

Chapter 2

Parabolic-Type Equations and Markov Processes

2.1 Introduction

During the last 30 years there have been a strong interest on stochastic processes on ultrametric spaces mainly due its connections with models of complex systems, such as glasses and proteins. These processes are very convenient for describing phenomena whose space of states display a hierarchical structure, see e.g. [9–13, 36, 61, 78, 80, 86, 94, 108, 111, 118, 122], and references therein. Avetisov et al. constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes, see [9–13]. From a mathematical point view, in these models the time-evolution of a complex system is described by a p -adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition density function of a Markov process on an ultrametric space. This process describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space (\mathbb{Q}_p) . This is the main motivation for developing a general theory of parabolic-type pseudodifferential equations.

This chapter is devoted to the study of several types of n -dimensional parabolic-type equations that are generalizations of the one-dimensional p -adic heat equation introduced in [111]. We also study some basic properties of the Markov processes associated with these equations. In Sect. 2.2, we introduce the operators \mathbf{W} which are generalizations of the Vladimirov and Taibleson operators. This type of operators was introduced by Chacón-Cortés and Zúñiga-Galindo in [25]. The \mathbf{W} operators are pseudodifferential operators having radial symbols. We attach to these symbols certain heat kernels, and show that they are transition density functions of Markov processes over \mathbb{Q}_p^n . We also study the Cauchy problem for the parabolic-type equations attached to operators \mathbf{W} by using semigroup theory. In Sect. 2.3, we introduce a class of elliptic pseudodifferential operators which are generalizations of the Vladimirov and Taibleson operators. This class of operators was introduced

by Zúñiga-Galindo in [122]. The symbol of an elliptic operator has the form $|f|_p^\beta$, with $\beta > 0$, where f is a polynomial that vanishes only at the origin. These symbols, in general, are not radial. We attach heat kernels to elliptic symbols and show that these heat kernels are transition density functions of Markov over \mathbb{Q}_p^n . The positivity and the decay at infinity of these heat kernels are delicate matters. Finally, we study the Cauchy problem for the heat equations attached to elliptic operators.

2.2 Operators W , Parabolic-Type Equations and Markov Processes

2.2.1 A Class of Non-local Operators

Take $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$, and fix a function

$$w : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$$

satisfying the following properties:

- (i) $w(y)$ is a radial (i.e. $w(y) = w(\|y\|_p)$), continuous and increasing function of $\|y\|_p$;
- (ii) $w(y) = 0$ if and only if $y = 0$;
- (iii) there exist constants $C_0 > 0$, $M \in \mathbb{Z}$, and $\alpha_1 > n$ such that

$$C_0 \|y\|_p^{\alpha_1} \leq w(\|y\|_p), \text{ for } \|y\|_p \geq p^M. \quad (2.1)$$

Note that condition (iii) implies that

$$\int_{\|y\|_p \geq p^M} \frac{d^n y}{w(\|y\|_p)} < \infty. \quad (2.2)$$

In addition, since $w(y)$ is a continuous function, (2.2) holds for any $M \in \mathbb{Z}$.

We define

$$(W\varphi)(x) = \kappa \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{w(y)} d^n y, \text{ for } \varphi \in \mathcal{D}, \quad (2.3)$$

where κ is a positive constant.

Lemma 4 For $1 \leq \rho \leq \infty$,

$$\mathcal{D}(\mathbb{Q}_p^n) \rightarrow L^\rho(\mathbb{Q}_p^n)$$

$$\varphi \rightarrow W\varphi$$

is a well-defined linear operator. Furthermore,

$$\mathcal{F}[W\varphi](\xi) = -\kappa \left(\int_{\mathbb{Q}_p^n} \frac{1 - \chi_p(y \cdot \xi)}{w(y)} d^n y \right) \mathcal{F}[\varphi](\xi). \quad (2.4)$$

Proof Note that

$$(W\varphi)(x) = \kappa \frac{1_{\mathbb{Q}_p^n \setminus B_{p^M}^n}(x)}{w(x)} * \varphi(x) - \kappa \varphi(x) \left(\int_{\|y\|_p \geq p^M} \frac{d^n y}{w(y)} \right), \quad (2.5)$$

for some constant $M = M(\varphi)$. Now, since $\varphi \in \mathcal{D} \subset L^\rho$, for $1 \leq \rho \leq \infty$, (2.2), the Young inequality implies that the first term on the right-hand side of (2.5) belongs to L^ρ for $1 \leq \rho \leq \infty$, and by (2.2) the second term in (2.5) also belongs to L^ρ for $1 \leq \rho \leq \infty$. Finally, formula (2.4) follows from Fubini's theorem, since

$$\left| \frac{\varphi(x - y) - \varphi(x)}{w(y)} \right| \in L^1(\mathbb{Q}_p^n \times \mathbb{Q}_p^n, d^n x d^n y).$$

■

We set

$$A_w(\xi) := \int_{\mathbb{Q}_p^n} \frac{1 - \chi_p(y \cdot \xi)}{w(y)} d^n y.$$

Lemma 5 *The function $A_w(\xi)$ has the following properties: (i) for $\|\xi\|_p = p^{-\gamma} \neq 0$, with $\gamma = \text{ord}(\xi)$,*

$$A_w(p^{-\gamma}) = (1 - p^{-n}) \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{w(p^j)} + \frac{p^{n\gamma+n}}{w(p^{\gamma+1})}; \quad (2.6)$$

(ii) it is radial, positive, continuous, and $A_w(0) = 0$, (iii) $A_w(p^{-\text{ord}(\xi)})$ is a decreasing function of $\text{ord}(\xi)$.

Proof We write $\xi = p^\gamma \xi_0$, with $\gamma = \text{ord}(\xi)$ and $\|\xi_0\|_p = 1$. Then

$$A_w(\xi) = \int_{\mathbb{Q}_p^n} \frac{1 - \chi_p(p^\gamma y \cdot \xi_0)}{w(\|y\|_p)} d^n y = p^{\gamma n} \int_{\mathbb{Q}_p^n} \frac{1 - \chi_p(z \cdot \xi_0)}{w(p^\gamma \|z\|_p)} d^n z. \quad (2.7)$$

We now note that

$$\mathbb{Q}_p^n \setminus \{0\} = \bigsqcup_{j \in \mathbb{Z}} p^j S_0^n$$

with

$$S_0^n = \{y \in \mathbb{Q}_p^n; \|y\|_p = 1\}.$$

By using this partition and (2.7), we have

$$\begin{aligned} A_w(\xi) &= \sum_{j \in \mathbb{Z}} p^{\gamma n} \int_{p^j S_0^n} \frac{1 - \chi_p(z \cdot \xi_0)}{w(p^\gamma \|z\|_p)} d^n z \\ &= \sum_{j \in \mathbb{Z}} \frac{p^{-jn + \gamma n}}{w(p^{-j + \gamma})} \left\{ (1 - p^{-n}) - \int_{S_0^n} \chi_p(p^j y \cdot \xi_0) d^n y \right\}. \end{aligned}$$

By using the formula

$$\int_{S_0^n} \chi_p(p^j y \cdot \xi_0) d^n y = \begin{cases} 1 - p^{-n} & \text{if } j \geq 0 \\ -p^{-n} & \text{if } j = -1 \\ 0 & \text{if } j < -1, \end{cases} \quad (2.8)$$

see e.g. [105, Lemma 4.1], we get

$$\begin{aligned} A_w(\xi) &= (1 - p^{-n}) \sum_{j=2}^{\infty} \frac{p^{n(\gamma+j)}}{w(p^{\gamma+j})} + \frac{p^{n\gamma+n}}{w(p^{\gamma+1})} \\ &= (1 - p^{-n}) \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{w(p^j)} + \frac{p^{n\gamma+n}}{w(p^{\gamma+1})}. \end{aligned} \quad (2.9)$$

From (2.9) follows that $A_w(\xi)$ is radial, positive, continuous outside of the origin, and that $A_w(p^{-ord(\xi)})$ is a decreasing function of $ord(\xi)$. To show that $A_w(p^{-\gamma})$ is a decreasing function of γ , we note that, by (2.9), $A_w(p^{-(\gamma+1)}) - A_w(p^{-\gamma}) = p^{n\gamma+n} \left(\frac{1}{w(p^{\gamma+2})} - \frac{1}{w(p^{\gamma+1})} \right) < 0$. The continuity at the origin follows from

$$A_w(0) := \lim_{\gamma \rightarrow \infty} (1 - p^{-n}) \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{w(p^j)} + \lim_{\gamma \rightarrow \infty} \frac{p^{n\gamma+n}}{w(p^{\gamma+1})} = 0,$$

since $\sum_{j=M}^{\infty} \frac{p^{nj}}{w(p^j)} < \infty$, cf. (2.2), and $\frac{1}{C_0} \geq \frac{p^{\alpha_1(\gamma+1)}}{w(p^{\gamma+1})} \geq \frac{p^{n\gamma+n}}{w(p^{\gamma+1})}$ for γ big enough, cf. (2.1). ■

Remark 6 We denote by $C(U, \mathbb{C})$, respectively by $C(U, \mathbb{R})$, the vector space of \mathbb{C} -valued, respectively of \mathbb{R} -valued, continuous functions defined on an open subset U of \mathbb{Q}_p^n . In some cases we use the notation $C(U)$, or just C , if there is no danger of confusion.

Proposition 7 (i) $(W\varphi)(x) = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} (A_w(\|\xi\|_p) \mathcal{F}_{x \rightarrow \xi} \varphi)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, and $W\varphi \in C(\mathbb{Q}_p^n) \cap L^p(\mathbb{Q}_p^n)$, for $1 \leq p \leq \infty$. The Operator W extends to an unbounded and densely defined operator in $L^2(\mathbb{Q}_p^n)$ with domain

$$\text{Dom}(W) = \{\varphi \in L^2; A_w(\|\xi\|_p) \mathcal{F}\varphi \in L^2\}. \quad (2.10)$$

(ii) $(-W, \text{Dom}(W))$ is self-adjoint and positive operator.

