Chapter 2
Basic Algebraic and Logical Definitions

2.1 Overview of This Chapter

This chapter contains basic notation, definitions and results, sometimes slightly beyond the immediate need of this book, as we want to put our work a bit more in perspective, and make it basically self-contained, for the convenience of the (expert) reader. Most of the material of this chapter (unless marked as "new") was published previously; see Schlechta (2004), Gabbay and Schlechta (2009a, b, d, e, 2011).

We begin with basic algebraic and logical definitions, including in particular many laws of nonmonotonic logics, in their syntactic and semantical variants, showing the connections between both sides; see Definition 2.2.6 and Tables 2.1 and 2.2.

We first recall some standard concepts, such as filters, and introduce our notations for various algebraic and logical notions. This part ends with Fact 2.2.2. We then introduce notation and rules particular to nonmonotonic logics. This might be the place to recall that such logics and their properties were constructed from two sides, the syntactical side, e.g., by D. Gabbay in Gabbay (1989), and the semantical side, e.g., for preferential structures by Y. Shoham in Yoav Shoham (1987). Both approaches were first brought together by S. Kraus, D. Lehmann, M. Magidor in Kraus et al. (1990). The semantical versions of these rules are partly due to one of us (see, e.g., Schlechta (2004)), and described systematically in Gabbay and Schlechta (2009b); see also Gabbay and Schlechta (2009d).

We would like to emphasize the importance of the definability preservation (dp) property. In the infinite case, not all model sets X are definable, i.e. there is some formula ϕ or theory T such that $X = M(\phi)$, the models of ϕ, or $X = M(T)$, the models of T. It is by no means evident that a model choice function μ, applied to a definable model set, gives us back again a definable model set (is definability preserving, or dp). If μ does not have this property, some representation results will not hold, which hold if μ is dp, and representation results become much more complicated; see Schlechta (2004) for positive and for impossibility results. In our present context, definability is again an important concept. Even if we have semantical interpolation, if language and operators are not strong enough, we cannot define the

© Springer International Publishing AG 2016

D.M. Gabbay and K. Schlechta, *A New Perspective on Nonmonotonic Logics*,
DOI 10.1007/978-3-319-46817-4_2

semantical interpolants, so we have semantical, but not syntactic interpolation. The reader is referred to Gabbay and Schlechta (2011), Chap. 5 for the discussion.

2.2 The Definitions

Notation 2.2.1
We use sometimes FOL as an abbreviation for first order logic, and NML for non-monotonic logic. To avoid LaTeX complications in bigger expressions, we replace $\widetilde{xxxxx}$ by $\overbrace{xxxxx}$.

Definition 2.2.1

(1) We use $:=$ and $:\Leftrightarrow$ to define the left-hand side by the right-hand side, as in the following two examples:
$X := \{x\}$ defines X as the singleton with element x.
$X < Y :\Leftrightarrow \forall x \in X \forall y \in Y(x < y)$ extends the relation $<$ from elements to sets.

(2) We use $\mathcal{P}$ to denote the power set operator.
$\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general Cartesian product, $X \times X'$ is the binary Cartesian product.
$card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in — the class of all sets.
Given a set of pairs $\mathcal{X}$, and a set X, we let $\mathcal{X} \restriction X := \{\langle x, i\rangle \in \mathcal{X} : x \in X\}$. (When the context is clear, we will sometimes simply write X for $\mathcal{X} \restriction X$.)
We will use the same notation $\restriction$ to denote the restriction of functions and in particular of sequences to a subset of the domain.
If Σ is a set of sequences over an index set X, and $X' \subseteq X$, we will abuse notation and write $\Sigma \restriction X'$ for $\{\sigma \restriction X' : \sigma \in \Sigma\}$.
Concatenation of sequences, e.g., of σ and σ', will be denoted by juxtaposition: $\sigma\sigma'$.

(3) $A \subseteq B$ will denote that A is a subset of B or equal to B, and $A \subset B$ that A is a proper subset of B; likewise for $A \supseteq B$ and $A \supset B$.
Given some fixed set U we work in, and $X \subseteq U$, $\boldsymbol{C}(X) := U - X$.

(4) If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X, we say that $\mathcal{Y}$ satisfies
($\cap$) iff it is closed under finite intersections,
($\bigcap$) iff it is closed under arbitrary intersections,
($\cup$) iff it is closed under finite unions,
($\bigcup$) iff it is closed under arbitrary unions,
($\boldsymbol{C}$) iff it is closed under complementation,
($-$) iff it is closed under set difference.

(5) We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.
We make ample and tacit use of the Axiom of Choice.

