided in Lemma 2.9 and Remark 2.15. At the current stage we provide the equilibrium argument. Assume we purchase cash flow $\mathbf{X}$ at time t at price $Q_t\,[\mathbf{X}]$. Hence, we generate the following payment cash flow by this acquisition

$$Q_t\,[\mathbf{X}]\;\mathbf{Z}^{(t)} = (0, \ldots, 0, Q_t\,[\mathbf{X}]\,, 0, \ldots, 0)\,, \tag{2.41}$$

if we pay the price for $\mathbf{X}$ at time t. From today's point of view this payment stream has value

$$Q_0\left[Q_t\,[\mathbf{X}]\;\mathbf{Z}^{(t)}\right], \tag{2.42}$$

since we have only information $\mathcal{F}_0$ at time 0 about the price $Q_t\,[\mathbf{X}]$ of $\mathbf{X}$ at time t. Equilibrium requires that

$$Q_0\,[\mathbf{X}] = Q_0\left[Q_t\,[\mathbf{X}]\;\mathbf{Z}^{(t)}\right], \tag{2.43}$$

since (based on today's information $\mathcal{F}_0$) the two payment streams should have the same value. That is, we agree *today* to either purchase and pay $\mathbf{X}$ today or to purchase and pay $\mathbf{X}$ at time t (at its current price $Q_t\,[\mathbf{X}]$ at that time). Since we use the same information $\mathcal{F}_0$ for these two contracts and they provide the same asset $\mathbf{X}$ the two contracts should have the same price.

Suppose now that we play the following game: We decide to purchase and pay cash flow $\mathbf{X}$ if and only if an event $F_t \in \mathcal{F}_t$ occurs. Since from today's point of view we do not know whether the event F_t will occur, we should have the following price equilibrium, see also (2.43),

$$Q_0 \left[\mathbf{X}\, 1_{F_t} \right] = Q_0 \left[Q_t \left[\mathbf{X} \right] \, \mathbf{Z}^{(t)} \, 1_{F_t} \right]. \tag{2.44}$$

Note that strictly speaking (2.44) is not well-defined because the cash flow $\mathbf{X}\, 1_{F_t}$ on the left-hand side is not $\mathbb{F}$-adapted. To make this argument rigorous one should restrict to the outstanding cash flow of $\mathbf{X}$ at time $t-1$. This is similar to the considerations (2.52) and (2.54). Nevertheless, we may rewrite (2.44) for the given deflator $\boldsymbol{\varphi}$ as follows

$$E \left[\sum_{k=0}^{n} \varphi_k \, X_k \, 1_{F_t} \right] = E \left[\varphi_t \, Q_t \left[\mathbf{X} \right] \, 1_{F_t} \right]. \tag{2.45}$$

Since $\varphi_t \; Q_t \left[\mathbf{X} \right]$ is $\mathcal{F}_t$-measurable and Eq. (2.45) should hold true for all $F_t \in \mathcal{F}_t$, this is exactly the definition of the conditional expectation given the σ-field $\mathcal{F}_t$. Henceforth, (2.45) implies (2.40), P-a.s., and justifies that (2.40) is an economically meaningful definition. A more financial mathematically based argumentation would say that deflated price processes need to be $(P, \mathbb{F})$-martingales in order to have an arbitrage-free pricing scheme, see Lemma 2.9 and Remark 2.15.

Remark. We would like to emphasize that we *first* choose the state price deflator $\boldsymbol{\varphi}$ and *then* calculate the price processes $(Q_t[\mathbf{X}])_{t=0,\ldots,n}$ of all cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ under the *same* state price deflator $\boldsymbol{\varphi}$. This provides a consistent pricing system (that naturally depends on the choice of $\boldsymbol{\varphi}$).

We close this section with remarks on discounting and forward rates. We have defined the discount factors at time 0 by

$$d_{0,m} = Q_0 \left[\mathbf{Z}^{(m)} \right] = E \left[\varphi_m \right], \qquad \text{for } m = 1, \ldots, n, \tag{2.46}$$

calculated from zero coupon bond cash flows $\mathbf{Z}^{(m)}$ with maturity dates m. For $t < m$, let $d_{t,m}$ be the *forward discount factor fixed at time 0* for discounting back from time m to time t. The terminology 'forward' refers to this fixing at an earlier time point. In this deterministic ($\mathcal{F}_0$-measurable) setup consistency requires

$$d_{0,t} \, d_{t,m} = d_{0,m}. \tag{2.47}$$

The left-hand side of (2.47) is the price at time 0 for receiving amount $d_{t,m}$ at time t, and an immediate reinvestment of this amount into 1 forward discount factor contract at time t at price $d_{t,m}$ (fixed at time 0) provides 1 unit of currency at maturity date m. The right-hand side of (2.47) is the price at time 0 of a zero coupon bond contract for directly receiving 1 unit of currency at time m. Hence, both strategies provide the same cash flow and are based on the same information $\mathcal{F}_0$, therefore consistency requires identity (2.47). Thus, the (consistent) forward discount factors for $t < m$ are determined by

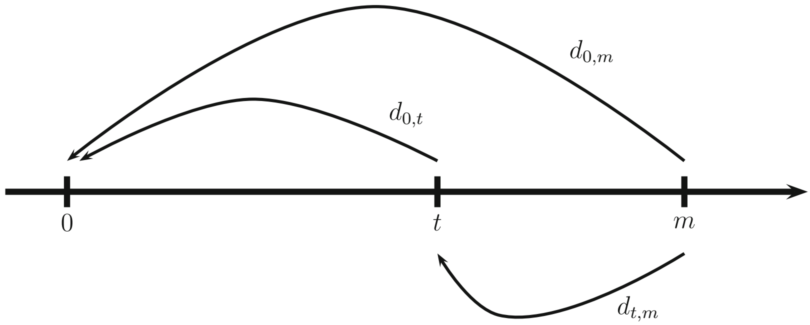

Fig. 2.4 $\mathcal{F}_0$-measurable forward discount factor $d_{t,m}$ for $t < m$

$$d_{t,m} = \frac{d_{0,m}}{d_{0,t}}. \tag{2.48}$$

This is the forward price of a zero coupon bond with maturity date m fixed at time 0, i.e. $\mathcal{F}_0$-measurable, to be paid at time t (Fig. 2.4).

On the other hand, the $\mathcal{F}_t$-measurable price at time t of a zero coupon bond with maturity date m is given by

$$Q_t\left[\mathbf{Z}^{(m)}\right] = \frac{1}{\varphi_t}\, E\left[\varphi_m \,|\, \mathcal{F}_t\right] = E\left[\left.\frac{\varphi_m}{\varphi_t}\right|\mathcal{F}_t\right]. \tag{2.49}$$

This price is in line with definition (2.40) for a single deterministic payment of size 1 at time m, see also Definition 2.7.

Notation. In financial mathematics one uses the following notation and identities for zero coupon bond prices, $t < m$,

$$P(t, m) = Q_t\left[\mathbf{Z}^{(m)}\right] = E\left[\left.\frac{\varphi_m}{\varphi_t}\right|\mathcal{F}_t\right] = E^*\left[\left.\exp\left\{-\sum_{s=t}^{m-1} r_s\right\}\right|\mathcal{F}_t\right], \tag{2.50}$$

where $(r_t)_{t=0,\dots,n}$ is the spot rate process and E^* is the expectation under an equivalent martingale measure $P^* \sim P$. The first identity in (2.50) is a definition, the second one is obtained from (2.40) and the last one is going to be derived in Sect. 2.5. Note that for $t = 0$ we have $d_{0,m} = P(0, m) = Q_0\left[\mathbf{Z}^{(m)}\right]$. In these notes we use $d_{t,m}$ for $\mathcal{F}_0$-measurable forward discount factors and $P(t, m)$ for $\mathcal{F}_t$-measurable zero coupon bond prices.

