

Chapter 2

Ergodicity for Functional Stochastic Equations Without Dissipativity

Dealing with diffusions with “pure delay” in which both the drift and the diffusion coefficients depend only on the arguments with delays, most of the existing results are not applicable. This chapter uses variation-of-constants formulae to overcome the difficulties due to the lack of the information at the current time, and establishes existence and uniqueness of invariant measures for functional stochastic equations that need not satisfy dissipative conditions. The functional stochastic equations considered in this chapter cover FSDEs of neutral type, FSPDEs driven by cylindrical Wiener processes, and FSDEs driven by Lévy processes without finite second moments. Sections 2.1–2.3 and 2.5 of this chapter are mainly based on the results in [13].

2.1 Introduction

One of the main approaches in the literature to date is to incorporate certain dissipativity to establish existence of invariant measures for functional stochastic equations. The dissipativity is normally assured by imposing conditions of the current time with certain decay conditions. Such an idea has been used extensively. For instance, utilizing the remote method (i.e., the dissipative method), Bao et al. [12] treated several classes of functional stochastic equations; applying an exponential-type estimate, Bo-Yuan [21] investigated stochastic differential delay equations (SDDEs) driven by Poisson jump processes; adopting the Arzelà–Ascoli tightness characterization, Es-Sarhir et al. [54] and Kinnally-Williams [79] considered FSDEs with super-linear drift terms and positivity constraints, respectively. Nevertheless, the existing literature is unable to deal with the following seemingly simple linear SDDE on the real line $\mathbb{R}$,

$$dX(t) = -X(t-1)dt + \sigma X(t-1)dW(t), \quad X_0 = \zeta, \quad (2.1)$$

where $\sigma \in \mathbb{R}$ and $(W(t))_{t \geq 0}$ is a real-valued standard Brownian motion. Observe that it is impossible to choose constants $\lambda_1 > \lambda_2 > 0$ such that

$$-2xy + \sigma^2 y^2 \leq -\lambda_1 x^2 + \lambda_2 y^2, \quad \forall x, y \in \mathbb{R} \quad (2.2)$$

holds even for some $\sigma \in \mathbb{R}$ sufficiently small. Indeed, if (2.2) holds, then

$$\lambda_1 \leq \beta \left(\frac{y}{x} + \frac{1}{\beta} \right)^2 - \frac{1}{\beta}, \quad x \neq 0, \quad (2.3)$$

where $\beta := \lambda_2 - \sigma^2 > 0$. It is readily seen that (2.3) no longer holds provided that $-\frac{2}{\beta} \leq \frac{y}{x} \leq 0$. That is, (2.1) does not obey a dissipative condition. Therefore, the techniques used in [12, 21, 54, 79] are not applicable to (2.1). We remark that the main problem is due to the lack of the information at the current time, so no dissipative conditions can be used.

In the well-known work [153], Yorke treated deterministic systems with pure delays. His work has stimulated much of the subsequent work resulting in a vast literature on the pure delay equations in the deterministic setup; see also Wu et al. [144] for a substantial extension to systems with Markov switching. Our consideration in this chapter is a generalization of the model in [153] in which both the drift and diffusion coefficients may involve only retarded elements. The right-hand sides of such differential equations may not involve information on the current time. Consequently, it is impossible to use any dissipative conditions.

In this chapter, we aim to obtain existence and uniqueness of invariant measures for several classes of functional stochastic equations, which include (2.1) as a special case. The crucial point is that we focus on functional stochastic systems without satisfying dissipative conditions. To overcome the difficulties, we use the variation-of-constants formulae, which have been applied successfully in [5, 60, 95, 119]. More precisely, [5] examined the exact almost sure growth rate of the running maximum of solutions for affine FSDEs, whereas in [60] and [95], the authors considered stationary solutions (see Remark 2.7) for finite-dimensional and infinite-dimensional retarded Ornstein–Uhlenbeck (O–U) processes, respectively. It is also worth pointing out that the variation-of-constants formula together with the semi-martingale characteristics has been utilized to study existence of invariant measures for a class of FSDEs driven by jump processes [119], which do not include (2.1), however.

The remainder of this chapter is organized as follows. Section 2.2 is devoted to existence and uniqueness of invariant measures for semi-linear FSDEs driven by Brownian motions using the Arzelà–Ascoli type tightness characterization. Sections 2.3 and 2.4 extend the main result derived in Section 2.2 to FSDEs of neutral type and FSPDEs driven by cylindrical Wiener processes, respectively. Section 2.5 focuses on existence and uniqueness of invariant measures for semi-linear FSDEs driven by Lévy processes, which need not have finite second moments. The key is to use the variation-of-constants formula together with the tightness criterion due to Kurtz [55].

