CHAPTER 2

Preliminaries

2.1. Basic examples of mod-convergence

Let us give a few examples of mod-ϕ convergence, which will guide our intuition throughout the book. In these examples, it will be useful sometimes to precise the speed of convergence in Definition 1.1.1.

DEFINITION 2.1.1. *We say that the sequence* $(X_n)_{n\in\mathbb{N}}$ *converges mod-*ϕ *at speed* $O((t_n)^{-v})$ *if the difference of the two sides of Equation* (1.1) *can be bounded by* $C_K\,(t_n)^{-v}$ *for any* z *in a given compact subset* K *of* $\mathcal{S}_{(c,d)}$*. We use the analogous definition with the* $o(\cdot)$ *notation.*

EXAMPLE 2.1.2. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of centered, independent and identically distributed real-valued random variables, with $\mathbb{E}[e^{zY}] = \mathbb{E}[e^{zY_1}]$ analytic and non-vanishing on a strip $\mathcal{S}_{(c,d)}$, possibly with $c = -\infty$ and/or $d = +\infty$. Set $S_n = Y_1 + \cdots + Y_n$. If the distribution of Y is infinitely divisible, then S_n converges mod-Y towards the limiting function $\psi \equiv 1$ with parameter $t_n = n$.

But there is another mod-convergence hidden in this framework (we now drop the assumption of infinite divisibility of the law of Y). The cumulant generating series of S_n is

$$\log \mathbb{E}[e^{zS_n}] = n\,\log \mathbb{E}[e^{zY}] = n\sum_{r=2}^{\infty} \frac{\kappa^{(r)}(Y)}{r!}\,z^r,$$

which is also analytic on $\mathcal{S}_{(c,d)}$ — the coefficients $\kappa^{(r)}(Y)$ are the *cumulants* of the variable Y (see [FW32]). Let $v \geq 3$ be an integer such that $\kappa^{(r)}(Y) = 0$ for each integer r strictly between 3 and $v-1$, and set $X_n = \frac{S_n}{n^{1/v}}$. It is always possible to take $v = 3$, but sometimes we can also consider higher values of v, for instance $v = 4$ as soon as Y is a symmetric random variable, and has therefore its odd moments

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and cumulants that vanish. One has

$$\log \varphi_n(z) = n^{\frac{v-2}{v}} \frac{\kappa^{(2)}(Y)}{2} z^2 + \frac{\kappa^{(v)}(Y)}{v!} z^v + \sum_{r=v+1}^{\infty} \frac{\kappa^{(r)}(Y)}{r!\, n^{\frac{r}{v}-1}} z^r,$$

and locally uniformly on $\mathbb{C}$ the right-most term is bounded by $\frac{C}{n^{1/v}}$. Consequently,

$$\psi_n(z) = \exp\left(-n^{\frac{v-2}{v}} \frac{\sigma^2 z^2}{2}\right) \varphi_n(z) \to \exp\left(\frac{\kappa^{(v)}(Y)}{v!} z^v\right) + O(n^{-1/v}),$$

that is, $(X_n)_{n\in\mathbb{N}}$ converges in the mod-Gaussian sense with parameters $t_n = \sigma^2 n^{(v-2)/v}$, speed $O(n^{-1/v})$ and limiting function $\psi(z) = \exp(\kappa^{(v)}(Y)\, z^v / v!)$. Note that this first example was used in [KNN15] to characterise the set of limiting functions in the setting of mod-ϕ convergence.

Through this chapter, we shall commonly rescale random variables in order to get estimates of fluctuations at different regimes. In order to avoid any confusion, we provide the reader with the following scheme, which details each possible scaling, and for each scaling, the regimes of fluctuations that can be deduced from the mod-ϕ convergence, as well as their scope. We also underline or frame the scalings and regimes that will be studied in this paper, and give references for the other kinds of fluctuations (Fig. 2.1).