Exercise 2.5 We revisit the discrete time Vasiček model presented in Exercise 2.3. Calculate for this model the zero coupon bond prices $P(t, m)$ at times $t < m$. We claim that these prices are given by

$$P(t,m) = Q_t\left[\mathbf{Z}^{(m)}\right] = \exp\left\{A(t,m) - r_t\ B(t,m)\right\}, \tag{2.51}$$

for appropriate functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ and $\mathcal{F}_t$-measurable spot rates r_t, see also (2.31).

Give an interpretation of r_t in terms of $P(t, t+1)$.

Remark. A zero coupon bond price representation of the form (2.51) is called an affine term structure, because its logarithm is an affine function of the observed spot rate r_t for all $t = 0, \ldots, m-1$, see also Filipović [Fi09] and Wüthrich–Merz [WM13]. □

2.4 The Meaning of Reserves

In the previous section we have considered the valuation of cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ at any time point $t = 0, \ldots, n$. However, in an insurance context we are mainly interested in the valuation of *future* cash flows $(0, \ldots, 0, X_{t+1}, \ldots, X_n)$ if we are currently at time t. For these future cash flows we need to build reserves on the liability side of the balance sheet, because they refer to the outstanding (loss) liabilities we still need to meet. This means that we need to predict and assign (market-)consistent values to X_k, $k > t$, based on the available information $\mathcal{F}_t$ at time t.

Note that from an economic point of view the terminology *reserves* is not completely correct (because reserves rather refers to shareholder value) and one should call the reserves instead *provisions* because they belong to the insured and the policyholder, respectively.

Postulate: Correct reserves should eliminate the possibility of "playing games" with insurance liabilities.

Throughout, we assume a fixed deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$ is given.

Assume that the insurance contract is represented by the (stochastic) cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. We define for $1 \le k \le n$ the *outstanding liabilities* of $\mathbf{X}$ at time $k-1$ by

$$\mathbf{X}_{(k)} = (0, \ldots, 0, X_k, \ldots, X_n) \in L^2_{n+1}(P, \mathbb{F}). \tag{2.52}$$

This is the remaining cash flow of $\mathbf{X}$ after time $k-1$. $\mathbf{X}_{(k)}$ represents the amounts for which we have to build reserves at time $k-1$, such that we are able to meet all future payments arising from this contract. The *reserves at time* $0 \le t \le k-1$ for the outstanding liabilities $\mathbf{X}_{(k)}$ are defined as

$$R_t^{(k)} = Q_t[\mathbf{X}_{(k)}] = \frac{1}{\varphi_t}\ E\left[\left.\sum_{s=k}^{n} \varphi_s\ X_s \right| \mathcal{F}_t\right]. \tag{2.53}$$

On the one hand, $R_t^{(k)}$ corresponds to the conditionally expected monetary value of the cash flow $\mathbf{X}_{(k)}$ viewed from time t. On the other hand, $R_t^{(k)}$ is used to predict the monetary value of the random variable $\mathbf{X}_{(k)}$. We will comment more on this prediction below.

We justify that (2.53) is a reasonable definition of the reserves. We argue for $R_t^{(k)}$ in a similar fashion as in the last section using an equilibrium argument: we want to avoid that we can play games with insurance contracts. In particular, we consider the following game: assume we have two insurance companies A and B that have exactly the same liability $\mathbf{X}$ and the following two business strategies.

- Company A keeps the contract until all payments are met.
- Company B decides (at time 0) to sell the run-off of the outstanding liabilities at time $t-1$ at price $R_{t-1}^{(t)}$ if an event $F_{t-1} \in \mathcal{F}_{t-1}$ occurs.

This implies that the two strategies generate the following cash flows:

	0	...	$t-1$	t	...	n
$\mathbf{X}^{(A)} =$	$(X_0,$	$\ldots,$	$X_{t-1},$	$X_t,$	$\ldots,$	$X_n)$
$\mathbf{X}^{(B)} =$	$(X_0,$	$\ldots,$	$X_{t-1} + R_{t-1}^{(t)} 1_{F_{t-1}},$	$X_t 1_{F_{t-1}^c},$	$\ldots,$	$X_n 1_{F_{t-1}^c})$

Hence, the price difference at time 0 of these two strategies is given by

$$Q_0\left[\mathbf{X}^{(A)} - \mathbf{X}^{(B)}\right] = E\left[-\varphi_{t-1} R_{t-1}^{(t)} 1_{F_{t-1}}\right] + E\left[\sum_{s=t}^{n} \varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.54}$$

As in (2.44), we have that the two strategies based on information $\mathcal{F}_0$ should have the same initial value, because they are based on the same information and they face the same liability $\mathbf{X}$. This implies requirement $Q_0\left[\mathbf{X}^{(A)} - \mathbf{X}^{(B)}\right] = 0$, and thus we should have for any event $F_{t-1} \in \mathcal{F}_{t-1}$ the following equality

$$E\left[\varphi_{t-1} R_{t-1}^{(t)} 1_{F_{t-1}}\right] = E\left[\sum_{s=t}^{n} \varphi_s X_s 1_{F_{t-1}}\right]. \tag{2.55}$$

Using the definition of conditional expectations, this gives the following definition of the reserves:

$$R_{t-1}^{(t)} = \frac{1}{\varphi_{t-1}} E\left[\sum_{s=t}^{n} \varphi_s \, X_s \,\middle|\, \mathcal{F}_{t-1}\right] = Q_{t-1}\left[\mathbf{X}_{(t)}\right], \tag{2.56}$$

which justifies (2.53) for $k = t$. The case $k > t$ is then easily obtained by iteration. Observe that the argument in (2.44) was not completely correct because of measurability issues. This is now solved in (2.54) by only considering outstanding liabilities after time $t-1$. However, there is still a minor issue in (2.54) because we did not prove that $R_k^{(t)}$, $k \le t-1$, is square integrable. As a consequence, our consistent pricing framework may allow for price processes that are not square integrable and the sale (or purchase) of such financial instruments may generate cash flows that are not square integrable. However, importantly, consistency is still preserved and the cash flow valuation framework may need to be extended to a bigger space, see also (2.23) and Lemma 2.9.

Observe that we have the following mean self-financing property:

Corollary 2.8 (Mean self-financing property) *The following recursion holds for all* $t = 1, \ldots, n-1$

$$E\left[\varphi_t \left(R_t^{(t+1)} + X_t\right)\middle|\, \mathcal{F}_{t-1}\right] = \varphi_{t-1}\, R_{t-1}^{(t)}. \tag{2.57}$$

Remark.

- The classical actuarial theory with $\varphi_t = (1+i)^{-t}$ for some positive interest rate i (see (2.30)) forms a consistent theory but the deflators are not market-consistent, because they are often far from observed economic behaviours at traded markets.
- Corollary 2.8 basically says that if we want to avoid arbitrage opportunities for reserves then we need to define them as conditional expectations of the stochastically discounted random cash flows.

Proof of Corollary 2.8. We have the following identity (using the $\mathcal{F}_t$-measurability of X_t and the tower property of conditional expectations, see Williams [Wi91], Chap. 9)

$$E\left[\varphi_t \left(R_t^{(t+1)} + X_t\right)\middle|\, \mathcal{F}_{t-1}\right] = E\left[\sum_{k=t}^{n} \varphi_k X_k \middle|\, \mathcal{F}_{t-1}\right] = \varphi_{t-1} R_{t-1}^{(t)}. \tag{2.58}$$

This completes the proof of the corollary. □

2.5 Equivalent Martingale Measures

Throughout, we assume a fixed deflator $\varphi \in L^2_{n+1}(P, \mathbb{F})$ is given.

Every deflated price process defined by (2.40) gives a $(P, \mathbb{F})$-martingale according to Lemma 2.9. As a consequence the pricing system implied by the given deflator φ satisfies the efficient market hypothesis and is consistent as described in Remark 2.15.