2.2 Ergodicity for FSDEs Driven by Brownian Motions

Let μ be an $\mathbb{R}^n \otimes \mathbb{R}^n$ -valued finite signed measure on $[-\tau, 0]$; that is, $\mu = (\mu_{ij})_{1 \leq i, j \leq n}$, where each μ_{ij} is a finite signed measure on $[-\tau, 0]$. Consider the following semi-linear FSDE

$$dX(t) = \left(\int_{[-\tau, 0]} \mu(d\theta) X(t + \theta) \right) dt + \sigma(X_t) dW(t), \quad t > 0, \quad X_0 = \xi. \quad (2.4)$$

Herein, $\sigma : \mathcal{C} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ satisfies (H2) in Chapter 1. By virtue of [99, Theorem 2.2, p. 150], (2.4) admits a unique strong solution $(X(t; \xi))_{t \geq -\tau}$ with the initial data $X_0 = \xi \in \mathcal{C}$. In what follows, we often write $X(t)$ and X_t in lieu of $X(t; \xi)$ and $X_t(\xi)$, respectively, for notational simplicity.

Let $\mathbb{C}$ be the set of all complex numbers and $\text{Re}(z)$ the real part of $z \in \mathbb{C}$. Set

$$v_\mu := \sup\{\text{Re}(\lambda) : \lambda \in \mathbb{C}, \Delta_\mu(\lambda) = 0\}. \quad (2.5)$$

Herein,

$$\Delta_\mu(\lambda) := \det \left(\lambda I_{n \times n} - \int_{[-\tau, 0]} e^{\lambda s} \mu(ds) \right),$$

where $\det(A)$ denotes the determinant of $A \in \mathbb{R}^n \otimes \mathbb{R}^n$, and $I_{n \times n} \in \mathbb{R}^n \otimes \mathbb{R}^n$ is the unitary matrix. $\Delta_\mu(\lambda) = 0$ is called the characteristic equation associated with the deterministic counterpart of (2.4) (i.e., $\sigma \equiv \mathbf{0}_{n \times m}$ therein, where $\mathbf{0}_{n \times m}$ is the $n \times m$ zero matrix).

In particular, when $\mu = A\delta_0$ with $A \in \mathbb{R}^n \otimes \mathbb{R}^n$ and δ_0 being the Dirac delta measure at the point 0, equation (2.4) reduces to the semi-linear FSDE:

$$dX(t) = AX(t)dt + \sigma(X_t)dW(t), \quad t > 0, \quad X_0 = \xi,$$

and v_μ is the largest real part of eigenvalues of A . Suppose that $(\Gamma(t))_{t \geq 0}$ solves the following $\mathbb{R}^n \otimes \mathbb{R}^n$ -valued equation

$$d\Gamma(t) = \left(\int_{[-\tau, 0]} \mu(d\theta) \Gamma(t + \theta) \right) dt \quad (2.6)$$

with the initial data $\Gamma(0) = I_{n \times n}$, $\Gamma(\theta) = \mathbf{0}_{n \times n}$ for $\theta \in [-\tau, 0)$. For more properties of Γ , please refer to Theorem B.3. The following lemma provides a variation-of-constants formula for (2.4).

Lemma 2.1 *Equation (2.4) admits a unique strong solution $(X(t; \xi))_{t \geq -\tau}$, which can be represented by*

$$\begin{aligned}
X(t; \xi) &= \Gamma(t)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t + \theta - s)\xi(s)ds \\
&\quad + \int_0^t \Gamma(t - s)\sigma(X_s(\xi))dW(s), \quad t > 0.
\end{aligned} \tag{2.7}$$

Proof Let $t \geq 0$ be fixed. By the chain rule, it follows that

$$d(\Gamma(t - s)X(s)) = d(\Gamma(t - s))X(s) + \Gamma(t - s)dX(s), \quad s \in [0, t].$$