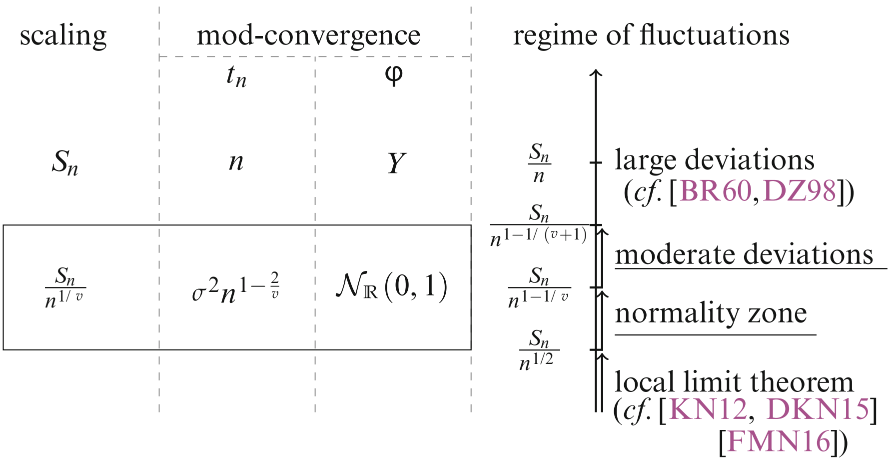

FIGURE 2.1. Panorama of the fluctuations of a sum of n i.i.d. random variables.

The content of this scheme will be fully explained in Chapter 4 (see in particular Section 4.4).

EXAMPLE 2.1.3. Denote X_n the number of disjoint cycles (including fixed points) of a random permutation that is chosen uniformly in the symmetric group $\mathfrak{S}(n)$. Feller's coupling (*cf.* [ABT03, Chapter 1]) shows that $X_n =_{(\mathrm{law})} \sum_{i=1}^{n} \mathcal{B}_{(1/i)}$, where $\mathcal{B}_p$ denotes a Bernoulli variable equal to 1 with probability p and to 0 with probability $1-p$, and the Bernoulli variables are independent in the previous expansion. So,

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \prod_{i=1}^{n} \left(1 + \frac{\mathrm{e}^z - 1}{i}\right) = \mathrm{e}^{H_n(\mathrm{e}^z-1)} \prod_{i=1}^{n} \frac{1 + \frac{\mathrm{e}^z-1}{i}}{\mathrm{e}^{\frac{\mathrm{e}^z-1}{i}}}$$

where $H_n = \sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + O(n^{-1})$. The Weierstrass infinite product in the right-hand side converges locally uniformly to an entire function, therefore (see [WW27]),

$$\mathbb{E}[\mathrm{e}^{zX_n}]\, \mathrm{e}^{-(\mathrm{e}^z-1)\log n} \to \mathrm{e}^{\gamma\,(\mathrm{e}^z-1)} \prod_{i=1}^{\infty} \frac{1 + \frac{\mathrm{e}^z-1}{i}}{\mathrm{e}^{\frac{\mathrm{e}^z-1}{i}}} = \frac{1}{\Gamma(\mathrm{e}^z)}$$

locally uniformly, *i.e.* one has mod-Poisson convergence with parameters $t_n = \log n$ and limiting function $1/\Gamma(\mathrm{e}^z)$. Moreover, the speed of convergence is a $O(n^{-1})$, hence, a $o((t_n)^{-v})$ for any integer v. We shall study generalisations of this example in Section 7.3.

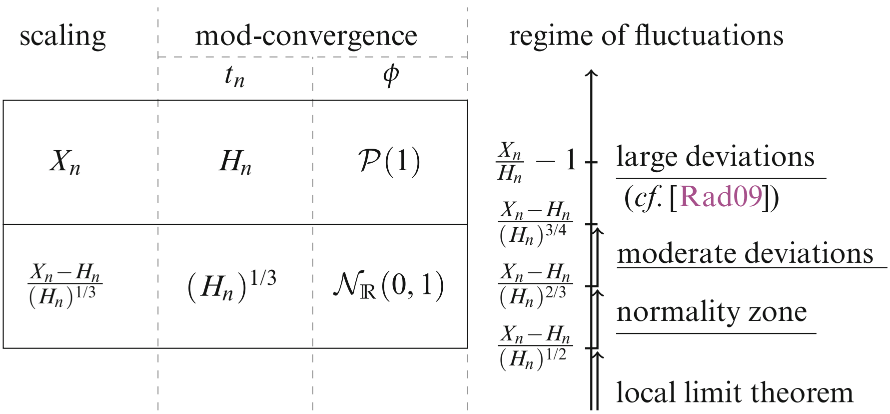

FIGURE 2.2. Panorama of the fluctuations of the number of cycles X_n of a random permutation of size n.