Lemma 2.9 *For every cash flow* $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ *the deflated price process defined by (2.40) satisfies:*

$$(\varphi_t Q_t [\mathbf{X}])_{t=0,\ldots,n} \quad \textit{is a } (P, \mathbb{F})\textit{-martingale.} \tag{2.59}$$

Proof Integrability of the components of $(\varphi_t Q_t[\mathbf{X}])_{t=0,\ldots,n}$ immediately follows from the assumed square integrability of the cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ we have, using the tower property of conditional expectations, see Williams [Wi91], Chap. 9,

$$E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right] = E\left[E\left[\sum_{k=0}^{n} \varphi_k X_k \middle| \mathcal{F}_{t+1}\right] \middle| \mathcal{F}_t\right] \tag{2.60}$$
$$= E\left[\sum_{k=0}^{n} \varphi_k X_k \middle| \mathcal{F}_t\right] = \varphi_t Q_t[\mathbf{X}].$$

This finishes the proof of the lemma. □

Remarks on Deflating and Discounting.

- From the martingale property derived in Lemma 2.9 we immediately have

$$Q_t[\mathbf{X}] = \frac{1}{\varphi_t} E\left[\varphi_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right] = E\left[\frac{\varphi_{t+1}}{\varphi_t} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right]. \tag{2.61}$$

  This implies for the span-deflated price process the identity

$$Q_t[\mathbf{X}] = E\left[Y_{t+1} Q_{t+1}[\mathbf{X}] \middle| \mathcal{F}_t\right], \tag{2.62}$$

  with *span-deflator* Y_{t+1} defined in (2.27). The span-deflator Y_{t+1} is $\mathcal{F}_{t+1}$-measurable, i.e. it is known only at the end of the time period $(t, t+1]$, and not at the beginning of that time period.
- We define the *span-discount* known at the beginning of the time period $(t, t+1]$, i.e. which is observable at time t:

$$D(\mathcal{F}_t) = E\left[Y_{t+1} | \mathcal{F}_t\right] = E\left[\frac{\varphi_{t+1}}{\varphi_t} \middle| \mathcal{F}_t\right] = P(t, t+1). \tag{2.63}$$

  It is often convenient to rewrite (2.62) in terms of the span-discount $D(\mathcal{F}_t)$ instead of the span-deflator Y_{t+1}. The reason is that the span-discount is previsible (known a priori and observable at the market) whereas the span-deflator is always a "hidden variable" that cannot directly be extracted from the actual market information. The basic idea is to change the probability measure P to P^* so that we can change from span-deflators Y_{t+1} to previsible span-discounts $D(\mathcal{F}_t)$.
- If the time interval $(t, t+1]$ is one year, then $D(\mathcal{F}_t) = P(t, t+1)$ is exactly the price of the zero coupon bond with a time to maturity of 1 year at time t, i.e., on this yearly time grid this corresponds to the one-year risk-free investment at time t. Therefore, $D(\mathcal{F}_t)^{-1}$ describes the development of the value of the bank account from time t to time $t+1$. That is, if we invest $B_0 = 1$ units of currency into the bank account at time 0, then the value of this investment at time $t \geq 1$ is given by the *annually risk-free roll-over*:

$$B_t = \prod_{s=0}^{t-1} D(\mathcal{F}_s)^{-1} = \prod_{s=0}^{t-1} E\left[Y_{s+1}|\mathcal{F}_s\right]^{-1} = \exp\left\{\sum_{s=0}^{t-1} r_s\right\}, \tag{2.64}$$

where we have defined

$$r_t = -\log E\left[Y_{t+1}|\mathcal{F}_t\right] = -\log D(\mathcal{F}_t) = -\log P(t, t+1). \tag{2.65}$$

The process $(r_t)_{t=0,\dots,n-1}$ is called the *spot rate process* in discrete time and we have already met it in Exercise 2.3. The process $(B_t)_{t=0,\dots,n}$ is called the value process of the *bank account*.

- The change of probability measure from P to P^* mentioned above will then allow us to change from deflators φ to discounting with the bank account numeraire $(B_t)_{t=0,\dots,n}$. This is going to be derived next.

We define the process $\boldsymbol{\xi} = (\xi_s)_{s=0,\dots,n}$ by $\xi_0 = 1$ and for $s = 1, \dots, n$ we set

$$\xi_s = \prod_{t=0}^{s-1} \frac{Y_{t+1}}{D(\mathcal{F}_t)} = \varphi_s\, B_s. \tag{2.66}$$

Corollary 2.10 $\boldsymbol{\xi} \gg 0$ *is a (normalized) density process w.r.t.* P *and* $\mathbb{F}$.

Proof Strict positivity $\gg$ is immediately clear. Moreover, $\boldsymbol{\xi}$ is a $(P, \mathbb{F})$-martingale (which immediately follows from Lemma 2.9 because $(B_t)_{t=0,\dots,n}$ is the price process of the bank account) with normalization $E[\xi_n] = 1$. This proves the claim. □

Since $\boldsymbol{\xi}$ is a density process w.r.t. P and $\mathbb{F}$ we can use it for the following change of measure. For $A \in \mathcal{F}_n = \mathcal{F}$ we define

$$P^*[A] = \int_A \xi_n\, dP = E\left[\xi_n 1_A\right]. \tag{2.67}$$

Lemma 2.11 *The following statements hold:*

(1) *P^* is a probability measure on $(\Omega, \mathcal{F})$ equivalent to P.*
(2) *We have a Radon–Nikodým derivative for $s \le n$*

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_s} = \xi_s, \qquad P\text{-a.s.} \tag{2.68}$$

(3) *Moreover, for $s \le t$ and $A \in \mathcal{F}_t$*

$$P^*\left[A|\,\mathcal{F}_s\right] = \frac{1}{\xi_s} E\left[\xi_t\, 1_A|\,\mathcal{F}_s\right], \qquad P\text{-a.s.} \tag{2.69}$$

Proof The proof of statement (1) follows from Corollary 2.10. The normalization implies that $P^*[\Omega] = E[\xi_n] = 1$, which says that P^* is a probability measure on

$(\Omega, \mathcal{F})$. Moreover, $\xi_n > 0$, P-a.s., implies that $P^* \sim P$, i.e. they are equivalent measures.

Next we prove statement (2). Note that for any set $C \in \mathcal{F}_s$

$$P^*[C] = E\left[\xi_n 1_C\right] = E\left[E\left[\xi_n \middle| \mathcal{F}_s\right] 1_C\right] = E\left[\xi_s 1_C\right], \tag{2.70}$$

using the martingale property of $\boldsymbol{\xi}$ in the last step. Therefore, ξ_s is the resulting density on $\mathcal{F}_s$.

Finally we prove (3). Note that we have for any set $C \in \mathcal{F}_s$, $s \le t \le n$, and using (2.70) in the 4th step

$$\begin{aligned} E^*\left[1_C\, 1_A\right] = E\left[1_C\, \xi_n\, 1_A\right] &= E\left[1_C\, E\left[\xi_n\, 1_A \middle| \mathcal{F}_s\right]\right] \\ &= E\left[\xi_s \left(1_C \frac{1}{\xi_s} E\left[\xi_n\, 1_A \middle| \mathcal{F}_s\right]\right)\right] \\ &= E^*\left[1_C \frac{1}{\xi_s} E\left[\xi_n\, 1_A \middle| \mathcal{F}_s\right]\right] \\ &= E^*\left[1_C \frac{1}{\xi_s} E\left[1_A\, E\left[\xi_n \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_s\right]\right] \\ &= E^*\left[1_C \frac{1}{\xi_s} E\left[\xi_t\, 1_A \middle| \mathcal{F}_s\right]\right]. \end{aligned} \tag{2.71}$$

Since this holds for all $C \in \mathcal{F}_s$ the claim follows by the definition of the conditional expectation w.r.t. P^*. □

Item (3) of Lemma 2.11 immediately provides the next corollary:

Corollary 2.12 *For $s \le t$ we have*

$$E^*\left[B_t^{-1} Q_t\left[\mathbf{X}\right] \middle| \mathcal{F}_s\right] = \frac{1}{\xi_s} E\left[\xi_t B_t^{-1} Q_t\left[\mathbf{X}\right] \middle| \mathcal{F}_s\right]. \tag{2.72}$$

If we apply Corollary 2.12 for $s = t - 1$ and use Lemma 2.9 we obtain

$$\begin{aligned} E^*\left[B_t^{-1} Q_t\left[\mathbf{X}\right] \middle| \mathcal{F}_{t-1}\right] &= \frac{1}{\xi_{t-1}} E\left[\xi_t B_t^{-1} Q_t\left[\mathbf{X}\right] \middle| \mathcal{F}_{t-1}\right] \\ &= \frac{1}{\xi_{t-1}} E\left[\varphi_t Q_t\left[\mathbf{X}\right] \middle| \mathcal{F}_{t-1}\right] \\ &= \frac{1}{\xi_{t-1}} \varphi_{t-1} Q_{t-1}\left[\mathbf{X}\right] = B_{t-1}^{-1} Q_{t-1}\left[\mathbf{X}\right]. \end{aligned} \tag{2.73}$$

We have just proved the following corollary, compare to Lemma 2.9.

