Preface

In recent years, p-adic analysis (or more generally non-Archimedean analysis) has received a lot of attention due to its connections with mathematical physics; see, e.g., [8–14, 20, 21, 25, 26, 36, 64, 65, 67–69, 75, 76, 80, 82, 85–87, 90, 94, 106–109, 111] and references therein. All these developments have been motivated by two physical ideas. The first is the conjecture (due to I. Volovich) in particle physics that at Planck distances the space-time has a non-Archimedean structure; see, e.g., [107, 112, 113]. The second idea comes from statistical physics, in particular, in connection with models describing relaxation in glasses, macromolecules, and proteins. It has been proposed that the non-exponential nature of those relaxations is a consequence of a hierarchical structure of the state space which can in turn be put in connection with p-adic structures; see, e.g., [6, 10–12, 46, 94]. Additionally, we should mention that in the middle of the 1980s the idea of using ultrametric spaces to describe the states of complex biological systems, which naturally possess a hierarchical structure, emerged in the works of Frauenfelder, Parisi, and Stain, among others; see, e.g., [6, 36, 46]. In protein physics, it is regarded as one of the most profound ideas put forward to explain the nature of distinctive life attributes.

On the other hand, stochastic processes on p-adic spaces, or more generally on ultrametric spaces, have been studied extensively in the last 30 years due to diverse physical and mathematical motivations; see, e.g., [1–4, 16, 17, 19, 28, 42, 43, 61–63, 71, 98, 116, 118, 122] and references therein.

In [10–12], Avetisov et al. introduced a new class of models for complex systems based on p-adic analysis. These models can be applied, for instance, to the study of the relaxation of certain biological complex systems. From a mathematical point of view, in these models, the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a Markov process on an ultrametric space, and this stochastic process is used to describe the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$). The first goal of this work is to study a very large class of heat-type pseudodifferential equations over p-adic and adelic spaces, which contains as special cases many of the equations that occur in the models of Avetisov et al. It is worth to mention here that the p-adic heat equation also appeared in certain works connected with the Riemann hypothesis [83].

The simplest type of master equation is the one-dimensional p-adic heat equation. This equation was introduced in the book of Vladimirov, Volovich, and Zelenov [111, Section XVI]. In [80, Chaps. 4 and 5] Kochubei presented a general theory for one-dimensional parabolic-type pseudodifferential equations with variable coefficients, whose fundamental solutions are transition density functions for Markov processes in the p-adic line; see also [97, 98, 108]. Varadarajan studied the heat equation on a division algebra over a non-Archimedean local field [108]. In [122], the author introduced p-adic analogs for the n-dimensional elliptic operators and studied the corresponding heat equations and the associated Markov processes; see also [23, 108]. In [26], building up on [25] and [80, 81], Chacón-Cortés and the author introduced a new type of nonlocal operators and a class of parabolic-type pseudodifferential equations with variable coefficients, which contains the one-dimensional p-adic heat equation of [111], the equations studied by Kochubei in [80], and the equations studied by Rodríguez-Vega in [97].

The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$; see Chap. 2. The p-adic norm satisfies $||x+y||_p \leq \max\left\{||x||_p, ||y||_p\right\}$, and the metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a complete ultrametric space. This space has a natural hierarchical structure, which is very useful in physical models that involve hierarchies. As a topological space, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line.

The p-adic heat equation is defined as

$$\frac{\partial u\,(x,t)}{\partial t} + (\boldsymbol{D}^{\alpha} u)\,(x,t) = 0, x \in \mathbb{Q}_p, t > 0 \tag{1}$$

where

$$(\boldsymbol{D}^{\alpha}\varphi)\,(x) = \mathcal{F}^{-1}_{\xi\to x}\left(|\xi|_p^{\alpha}\,\mathcal{F}_{x\to\xi}\varphi\right), \alpha > 0$$

is the Vladimirov operator and $\mathcal{F}$ denotes the p-adic Fourier transform. This equation is the p-adic counterpart of the classical heat equation, which describes a particle performing a random motion (the Brownian motion); a 'similar' statement is valid for the p-adic heat equation. More precisely, the fundamental solution of (1) is the transition density of a bounded right-continuous Markov process without second kind discontinuities.

A well-known and accepted scientific paradigm in physics of complex systems (such as glasses and proteins) asserts that the dynamics of a large class of complex systems is described as a random walk on a complex energy landscape; see, e.g., [6, 46, 47, 114], [82] and references therein. A landscape is a continuous real-valued function that represents the energy of a system. The term complex landscape means that energy function has many local minima. In the case of complex landscapes, in which there are many local minima, a "simplification method" called interbasin kinetics is applied. The idea is to study the kinetics generated by transitions between groups of states (basins). In this setting, the minimal basins correspond to local