Once again, there is another mod-convergence hidden in this example (Fig. 2.2). Indeed, consider $Y_n = \frac{X_n - H_n}{(H_n)^{1/3}}$. Its generating function has asymptotics

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{zY_n}] &= \mathrm{e}^{H_n\left(\mathrm{e}^{\frac{z}{(H_n)^{1/3}}}-1\right)-z(H_n)^{2/3}}\,(1+o(1))\\ &= \mathrm{e}^{(H_n)^{1/3}\,\frac{z^2}{2}}\,\exp\left(\frac{z^3}{6}\right)\,(1+o(1)).\end{aligned}$$

Therefore, one has mod-Gaussian convergence of Y_n with parameters $t_n = (H_n)^{1/3}$ and limiting function $\exp(z^3/6)$.

This is in fact a particular case of a more general phenomenon: every sequence that converges mod-ϕ converges with a different rescaling in the mod-Gaussian sense.

PROPOSITION 2.1.4. *Assume X_n converges mod-ϕ with parameters t_n and limiting function ψ, where ϕ is not the Gaussian distribution. Let*

$$m = \min_{i\geq 3}\{i \mid \eta^{(i)}(0) \neq 0\}.$$

Then, the sequence of random variables $Y_n = (X_n - t_n\eta'(0))/(t_n)^{1/m}$ converges in the mod-Gaussian sense with parameters $(t_n)^{1-2/m}\eta''(0)$ towards the limiting function $\Psi(z) = \exp\left(\eta^{(m)}(0)z^m/m!\right)$.

PROOF. This follows from a simple computation

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{z(X_n - t_n\eta'(0))}{(t_n)^{1/m}}\right)\right]\\ &= \exp\left(\frac{-t_n\eta'(0))}{(t_n)^{1/m}}\right)\,\exp\left(t_n\,\eta\left(\frac{z}{(t_n)^{1/m}}\right)\right)\,\psi\left(\frac{z}{(t_n)^{1/m}}\right)\,(1+o(1)).\end{aligned}$$

The factor $\psi(\frac{z}{(t_n)^{1/m}})$ tends to 1 and we do a Taylor expansion of $\eta(\frac{z}{(t_n)^{1/m}})$. We get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{z(X_n - t_n\eta'(0))}{(t_n)^{1/m}}\right)\right]\\ &= \exp\left((t_n)^{1-2/m}\eta''(0)\frac{z^2}{2} + \eta^{(m)}(0)\frac{z^m}{m!} + o(1)\right)(1+o(1)). \qquad \square\end{aligned}$$

Naturally, the mod-ϕ convergence gives more information than the implied mod-Gaussian convergence: our deviation results — see Theorems 3.2.2 and 4.2.1 — for the former involve deviation probabilities of X_n at scale $O(t_n)$, while with the mod-Gaussian convergence, we get deviation probabilities of Y_n at scale $O((t_n)^{1-2/m})$, that is deviations of X_n at scale $O((t_n)^{1-1/m})$.

2.2. Legendre-Fenchel transforms

We now present the definition and some simple properties of the Legendre-Fenchel transform, a classical tool in large deviation theory (see *e.g.* [DZ98, Section 2.2]) that we shall use a lot in this paper. The Legendre-Fenchel transform is the following operation on (convex) functions:

DEFINITION 2.2.1. *The Legendre-Fenchel transform of a function η is defined by:*

$$F(x) = \sup_{h \in \mathbb{R}} (hx - \eta(h)).$$

This is an involution on convex lower semi-continuous functions.