(iii) W is the infinitesimal generator of a contraction C_0 semigroup $(\mathcal{T}(t))_{t \geq 0}$. Moreover, the semigroup $(\mathcal{T}(t))_{t \geq 0}$ is bounded holomorphic with angle $\pi/2$.

Proof (i) It follows from Lemma 4 and the fact that $A_w(\|\xi\|_p)$ is continuous, cf. Lemma 5. (ii) follows from the fact that W is a pseudodifferential operator and that the Fourier transform preserves the inner product of L^2 . (iii) It follows of well-known results, see e.g. [41, Chap. 2, Sect. 3] or [24]. For the property of the semigroup of being holomorphic, see e.g. [41, Chap. 2, Sect. 4.7]. ■

2.2.2 Some Additional Results

Lemma 8 Assume that there exist positive constants $\alpha_1, \alpha_2, C_0, C_1$, with $\alpha_1 > n$, $\alpha_2 > n$, and $\alpha_3 \geq 0$, such that

$$C_0 \|\xi'\|_p^{\alpha_1} \leq w(\|\xi'\|_p) \leq C_1 \|\xi'\|_p^{\alpha_2} e^{\alpha_3 \|\xi'\|_p}, \text{ for any } \xi' \in \mathbb{Q}_p^n. \quad (2.11)$$

Then there exist positive constants C_2, C_3 , such that

$$C_2 \|\xi\|_p^{\alpha_2-n} e^{-\alpha_3 p \|\xi\|_p^{-1}} \leq A_w(\|\xi\|_p) \leq C_3 \|\xi\|_p^{\alpha_1-n}$$

for any $\xi \in \mathbb{Q}_p^n$, with the convention that $e^{-\alpha_3 p \|\xi\|_p^{-1}} := \lim_{\|\xi\|_p \rightarrow 0} e^{-\alpha_3 p \|\xi\|_p^{-1}} = 0$. Furthermore, if $\alpha_3 > 0$, then $\alpha_1 \geq \alpha_2$, and if $\alpha_3 = 0$, then $\alpha_1 = \alpha_2$.

Proof By using the lower bound for w given in (2.11), and $\|\xi\|_p = p^{-\gamma}$,

$$A_w(\|\xi\|_p) \leq \frac{(1-p^{-n})}{C_0} \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{p^{j\alpha_1}} + \frac{p^{n\gamma+n}}{p^{\alpha_1(\gamma+1)}} \leq C_3 \|\xi\|_p^{\alpha_1-n}.$$

On the other hand, $A_w(\|\xi\|_p) \geq \frac{p^{n\gamma+n}}{w(p^{\gamma+1})}$, and by using the upper bound for w given in (2.11),

$$A_w(\|\xi\|_p) \geq \frac{p^{n\gamma+n}}{w(p^{\gamma+1})} \geq \frac{p^{n\gamma+n}}{C_1 p^{\alpha_2(\gamma+1)} e^{\alpha_3 p^{\gamma+1}}} \geq C_2 \|\xi\|_p^{\alpha_2-n} e^{-\alpha_3 p \|\xi\|_p^{-1}}.$$

■

Definition 9 We say that W (or A_w) is of exponential type if inequality (2.11) is only possible for $\alpha_3 > 0$ with $\alpha_1, \alpha_2, C_0, C_1$ positive constants and $\alpha_1 > n, \alpha_2 > n$. If (2.11) holds for $\alpha_3 = 0$ with $\alpha_1, \alpha_2, C_0, C_1$ positive constants and $\alpha_1 > n, \alpha_2 > n$, we say that W (or A_w) is of polynomial type.

We note that if W is of polynomial type then $\alpha_1 = \alpha_2 > n$ and C_0, C_1 are positive constants with $C_1 \geq C_0$.

Lemma 10 *With the hypotheses of Lemma 8,*

$$e^{-t\kappa A_w(\|\xi\|_p)} \in L^\rho(\mathbb{Q}_p^n) \text{ for } 1 \leq \rho < \infty \text{ and } t > 0.$$

Proof Since $e^{-tA_w(\|\xi\|_p)}$ is a continuous function, it is sufficient to show that there exists $M \in \mathbb{N}$ such that

$$I_M(t) := \int_{\|\xi\|_p > p^M} e^{-\rho\kappa t A_w(\|\xi\|_p)} d^n \xi < \infty, \text{ for } t > 0.$$

Take $M \in \mathbb{N}$, by Lemma 8, we have

$$C_2 \|\xi\|_p^{\alpha_2-n} e^{-\alpha_3 p \|\xi\|_p^{-1}} > C_2 \|\xi\|_p^{\alpha_2-n} e^{-\alpha_3 p^{-M+1}} \text{ for } \|\xi\|_p > p^M,$$

and (with $B = C_2 \rho \kappa e^{-\alpha_3 p^{-M+1}}$),

$$I_M(t) \leq \int_{\|\xi\|_p > p^M} e^{-tB \|\xi\|_p^{\alpha_2-n}} d^n \xi \leq C(M, \kappa, \rho) t^{\frac{-n}{\alpha_2-n}}, \text{ for } t > 0.$$

■

2.2.3 *p*-Adic Description of Characteristic Relaxation in Complex Systems

In [11] Avetisov et al. developed a new approach to the description of relaxation processes in complex systems (such as glasses, macromolecules and proteins) on the basis of *p*-adic analysis. The dynamics of a complex system is described by a

random walk in the space of configurational states, which is approximated by an ultrametric space (\mathbb{Q}_p) . Mathematically speaking, the time-evolution of the system is controlled by a master equation of the form

$$\frac{\partial f(x, t)}{\partial t} = \int_{\mathbb{Q}_p} \{v(x | y)f(y, t) - v(y | x)f(x, t)\} dy, \quad x \in \mathbb{Q}_p, t \in \mathbb{R}_+, \quad (2.12)$$

where the function $f(x, t) : \mathbb{Q}_p \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a probability density distribution, and the function $v(x | y) : \mathbb{Q}_p \times \mathbb{Q}_p \rightarrow \mathbb{R}_+$ is the probability of transition from state y to the state x per unit time. The transition from a state y to a state x can be perceived as overcoming the energy barrier separating these states. In [11] an Arrhenius type relation was used:

$$v(x | y) \sim A(T) \exp \left\{ -\frac{U(x | y)}{kT} \right\},$$

where $U(x | y)$ is the height of the activation barrier for the transition from the state y to state x , k is the Boltzmann constant and T is the temperature. This formula establishes a relation between the structure of the *energy landscape* $U(x | y)$ and the transition function $v(x | y)$. The case $v(x | y) = v(y | x)$ corresponds to a *degenerate energy landscape*. In this case the master equation (2.12) takes the form

$$\frac{\partial f(x, t)}{\partial t} = \int_{\mathbb{Q}_p} v(|x - y|_p) \{f(y, t) - f(x, t)\} dy,$$

where $v(|x - y|_p) = \frac{A(T)}{|x - y|_p} \exp \left\{ -\frac{U(|x - y|_p)}{kT} \right\}$. By choosing U conveniently, several energy landscapes can be obtained. Following [11], there are three basic landscapes: (i) (logarithmic) $v(|x - y|_p) = \frac{1}{|x - y|_p \ln^\alpha(1 + |x - y|_p)}$, $\alpha > 1$; (ii) (linear) $v(|x - y|_p) = \frac{1}{|x - y|_p^{\alpha+1}}$, $\alpha > 0$; (iii) (exponential) $v(|x - y|_p) = \frac{e^{-\alpha|x - y|_p}}{|x - y|_p}$, $\alpha > 0$.

Thus, it is natural to study the following Cauchy problem:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \kappa \int_{\mathbb{Q}_p^n} \frac{u(x - y, t) - u(x, t)}{w(y)} d^n y, & x \in \mathbb{Q}_p^n, t \in \mathbb{R}_+, \\ u(x, 0) = \varphi \in \mathcal{D}(\mathbb{Q}_p^n), \end{cases}$$

where $w(y)$ is a radial function belonging to a class of functions that contains functions like:

- (i) $w(\|y\|_p) = \Gamma_p^n(-\alpha) \|y\|_p^{\alpha+n}$, here $\Gamma_p^n(\cdot)$ is the n -dimensional p -adic Gamma function, and $\alpha > 0$;
- (ii) $w(\|y\|_p) = \|y\|_p^\beta e^{\alpha\|y\|_p}$, $\alpha > 0$.

By imposing condition (2.11) to w , we include the linear and exponential energy landscapes in our study. On the other hand, take $w(\|y\|_p)$ satisfying (2.11) and take $f(\|y\|_p)$ a continuous and increasing function such that

$$0 < \sup_{y \in \mathbb{Q}_p^n} f(\|y\|_p) < \infty \text{ and } 0 < \inf_{y \in \mathbb{Q}_p^n} f(\|y\|_p) < \infty.$$

Then $f(\|y\|_p) w(\|y\|_p)$ satisfies (2.11). This fact shows that the class of operators \mathbf{W} is very large.

Finally we note that $\|y\|_p^\beta \ln^\alpha(1 + \|y\|_p)$, $\beta > n$, $\alpha \in \mathbb{N}$, does not satisfies $\|y\|_p^{\alpha_1} \leq \|y\|_p^\beta \ln^\alpha(1 + \|y\|_p)$ for any $y \in \mathbb{Q}_p^n$, and hence our results do not include the case of logarithmic landscapes.