Definition 2.2.2
$\prec^*$ will denote the transitive closure of the relation $\prec$. If $<$, $\prec$, or a similar relation is given, $a \perp b$ will express that a and b are $<$-incomparable (or $\prec$-incomparable); context will tell. Given any relation $<$, $\leq$ will stand for $<$ or $=$; conversely, given $\leq$, $<$ will stand for $\leq$, but not $=$; similarly for $\prec$, etc.

Definition 2.2.3
Fix a base set X.
A filter (resp. weak filter) on or over X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $(F1) - (F3)$ (resp. (F1), (F2), $(F3')$) hold:
(F1) $X \in \mathcal{F}$,
(F2) $A \subseteq B \subseteq X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$,
(F3) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,
$(F3')$ $A, B \in \mathcal{F}$ imply $A \cap B \neq \emptyset$.
So a weak filter satisfies $(F3')$ instead of (F3). Note that $\mathcal{P}(X)$ is a filter, but not a weak one, but all proper filters are also weak filters.
A filter is called a principal filter iff there is $X' \subseteq X$ s.t. $\mathcal{F} = \{A : X' \subseteq A \subseteq X\}$.

A (weak) ideal on or over X is a set $\mathcal{I} \subseteq \mathcal{P}(X)$ such that $(I1) - (I3)$ (or (I1), (I2), $(I3')$) hold:
(I1) $\emptyset \in \mathcal{I}$,
(I2) $A \subseteq B \subseteq X$, $B \in \mathcal{I}$ imply $A \in \mathcal{I}$,
(I3) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
$(I3')$ $A, B \in \mathcal{I}$ imply $A \cup B \neq X$.
So a weak ideal satisfies $(I3')$ instead of (I3).
A filter is an abstract notion of size; elements of a filter on X are called big subsets of X; their complements are called small, and the rest have medium size. The dual applies to ideals; this is justified by the following trivial fact:

Fact 2.2.1
If $\mathcal{F}$ is a (weak) filter on X, then $\mathcal{I} := \{X - A : A \in \mathcal{F}\}$ is a (weak) ideal on X; if $\mathcal{I}$ is a (weak) ideal on X, then $\mathcal{F} := \{X - A : A \in \mathcal{F}\}$ is a (weak) filter on X.

Definition 2.2.4

(1) V will be the set of truth values when there are more than the classical ones, TRUE and FALSE.
We work here in a classical propositional language $\mathcal{L}$; a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ, etc., theories T, S, etc.
$v(\mathcal{L})$ or simply L will be the set of propositional variables of $\mathcal{L}$.
$F(\mathcal{L})$ will be the set of formulas of $\mathcal{L}$.
A propositional model m will be a function from the set of propositional variables to the set of truth values V when we have more than two truth values.
$M_{\mathcal{L}}$, or simply M when the context is clear, will be the set of (classical) models for $\mathcal{L}$; $M(T)$ or M_T is the set of models of T; likewise for $M(\phi)$ for a formula ϕ.

(2) $\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T \text{ a theory in } \mathcal{L}\}$, the set of definable model sets.
Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ contains singletons and is closed under arbitrary intersections and finite unions.
An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called definability preserving, (dp) or (μdp) in short, iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y}$, $f(X) \in \boldsymbol{D}_{\mathcal{L}}$.
We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ — as needed for theory revision — with the obvious meaning.
(3) $\vdash$ will be classical derivability, and
$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under $\vdash$.
(4) $Con(.)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is classically consistent; likewise for $Con(T)$. $Con(T, T')$ will stand for $Con(T \cup T')$, etc.
(5) Given a consequence relation $\mid\!\sim$, we define
$\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$.
(There is no fear of confusion with $\overline{T}$, as it just is not useful to close twice under classical logic.)
(6) $T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.
(7) If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$; likewise for $Th(m), m \in M_{\mathcal{L}}$. ($\models$ will usually be classical validity.)

In the following, the X_i are arbitrary (nonempty) sets, standing for the set of truth values, and J is intuitively the set of propositional variables. So any element of $\Pi\{X_i : i \in J\}$ stands for a propositional model. Inessential variables in a set Σ are those which do not have any influence on the truth of the formula whose model set is Σ.

Definition 2.2.5
Let $\Sigma \subseteq \Pi := \Pi\{X_i : i \in J\}$. Define:
(1) For $\sigma, \sigma' \in \Sigma$, $J' \subseteq J$, define:
$\sigma \sim_{J'} \sigma' :\Leftrightarrow \forall x \in J' \sigma(x) = \sigma'(x)$.
(2) $I(\Sigma) := \{i \in J : \Sigma = \Sigma \restriction (J - \{i\}) \times X_i\}$ (up to reordering) (they are the irrelevant or inessential i) and
$R(\Sigma) := J - I(\Sigma)$ (the relevant or essential i).