Assume that η is the logarithm of the moment generating series of a random variable. In this case, by Hölder's inequality, η is a convex function (Fig. 2.3). Then F is always non-negative, and the unique h maximizing $hx - \eta(h)$, if it exists, is then defined by the implicit equation $\eta'(h) = x$ (note that h depends on x, but we have chosen not to write $h(x)$ to make notation lighter). This implies the following useful identities:

$$F(x) = xh - \eta(h) \quad ; \quad F'(x) = h \quad ; \quad F''(x) = h'(x) = \frac{1}{\eta''(h)}.$$

EXAMPLE 2.2.2. If $\eta(z) = mz + \frac{\sigma^2 z^2}{2}$ (Gaussian variable with mean m and variance σ^2), then

$$h = \frac{x-m}{\sigma^2} \qquad ; \qquad F_{\mathcal{N}_{\mathbb{R}}(m,\sigma^2)}(x) = \frac{(x-m)^2}{2\sigma^2}$$

whereas if $\eta(z) = \lambda(e^z - 1)$ (Poisson law with parameter λ), then

$$h = \log \frac{x}{\lambda} \qquad ; \qquad F_{\mathcal{P}(\lambda)}(x) = \begin{cases} x \log \frac{x}{\lambda} - (x - \lambda) & \text{if } x > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

FIGURE 2.3. The Legendre-Fenchel transforms of a Gaussian law and of a Poisson law.

2.3. Gaussian integrals

Some computations involving the Gaussian density are used several times throughout the paper, so we decided to present them together here.

LEMMA 2.3.1 (Gaussian integrals).

(1) moments:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2}}\, x^{2m}\, dx = (2m-1)!! = (2m-1)(2m-3)\cdots 3\,1,$$

and the odd moments vanish.

(2) Fourier transform: with $g(x) = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$*, one has*

$$g^*(\zeta) = \int_{\mathbb{R}} g(x)\, \mathrm{e}^{\mathrm{i}x\zeta}\, dx = \mathrm{e}^{-\frac{\zeta^2}{2}}.$$

More generally, with the Hermite polynomials defined by

$$H_r(x) = (-1)^r\, \mathrm{e}^{\frac{x^2}{2}}\, \frac{\partial^r}{\partial x^r}(\mathrm{e}^{-\frac{x^2}{2}}),$$

one has

$$(g\, H_r)^*(\zeta) = (\mathrm{i}\zeta)^r\, \mathrm{e}^{-\frac{\zeta^2}{2}}.$$

(3) tails: if $a \to +\infty$*, then*

$$\int_0^\infty \mathrm{e}^{-\frac{(y+a)^2}{2}}\, dy = \frac{\mathrm{e}^{-\frac{a^2}{2}}}{a}\left(1 - \frac{1}{a^2} + O\left(\frac{1}{a^4}\right)\right)$$

$$\int_0^\infty y\, \mathrm{e}^{-\frac{(y+a)^2}{2}}\, dy = \frac{\mathrm{e}^{-\frac{a^2}{2}}}{a^2}\left(1 + O\left(\frac{1}{a^2}\right)\right)$$

$$\int_0^\infty y^2\, \mathrm{e}^{-\frac{(y+a)^2}{2}}\, dy = O\left(\frac{\mathrm{e}^{-\frac{a^2}{2}}}{a^3}\right)$$

$$\int_0^\infty y^3\, \mathrm{e}^{-\frac{(y+a)^2}{2}}\, dy = O\left(\frac{\mathrm{e}^{-\frac{a^2}{2}}}{a^2}\right)$$

In particular, the tail of the standard Gaussian distribution is

$$\frac{1}{\sqrt{2\pi}}\int_a^\infty \mathrm{e}^{-\frac{x^2}{2}}\, dx \simeq \frac{1}{a\sqrt{2\pi}}\, \mathrm{e}^{-\frac{a^2}{2}}.$$

(4) complex transform: for $\beta > 0$,

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{\beta^2}{2}}}{2\pi} \frac{\mathrm{e}^{-\frac{w^2}{2}}}{\beta + \mathrm{i}w}\,dw = \int_{\beta}^{\infty} \frac{\mathrm{e}^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\,d\alpha = \mathbb{P}[\mathcal{N}_{\mathbb{R}}(0,1) \geq \beta].$$