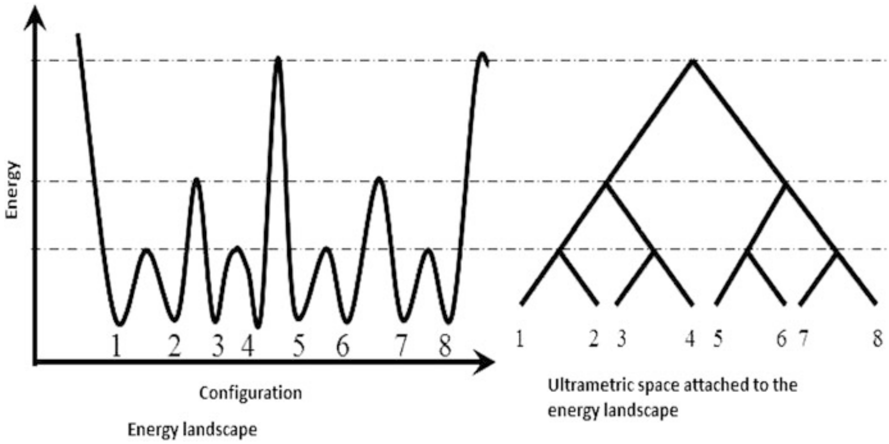

Fig. 1

minima of energy, and the large basins (superbasins or union of basins) have a hierarchical structure. A key idea is that a complex landscape is approximated by a disconnectivity graph (an ultrametric space) and by a function on this graph that gives the distribution function of activation energies. For the construction of the disconnectivity graph, we can imagine that the energy landscape is a water tank and that we are pumping water into it. The energy landscape of the system is flooded with water, and water forms pools around the local minima. By increasing the level of water, pools merge, until only one big pool remains. This procedure allows us to construct a directed tree of basins (pools). The activation barrier function on this tree is constructed as follows: minimal pools are assigned their depth (local minima energy), and larger pools are assigned energy levels at which these pools emerge (the activation barrier between basins). This procedure is illustrated in Fig. 1. For "real" energy landscapes and the corresponding graphs, the reader may see [114]. The next step is to construct a model of hierarchical dynamics based on the disconnectivity graph. By using the postulates of the interbasin kinetics, one gets that the transitions between basins are described by the following equations:

$$\frac{\partial f\,(i,t)}{\partial t} = \sum_{j} T\,(j,i) f\,(j,t)\, v(j) - \sum_{j \neq i} T\,(i,j) f\,(i,t)\, v\,(j)\,,$$

where the indices i,j number the states of the system (which correspond to local minima of energy), $T\,(i,j) \geq 0$ is the probability per unit time of a transition from i to j, and the $v(j) > 0$ are the basin volumes. Under suitable physical and mathematical hypotheses, the above master equation has the following "continuous limit":

$$\frac{\partial f\,(x,t)}{\partial t} = \int_{\mathbb{Q}_p} [w\,(x\,|y) f\,(y,t) - w\,(y\,|x) f\,(x,t)]\, dy,$$

where $x \in \mathbb{Q}_p, t \geq 0$. The function $f(x,t) : \mathbb{Q}_p \times \mathbb{R}_+ \to \mathbb{R}_+$ is a probability density distribution, so that $\int_B f(x,t)\, dx$ is the probability of finding the system in a domain $B \subset \mathbb{Q}_p$ at the instant t. The function $w(x\,|y) : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_+$ is the probability of the transition from state y to state x per unit of time. This family of parabolic-type equations contains the p-adic heat equation as a particular case.

The first part of this work is dedicated to present a snapshot of the theory, still under construction, of pseudodifferential equations of parabolic type and their Markov processes on p-adic and adelic spaces. In Chap. 1, we review, without proofs, some basic aspects of p-adic analysis and p-adic manifolds that we use in this work. An interesting novelty of this work is that some of the equations studied here require using geometric methods, for instance, integration on p-adic manifolds. Chapters 2 and 3 are dedicated to the study of parabolic-type equations and their Markov processes. The results presented in these chapters continue and extend, in a considerable form, the corresponding results presented in the books [80] and [111]. Chapter 4 deals with the heat equation on the ring of adeles. We present only the essential techniques and results; many important related topics were left aside. For instance, we do not include pseudodifferential operators and wavelets on general locally compact ultrametric spaces, not necessarily having a group structure; see, e.g., [77, 82] and [66]; wavelet analysis on adeles and pseudodifferential operators; see, e.g., [74] and [73]; and p-adic Brownian motion and stochastic pseudodifferential equations; see, e.g., [21, 70, 71], among other several important topics.

The second part, Chaps. 5 and 6, is dedicated to pseudodifferential equations whose symbols involve polynomials. In the 1950s Gel'fand and Shilov showed that fundamental solutions for certain types of partial differential operators with constant coefficients can be obtained by using local zeta functions [49]. The existence of fundamental solutions for general differential operators with constant coefficients was established by Atiyah [15] and Bernstein [18] using local zeta functions. A similar program can be carried out in the p-adic setting; see, e.g., [120]; see also [123]. The goal of Chap. 5 is to prove the existence of fundamental solutions for pseudodifferential operators via local zeta functions. Chapter 6 deals with a new class of non-Archimedean pseudodifferential equations of Klein-Gordon type. These equations have many similar properties to the classical Klein-Gordon equations; see, e.g., [31, 32, 103]. Finally, in Chap. 7, we present some open problems connecting non-Archimedean pseudodifferential equations with number theory, probability, and physics.

These notes were intended as a one-semester doctoral course at CINVESTAV for students interested in doing research in p-adic analysis connected with physics of complex systems, probability, and number theory. There are many open problems in this area. The central goal of these notes was to prepare fastly my students to do research.

I wish to thank all my coauthors (Oscar F. Casas-Sánchez, Leonardo Chacón-Cortes, Jeanneth Galeano-Peñaloza, John J. Rodríguez-Vega, and Sergii M. Torba) and my current doctoral students and postdocs (Victor Aguilar, Samuel Arias, Edilberto Arroyo, Maria Luisa Mendoza, and Anselmo Torresblanca) for helping

me on this project. I also wish to thank my family (my wife Monica, my daughter Daniela, and my son Felipe) for their constant support.

Finally, I wish to thank the CONACYT for supporting my research activities during the last 10 years through several grants (my latest CONACYT grant no. 250845) and through the Mexican Research Chairs Program (SNI level III since 2016).

Mexico W.A. Zúñiga-Galindo

Springer

http://www.springer.com/978-3-319-46737-5

Pseudodifferential Equations Over Non-Archimedean Spaces
Zúñiga-Galindo, W.A.
2016, XVI, 175 p. 1 illus., Softcover
ISBN: 978-3-319-46737-5