2.2.4 Heat Kernels

In this section we assume that function w satisfies conditions (2.11). We define

$$Z(x, t; w, \kappa) := Z(x, t) = \int_{\mathbb{Q}_p^n} e^{-\kappa t A_w(\|\xi\|_p)} \chi_p(-x \cdot \xi) d^n \xi \text{ for } t > 0 \text{ and } x \in \mathbb{Q}_p^n.$$

Note that by Lemma 10, $Z(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1}[e^{-\kappa t A_w(\|\xi\|_p)}] \in C \cap L^2$ for $t > 0$. We call a such function a *heat kernel*. When considering $Z(x, t)$ as a function of x for t fixed we will write $Z_t(x)$.

Lemma 11 (i) *There exists a positive constant C , such that*

$$Z(x, t) < Ct \|x\|_p^{-\alpha_1}, \text{ for } x \in \mathbb{Q}_p^n \setminus \{0\} \text{ and } t > 0.$$

(ii) $Z_t(x) \in L^1(\mathbb{Q}_p^n)$ for every $t > 0$.

Proof (i) Let $\|x\|_p = p^\beta$. Since $Z(x, t) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$, by using $\mathbb{Q}_p^n \setminus \{0\} = \bigsqcup_{j \in \mathbb{Z}} p^j S_0^n$ and formula (2.8), we get

$$Z(x, t) = \|x\|_p^{-n} \left[(1 - p^{-n}) \sum_{j=0}^{\infty} e^{-\kappa A_w(p^{-\beta-j})t} p^{-nj} - e^{-\kappa A_w(p^{-\beta+1})t} \right].$$

By using that $e^{-\kappa A_w(p^{-\beta-j})t} \leq 1$ for $j \in \mathbb{N}$, we have

$$Z(x, t) \leq \|x\|_p^{-n} \left[1 - e^{-\kappa A_w(p^{-\beta+1})t} \right].$$

We now apply the mean value theorem to the real function $f(u) = e^{-\kappa A_w(p^{-\beta+1})u}$ on $[0, t]$ with $t > 0$, and Lemma 8,

$$Z(x, t) \leq C_0 \|x\|_p^{-n} t A_w(p^{-\beta+1}) \leq Ct \|x\|_p^{-\alpha_1}.$$

(ii) Notice that

$$\int_{\mathbb{Q}_p^n} Z_t(x) d^n x = \int_{B_0^n} Z_t(x) d^n x + \int_{\mathbb{Q}_p^n \setminus B_0^n} Z_t(x) d^n x,$$

the existence of the first integral follows from the continuity of $Z_t(x)$, for the second integral we use the bound obtained in (i). ■

Lemma 12 $Z(x, t) \geq 0$, for $x \in \mathbb{Q}_p^n$ and $t > 0$.

Proof Since $e^{-\kappa t A_w(\|\xi\|_p)}$ is radial, by using $\mathbb{Q}_p^n \setminus \{0\} = \bigsqcup_{j \in \mathbb{Z}} p^j S_0^n$ and formula (2.8), we have

$$\begin{aligned} Z(x, t) &= \sum_{i=-\infty}^{\infty} e^{-\kappa t A_w(p^i)} \int_{\|\xi\|_p = p^i} \chi_p(-x \cdot \xi) d^n \xi \\ &= \sum_{i=-\infty}^{\infty} p^{ni} \left[e^{-\kappa t A_w(p^i)} - e^{-\kappa t A_w(p^{i+1})} \right] \Omega(\|p^{-i} x\|_p) \geq 0 \end{aligned}$$

since A_w is increasing function of i , cf. Lemma 5. ■

Theorem 13 *The function $Z(x, t)$ has the following properties:*

- (i) $Z(x, t) \geq 0$ for any $t > 0$;
- (ii) $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = 1$ for any $t > 0$;
- (iii) $Z_t(x) \in C(\mathbb{Q}_p^n, \mathbb{R}) \cap L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n)$ for any $t > 0$;
- (iv) $Z_t(x) * Z_{t'}(x) = Z_{t+t'}(x)$ for any $t, t' > 0$;
- (v) $\lim_{t \rightarrow 0^+} Z(x, t) = \delta(x)$ in $\mathcal{D}'(\mathbb{Q}_p^n)$, where δ denotes the Dirac distribution.

Proof (i) It follows from Lemma 12. (ii) Since $Z_t(x), \mathcal{F}_{x \rightarrow \xi}(Z_t(x)) = e^{-\kappa t A_w(\|\xi\|_p)} \in C \cap L^1$, for any $t > 0$, cf. Lemma 10 and Lemma 11 (ii), the result follows from the inversion formula for the Fourier transform. (iii) It follows from Lemma 10 and Lemma 11 (ii). (iv) By the previous property $Z_t(x) \in L^1$ for any $t > 0$, then

$$\begin{aligned} Z_t(x) * Z_{t'}(x) &= \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{-\kappa t A_w(\|\xi\|_p)} e^{-\kappa t' A_w(\|\xi\|_p)} \right) \\ &= \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{-\kappa(t+t') A_w(\|\xi\|_p)} \right) = Z_{t+t'}(x). \end{aligned}$$

(v) Since we have $e^{-\kappa t A_w(\|\xi\|_p)} \in C(\mathbb{Q}_p^n, \mathbb{R}) \cap L^1$ for $t > 0$, cf. Lemma 10, the inner product

$$\left\langle e^{-\kappa t A_w(\|\xi\|_p)}, \phi \right\rangle = \int_{\mathbb{Q}_p^n} e^{-\kappa t A_w(\|\xi\|_p)} \overline{\phi(\xi)} d^n \xi$$

defines a distribution on \mathbb{Q}_p^n , then, by the dominated convergence theorem,

$$\lim_{t \rightarrow 0^+} \left\langle e^{-\kappa t A_w(\|\xi\|_p)}, \phi \right\rangle = \langle 1, \phi \rangle$$

and thus

$$\lim_{t \rightarrow 0^+} \langle Z(x, t), \phi \rangle = \lim_{t \rightarrow 0^+} \left\langle e^{-\kappa t A_w(\|\xi\|_p)}, \mathcal{F}^{-1} \phi \right\rangle = \langle 1, \mathcal{F}^{-1} \phi \rangle = \langle \delta, \phi \rangle.$$

■

2.2.5 Markov Processes Over \mathbb{Q}_p^n

Along this section we consider $(\mathbb{Q}_p^n, \|\cdot\|_p)$ as complete non-Archimedean metric space and use the terminology and results of [39, Chapters 2, 3]. Let \mathcal{B} denote the Borel σ -algebra of \mathbb{Q}_p^n . Thus $(\mathbb{Q}_p^n, \mathcal{B}, d^n x)$ is a measure space.

We set

$$p(t, x, y) := Z(x - y, t) \text{ for } t > 0, x, y \in \mathbb{Q}_p^n,$$

and

$$P(t, x, B) = \begin{cases} \int_B p(t, y, x) d^n y & \text{for } t > 0, \quad x \in \mathbb{Q}_p^n, \quad B \in \mathcal{B} \\ \mathbf{1}_B(x) & \text{for } t = 0. \end{cases}$$

Lemma 14 *With the above notation the following assertions hold:*

- (i) $p(t, x, y)$ is a normal transition density;
- (ii) $P(t, x, B)$ is a normal transition function.

Proof The result follows from Theorem 13, see [39, Section 2.1] for further details. ■

Lemma 15 *The transition function $P(t, x, B)$ satisfies the following two conditions:*

(i) *for each $u \geq 0$ and compact B*

$$\lim_{x \rightarrow \infty} \sup_{t \leq u} P(t, x, B) = 0 \text{ [Condition } L(B)\text{]};$$

(ii) *for each $\epsilon > 0$ and compact B*

$$\lim_{t \rightarrow 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p^n \setminus B_\epsilon^n(x)) = 0 \text{ [Condition } M(B)\text{]}.$$

Proof (i) By Lemma 11 and the fact that $\|\cdot\|_p$ is an ultranorm, we have

$$P(t, x, B) \leq Ct \int_B \|x - y\|_p^{-\alpha_1} d^n y = tC \|x\|_p^{-\alpha_1} \text{vol}(B) \text{ for } x \in \mathbb{Q}_p^n \setminus B.$$

$$\text{Therefore } \lim_{x \rightarrow \infty} \sup_{t \leq u} P(t, x, B) = 0.$$

(ii) Again, by Lemma 11, the fact that $\|\cdot\|_p$ is an ultranorm, and $\alpha_1 > n$, we have

$$\begin{aligned} P(t, x, \mathbb{Q}_p^n \setminus B_\epsilon^n(x)) &\leq Ct \int_{\|x-y\|_p > \epsilon} \|x - y\|_p^{-\alpha_1} d^n y = Ct \int_{\|z\|_p > \epsilon} \|z\|_p^{-\alpha_1} d^n z \\ &= C'(\alpha_1, \epsilon, n) t. \end{aligned}$$

Therefore

$$\lim_{t \rightarrow 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p^n \setminus B_\epsilon^n(x)) \leq \lim_{t \rightarrow 0^+} \sup_{x \in B} C'(\alpha_1, \epsilon, n) t = 0.$$

■

Theorem 16 *$Z(x, t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.*

Proof The result follows from [39, Theorem 3.6] by using that $(\mathbb{Q}_p^n, \|x\|_p)$ is semi-compact space, i.e. a locally compact Hausdorff space with a countable base, and $P(t, x, B)$ is a normal transition function satisfying conditions $L(B)$ and $M(B)$, cf. Lemmas 14, 15. ■

2.2.6 The Cauchy Problem

Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \mathbf{W}u(x, t) = 0, & x \in \mathbb{Q}_p^n, t \in [0, \infty), \\ u(x, 0) = u_0(x), & u_0(x) \in \text{Dom}(\mathbf{W}), \end{cases} \quad (2.13)$$

where $(\mathbf{W}\phi)(x) = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} (A_w(\|\xi\|_p) \mathcal{F}_{x \rightarrow \xi} \phi)$ for $\phi \in \text{Dom}(\mathbf{W})$, see (2.10), and $u : \mathbb{Q}_p^n \times [0, \infty) \rightarrow \mathbb{C}$ is an unknown function. We say that a function $u(x, t)$ is a *solution* of (2.13), if $u(x, t) \in C([0, \infty), \text{Dom}(\mathbf{W})) \cap C^1([0, \infty), L^2(\mathbb{Q}_p^n))$ and u satisfies (2.13) for all $t \geq 0$.