Corollary 2.13 *For every cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ the bank account numeraire discounted price process satisfies:*

$$\left(B_t^{-1} Q_t\left[\mathbf{X}\right]\right)_{t=0,\ldots,n} \quad \text{is a } (P^*, \mathbb{F})\text{-martingale.} \tag{2.74}$$

Remark 2.14 (**Real world and equivalent martingale measures**)

- Every price process $(Q_t\left[\mathbf{X}\right])_{t=0,\ldots,n}$ constructed by (2.40) is a $(P, \mathbb{F})$-martingale if deflated with the fixed chosen deflator φ, see Lemma 2.9. This price process is also a $(P^*, \mathbb{F})$-martingale if it is discounted with the bank account numeraire $(B_t)_{t=0,\ldots,n}$ and under the change of measure (2.67), see Corollary 2.13.
- The probability measure P is called the *real world probability measure*, *objective measure* or *physical measure* and it describes the probability law as it can be observed in real world. If, for pricing purposes, we switch to the bank account numeraire discounting we also need to transform the underlying probability measure, and the resulting (equivalent) probability measure P^* is called the *equivalent martingale measure*, *pricing measure*, or *risk neutral measure*.
- If we work with financial instruments only, then it is often easier to work under P^*. If we additionally have insurance products, then one usually works under P. Therefore, actuaries need to fully understand the connection between these two measures.
- For the equivalent martingale measure P^* we always choose the bank account numeraire $(B_t)_{t=0,\ldots,n}$ for discounting. In general, if $(A_t)_{t=0,\ldots,n}$ is any strictly positive and normalized price process, then we could also choose this price process as a numeraire and find the appropriate equivalent measure $P^A \sim P$ such that all price processes $(A_t^{-1} Q_t\left[\mathbf{X}\right])_{t=0,\ldots,n}$ are $(P^A, \mathbb{F})$-martingales. For more on this subject we refer to Sect. 4.3, and to Wüthrich–Merz [WM13], Sect. 11.2.

In the one-period model we obtain identity

$$Q_0\left[\mathbf{X}\right] = D(\mathcal{F}_0)\; E^*\left[\,Q_1\left[\mathbf{X}\right]\right] = E\left[\,Y_1\; Q_1\left[\mathbf{X}\right]\right]. \tag{2.75}$$

This shows the difference between discounting with $\mathcal{F}_0$-measurable span-discount $D(\mathcal{F}_0)$ under P^* and deflating with $\mathcal{F}_1$-measurable span-deflator Y_1 under P.

Remark 2.15 (**Fundamental theorem of asset pricing (FTAP)**)

- The *efficient market hypothesis* in its strong form assumes that all deflated price processes
$$\widetilde{Q}_t = \varphi_t\; Q_t\left[\mathbf{X}\right], \qquad t = 0, \ldots, n, \tag{2.76}$$
form $(P, \mathbb{F})$-martingales. This implies for the expected net gains, $t > s$,
$$E\left[\widetilde{Q}_t - \widetilde{Q}_s \middle| \mathcal{F}_s\right] = 0, \tag{2.77}$$
which means that there cannot be strictly positive expected net gains without any downside risks.

- The *efficient market hypothesis* in its weak form assumes that "there is no free lunch", i.e. there do not exist (appropriately defined) self-financing trading strategies with positive expected gains and without any downside risks. In a finite and discrete time model this is equivalent to the existence of an equivalent martingale measure for the bank account numeraire discounted price processes (see e.g. Theorem 2.6 in Lamberton–Lapeyre [LL91]). The proof for a finite probability space is essentially an exercise in linear algebra (see the toy model in Sect. 2.2.3); in more general settings the characterization is more delicate and typically referred to as the fundamental theorem of asset pricing (FTAP), see Delbaen–Schachermayer [DS94, DS06] and Föllmer–Schied [FS11].
- Summarizing: the existence of an equivalent martingale measure rules out appropriately defined arbitrage (which is the easier direction). The opposite direction that no-arbitrage defined in the right way implies the existence of an equivalent martingale measure is more delicate. In this lecture we will always identify no-arbitrage (defined in the right way) with the existence of an equivalent martingale measure for the bank account numeraire in the sense of Corollary 2.13 and Lemma 2.9, respectively.
- In complete markets, the equivalent martingale measure is unique and we can perfectly replicate any cash flow by traded instruments (for an example see Sect. 2.2.3). The uniqueness of the equivalent martingale measure also implies uniqueness of the state price deflator.
- In incomplete markets, where we have cash flows that cannot be perfectly replicated by traded instruments, we typically have more than one equivalent martingale measure (and state price deflator), and we need an economic model to decide which measure is appropriate for calculating prices of non-traded instruments, see Föllmer–Schied [FS11] or Malamud et al. [MTW08]. This in particular applies to insurance products.

Toy Example (Revisited).
We revisit the toy example of Sect. 2.2.3. We transform our probability measure according to Lemma 2.11 (here we work in a one-period model with $Q_0 = Q$): the equivalent martingale measure is given by

$$p^*(\omega_i) = \xi_1(\omega_i)\ p(\omega_i) = \frac{\varphi_1(\omega_i)}{E\left[\varphi_1\right]}\ p(\omega_i) = \frac{Q(\omega_i)}{Q\left[\mathbf{Z}^{(1)}\right]}. \tag{2.78}$$

Hence, from (2.35) and (2.38) we obtain

$$Q\left[\mathbf{X}\right] = E\left[\varphi_1\ X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.79}$$

$$Q\left[\mathbf{X}\right] = B_1^{-1}\ E^*\left[X_1\right] = \sum_{i=1}^{2} Q(\omega_i)\ X_1(\omega_i), \tag{2.80}$$

with

$$B_1^{-1} = E\left[\varphi_1\right] = Q\left[\mathbf{Z}^{(1)}\right], \tag{2.81}$$

which is deterministic at time 0. Hence under P^* we have

$$Q\left[\mathbf{X}\right] = B_1^{-1}\, E^*\left[X_1\right] = Q\left[\mathbf{Z}^{(1)}\right] E^*\left[X_1\right]. \tag{2.82}$$

This leads to the following table with $p^*(\omega_1) = 0.368$:

	$\mathbf{Z}^{(1)}$	Asset A	Asset B
Market price $Q_0[\cdot]$ at time 0	0.95	1.65	1.00
Payout if state ω_1 occurs	1	3	2
Payout if state ω_2 occurs	1	1	0.5
P^* expected payout	1	1.737	1.053
P^* expected return	5.26 %	5.26 %	5.26 %

which is the martingale property of the discounted cash flow $Q\left[\mathbf{Z}^{(1)}\right] X_1$ w.r.t. the equivalent martingale measure P^*. □

Exercise 2.6 We revisit the discrete time Vasiček model given in Exercise 2.3. The spot rate dynamics $(r_t)_{t=0,\ldots,n}$ was given by $r_0 > 0$ (fixed) and for $t \geq 1$

$$r_t = b + \beta r_{t-1} + \rho \varepsilon_t, \tag{2.83}$$

for given $b, \beta, \rho > 0$, and $(\varepsilon_t)_{t=0,\ldots,n}$ is $\mathbb{F}$-adapted with ε_t independent of $\mathcal{F}_{t-1}$ for all $t = 1, \ldots, n$ and standard Gaussian distributed under the *real world probability measure* P.