Fact 2.2.2
(1) $\Sigma = \Sigma \restriction R(\Sigma) \times \Pi \restriction I(\Sigma)$ (up to reordering)
(2) $\sigma \restriction R(\Sigma) = \sigma' \restriction R(\Sigma) \wedge \sigma \in \Sigma \Rightarrow \sigma' \in \Sigma$.

Proof
(1) Enumerate $I(\Sigma)$, $I(\Sigma) = \{i : i < \kappa\}$. Define $\Sigma_j := \Sigma \restriction (R(\Sigma) \cup \{i : i < j\})$. We show by induction that $\Sigma_j = \Sigma \restriction R(\Sigma) \times \Pi \restriction (I(\Sigma) \cap j)$ for $j \leq \kappa$ (up to reordering).
$j = 0$ is trivial — there is nothing to show.
$j \to j + 1$: This follows from the induction hypothesis and the definition of $I(\Sigma)$.

j is a limit ordinal: Any sequence of length j can be written as the coherent union of shorter sequences, and these are in both sets, as the result holds for $j' < j$ by induction hypothesis.
(2) Trivial.
□

To put our work more into perspective, we repeat now material from Gabbay and Schlechta (2009b). This gives the main definitions and rules for nonmonotonic logics; see Tables 2.1 and 2.2, "Logical rules, definitions and connections".

Definition 2.2.6
The definitions are given in Table 2.1, "Logical rules, definitions and connections Part I" and 2.2, "Logical rules, definitions and connections Part II", which also show connections between different versions of rules, the semantics, and rules about size. (The tables are split in two, as they would not fit onto one page otherwise.)

Explanation of the tables:

(1) The first table gives the basic properties, the second table those for Cumulativity and Rational Monotony.
(2) The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e. set of formulas) version.
(3) "Corr." stands for "Correspondence".'
(4) The third column, "Corr.", is to be understood as follows:
Let a logic $\vdash\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined and satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.
If $\vdash\sim$ satisfies a rule on the left-hand side, then — provided the additional properties noted in the middle for $\Rightarrow$ hold, too — f will satisfy the property on the right-hand side.
Conversely:
If $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\vdash\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\vdash\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.
If f satisfies a property on the right-hand side, then — provided the additional properties noted in the middle for $\Leftarrow$ hold, too — $\vdash\sim$ will satisfy the property on the left-hand side.

(5) We use the following abbreviations for those supplementary conditions in the "Correspondence" columns:
"$T = \phi$" means that if one of the theories (the one named the same way in Definition 2.2.6) is equivalent to a formula, we do not need (μdp). $-(\mu dp)$ stands for "without (μdp)".

Table 2.1 Logical rules, definitions and connections Part I

Logical rule		Corr.	Model set	Corr.	Size Rules
Basics					
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\sim \beta$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq)$ $f(X) \subseteq X$	trivial	(Opt)
(REF) Reflexivity $T \cup \{\alpha\} \mid\sim \alpha$					
(LLE) Left logical equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\sim \beta \Rightarrow$ $\alpha' \mid\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$		(trivally true)		
(RW) Right weakening $\alpha \mid\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $\alpha \mid\sim \beta'$	(RW) $T \mid\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $T \mid\sim \beta'$		(upward closure)	trivial	(iM)
(wOR) $\alpha \mid\sim \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$	$\Leftrightarrow$	$(eM\mathcal{I})$
$(disjOR)$ $\alpha \vdash \neg\alpha', \alpha \mid\sim \beta,$ $\alpha' \mid\sim \beta \Rightarrow \alpha \vee \alpha' \mid\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(I \cup disj)$
(CP) Consistency preservation $\alpha \mid\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ $\Leftarrow$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I_1)
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I_1)
	(AND_1) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$				(I_2)