In this section, we understand the notions of continuity in t , differentiability in t and equalities in the $L^2(\mathbb{Q}_p^n)$ sense, as it is customary in the semigroup theory.

We know from Proposition 7 that the operator \mathbf{W} generates a C_0 semigroup $(\mathcal{T}(t))_{t \geq 0}$, then Cauchy problem (2.13) is well-posed, i.e. it is uniquely solvable with the solution continuously dependent on the initial datum, and its solution is given by $u(x, t) = \mathcal{T}(t)u_0(x)$, for $t \geq 0$, see e.g. [24, Theorem 3.1.1]. However the general theory does not give an explicit formula for the semigroup $(\mathcal{T}(t))_{t \geq 0}$. We show that the operator $\mathcal{T}(t)$ for $t > 0$ coincides with the operator of convolution with the heat kernel $Z_t * \cdot$. In order to prove this, we first construct a solution of Cauchy problem (2.13) with the initial value from \mathcal{D} without using the semigroup theory. Then we extend the result to all initial values from $\text{Dom}(\mathbf{W})$, see Propositions 18–20.

2.2.6.1 Homogeneous Equations with Initial Values in \mathcal{D}

To simplify the notation, set $Z_0 * u_0 = (Z_t(x) * u_0(x))|_{t=0} := u_0$. We define the function

$$u(x, t) = Z_t(x) * u_0(x), \text{ for } t \geq 0. \quad (2.14)$$

Since $Z_t(x) \in L^1$ for $t > 0$ and $u_0 \in \mathcal{D}(\mathbb{Q}_p^n) \subset L^\infty(\mathbb{Q}_p^n)$, the convolution exists and is a continuous function, see e.g. [100, Theorem 1.1.6].

Lemma 17 *Take $u_0 \in \mathcal{D}$ with the support of \widehat{u}_0 contained in B_R^n , and $u(x, t)$, $t \geq 0$ defined as in (2.14). Then the following assertions hold:*

- (i) $u(x, t)$ is continuously differentiable in time for $t \geq 0$ and the derivative is given by

$$\frac{\partial u(x, t)}{\partial t} = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{-\kappa t A_w(\|\xi\|_p)} A_w(\|\xi\|_p) 1_{B_R^n}(\xi) \right) * u_0(x);$$

(ii) $u(x, t) \in \text{Dom}(W)$ for any $t \geq 0$ and

$$(Wu)(x, t) = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{-\kappa t A_w(\|\xi\|_p)} A_w(\|\xi\|_p) 1_{B_R^n}(\xi) \right) * u_0(x).$$

Proof (i) The proof is similar to the one given for Lemma 110 in Chap. 4. (ii) Note that $e^{-\kappa t A_w(\|\xi\|_p)} \widehat{u}_0(\xi)$, $A_w(\|\xi\|_p) e^{-\kappa t A_w(\|\xi\|_p)} \widehat{u}_0(\xi) \in C \cap L^2 \cap L^1$ for $t \geq 0$, i.e. $u(x, t) \in \text{Dom}(W)$ for $t \geq 0$. Now

$$\begin{aligned} (Wu)(x, t) &= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_w(\|\xi\|_p) \mathcal{F}_{\xi \rightarrow x} (u(x, t)) \right) \\ &= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_w(\|\xi\|_p) e^{-\kappa t A_w(\|\xi\|_p)} \widehat{u}_0(\xi) \right) \\ &= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_w(\|\xi\|_p) e^{-\kappa t A_w(\|\xi\|_p)} 1_{B_R^n}(\xi) \widehat{u}_0(\xi) \right) \\ &= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{-\kappa t A_w(\|\xi\|_p)} A_w(\|\xi\|_p) 1_{B_R^n}(\xi) \right) * u_0(x). \end{aligned}$$

■

As a direct consequence of Lemma 17 we obtain the following result.

Proposition 18 Assume that $u_0 \in \mathcal{D}$. Then function $u(x, t)$ defined in (2.14) is a solution of Cauchy problem (2.13).

2.2.6.2 Homogeneous Equations with Initial Values in L^2

We define

$$T(t)u = \begin{cases} Z_t * u, & t > 0 \\ u, & t = 0, \end{cases} \quad (2.15)$$

for $u \in L^2$.

Lemma 19 The operator $T(t) : L^2(\mathbb{Q}_p^n) \longrightarrow L^2(\mathbb{Q}_p^n)$ is bounded for any fixed $t \geq 0$.

Proof For $t > 0$, the result follows from the Young inequality by using the fact that $Z_t \in L^1$, cf. Theorem 13 (iii). ■

Proposition 20 The following assertions hold.

- (i) The operator W generates a C_0 semigroup $(\mathcal{T}(t))_{t \geq 0}$. The operator $\mathcal{T}(t)$ coincides for each $t \geq 0$ with the operator $T(t)$ given by (2.15).
- (ii) Cauchy problem (2.13) is well-posed and its solution is given by $u(x, t) = Z_t * u_0$, $t \geq 0$.

Proof (i) By Proposition 7 (iii) the operator \mathbf{W} generates a C_0 semigroup $(\mathcal{T}(t))_{t \geq 0}$. Hence Cauchy problem (2.13) is well-posed, see e.g. [24, Theorem 3.1.1]. By Proposition 18, $\mathcal{T}(t)|_{\mathcal{D}} = T(t)|_{\mathcal{D}}$ and both operators $\mathcal{T}(t)$ and $T(t)$ are defined on the whole L^2 and bounded, cf. Lemma 19. By the continuity we conclude that $\mathcal{T}(t) = T(t)$ on L^2 . Now the statements follow from well-known results of the semigroup theory, see e.g. [24, Theorem 3.1.1], [41, Chap. 2, Proposition 6.2]. ■

2.2.6.3 Non-homogeneous Equations

Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \mathbf{W}u(x, t) = g(x, t), & x \in \mathbb{Q}_p^n, t \in [0, T], T > 0, \\ u(x, 0) = u_0(x), & u_0(x) \in \text{Dom}(\mathbf{W}). \end{cases} \quad (2.16)$$

We say that a function $u(x, t)$ is a solution of (2.16), if $u(x, t)$ belongs to $C([0, T], \text{Dom}(\mathbf{W})) \cap C^1([0, T], L^2(\mathbb{Q}_p^n))$ and if $u(x, t)$ satisfies equation (2.16) for $t \in [0, T]$.

Theorem 21 *Assume that $u_0 \in \text{Dom}(\mathbf{W})$ and $g \in C([0, \infty), L^2(\mathbb{Q}_p^n)) \cap L^1((0, \infty), \text{Dom}(\mathbf{W}))$. Then Cauchy problem (2.16) has a unique solution given by*

$$u(x, t) = \int_{\mathbb{Q}_p^n} Z(x - \xi, t) u_0(\xi) d^n \xi + \int_0^t \int_{\mathbb{Q}_p^n} Z(x - \xi, t - \theta) g(\xi, \theta) d^n \xi d\theta.$$

Proof The result follows from Proposition 20 by using some well-known results of the semigroup theory, see e.g. [24, Proposition 4.1.6]. ■

2.2.7 The Taibleson Operator and the p -Adic Heat Equation

We set

$$\Gamma_p^{(n)}(\alpha) := \frac{1 - p^{\alpha-n}}{1 - p^{-\alpha}}, \text{ for } \alpha \in \mathbb{R} \setminus \{0\}.$$

This function is called the p -adic Gamma function. The function

$$k_\alpha(x) = \frac{||x||_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \alpha \in \mathbb{R} \setminus \{0, n\}, \quad x \in \mathbb{Q}_p^n,$$

is called *the multi-dimensional Riesz Kernel*; it determines a distribution on $\mathcal{D}(\mathbb{Q}_p^n)$ as follows. If $\alpha \neq 0, n$, and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, then

$$\begin{aligned} (k_\alpha(x), \varphi(x)) &= \frac{1-p^{-n}}{1-p^{\alpha-n}}\varphi(0) + \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\|x\|_p > 1} \|x\|_p^{\alpha-n} \varphi(x) d^n x \\ &\quad + \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\|x\|_p \leq 1} \|x\|_p^{\alpha-n} (\varphi(x) - \varphi(0)) d^n x. \end{aligned} \quad (2.17)$$

Then $k_\alpha \in \mathcal{D}'(\mathbb{Q}_p^n)$, for $\mathbb{R} \setminus \{0, n\}$. In the case $\alpha = 0$, by passing to the limit in (2.17), we obtain

$$(k_0(x), \varphi(x)) := \lim_{\alpha \rightarrow 0} (k_\alpha(x), \varphi(x)) = \varphi(0),$$

i.e., $k_0(x) = \delta(x)$, the Dirac delta function, and therefore $k_\alpha \in \mathcal{D}'(\mathbb{Q}_p^n)$, for $\mathbb{R} \setminus \{n\}$.