The deflator $\boldsymbol{\varphi}$ was then defined by

$$\varphi_t = \exp\left\{-\sum_{k=1}^{t}\left[r_{k-1} + \frac{\lambda^2}{2} r_{k-1}^2\right] - \sum_{k=1}^{t} \lambda r_{k-1} \varepsilon_k\right\}, \tag{2.84}$$

for given $\lambda \in \mathbb{R}$.

- Calculate the span-discount $D(\mathcal{F}_t) = P(t, t+1)$ from the span-deflator

$$Y_{t+1} = \frac{\varphi_{t+1}}{\varphi_t} = \exp\left\{-\left[r_t + \frac{\lambda^2}{2} r_t^2\right] - \lambda r_t \varepsilon_{t+1}\right\}, \tag{2.85}$$

  and show that the model is well-defined.
- Prove that the density process $\boldsymbol{\xi} = (\xi_t)_{t=0,\ldots,n}$ is given by

$$\xi_t = \exp\left\{-\sum_{k=1}^{t} \frac{\lambda^2}{2} r_{k-1}^2 - \sum_{k=1}^{t} \lambda r_{k-1} \varepsilon_k\right\}, \tag{2.86}$$

  where an empty sum is set equal to zero.

- Prove that
$$\varepsilon_t^* = \varepsilon_t + \lambda r_{t-1} \tag{2.87}$$
has, conditionally given $\mathcal{F}_{t-1}$, a standard Gaussian distribution under the *equivalent martingale measure* $P^* \sim P$, obtained from the density process $\boldsymbol{\xi}$ as in Lemma 2.11.
Hint: use the moment generating function and Lemma 2.11.
- Prove that (2.87) implies for the spot rate process $(r_t)_{t=0,\dots,n}$: $r_0 > 0$ (fixed) and for $t \geq 1$
$$r_t = b + (\beta - \lambda\rho) r_{t-1} + \rho \varepsilon_t^*, \tag{2.88}$$
where $(\varepsilon_t^*)_{t=0,\dots,n}$ is $\mathbb{F}$-adapted with ε_t^* independent of $\mathcal{F}_{t-1}$ for all $t = 1, \dots, n$ and standard Gaussian distributed under the equivalent martingale measure P^*.
- Calculate the zero coupon bond prices $t < m$ (see also Exercise 2.5)
$$P(t,m) = E^*\left[\exp\left\{-\sum_{s=t}^{m-1} r_s\right\}\middle|\mathcal{F}_t\right] = \exp\left\{A(t,m) - r_t\, B(t,m)\right\}. \tag{2.89}$$

□

Remark on Exercise 2.6. In (2.86) we calculate the density process $(\xi_t)_{t=0,\dots,n}$ for the discrete time Vasiček model. It depends on the parameter $\lambda \in \mathbb{R}$. We see that if $\lambda = 0$, the density process is identically equal to 1, and henceforth $P^* = P$. Therefore, λ models the difference between the real world probability measure P and the equivalent martingale measure P^* which is in economic theory explained through the market risk aversion. Therefore, λ is often called the *market price of risk* parameter and explains the aggregate market risk aversion (in our Vasiček model). In general, a higher (market) risk aversion explains lower prices because the more risk averse someone is, the less he is willing to accept risky positions.

Conclusions:

- We have found three different ways to value cash flows **X**:

1. via linear, positive and normalized functionals Q,
2. via deflators $\boldsymbol{\varphi}$ under the real world measure P,
3. via the bank account numeraire $(B_t)_t$ under equivalent martingale measures P^*.

- The advantage of using equivalent martingale measures is that the discount factor is a priori known (previsible), which means that we have a state independent discount factor (for the one-period risk-free roll-over). The main disadvantage of using equivalent martingale measures is that the concept is not straightforward for the calibration of real world events and prediction cannot be done under equivalent martingale measures.

- By contrast, deflators are calculated using the real world probability measure (expressing market risk aversion). Moreover, as shown below, they clearly describe the dependence structures (also between deflators and cash flows). From a practical point of view, deflators allow us to model embedded (financial) options and guarantees in insurance policies, and they are therefore preferred especially by actuaries who value life insurance products that contain both financial and insurance technical risk factors.

2.6 Insurance Technical and Financial Variables

2.6.1 Choice of Numeraire

Choose a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. For practical purposes in insurance applications it makes sense to factorize the payments X_k into an appropriate financial basis $\mathcal{U}_k$, $k = 0, \ldots, n$, and the number of units Λ_k of this basis needed. Assume that we can split the payments X_k as follows

$$X_k = \Lambda_k \, U_k^{(k)}, \quad k = 0, \ldots, n, \tag{2.90}$$

where the variable $U_t^{(k)}$ denotes the price of one unit of the financial instrument $\mathcal{U}_k$ at time $t = 0, \ldots, n$, and (for non-zero X_k and $U_k^{(k)}$, respectively)

$$\Lambda_k = \frac{X_k}{U_k^{(k)}}, \quad k = 0, \ldots, n, \tag{2.91}$$

gives the number of units (insurance technical variable) that we need to hold in order to replicate X_k with financial instrument $\mathcal{U}_k$. This means that we measure insurance liabilities in units $\mathcal{U}_k$ (financial instruments) which have price processes $(U_t^{(k)})_{t=0,\ldots,n}$, and in insurance technical variables Λ_k.

We emphasize that one should clearly distinguish between the financial instrument $\mathcal{U}_k$ (which can be understood as a contract) and its price process given by

$$(U_t^{(k)})_{t=0,\ldots,n} = \left(U_0^{(k)}, U_1^{(k)}, \ldots, U_k^{(k)}, \ldots, U_n^{(k)}\right). \tag{2.92}$$

The financial instrument $\mathcal{U}_k$ can either be an asset or a liability and it can be purchased or sold at given prices $U_t^{(k)}$ in time points t.

Assume that the price process $(U_t^{(k)})_{t=0,\ldots,k} \gg 0$ is strictly positive (up to time k). Then $(U_t^{(k)})_{t=0,\ldots,k}$ should be used as a *numeraire* to study the liability X_k, thus, every payment X_k should be studied in its appropriate unit (and numeraire). We have already met this idea in Remark 2.14.

Examples of Units for Numeraires.

- cash in different currencies like CHF, USD, EUR
- indexed cash typically described by an inflation index, salary index, claims inflation index, medical expenses index, etc.
- company share, stock, private equity, real estate, etc. (be careful with strict positivity of price processes)
- asset portfolio (with strictly positive price process)

Examples of Insurance Technical Events.

- death, survival, disability
- car accident, fire event, burglary, nuclear power accident
- medical expenses, workman's compensation

We would like to factorize the filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ into a product space such that we get an independent decoupling into:

$$\mathbb{T} = (\mathcal{T}_t)_{t=0,\ldots,n} \quad \text{filtration of insurance technical events,} \tag{2.93}$$

$$\mathbb{G} = (\mathcal{G}_t)_{t=0,\ldots,n} \quad \text{filtration of financial events,} \tag{2.94}$$

with for all $t = 0, \ldots, n$

$$\mathcal{F}_t = \sigma\left(\mathcal{T}_t, \mathcal{G}_t\right) = \text{smallest } \sigma\text{-field containing all events of } \mathcal{T}_t \text{ and } \mathcal{G}_t. \tag{2.95}$$

We assume that under P the two filtrations $\mathbb{T}$ and $\mathbb{G}$ are independent, thus, $\mathbb{F}$ can be decoupled into two independent filtrations, one describing insurance technical events $\mathbb{T}$ and one describing financial events $\mathbb{G}$. That is, the real world probability measure P admits a product representation (by a slight abuse of notation)

$$P = P_{\mathbb{T}} \times P_{\mathbb{G}}, \tag{2.96}$$

with $P_{\mathbb{T}}$ describing insurance technical risks $\mathbf{\Lambda} = (\Lambda_0, \ldots, \Lambda_n)$ which will be $\mathbb{T}$-adapted and with $P_{\mathbb{G}}$ describing financial price processes $(U_t^{(k)})_{t=0,\ldots,n}$ which will be $\mathbb{G}$-adapted. This decoupling will be crucial in the sequel of this manuscript and is explained in the next assumption.