	(AND_n) $\alpha \vdash \beta_1, \ldots, \alpha \vdash \beta_{n-1} \Rightarrow$ $\alpha \not\vdash (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$				(I_n)
(AND) $\alpha \vdash \beta, \alpha \vdash \beta' \Rightarrow$ $\alpha \vdash \beta \wedge \beta'$	(AND) $T \vdash \beta, T \vdash \beta' \Rightarrow$ $T \vdash \beta \wedge \beta'$		(closure under finite intersection)	trivial	(I_ω)
(CCL) Classical closure	(CCL) $\overline{\overline{T}}$ classically closed		(trivially true)	trivial	$(iM) + (I_\omega)$
(OR) $\alpha \vdash \beta, \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \vdash \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\alpha}} \cup \{\alpha'\}\}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T}} \cup T'$	$\Rightarrow$ $\Leftarrow (\mu dp) + (\mu \subseteq)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow (\mu \subseteq)$ $T' = \phi$ $\Leftarrow$ $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$ $(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
(CUT) $T \vdash \alpha; T \cup \{\alpha\} \vdash \beta \Rightarrow$ $T \vdash \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega)$

Table 2.2 Logical rules, definitions and connections Part II

Logical rules, definitions and connections Part II					
Logical rule		Corr.	Model set	Corr.	Size-rule
Cumulativity					
(wCM) $\alpha \vdash \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \vdash \beta$				trivial	$(eM\mathcal{F})$
(CM_2) $\alpha \vdash \beta, \alpha \vdash \beta' \Rightarrow \alpha \wedge \beta \nvdash \neg\beta'$					(I_2)
(CM_n) $\alpha \vdash \beta_1, \ldots, \alpha \vdash \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \nvdash \neg\beta_n$					(I_n)
(CM) Cautious monotony $\alpha \vdash \beta, \alpha \vdash \beta' \Rightarrow$ $\alpha \wedge \beta \vdash \beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	$\Leftrightarrow$	$(\mathcal{M}_\omega^+)(4)$
or $(ResM)$ Restricted monotony $T \vdash \alpha, \beta \Rightarrow T \cup \{\alpha\} \vdash \beta$		$\Rightarrow$ $\Leftarrow$	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\alpha \vdash \beta \Rightarrow$ $(\alpha \vdash \beta' \Leftrightarrow \alpha \wedge \beta \vdash \beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}_\omega^+)(4)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$

Rationality					
$(RatM)$ Rational monotony $\alpha \vdash \beta, \alpha \not\vdash \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \vdash \beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	$\Leftrightarrow$	$(\mathcal{M}^{++})$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{T'} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
(DR) $\alpha \vee \beta \vdash \gamma \Rightarrow$ $\alpha \vdash \gamma$ or $\beta \vdash \gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ $\Leftarrow$	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$		
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X . a \notin f(\{a, b\})$		

(6) $A = B \parallel C$ will abbreviate $A = B$, or $A = C$, or $A = B \cup C$.

Further comments:

(1) (PR) is also called $infinite\ conditionalization$; We choose this name for its central role for preferential structures (PR) or (μPR).
(2) The system of rules (AND) (OR) (LLE) (RW) (SC) (CP) (CM) (CUM) is also called system P (for preferential). Adding $(RatM)$ gives the system R (for rationality or rankedness).
Roughly, smooth preferential structures generate logics satisfying system P, while ranked structures generate logics satisfying system R.

(3) A logic satisfying (REF), $(ResM)$, and (CUT) is called a $consequence\ relation$.
(4) (LLE) and (CCL) will hold automatically, whenever we work with model sets.
(5) (AND) is obviously closely related to filters, and corresponds to closure under finite intersections. (RW) corresponds to upward closure of filters.
More precisely, validity of both depend on the definition, and the direction we consider.
Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a principal filter, $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition, if $X = M(T)$, then $T \ \mid\sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.
Conversely, we can define for $X = M(T)$
$\mathcal{X} := \{X' \subseteq X : \exists \phi(X' = X \cap M(\phi)$ and $T \ \mid\sim \phi)\}$.
(AND) then makes $\mathcal{X}$ closed under finite intersections, and (RW) makes $\mathcal{X}$ upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete $\mathcal{X}$ by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.
Alternatively, we can define
$\mathcal{X} := \{X' \subseteq X : \bigcap\{X \cap M(\phi) : T \ \mid\sim \phi\} \subseteq X'\}$.

(6) (SC) corresponds to the choice of a subset.
(7) (CP) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.
(8) (PR) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \ \mid\sim \sigma$, so $\phi \ \mid\sim \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.
(9) (CUM) (whose more interesting half in our context is (CM)) may best be seen as a normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones from holding. (This is, of course, a meta-level argument concerning an object-level rule. But also object-level rules should — at least generally — have an intuitive justification, which will then come from a meta-level argument.)



http://www.springer.com/978-3-319-46815-0

A New Perspective on Nonmonotonic Logics

Gabbay, D.; Schlechta, K.

2016, X, 365 p. 80 illus., Hardcover
ISBN: 978-3-319-46815-0