It follows from (2.17) that for $\alpha > 0$,

$$(k_{-\alpha}(x), \varphi(x)) = \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|x\|_p^{-\alpha-n} (\varphi(x) - \varphi(0)) d^n x. \quad (2.18)$$

Definition 22 The Taibleson pseudodifferential operator D_T^α , $\alpha > 0$, is defined as

$$(D_T^\alpha \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (\|\xi\|_p^\alpha \mathcal{F}_{x \rightarrow \xi} \varphi), \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

By using (2.18) and the fact that $(\mathcal{F}k_{-\alpha})(x)$ equals $\|x\|_p^\alpha$, $\alpha \neq -n$, in $\mathcal{D}'(\mathbb{Q}_p^n)$, we have

$$\begin{aligned} (D_T^\alpha \varphi)(x) &= (k_{-\alpha} * \varphi)(x) \\ &= \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n} (\varphi(x-y) - \varphi(x)) d^n y. \end{aligned} \quad (2.19)$$

Then the Taibleson operator belongs to the class of operators \mathbf{W} introduced before. The right-hand side of (2.19) makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying

$$\int_{\|x\|_p \geq 1} \|x\|_p^{-\alpha-n} |\varphi(x)| d^n x < \infty.$$

A similar observation is valid in general for operators of \mathbf{W} type. The equation

$$\frac{\partial u(x, t)}{\partial t} + \kappa(D_T^\alpha u)(x, t) = 0, \quad x \in \mathbb{Q}_p^n, \quad t \geq 0,$$

where κ is a positive constant, is a multi-dimensional analog of the p -adic heat equation introduced in [111].

2.3 Elliptic Pseudodifferential Operators, Parabolic-Type Equations and Markov Processes

In this section we consider following Cauchy problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (f(\partial, \beta) u)(x, t) = 0, & x \in \mathbb{Q}_p^n, n \geq 1, t \geq 0 \\ u(x, 0) = \varphi(x), \end{cases} \quad (2.20)$$

where $f(\partial, \beta)$ is an elliptic pseudodifferential operator of the form

$$(f(\partial, \beta) \phi)(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(|f(\xi)|_p^\beta \mathcal{F}_{x \rightarrow \xi} \phi(x, t) \right).$$

Here β is a positive real number, and $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ is a homogeneous polynomial of degree d satisfying the property $f(\xi) = 0 \Leftrightarrow \xi = 0$. We establish the existence of a unique solution to Cauchy problem (2.20) in the case in which $\varphi(x)$ is a continuous and an integrable function. Under these hypotheses we show the existence of a solution $u(x, t)$ that is continuous in x , for a fixed $t \in [0, T]$, bounded, and integrable function. In addition the solution can be presented in the form

$$u(x, t) = Z(x, t) * \varphi(x)$$

where $Z(x, t)$ is the *fundamental solution* (also called the *heat kernel*) to Cauchy's Problem 2.20:

$$Z(x, t, f, \beta) := Z(x, t) = \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) e^{-t|f(\xi)|_p^\beta} d^n \xi, \quad \xi \in \mathbb{Q}_p^n, t > 0. \quad (2.21)$$

The fundamental solution is a transition density of a Markov process with space state \mathbb{Q}_p^n .

2.3.1 Elliptic Operators

Let $h(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be a non-constant polynomial. In this section we work with operators of the form $h(\partial, \beta) \phi = \mathcal{F}^{-1} \left(|h|_p^\beta \mathcal{F} \phi \right)$, $\beta > 0$, $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$. We

will say that $h(\partial, \beta)$ is a pseudodifferential operator with symbol $|h|_p^\beta = |h(\xi)|_p^\beta$. The operator $h(\partial, \beta)$ has a self-adjoint extension with dense domain in L^2 .

Definition 23 Let $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be a non-constant polynomial. We say that $f(\xi)$ is an elliptic polynomial of degree d , if it satisfies: (i) $f(\xi)$ is a homogeneous polynomial of degree d , and (ii) $f(\xi) = 0 \Leftrightarrow \xi = 0$.

Lemma 24 (i) There are infinitely many elliptic polynomials. (ii) For any $n \in \mathbb{N} \setminus \{0\}$ and $p \neq 2$, there exists an elliptic polynomial $h(\xi_1, \dots, \xi_n)$ with coefficients in \mathbb{Z}_p^\times and degree $2d(n) := 2d$ such that

$$|h(\xi_1, \dots, \xi_n)|_p = \|(\xi_1, \dots, \xi_n)\|_p^{2d}. \quad (2.22)$$

Proof (i) Assume that $h(\xi_1, \dots, \xi_n)$ is an elliptic polynomial of degree d . Take $\tau \in \mathbb{Q}_p^\times$ such that the equation $x^2 = \tau$ has no solutions in \mathbb{Q}_p^\times , then $h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d}$ is an elliptic polynomial of degree $2d$. Since there are elliptic quadratic forms for $1 \leq n \leq 4$, see e.g. [22, Chapter 1], one concludes the existence of infinitely many elliptic polynomials. (ii) By choosing $\tau \in \mathbb{Z}_p^\times$, it follows from (i) that if $h(\xi_1, \dots, \xi_n)$ is an elliptic polynomials of degree d with coefficients in \mathbb{Z}_p^\times , then $h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d}$ is elliptic with coefficients in \mathbb{Z}_p^\times . We pick d such that p does not divide d . We prove by induction on n that $h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d}$ satisfies (2.22). Assume, as induction hypothesis, that $h(\xi_1, \dots, \xi_n)$ satisfies (2.22). If $\left| h(\xi_1, \dots, \xi_n)^2 \right|_p \neq \left| \xi_{n+1}^{2d} \right|_p$, then $\left| h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d} \right|_p = \|(\xi_1, \dots, \xi_{n+1})\|_p^{2d}$. If $\left| h(\xi_1, \dots, \xi_n)^2 \right|_p = \left| \xi_{n+1}^{2d} \right|_p$, taking $\xi_{n+1} = p^m u_{n+1}$, with $u_{n+1} \in \mathbb{Z}_p^\times$, we have

$$\left| h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d} \right|_p = p^{-2md} \left| h(p^{-m}\xi_1, \dots, p^{-m}\xi_n)^2 - \tau u_{n+1}^{2d} \right|_p.$$

We note that $h(p^{-1}\xi_1, \dots, p^{-1}\xi_n)^2 - \tau u_{n+1}^{2d} \in \mathbb{Z}_p^\times$, otherwise

$$h(p^{-m}\xi_1, \dots, p^{-m}\xi_n)^2 - \tau u_{n+1}^{2d} \equiv 0 \pmod{p}$$

and by using that p does not divide $2d$, i.e. $p \neq 2$ and the Hensel lemma, there exists a nontrivial solution of $h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d} = 0$, which is impossible. Finally, by using $\left| h(\xi_1, \dots, \xi_n)^2 \right|_p = \|(\xi_1, \dots, \xi_n)\|_p^{2d} = \left| \xi_{n+1} \right|_p^{2d} = p^{-2md}$, we have $\left| h(\xi_1, \dots, \xi_n)^2 - \tau\xi_{n+1}^{2d} \right|_p = \|(\xi_1, \dots, \xi_{n+1})\|_p^{2d}$. \blacksquare

Lemma 25 Let $f(\xi) \in \mathbb{Q}_p[\xi]$, $\xi = (\xi_1, \dots, \xi_n)$, be an elliptic polynomial of degree d . Then there exist positive constants $C_0 = C_0(f)$, $C_1 = C_1(f)$ such that

$$C_0 \|\xi\|_p^d \leq |f(\xi)|_p \leq C_1 \|\xi\|_p^d, \text{ for every } \xi \in \mathbb{Q}_p^n. \quad (2.23)$$

Proof Without loss of generality we may assume that $\xi \neq 0$. Let $\tilde{\xi} \in \mathbb{Q}_p^\times$ be an element such that $\left| \tilde{\xi} \right|_p = \|\xi\|_p \neq 0$. We first note that

$$|f(\xi)|_p = \left| \tilde{\xi} \right|_p^d \left| f(\tilde{\xi}^{-1}\xi) \right|_p, \quad (2.24)$$

with $\tilde{\xi}^{-1}\xi \in S_0^n = \{z \in \mathbb{Z}_p^n : \|z\|_p = 1\}$. Now $|f|_p$ is continuous on S_0^n , that is a compact subset of \mathbb{Z}_p^n , then $\inf_{z \in S_0^n} |f(z)|_p$, and $\sup_{z \in S_0^n} |f(z)|_p$ are attained on S_0^n , and since $|f|_p > 0$ on S_0^n , we have $\sup_{z \in S_0^n} |f(z)|_p \geq \inf_{z \in S_0^n} |f(z)|_p > 0$. Therefore from (2.24) we have

$$\left(\inf_{z \in S_0^n} |f(z)|_p \right) \left| \tilde{\xi} \right|_p^d \leq |f(\xi)|_p \leq \left(\sup_{z \in S_0^n} |f(z)|_p \right) \left| \tilde{\xi} \right|_p^d.$$

■

Along this section $f(\xi)$ will denote an elliptic polynomial of degree d . Now, since $cf(\xi)$ is elliptic for any $c \in \mathbb{Q}_p^\times$ when $f(\xi)$ is elliptic, we will assume that all the elliptic polynomials have coefficients in \mathbb{Z}_p .