Assumption 2.16 Assume that $\mathbb{T}$ and $\mathbb{G}$ are two independent filtrations on the probability space $(\Omega, \mathcal{F}, P)$ that generate filtration $\mathbb{F}$ according to (2.95). Assume that the cash flows $\mathbf{X}$ of interest are of the form

$$\mathbf{X} = (\Lambda_0\, U_0^{(0)}, \ldots, \Lambda_n\, U_n^{(n)}), \tag{2.97}$$

with $\mathbf{\Lambda} \in L^2_{n+1}(P, \mathbb{T})$ and $(U_t^{(k)})_{t=0,\ldots,n} \in L^2_{n+1}(P, \mathbb{G})$ for all $k = 0, \ldots, n$. Moreover, assume that the chosen (fixed) deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ factorizes $\varphi_k = \varphi_k^{\mathbb{T}} \varphi_k^{\mathbb{G}}$ for all $k = 0, \ldots, n$ such that $\boldsymbol{\varphi}^{\mathbb{T}} = (\varphi_k^{\mathbb{T}})_{k=0,\ldots,n}$ is $\mathbb{T}$-adapted and $\boldsymbol{\varphi}^{\mathbb{G}} = (\varphi_k^{\mathbb{G}})_{k=0,\ldots,n}$ is $\mathbb{G}$-adapted.

The valuation of these cash flows $\mathbf{X} = (\Lambda_0\, U_0^{(0)}, \ldots, \Lambda_n\, U_n^{(n)}) \in L^2_{n+1}(P, \mathbb{F})$ is then under Assumption 2.16 given by

$$\begin{aligned} \varphi_t\, Q_t\,[\mathbf{X}] &= E\left[\left.\sum_{k=0}^{n} \varphi_k\, X_k \right| \mathcal{F}_t\right] \\ &= E\left[\left.\sum_{k=0}^{n} \varphi_k^{\mathbb{T}}\, \Lambda_k\, \varphi_k^{\mathbb{G}}\, U_k^{(k)} \right| \mathcal{T}_t, \mathcal{G}_t\right] \\ &= \sum_{k=0}^{n} E\left[\varphi_k^{\mathbb{T}} \Lambda_k \middle| \mathcal{T}_t\right]\, E\left[\varphi_k^{\mathbb{G}} U_k^{(k)} \middle| \mathcal{G}_t\right]. \end{aligned} \tag{2.98}$$

Remarks.

- The term $E[\varphi_k^{\mathbb{T}} \Lambda_k | \mathcal{T}_t]$ describes the price of the insurance technical cover in units of the corresponding numeraire instrument $\mathcal{U}_k$ at time t. $\boldsymbol{\varphi}^{\mathbb{T}}$ defines the *insurance technical loading*, the so-called *probability distortion*, of the insurance technical price. This is further outlined in Sect. 2.6.2.
- The term $E[\varphi_k^{\mathbb{G}} U_k^{(k)} | \mathcal{G}_t]$ relates to the price of one unit of the financial instrument $\mathcal{U}_k$ at time t, see also Sect. 2.6.2 on probability distortions below. From this we conclude that $\boldsymbol{\varphi}^{\mathbb{G}}$ should be obtained from financial market data, because it should reflect asset prices at (traded) financial markets appropriately. For example, we can use the Vasiček model, proposed in Exercise 2.3, and calibrate the model to financial market data, see Wüthrich–Bühlmann [WB08] and Wüthrich–Merz [WM13].
- We have separated the pricing problem into two independent pricing problems, one for pricing insurance technical cover in units of a numeraire instrument and one for pricing units of financial instruments. This split looks very natural, but in practice one needs to be careful with its applications. Especially in non-life insurance, it is very difficult to find such an "orthogonal" (independent) split, since the severities of the claims often depend on the financial market and the split is far from non-trivial. For example, if we consider workman's compensation (which pays the salary when someone is injured or sick), it is very difficult to describe the dependence structure between (1) the salary height, (2) the length of the sickness (which may have a mental cause), (3) the state of the job market, (4) the state of the financial market, and (5) the political environment.
- The financial economy including insurance products could also be defined in other ways that would allow for similar splits. For an example we refer to Malamud et al. [MTW08]. There one starts with a complete financial market model described by the financial filtration. Then one introduces insurance products that enlarge the underlying financial filtration. This enlargement in general makes the market incomplete (but still arbitrage-free) and adds idiosyncratic risks to the economic model. Finally, one defines the "hedgeable" filtration that exactly describes the part of the insurance claims that can be described via financial market movements. The remaining parts are then the insurance technical risks. For an analysis of this split in terms of projections we also refer to Happ et al. [HMW15].

2.6.2 Probability Distortion

In this section we discuss the factorization of the deflator $\varphi_k = \varphi_k^{\mathbb{T}} \varphi_k^{\mathbb{G}}$ given in Assumption 2.16. The choice of the probability distortion $\boldsymbol{\varphi}^{\mathbb{T}}$ needs some care in order to obtain a reasonable model, as we will see shortly.

(1) Firstly, we choose $\boldsymbol{\varphi}^{\mathbb{T}} \gg 0$ and $\boldsymbol{\varphi}^{\mathbb{G}} \gg 0$ which is in line with $\boldsymbol{\varphi} \gg 0$. Moreover, $\boldsymbol{\varphi}^{\mathbb{T}} \in L^2_{n+1}(P, \mathbb{T})$ is a necessary assumption which follows from $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ and the independence and $\mathbb{T}$-adaptedness in Assumption 2.16.

(2) Secondly, to avoid ambiguity, we set for all $t = 0, \ldots, n$

$$E\left[\varphi_t^{\mathbb{T}}\right] = 1. \tag{2.99}$$

Otherwise, the decoupling into a product $\varphi_t = \varphi_t^{\mathbb{T}} \varphi_t^{\mathbb{G}}$ is not unique, which can easily be seen by multiplying and dividing both terms by the same positive constant.

(3) Thirdly, we assume that the sequence $(\varphi_t^{\mathbb{T}})_{t=0,\ldots,n}$ is a $(P, \mathbb{T})$-martingale, i.e.

$$E\left[\left.\varphi_{t+1}^{\mathbb{T}}\right| \mathcal{T}_t\right] = \varphi_t^{\mathbb{T}}. \tag{2.100}$$

Of course, the normalization (2.99) is then an easy consequence of the requirement

$$\varphi_0^{\mathbb{T}} = 1. \tag{2.101}$$

Under Assumption 2.16 and assuming (1)–(3) for the probability distortion $\boldsymbol{\varphi}^{\mathbb{T}}$ we see that

$$(\varphi_t^{\mathbb{T}})_{t=0,\ldots,n} \text{ is a density process w.r.t. } \mathbb{T} \text{ and } P. \tag{2.102}$$

This allows us to define an equivalent probability measure $P^*_{\mathbb{T}} \sim P$ on $(\Omega, \mathcal{T}_n, P)$ via the Radon–Nikodým derivative

$$\left.\frac{dP^*_{\mathbb{T}}}{dP}\right|_{\mathcal{T}_n} = \varphi_n^{\mathbb{T}}. \tag{2.103}$$

Moreover, we define the price process for the insurance technical variable Λ_k as follows: for $t \leq k$