PROOF. Recall that the generating series of Hermite polynomials ([Sze75, Chapter 5]) is

$$\sum_{r=0}^{\infty} H_r(x)\,\frac{t^r}{r!} = \mathrm{e}^{\frac{x^2}{2}} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \frac{\partial^r}{\partial x^r}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) = \mathrm{e}^{\frac{x^2}{2}}\,\mathrm{e}^{-\frac{(x-t)^2}{2}} = \mathrm{e}^{-\frac{t^2}{2}+tx}.$$

Integrating against $g(x)\,\mathrm{e}^{\mathrm{i}x\zeta}\,dx$ yields

$$\begin{aligned}\sum_{r=0}^{\infty}(g(x)\,H_r(x))^*(\zeta)\,\frac{t^r}{r!} &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-t)^2}{2}+\mathrm{i}\zeta x}\,dx \\ &= \frac{\mathrm{e}^{\mathrm{i}\zeta t}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{y^2}{2}+\mathrm{i}\zeta y}\,dy = \mathrm{e}^{\mathrm{i}\zeta t-\frac{\zeta^2}{2}} \\ &= \sum_{r=0}^{\infty}(\mathrm{i}\zeta)^r \mathrm{e}^{-\frac{\zeta^2}{2}}\,\frac{t^r}{r!}\end{aligned}$$

whence the identity (2) for Fourier transforms.

With $r = 0$, one gets the Fourier transform of the Gaussian distribution $g^*(\zeta) = \mathrm{e}^{-\frac{\zeta^2}{2}}$, hence the moments (1) by derivation at $\zeta = 0$. The estimate of tails (3) is obtained by an integration by parts; notice that similar techniques yield the tails of distributions $x^m\,\mathrm{e}^{-x^2/2}\,dx$ with $m \geq 1$. Finally, as for the complex transform (4), remark that

$$F(\beta) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{\beta^2}{2}}}{2\pi} \frac{\mathrm{e}^{-\frac{w^2}{2}}}{\beta + \mathrm{i}w}\,dw = \frac{1}{2\mathrm{i}\pi}\int_{\Gamma=\beta+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{\frac{(z-\beta)^2-\beta^2}{2}}}{z}\,dz,$$

the second integral being along the complex curve $\Gamma = \beta + \mathrm{i}\mathbb{R}$. By standard complex analysis arguments, this integral is the same along any line $\Gamma' = \beta' + i\mathbb{R}$ (for $\beta' > 0$). Namely

$$F(\beta) = \frac{1}{2\mathrm{i}\pi}\int_{\Gamma'=\beta'+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{\frac{(z-\beta)^2-\beta^2}{2}}}{z}\,dz.$$

Since $\lim_{\beta\to+\infty} F(\beta) = 0$,

$$F(\beta) = -\int_{\beta}^{\infty} F'(\alpha)\,d\alpha = \int_{\beta}^{\infty}\left(\frac{1}{2\mathrm{i}\pi}\int_{\Gamma'=\beta'+\mathrm{i}\mathbb{R}} \mathrm{e}^{\frac{(z-\alpha)^2-\alpha^2}{2}}\,dz\right)d\alpha.$$

Again, the integration line Γ' in the second integral can be replaced by $\Gamma = \alpha + \mathrm{i}\mathbb{R}$ and we get

$$F(\beta) = \int_{\beta}^{\infty} \left(\frac{1}{2\mathrm{i}\pi} \int_{\Gamma=\alpha+\mathrm{i}\mathbb{R}} \mathrm{e}^{\frac{(z-\alpha)^2-\alpha^2}{2}}\, dz \right) d\alpha. = \int_{\beta}^{\infty} \frac{\mathrm{e}^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\, d\alpha,$$

which is the tail $\mathbb{P}[\mathcal{N}_{\mathbb{R}}(0,1) \geq \beta]$ of a standard Gaussian law. $\square$

Also, there will be several instances of the Laplace method for asymptotics of integrals, but each time in a different setting; so we found it more convenient to reprove it each time.

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