Lemma 26 *Let $A \subseteq \mathbb{Q}_p^n$ be an open compact subset such that $0 \notin A$. There exist a finite number of points $\tilde{\xi}_i \in A$, $i = 1, \dots, L_0$, and a constant $M := M(A, f) \in \mathbb{N} \setminus \{0\}$ such that*

$$A = \bigsqcup_{i=1}^{L_0} \tilde{\xi}_i + (p^M \mathbb{Z}_p)^n \text{ and } |f(\xi)|_p \big|_{\tilde{\xi}_i + (p^M \mathbb{Z}_p)^n} = \left| f(\tilde{\xi}_i) \right|_p, i = 1, \dots, L_0.$$

Proof By (2.23), for $\xi \in A$,

$$|f(\xi)|_p \geq C_0 \|\xi\|_p^d \geq C_0 \inf_{\xi \in A} \|\xi\|_p^d \geq p^{-M'(A, f)},$$

where $M' := M'(A, f)$ is a positive integer constant. Now for $\tilde{\xi}_i \in A$ and $y \in \mathbb{Z}_p^n$,

$$f(\tilde{\xi}_i + p^M y) = f(\tilde{\xi}_i) + p^M T(\tilde{\xi}_i, y),$$

where $T(\tilde{\xi}_i, y)$ is a polynomial function in $\tilde{\xi}_i, y$, with

$$\sup_{\tilde{\xi}_i \in A, y \in \mathbb{Z}_p^n} \left| T(\tilde{\xi}_i, y) \right|_p \leq p^\delta.$$

We set $M = M' + \delta + 1$. Then

$$\left| f\left(\tilde{\xi}_i + p^M y\right) \right|_p = \left| f\left(\tilde{\xi}_i\right) + p^M T\left(\tilde{\xi}_i, y\right) \right|_p = \left| f\left(\tilde{\xi}_i\right) \right|_p.$$

Now, since A is open compact, there exist a finite number of points $\tilde{\xi}_i \in A$, such that $A = \bigsqcup_i \tilde{\xi}_i + (p^M \mathbb{Z}_p)^n$. ■

Remark 27 Lemma 26 is valid for arbitrary polynomials satisfying only $f(\xi) = 0 \Leftrightarrow \xi = 0$. Indeed, by using that A is compact and that $|f(\xi)|_p$ is continuous, there exists a constant M' such that $|f(\xi)|_p \geq p^{-M'}$ for $\xi \in A$.

Definition 28 If $f(\xi) \in \mathbb{Z}_p[\xi]$ is an elliptic polynomial of degree d , then we say that $|f|_p^\beta$ is an elliptic symbol, and that $f(\partial, \beta)$ is an elliptic pseudodifferential operator of order d .

By Lemma 24, the Taibleson operator is elliptic for $p \neq 2$. However, there are elliptic symbols which are not radial functions. For instance, $\left| \xi_1^2 - p\xi_2^2 \right|_p^\beta = \left[\max \left\{ |\xi_1|_p^2, p^{-1} |\xi_2|_p^2 \right\} \right]^\beta$. Then, there are two different generalizations of the Taibleson operator (or Vladimirov operator): the W operators which are pseudodifferential operators with radial symbols, and the elliptic operators which include pseudodifferential operators with non-radial symbols.

2.3.2 Decaying of the Fundamental Solution at Infinity

Lemma 29 For every $t > 0$, $|Z(x, t)| \leq Ct^{\frac{-n}{d\beta}}$, where C is a positive constant. Furthermore, $e^{-t|f(\xi)|_p^\beta} \in L^1$ as a function of ξ , for every $t > 0$.

Proof Let an integer m be such that $p^{m-1} \leq (Ct)^{\frac{1}{d\beta}} \leq p^m$. By applying (2.23),

$$\begin{aligned} |Z(x, t)| &\leq \int_{\mathbb{Q}_p^n} e^{-Ct\|\xi\|_p^{d\beta}} d^n \xi \leq \int_{\mathbb{Q}_p^n} e^{-p^{-(m-1)d\beta}\|\xi\|_p^{d\beta}} d^n \xi \\ &= \int_{\mathbb{Q}_p^n} e^{-\|p^{-(m-1)}\xi\|_p^{d\beta}} d^n \xi \leq p^n C^{-\frac{n}{d\beta}} \left(\int_{\mathbb{Q}_p^n} e^{-\|z\|_p^{d\beta}} d^n z \right) t^{-\frac{n}{d\beta}}. \end{aligned}$$

The result follows from the fact that $e^{-\|z\|_p^{d\beta}}$ is an integrable function. ■

Define

$$Z_L(x, t, f, \beta) := Z_L(x, t) = \int_{(p^{-L}\mathbb{Z}_p)^n} \chi_p(-x \cdot \xi) e^{-|f(\xi)|_p^\beta} d^n \xi, \quad L \in \mathbb{N},$$

where $\beta > 0$, $t > 0$, and $f(\xi) \in \mathbb{Z}_p[\xi_1, \dots, \xi_n]$ is an elliptic polynomial of degree d .

Lemma 30 *If $\|x\|_p \geq p^{M+1}$ and $tp^{Md\beta} \|x\|_p^{-d\beta} \leq 1$, where M is the constant defined in Lemma 26, then there exists a positive constant C such that*

$$|Z_0(x, t)| \leq Ct \|x\|_p^{-d\beta-n}.$$

Proof By applying Fubini's Theorem,

$$Z_0(x, t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^l \int_{\mathbb{Z}_p^n} \chi_p(-x \cdot \xi) |f(\xi)|_p^{\beta l} d^n \xi. \quad (2.25)$$

By using the fact that $\|x\|_p > p^{M+1} > 1$, $\int_{\mathbb{Z}_p^n} \chi_p(-x \cdot \xi) d\xi = 0$, and thus (2.25) can be rewritten as

$$Z_0(x, t) = \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} t^l \int_{\mathbb{Z}_p^n} \chi_p(-x \cdot \xi) |f(\xi)|_p^{\beta l} d^n \xi. \quad (2.26)$$

We set

$$I(j, l) := I(x, f, \beta, j, l) = \int_{\mathbb{Z}_p^n} \chi_p(-p^j x \cdot \xi) |f(\xi)|_p^{\beta l} d^n \xi, \quad \text{for } j \geq 0, l \geq 1,$$

and

$$\tilde{I}(j, l, S_0^n) := \tilde{I}(x, f, \beta, j, l, S_0^n) = \int_{S_0^n} \chi_p(-p^j x \cdot \xi) |f(\xi)|_p^{\beta l} d^n \xi,$$

for $j \geq 0, l \geq 1$. By decomposing \mathbb{Z}_p^n as the disjoint union of $(p\mathbb{Z}_p)^n$ and S_0^n ,

$$\begin{aligned} I(0, l) &= \int_{\mathbb{Z}_p^n} \chi_p(-x \cdot \xi) |f(\xi)|_p^{\beta l} d\xi \\ &= \int_{(p\mathbb{Z}_p)^n} \chi_p(-x \cdot \xi) |f(\xi)|_p^{\beta l} d\xi + \int_{S_0^n} \chi_p(-x \cdot \xi) |f(\xi)|_p^{\beta l} d^n \xi \\ &= p^{-n-\beta dl} I(1, l) + \tilde{I}(0, l, S_0^n). \end{aligned}$$

By iterating this formula k -times, we obtain

$$I(0, l) = \sum_{j=0}^k p^{-j(n+\beta dl)} \tilde{I}(j, l, S_0^n) + p^{-(k+1)(n+\beta dl)} I(k+1, l).$$

Hence $I(0, l)$ admits the expansion

$$I(0, l) = \sum_{j=0}^{\infty} p^{-j(n+\beta dl)} \tilde{I}(j, l, S_0^n). \quad (2.27)$$

On the other hand, since S_0^n is open compact and f is elliptic, by applying Lemma 26, we obtain

$$\tilde{I}(j, l, A) = \sum_{i=1}^{L_0} p^{-Mn} \chi_p \left(-p^j x \cdot \tilde{\xi}_i \right) \left| f \left(\tilde{\xi}_i \right) \right|_p^{\beta l} \int_{\mathbb{Z}_p^n} \chi_p \left(-p^{j+M} x \cdot y \right) d^n y, \quad (2.28)$$

Now by using

$$\int_{\mathbb{Z}_p^n} \chi_p \left(-p^{j+M} x \cdot y \right) d^n y = \begin{cases} 0 & \text{if } j < -M - \text{ord}(x) \\ 1 & \text{if } j \geq -M - \text{ord}(x), \end{cases}$$

with $\text{ord}(x) = \min_i \text{ord}(x_i)$, we can rewrite $\tilde{I}(j, l, A)$ as

$$\begin{cases} p^{-Mn} \sum_{i=1}^{L_0} \chi_p \left(-p^j x \cdot \tilde{\xi}_i \right) \left| f \left(\tilde{\xi}_i \right) \right|_p^{\beta l} & \text{if } j \geq -M - \text{ord}(x) \\ 0 & \text{otherwise.} \end{cases} \quad (2.29)$$

We set $\alpha := \alpha(x) = -M - \text{ord}(x) \geq 1$ because $\|x\|_p = p^{-\text{ord}(x)} \geq p^{M+1}$. With this notation, by combining (2.27)–(2.29) and using that $f(\xi)$ has coefficients in \mathbb{Z}_p and $\tilde{\xi}_i \in \mathbb{Z}_p^n$, $i = 1, \dots, L$,

$$\begin{aligned} |I(0, l)| &\leq p^{-Mn} \left(\sum_{i=1}^{L_0} \left| f \left(\tilde{\xi}_i \right) \right|_p^{\beta l} \right) \sum_{j=\alpha}^{\infty} p^{-j(n+\beta dl)} \\ &\leq \left(\frac{L_0}{1 - p^{-(n+\beta dl)}} \right) \|x\|_p^{-n} p^{-\alpha \beta dl}. \end{aligned}$$

By using this estimation for $|I(0, t)|$ in (2.26),

$$|Z_0(x, t)| \leq \left(\frac{L_0}{1 - p^{-(n+\beta d)}} \right) \|x\|_p^{-n} \left(e^{p^{M\beta} t \|x\|^{-d\beta}} - 1 \right),$$

finally, by using the hypothesis $tp^{M\beta} \|x\|_p^{-d\beta} \leq 1$, we have

$$|Z_0(x, t)| \leq Ct \|x\|_p^{-d\beta-n}.$$

■

Proposition 31 *If $p^{M\beta} t \|x\|_p^{-d\beta} \leq 1$, then $|Z(x, t)| \leq Ct \|x\|_p^{-d\beta-n}$, for $x \in \mathbb{Q}_p^n$ and $t > 0$.*