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{\mathbb{T}}}\, E\left[\left.\varphi_k^{\mathbb{T}} \Lambda_k\right| \mathcal{T}_t\right]. \tag{2.104}$$

Lemma 2.17 *Assume Assumption 2.16 and (2.102) hold true. The probability distorted process*

$$\left(\varphi_t^{\mathbb{T}} \Lambda_{t,k}\right)_{t=0,\ldots,k} \quad \textit{forms a } (P, \mathbb{T})\textit{-martingale.} \tag{2.105}$$

The process

$$\left(\Lambda_{t,k}\right)_{t=0,\ldots,k} \quad \textit{forms a } (P^*_{\mathbb{T}}, \mathbb{T})\textit{-martingale.} \tag{2.106}$$

Proof of Lemma 2.17. The first claim follows similarly to Lemma 2.9 and uses the tower property of conditional expectations, see Williams [Wi91]. The second claim follows similarly to Corollary 2.13 and equality (2.73). Note that here the numeraire is equal to 1 (due to our choice of the density process). □

An immediate consequence of Lemma 2.17 is the following corollary:

Corollary 2.18 *Under the assumptions of Lemma 2.17 we have*

$$\Lambda_{t,k} = \frac{1}{\varphi_t^{\mathbb{T}}} \, E\left[\varphi_k^{\mathbb{T}} \Lambda_k \middle| \mathcal{T}_t\right] = E^*_{\mathbb{T}}\left[\Lambda_k \middle| \mathcal{T}_t\right]. \tag{2.107}$$

This has further consequences:

Theorem 2.19 *Under the assumptions of Lemma 2.17 and (2.59) we obtain that the price process $(U_t^{(k)})_{t=0,\ldots,k}$ of the financial instrument $\mathcal{U}_k$ satisfies for $t < k$*

$$U_t^{(k)} = \frac{1}{\varphi_t^{\mathbb{G}}} \, E\left[\varphi_{t+1}^{\mathbb{G}} U_{t+1}^{(k)} \middle| \mathcal{G}_t\right]. \tag{2.108}$$

Proof of Theorem 2.19. We define the cash flow $\mathbf{X} = U_k^{(k)} \mathbf{Z}^{(k)} = (0, \ldots, 0, U_k^{(k)}, 0, \ldots, 0) \in L^2_{n+1}(P, \mathbb{F})$. Note that in fact the cash flow $\mathbf{X}$ is in $L^2_{n+1}(P, \mathbb{G})$. The martingale property (2.59), Assumption 2.16 and Corollary 2.18 imply for $t < s \le k$

$$\begin{aligned}
\varphi_t Q_t[\mathbf{X}] = E\left[\varphi_s Q_s[\mathbf{X}] \middle| \mathcal{F}_t\right] &= E\left[\varphi_s U_s^{(k)} \middle| \mathcal{F}_t\right] \\
&= E\left[\varphi_s^{\mathbb{T}} \varphi_s^{\mathbb{G}} U_s^{(k)} \middle| \mathcal{F}_t\right] \\
&= E\left[\varphi_s^{\mathbb{T}} \middle| \mathcal{T}_t\right] E\left[\varphi_s^{\mathbb{G}} U_s^{(k)} \middle| \mathcal{G}_t\right] \\
&= \varphi_t^{\mathbb{T}} \, E\left[\varphi_s^{\mathbb{G}} U_s^{(k)} \middle| \mathcal{G}_t\right].
\end{aligned} \tag{2.109}$$

This implies that

$$U_t^{(k)} = Q_t[\mathbf{X}] = \frac{1}{\varphi_t^{\mathbb{G}}} E\left[\varphi_s^{\mathbb{G}} U_s^{(k)} \middle| \mathcal{G}_t\right]. \tag{2.110}$$

Henceforth $(\varphi_t^{\mathbb{G}} U_t^{(k)})_{t=0,\ldots,k}$ is a $(P, \mathbb{G})$-martingale. □

Of course, Theorem 2.19 is not really a surprise because it (only) says that a pure financial price process needs to have the martingale property w.r.t. the financial market model $(P, \mathbb{G})$ in order to be free of arbitrage.

Corollary 2.18 and Theorem 2.19 imply that we can study the insurance technical variables $\mathbf{\Lambda}$ and the price processes of the financial instruments $\mathcal{U}_k$ independently. The valuation of the outstanding loss liabilities

$$\mathbf{X}_{(k)} = (0, \ldots, 0, \Lambda_k U_k^{(k)}, \ldots, \Lambda_n U_n^{(n)}) \in L^2_{n+1}(P, \mathbb{F}) \tag{2.111}$$

at time $t \leq k$ can then easily be done, and the reserves are given by

$$R_t^{(k)} = Q_t \left[\mathbf{X}_{(k)} \right] = \sum_{s=k}^{n} \frac{1}{\varphi_t^{\mathbb{T}}} E \left[\varphi_s^{\mathbb{T}} \Lambda_s \middle| \mathcal{T}_t \right] \frac{1}{\varphi_t^{\mathbb{G}}} E \left[\varphi_s^{\mathbb{G}} U_s^{(s)} \middle| \mathcal{G}_t \right] \tag{2.112}$$
$$= \sum_{s=k}^{n} \Lambda_{t,s} \, U_t^{(s)}.$$

Conclusions. Under the product space Assumption 2.16, the assumption (2.102) that the insurance technical deflator is a density process w.r.t. $\mathbb{T}$ and P, and under the no-arbitrage assumption (2.59) we obtain that we can separate the valuation problem into two independent valuation problems:

(1) the insurance technical processes $(\Lambda_{t,k})_{t=0,\ldots,k}$, $k = 0, \ldots, n$, describe the probability distorted developments of the predictions of the insurance technical variables Λ_k if we increase the information $\mathcal{T}_t \to \mathcal{T}_{t+1}$;
(2) the financial processes $(U_t^{(k)})_{t=0,\ldots,k}$, $k = 0, \ldots, n$, describe the price processes of the financial instruments $\mathcal{U}_k$ in the financial market model $(\Omega, \mathcal{G}_n, P, \mathbb{G})$.

Example 2.7 (*Best-estimate predictions and reserves*) Choose $\boldsymbol{\varphi}^{\mathbb{T}} \equiv 1$. Hence, $\boldsymbol{\varphi}^{\mathbb{T}}$ gives an admissible probability distortion (normalized martingale). This choice implies for the insurance technical process at time $t \leq k$

$$\Lambda_{t,k} = E \left[\Lambda_k \middle| \mathcal{T}_t \right], \tag{2.113}$$

i.e. $\Lambda_{t,k}$ is simply the "best-estimate" prediction of Λ_k based on the information $\mathcal{T}_t$ (conditional expectation which has minimal conditional prediction variance). If we use this probability distortion in the reserves definition (2.112), then we call $R_t^{(k)}$ best-estimate reserves at time $t < k$ for the outstanding liabilities $\mathbf{X}_{(k)}$. □

Exercise 2.8 (*Esscher premium*) We choose a positive random variable Y on the underlying filtered probability space $(\Omega, \mathcal{T}_n, P, \mathbb{T})$ such that for some $\alpha > 0$ the following moment generating function exists

$$M_Y(2\alpha) = E\left[\exp\{2\alpha Y\}\right] < \infty. \tag{2.114}$$

Then we define the probability distortion

$$\varphi_t^{\mathbb{T}} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{E\left[\exp\{\alpha Y\}\right]} = \frac{E\left[\exp\{\alpha Y\} \middle| \mathcal{T}_t\right]}{M_Y(\alpha)}. \tag{2.115}$$

(1) Prove that $\boldsymbol{\varphi}^{\mathbb{T}} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{\mathbb{T}} \in L^2_{n+1}(P, \mathbb{T})$.
(2) Show that $(\varphi_t^{\mathbb{T}})_{t=0,\ldots,n}$ is a density process w.r.t. $\mathbb{T}$ and P.

