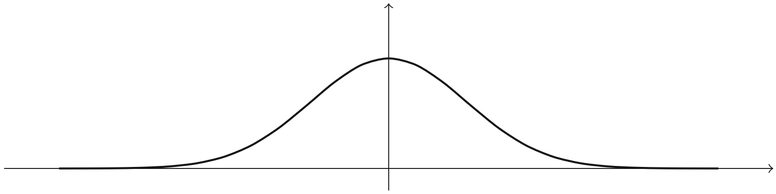
Preface

The central limit theorem (see [dM56, Lap40, Lya00, Lya01, Lin22, Lév25]) for independent and identically distributed (i.i.d.) random variables is one of the most ubiquitous theorems in probability theory: suitably renormalised, the sum of such variables converges towards a standard normal variable.



We refer to [Fis11] for an historical account of this classical result. Roughly speaking, it shows the universality of fluctuations around the first-order limit given by the law of large numbers. Although independence is often used as a first approximation, many natural phenomena exhibit some complex dependency structure. Therefore, a large body of literature is devoted to relaxing the independence hypothesis. Thus, central limit theorems have been given for random variables with an underlying structure such as martingales ([HH80, Chapter 3]), Markov chains (see, *e.g.* [Cog72, IL71, GL78], and [Jon04] for a survey), mixing sequences (see [Ros56, Phi69, Dav73, LL96], and [Bra05] for a survey on mixing conditions), lattice models ([GJL75, EN78, New80]), m-dependence (*cf.* [RH48, Dia55, Ber73]), dependency graphs ([PL83, Jan88, BR89, Mik91]), exchangeable pairs ([BC05, CGS11, Ros11]) and determinantal point processes (*cf.* [HKPV09, Section 4.6]).

Concretely, the central limit theorem for X_n expresses the limit of the tail probability when t is a fixed real number:

$$\lim_{n\to\infty} \mathbb{P}[X_n - \mathbb{E}[X_n] \geq t\sigma_n] = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} \mathrm{e}^{-\frac{s^2}{2}}\, ds. \tag{1}$$

Such estimates are crucial in statistics to build confidence intervals. The convergence in law towards a Gaussian distribution is often complemented by other asymptotic results:

- the *speed of convergence* (uniformly in t) in Equation (1) (see [Ber41] and [Fel71, Chapter XVI]);
- the behaviour of the left-hand side of Eq. (1) when t tends to infinity together with n (*moderate and large deviation results*, see [DZ98]);
- a subsequent question is how fast can t grow, so that the limit given in Eq. (1) is still valid (*normality zone*);
- in another direction, *local limit theorems* describe the probability of $X_n - \mathbb{E}[X_n]$ to be in an interval of constant scale (see [Gne48]).

The canonical way to establish the central limit theorem for i.i.d. random variables is to use characteristic functions and Lévy's continuity theorem. This is also used in some of the above-mentioned extensions, in addition to other techniques, mainly based on the method of moments and Stein's method (see for instance the first chapters of [Str11]).

In this monograph, we focus on the characteristic function approach, for which we propose a *renormalisation theory* — called mod-ϕ convergence. If the characteristic function converges after a suitable renormalisation, we prove some precise moderate and large deviation results, which enables us to describe the normality zone. Results for the speed of convergence and local limit theorems will be discussed in a companion work [FMN16]. The idea of using estimates on characteristic functions to obtain central limit theorems and deviation probabilities is of course not new, but we provide here a general framework, together with many examples. These examples come from various mathematical fields: classical probability theory, number theory (statistics of additive arithmetic functions), combinatorics (statistics of random permutations), random matrix theory (characteristic polynomials of random matrices in compact Lie groups), graph theory (number of subgraphs in a random Erdős-Rényi graph) and non-commutative probability theory (asymptotics of random character values of symmetric groups).

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