Proof By Lemma 29, $\chi_p(-x \cdot \xi) e^{-t|f(\xi)|_p^\beta} \in L^1$ as a function of ξ , for $x \in \mathbb{Q}_p^n$ and $t > 0$ fixed. Then, by using the dominated convergence theorem,

$$Z(x, t) = \lim_{L \rightarrow \infty} Z_L(x, t) = \lim_{L \rightarrow \infty} \int_{(p^{-L}\mathbb{Z}_p)^n} \chi_p(-x \cdot \xi) e^{-t|f(\xi)|_p^\beta} d^n \xi.$$

By a change of variables we have

$$Z_L(x, t) = p^{Ln} \int_{\mathbb{Z}_p^n} \chi_p(-p^{-L}x \cdot \xi) e^{-p^{L\beta d} t |f(\xi)|_p^\beta} d\xi = p^{Ln} Z_0(p^{-L}x, p^{L\beta d} t).$$

Now by applying the Lemma 30,

$$|Z_L(x, t)| \leq Cp^{Ln} \left(\frac{tp^{L\beta d}}{\|xp^{-L}\|_p^{n+\beta d}} \right) \leq C \frac{t}{\|x\|_p^{n+\beta d}},$$

where C is a constant independent of L . Therefore

$$|Z(x, t)| = \lim_{L \rightarrow \infty} |Z_L(x, t)| \leq Ct \|x\|_p^{-n-\beta d},$$

if $p^{M\beta} t \|x\|_p^{-d\beta} \leq 1$.

■

Theorem 32 *For any $x \in \mathbb{Q}_p^n$ and any $t > 0$,*

$$|Z(x, t)| \leq At \left(\|x\|_p + t^{\frac{1}{\beta d}} \right)^{-d\beta-n},$$

where A is a positive constant.

Proof If $t^{\frac{1}{\beta d}} \leq p^{-M} \|x\|_p$, then $t^{\frac{1}{\beta d}} \leq \|x\|_p$ because $M \geq 1$, and by applying Proposition 31,

$$\begin{aligned} |Z(x, t)| &\leq Ct \|x\|_p^{-d\beta-n} \leq Ct \left(\frac{1}{2} \|x\|_p + \frac{1}{2} t^{\frac{1}{\beta d}} \right)^{-d\beta-n} \\ &\leq \frac{2^{d\beta+n} Ct}{\left(\|x\|_p + t^{\frac{1}{\beta d}} \right)^{d\beta+n}}. \end{aligned}$$

Now if $\|x\|_p < t^{\frac{1}{\beta d}}$, by applying Lemma 29,

$$2^{d\beta+n} Ct \left(\|x\|_p + t^{\frac{1}{\beta d}} \right)^{-d\beta-n} \geq Ct^{-\frac{n}{\beta d}} \geq |Z(x, t)|.$$

■

When considering $Z(x, t)$ as a function of x for t fixed we will write $Z_t(x)$ as before.

Corollary 33 *With the hypothesis of Theorem 32, the following assertions hold: (i) $Z_t(x) \in L^\rho(\mathbb{Q}_p^n)$, for $1 \leq \rho \leq \infty$, for $t > 0$; (ii) $Z_t(x)$ is a continuous function in x , for $t > 0$ fixed.*

Proof (i) The first part follows directly from the estimation given in Theorem 32. (ii) The continuity follows from the fact that $Z_t(x)$ is the Fourier transform of $e^{-t|f(\xi)|_p^\beta}$, $t > 0$, that is an integrable function by Lemma 25. ■

2.3.3 Positivity of the Fundamental Solution

Theorem 34 $Z(x, t) \geq 0$ for every $x \in \mathbb{Q}_p^n$ and every $t > 0$.

Proof We start by making the following observation about the fiber of $f : \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ at $\lambda \in \mathbb{Q}_p$.

(Claim A) $f^{-1}(\lambda)$ is a compact subset of \mathbb{Q}_p^n .

Since f is continuous $f^{-1}(\lambda)$ is a closed subset of \mathbb{Q}_p^n . By applying (2.23),

$$f^{-1}(\lambda) \subseteq \left\{ \xi \in \mathbb{Q}_p^n, \|\xi\|_p \leq \left(\frac{|\lambda|}{C_0} \right)^{\frac{1}{d}} \right\},$$

and thus $f^{-1}(\lambda)$ is a bounded subset of \mathbb{Q}_p^n .

(Claim B) The critical set $C_f = \{\xi \in \mathbb{Q}_p^n; \nabla f(\xi) = 0\}$ of the mapping f is reduced to the origin of \mathbb{Q}_p^n .

This claim follows from the Euler identity

$$\frac{1}{d} \sum_{i=1}^n \xi_i \frac{\partial f(\xi)}{\partial \xi_i} = f(\xi),$$

and the fact that f is an elliptic polynomial.

On the other hand, since $\chi_p(-x \cdot \xi) 1_{f^{-1}(\lambda)}(\xi) \in \mathcal{D}(\mathbb{Q}_p^n)$, as a function of ξ , for $\lambda \neq 0$, by applying integration on fibers to $Z(x, t)$, see Chap. 1, formula (1.3), with $t > 0$ fixed,

$$Z(x, t) = \int_{\mathbb{Q}_p \setminus \{0\}} e^{-t|\lambda|_p^\beta} \left(\int_{f(\xi)=\lambda} \chi_p(-x \cdot \xi) |\gamma_{\text{GL}}| \right) d\lambda,$$

where $|\gamma_{\text{GL}}|$ is the measure induced by the Gel'fand-Leray form along the fiber $f^{-1}(\lambda)$. Hence in order to prove the theorem, it is sufficient to show that

$$F(\lambda, x) := \left(\int_{f(\xi)=\lambda} \chi_p(-x \cdot \xi) |\gamma_{\text{GL}}| \right) \geq 0, \text{ for every } x \in \mathbb{Q}_p^n \setminus \{0\}.$$

Let $\tilde{\xi}$ be a fixed point of $f^{-1}(\lambda)$, $\lambda \in \mathbb{Q}_p \setminus \{0\}$. By Claim B we may assume, after renaming the variables if necessary, that $\frac{\partial f}{\partial \xi_n}(\tilde{\xi}) \neq 0$. We set $y = \phi(\xi)$ with

$$y_j := \begin{cases} \xi_j & j = 1, \dots, n-1 \\ f(\tilde{\xi} + p^e \xi) - f(\tilde{\xi}) & j = n. \end{cases}$$

By applying the non-Archimedean implicit function theorem (see Chap. 1, Theorem 1) there exist $e, l \in \mathbb{N}$ such that $y = \phi(\xi)$ is a bianalytic mapping from \mathbb{Z}_p^n onto $(p^l \mathbb{Z}_p)^n$. Then

$$\xi = \phi^{-1}(y) = \left(y_1, \dots, y_{n-1}, \sum_{j=1}^{\infty} G_j(y) \right),$$

where $G_j(y)$ is a form of degree j , and $G_1(y) \neq 0$. By shrinking the neighborhoods around $\tilde{\xi}$ and the origin, i.e., by taking e and l big enough, we may assume that the following conditions hold:

(C) the Jacobian $J_{\phi^{-1}}$ of ϕ^{-1} satisfies $|J_{\phi^{-1}}(y)|_p = |J_{\phi^{-1}}(0)|_p$, for every $y \in (p^l \mathbb{Z}_p)^n$;

$$(D) \text{ ord} \left(x_n \sum_{j=1}^{\infty} G_j(y) \right) \geq 0, \text{ for any } y \in (p^l \mathbb{Z}_p)^n.$$

Since $f^{-1}(\lambda)$, $\lambda \in \mathbb{Q}_p \setminus \{0\}$, is a compact subset by Claim A, $F(\lambda, x)$ can be expressed as a finite sum of integrals of the form

$$\int_{\tilde{\xi} + p^e \mathbb{Z}_p^n \cap f^{-1}(\lambda)} \chi_p(-x \cdot \xi) |\gamma_{GL}|.$$

Now by changing variables $\xi = \phi^{-1}(y)$, and using (C), (D), we obtain

$$\begin{aligned} & \int_{\tilde{\xi} + p^e \mathbb{Z}_p^n \cap f^{-1}(\lambda)} \chi_p(-x \cdot \xi) |\gamma_{GL}| \\ &= |J_{\phi^{-1}}(0)|_p \int_{p^l \mathbb{Z}_p^{n-1}} \chi_p \left(- \sum_{j=1}^{n-1} x_j \xi_j - x_n \sum_{l=1}^{\infty} G_j(y) \right) d^{n-1}y \\ &= |J_{\phi^{-1}}(0)|_p \int_{p^l \mathbb{Z}_p^{n-1}} \chi_p \left(- \sum_{j=1}^{n-1} x_j \xi_j \right) d^{n-1}y \\ &= \left(p^{-l(n-1)} |J_{\phi^{-1}}(0)|_p \right) 1_{p^{-l} \mathbb{Z}_p^{n-1}}(x) \geq 0, \end{aligned}$$

where $1_{p^{-l} \mathbb{Z}_p^{n-1}}(x)$ denotes the characteristic function of $p^{-l} \mathbb{Z}_p^{n-1}$. Therefore $F(\lambda, x) \geq 0$. ■

2.3.4 Some Additional Results

We denote by $C_b := C_b(\mathbb{Q}_p^n, \mathbb{R})$ the \mathbb{R} -vector space of all functions $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{R}$ which are continuous and satisfy $\|\varphi\|_{L^\infty} = \sup_{x \in \mathbb{Q}_p^n} |\varphi(x)| < \infty$.