Assume that $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and $t < k$. Prove under Assumption 2.16 and (2.59) that

$$Q_t[\mathbf{X}_k] = \frac{1}{E\left[\exp\{\alpha\Lambda_k\}\middle|\mathcal{T}_t\right]} E\left[\Lambda_k\, e^{\alpha\Lambda_k}\middle|\mathcal{T}_t\right] U_t^{(k)}. \tag{2.116}$$

If we define the conditional moment generating function by

$$M_{\Lambda_k|\mathcal{T}_t}(\alpha) = E\left[\exp\{\alpha\Lambda_k\}\middle|\mathcal{T}_t\right], \tag{2.117}$$

then the term

$$\Lambda_{t,k} = \frac{d}{dr}\log M_{\Lambda_k|\mathcal{T}_t}(r)\Big|_{r=\alpha} = M_{\Lambda_k|\mathcal{T}_t}(\alpha)^{-1} E\left[\Lambda_k\, e^{\alpha\Lambda_k}\middle|\mathcal{T}_t\right] \tag{2.118}$$

describes the Esscher premium of Λ_k at time $t < k$, see Bühlmann [Bü80] and Gerber–Pafumi [GP98].

(3) Prove that the Esscher premium (2.118) is strictly increasing in α.

Remark: α plays the role of the risk aversion. □

Exercise 2.9 (*Expected shortfall*) Choose an absolutely continuous and integrable random variable Y on the filtered probability space $(\Omega, \mathcal{T}_n, P, \mathbb{T})$. Denote the distribution function of Y by $F_Y(x) = P[Y \le x]$ and the generalized inverse by $F_Y^{\leftarrow}$, where $F_Y^{\leftarrow}(u) = \inf\{x | F_Y(x) \ge u\}$. Henceforth, the Value-at-Risk of Y at level $1-\alpha \in (0,1)$ is given by

$$\mathrm{VaR}_{1-\alpha}(Y) = F_Y^{\leftarrow}(1-\alpha). \tag{2.119}$$

We obtain, see also Sect. 1.2.1 in Wüthrich [Wü13],

$$\begin{aligned} P\left[Y > \mathrm{VaR}_{1-\alpha}(Y)\right] &= 1 - P\left[Y \le \mathrm{VaR}_{1-\alpha}(Y)\right] \\ &= 1 - F_Y\left(\mathrm{VaR}_{1-\alpha}(Y)\right) \\ &= 1 - F_Y\left(F_Y^{\leftarrow}(1-\alpha)\right) = \alpha. \end{aligned} \tag{2.120}$$

Choose $c \in (0,1)$ and define (note that Y is $\mathcal{T}_n$-measurable)

$$\varphi_n^{\mathbb{T}} = (1-c) + \frac{c}{\alpha} 1_{\{Y > \mathrm{VaR}_{1-\alpha}(Y)\}}, \tag{2.121}$$

and for $t < n$

$$\varphi_t^{\mathbb{T}} = E\left[\varphi_n^{\mathbb{T}}\middle|\mathcal{T}_t\right]. \tag{2.122}$$

(1) Prove that $\boldsymbol{\varphi}^{\mathbb{T}} \gg 0$. Moreover, prove that $\boldsymbol{\varphi}^{\mathbb{T}} \in L^2_{n+1}(P, \mathbb{T})$.
(2) Show that $(\varphi_t^{\mathbb{T}})_{t=0,\ldots,n}$ is a density process w.r.t. $\mathbb{T}$ and P.

(3) Assume that $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0)$ with $X_k = \Lambda_k U_k^{(k)}$. Choose $Y = \Lambda_k$ and $t < k$. Under Assumption 2.16 and (2.59) show that

$$Q_t[\mathbf{X}_k] = \left\{ \beta_t E[\Lambda_k | \mathcal{T}_t] + (1-\beta_t) \frac{E\left[\Lambda_k 1_{\{\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\}} \middle| \mathcal{T}_t\right]}{P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) \middle| \mathcal{T}_t\right]} \right\} U_t^{(k)}, \tag{2.123}$$

with so-called credibility weights

$$\beta_t = \frac{1-c}{(1-c) + c\, \frac{P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) | \mathcal{T}_t\right]}{\alpha}}. \tag{2.124}$$

We define the conditional probability

$$\alpha_t = P\left[\Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k) \middle| \mathcal{T}_t\right], \tag{2.125}$$

which says

$$\mathrm{VaR}_{1-\alpha}(\Lambda_k) = \mathrm{VaR}_{1-\alpha_t}(\Lambda_k | \mathcal{T}_t), \tag{2.126}$$

where $\mathrm{VaR}_{1-\alpha_t}(\Lambda_k|\mathcal{T}_t)$ denotes the Value-at-Risk of $\Lambda_k|\mathcal{T}_t$ at level $1-\alpha_t$. Henceforth, the credibility weight is given by

$$\beta_t = \frac{1-c}{(1-c) + c\, \frac{\alpha_t}{\alpha}}. \tag{2.127}$$

The last term in the bracket of (2.123) can be interpreted as the expected shortfall of $\Lambda_k|\mathcal{T}_t$ at level $1-\alpha_t$. We highlight this for $t = 0$. Then we have $\alpha_0 = \alpha$ (note that $\mathcal{T}_0 = \{\emptyset, \Omega\}$), which implies $\beta_t = 1 - c$ and

$$\begin{aligned} Q_0[\mathbf{X}_k] &= \left\{(1-c)E[\Lambda_k] + cE\left[\Lambda_k | \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right]\right\} U_0^{(k)} \qquad (2.128)\\ &= \left\{E[\Lambda_k] + c\left(E\left[\Lambda_k | \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right] - E[\Lambda_k]\right)\right\} U_0^{(k)}. \end{aligned}$$

Henceforth, the reserve for Λ_k at time 0 is given by its expected value $E[\Lambda_k]$ plus a loading where $c \in (0, 1)$ plays the role of the cost-of-capital rate and

$$E\left[\Lambda_k | \Lambda_k > \mathrm{VaR}_{1-\alpha}(\Lambda_k)\right] - E[\Lambda_k] \tag{2.129}$$

is the capital-at-risk (unexpected loss) measured by the expected shortfall on the security level $1-\alpha$. This is in line with the actual solvency considerations, see for example SST [SST06], Pelsser [Pe10], Salzmann–Wüthrich [SW10] and Sect. 4.5. □

2.7 Conclusions on Chapter 2

We have developed the theoretical foundations of *market-consistent actuarial valuation* based on (potentially) *distorted expected values*, see (2.98) and (2.112). The distorted probabilities lead to the *price of risk*. The framework as developed above is not yet the full story, since it only gives the price of risk, the so-called probability distorted risk premium of assets and insurance liabilities. However, it does not provide sufficient information about the risk bearing and risk mitigation. That is, we have not described how the risk bearing should be organized in order to protect against insolvencies in adverse scenarios, but we have only calculated its market-consistent price.

An insurance company can take the following measures to protect itself against the financial impacts of adverse scenarios:

1. buying options and reinsurance, if available,
2. hedging options internally,
3. setting up sufficient risk bearing capital (solvency margin).

In practice, one has to be very careful in each application whether the price of risk resulting from a mathematical model is already sufficient to finance adverse scenarios, in particular, model risk is often not considered appropriately.

Remark on the Existing Literature. There is a wide range of literature on the definition of market-consistent values. Usually these definitions are not mathematically rigorous and they often (slightly) differ from each other, e.g. they state that market-consistent values should be realistic values or that they should serve for the exchange of two portfolios, etc. One has to be prudent with these definitions in a modelling context because often they are not sufficiently precise.

Our approach gives a mathematical framework for market-consistent valuation that is fully consistent and that respects the theory of classical financial mathematics. Charges for the risk bearing are integrated via distorted probabilities, however (as mentioned above) this does not solve the question of the organization and mitigation of risk, yet. In the next couple of chapters we mainly focus on risk mitigation techniques which leads us to the description of the valuation portfolio and of asset and liability management (ALM) techniques in a full balance sheet approach.

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