Proposition 35 *The fundamental solution has the following properties:*

- (i) $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = 1$, for any $t > 0$;
- (ii) if $\varphi \in C_b$, then $\lim_{(x,t) \rightarrow (x_0,0)} \int_{\mathbb{Q}_p^n} Z(x-y, t) \varphi(y) d^n y = \varphi(x_0)$;
- (iii) $Z(x, t+t') = \int_{\mathbb{Q}_p^n} Z(x-y, t) Z(y, t') d^n y$, for $t, t' > 0$.

Proof

- (i) It follows from Corollary 33 and the Fourier inversion formula.
- (ii) We set $u(x, t) = \int_{\mathbb{Q}_p^n} Z(x - y, t) \varphi(y) d^n y$. We have to show that

$$\lim_{(x,t) \rightarrow (x_0,0)} u(x, t) = \varphi(x_0),$$

for any fixed $x_0 \in \mathbb{Q}_p^n$. Since φ is continuous at x_0 there exists a ball $B_{-e}^n(x_0) = \{y \in \mathbb{Q}_p^n; \|y - x_0\|_p \leq p^{-e}\}$, such that $|\varphi(y) - \varphi(x_0)| < \frac{\epsilon}{2}$, for every $y \in B_{-e}^n(x_0)$. Then $|u(x, t) - \varphi(x_0)| \leq |I_1| + |I_2|$, where

$$|I_1| := \left| \int_{\|y-x_0\|_p \leq p^{-e}} Z(x-y, t) [\varphi(y) - \varphi(x_0)] d^n y \right|,$$

$$|I_2| := \left| \int_{\|y-x_0\|_p > p^{-e}} Z(x-y, t) [\varphi(y) - \varphi(x_0)] d^n y \right|.$$

By using the continuity of φ and (i),

$$|I_1| < \frac{\epsilon}{2}, \text{ for } y \in B_{-e}^n(x_0).$$

By applying Theorem 32 to $|I_2|$,

$$|I_2| \leq 2Ct \|\varphi\|_{L^\infty} \int_{\|y-x_0\|_p > p^{-e}} \|x-y\|_p^{-d\beta-n} d^n y.$$

Now, since we are interested in the values of x close to x_0 , we may assume that $\|x - x_0\|_p < p^{-e}$, then by the ultrametric triangle inequality,

$$\|x - y\|_p = \max(\|x - x_0\|_p, \|y - x_0\|_p) = \|y - x_0\|_p,$$

and

$$|I_2| \leq 2Ct \|\varphi\|_{L^\infty} \int_{\|z\|_p > p^{-e}} \|z\|_p^{-d\beta-n} d^n z \leq C_1 t \|\varphi\|_{L^\infty},$$

for $t > 0$, where C_1 is a positive constant. Note that $\|\varphi\|_{L^\infty} = 0$, implies $\varphi \equiv 0$, since φ is a continuous function. In this case the theorem is valid. For this reason we assume that $\|\varphi\|_{L^\infty} > 0$. Hence

$$|I_2| < \frac{\epsilon}{2}, \text{ for } (t, x) \text{ satisfying } \|x - x_0\|_p < p^{-e}, 0 < t < \frac{\epsilon}{2C_1 \|\varphi\|_{L^\infty}}.$$

(iii) By using that $e^{-t|f(\xi)|_p^\beta} \in L^1$, for every $t > 0$,

$$\int_{\mathbb{Q}_p^n} Z(x-y, t) Z(y, t') d^n y = \mathcal{F}^{-1} \left(e^{-t|f(\xi)|_p^\beta} e^{-t'|f(\xi)|_p^\beta} \right) = Z(x, t+t'),$$

for $t, t' > 0$.

■

2.3.5 The Cauchy Problem

In this section we study the following Cauchy problem:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + (f(\partial, \beta) u)(x, t) = 0, & t > 0 \\ u(x, 0) = \varphi(x), \end{cases} \quad (2.30)$$

where $\varphi \in L^1 \cap C_b$.

Lemma 36 *If $\varphi \in L^1$, then the function*

$$u(x, t) = \int_{\mathbb{Q}_p^n} Z(x-y, t) \varphi(y) d^n y \quad (2.31)$$

is a classical solution of the equation

$$\frac{\partial u(x, t)}{\partial t} + (f(\partial, \beta) u)(x, t) = 0, t > 0.$$

In addition, $u(x, t) \in L^\rho$, for $1 \leq \rho \leq \infty$, for every fixed $t > 0$.

Proof It is clear that one may differentiate in (2.31) under the integral sign:

$$\frac{\partial u(x, t)}{\partial t} = \int_{\mathbb{Q}_p^n} \varphi(y) \frac{\partial}{\partial t} Z(x-y, t) d^n y = \frac{\partial Z(x, t)}{\partial t} * \varphi(x), \text{ for } t > 0. \quad (2.32)$$

On the other hand, since $Z(x, t) \in L^\rho$, $1 \leq \rho \leq \infty$, for any fixed $t > 0$ (cf. Corollary 33), and $\varphi \in L^1$, then $u(x, t) \in L^\rho$, $1 \leq \rho \leq \infty$, for any fixed $t > 0$, and its Fourier transform with respect x is $e^{-t|f(\xi)|_p^\beta} (\mathcal{F}\varphi)(\xi) \in L^\rho$, $1 \leq \rho \leq \infty$, because $(\mathcal{F}\varphi)(\xi) \in L^\infty$, by the Riemann-Lebesgue Theorem, and the fact that f is elliptic. Now by using Lemma 25 we have $|f(\xi)|_p^\beta e^{-t|f(\xi)|_p^\beta} \in L^1 \cap L^2$ for any fixed $t > 0$.

Then $(f(\partial, \beta) u_0)(x, t)$ is given by

$$\begin{aligned} (f(\partial, \beta) u)(x, t) &= \mathcal{F}_{\xi \rightarrow x}^{-1} \left(|f(\xi)|_p^\beta e^{-t|f(\xi)|_p^\beta} \right) * \varphi(x) \\ &= -\mathcal{F}_{\xi \rightarrow x}^{-1} \left(\frac{\partial}{\partial t} e^{-t|f(\xi)|_p^\beta} \right) * \varphi(x), \end{aligned}$$

for $t > 0$, and since one may differentiate in (2.21) under the integral,

$$(f(\partial, \beta) u)(x, t) = -\frac{\partial Z(x, t)}{\partial t} * \varphi(x). \quad (2.33)$$

Now the result follows directly from (2.32) and (2.33). \blacksquare

Lemma 37 *Let $u(x, t)$ be as in Lemma 36. Then the following assertions hold: (i) $u(x, t)$ is continuous for any $t \geq 0$; (ii) $|u(x, t)| \leq \|\varphi\|_{L^\infty}$ for any $t \geq 0$.*

Proof (i) For $t > 0$, since $|f(\xi)|_p^\beta e^{-t|f(\xi)|_p^\beta} (\mathcal{F}\varphi)(\xi) \in L^1$, $u(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(|f(\xi)|_p^\beta e^{-t|f(\xi)|_p^\beta} (\mathcal{F}\varphi)(\xi) \right)$ is continuous. The continuity at $t = 0$ follows from the fact that $u(x, 0) = \varphi(x) = \lim_{t \rightarrow 0} u(x, t)$, cf. Proposition 35 (ii). For $t > 0$, the result follows from the Young inequality. \blacksquare

Theorem 38 *If $\varphi \in L^1 \cap C_b$, then the Cauchy problem*

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + (f(\partial, \beta) u)(x, t) = 0, & x \in \mathbb{Q}_p^n, t > 0 \\ u(x, 0) = \varphi(x) \end{cases}$$

has a classical solution given by

$$u(x, t) = \int_{\mathbb{Q}_p^n} Z(x - y, t) \varphi(y) d^n y.$$

Furthermore, the solution has the following properties:

- (1) $u(x, t)$ is a continuous function in x , for every fixed $t \geq 0$;
- (2) $\sup_{(x, t) \in \mathbb{Q}_p^n \times [0, +\infty)} |u(x, t)| \leq \|\varphi\|_{L^\infty}$;
- (3) $u(x, t) \in L^\rho$, $1 \leq \rho \leq \infty$, for any fixed $t > 0$.

Proof The result follows from Lemmas 36, 37. \blacksquare

2.3.6 Markov Processes Over \mathbb{Q}_p^n

Theorem 39 *$Z(x, t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.*

Proof By Proposition 35 (iii) the family of operators

$$(\Theta(t)f)(x) = \int_{\mathbb{Q}_p^n} Z(x-y, t) f(y) d^n y$$

has the semigroup property. We know that $Z(x, t) \geq 0$ and $\Theta(t)$ preserves the function $f(x) \equiv 1$ (cf. Proposition 35 (i)). Thus $\Theta(t)$ is a Markov semigroup. The requiring properties of the corresponding Markov process follow from Theorem 32 and general theorems of the theory of Markov processes [39], see also Sect. 2.2.5. ■

Remark 40 By using the results of [42], it is possible to show that there exists a Lévy process with state space \mathbb{Q}_p^n and transition function

$$P(t, x, E) = \begin{cases} Z_t(x) * 1_E(x) & \text{for } t > 0, x \in \mathbb{Q}_p^n \\ 1_E(x) & \text{for } t = 0, x \in \mathbb{Q}_p^n, \end{cases}$$

where E is an element of the family of subsets of \mathbb{Q}_p^n formed by finite unions of disjoint balls and the empty set. However, for the sake of simplicity we state our results in the framework of Markov processes.

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