Integrating from 0 to t , we deduce from (2.4) and (2.6) that

$$\begin{aligned}
&X(t) - \Gamma(t)\xi(0) \\
&= - \int_0^t \left(\int_{[-\tau, 0]} \mu(d\theta) \Gamma(t - s + \theta) \right) X(s)ds \\
&\quad + \int_0^t \Gamma(t - s) \left(\int_{[-\tau, 0]} \mu(d\theta) X(s + \theta) \right) ds + \int_0^t \Gamma(t - s)\sigma(X_s)dW(s) \\
&= - \int_{[-\tau, 0]} \mu(d\theta) \int_0^t \Gamma(t - s + \theta)X(s)ds \\
&\quad + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^{t+\theta} \Gamma(t - s + \theta)X(s)ds + \int_0^t \Gamma(t - s)\sigma(X_s)dW(s) \\
&= \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t - s + \theta)X(s)ds - \int_{[-\tau, 0]} \mu(d\theta) \int_{t+\theta}^t \Gamma(t - s + \theta)X(s)ds \\
&\quad + \int_0^t \Gamma(t - s)\sigma(X_s)dW(s) \\
&= \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t + \theta - s)\xi(s)ds + \int_0^t \Gamma(t - s)\sigma(X_s)dW(s),
\end{aligned}$$

where in the second step we have used substitution, and in the last two steps utilized additive property of integral together with the observation $\Gamma(0) = I_{n \times n}$, $\Gamma(\theta) = \mathbf{0}_{n \times n}$, $\theta \in [-\tau, 0]$. Thus, (2.7) follows.

Remark 2.2 With regard to the variation-of-constants formula for semi-linear FSDEs driven by a general semi-martingale, we refer to Reiß et al. [118]. While, in Lemma 2.1, we provide an alternative proof for the variation-of-constants formula (2.7) to make the content self-contained.

In what follows, we assume $v_{\mu} < 0$. By Theorem B.3, for any $\lambda \in (0, -v_{\mu})$, there exists a constant $c_{\lambda} > 0$ such that

$$\|\Gamma(t)\| \leq c_{\lambda} e^{-\lambda t}, \quad t \geq -\tau, \tag{2.8}$$

where $\|\Gamma(t)\|$ denotes the operator norm of the matrix $\Gamma(t) \in \mathbb{R}^n \otimes \mathbb{R}^n$. We remark that the optimal constant c_λ is increasing for $\lambda \in (0, -v_\mu)$. If, in particular, $\mu = A\delta_0$ for a symmetric $n \times n$ -matrix A , (2.8) holds with $c_\lambda = 1$ and $\lambda \in (0, -v_\mu]$. In general, see Theorem B.5 in the Appendix B.2 for an upper bound of c_λ .

The following lemma plays a crucial role in obtaining the long-term behavior of segment process $(X_t(\xi))_{t \geq 0}$.

Lemma 2.3 *Let $(u(t))_{t \geq 0}$ be an $\mathbb{R}^n \otimes \mathbb{R}^m$ -valued predictable process. Then, for any $p \geq 1$ and $t > 0$,*

$$\left(\mathbb{E} \left| \int_0^t \Gamma(t-s)u(s)dW(s) \right|^{2p} \right)^{1/p} \leq p(2p-1) \int_0^t (\mathbb{E} \|\Gamma(t-s)u(s)\|_{HS}^{2p})^{1/p} ds$$

provided that the integral on the right-hand side is finite for each $t > 0$.

Proof Observe that this estimate cannot be obtained directly from [36, Lemma 7.7, p. 174] due to the fact that $\int_0^t \Gamma(t-s)u(s)dW(s)$ is not a martingale. Nevertheless, some ideas of the aforementioned reference can be used. To make it self-contained, we provide an outline of the proof. Let

$$M(r) = \int_0^r \Gamma(t-s)u(s)dW(s), \quad r \in [0, t].$$

In what follows, we let $r \in [0, t]$. By Itô's formula and then Hölder's inequality, it follows that

$$\begin{aligned} \mathbb{E}|M(r)|^{2p} &\leq p(2p-1) \int_0^r \mathbb{E}(|M(s)|^{2(p-1)} \|\Gamma(t-s)u(s)\|_{HS}^2) ds \\ &\leq p(2p-1) \sup_{0 \leq s \leq r} (\mathbb{E}|M(s)|^{2p})^{(p-1)/p} \int_0^r (\mathbb{E} \|\Gamma(t-s)u(s)\|_{HS}^{2p})^{1/p} ds. \end{aligned}$$

Hence, one has

$$\sup_{0 \leq s \leq r} \mathbb{E}|M(s)|^{2p} \leq p(2p-1) \sup_{0 \leq s \leq r} (\mathbb{E}|M(s)|^{2p})^{(p-1)/p} \int_0^r (\mathbb{E} \|\Gamma(t-s)u(s)\|_{HS}^{2p})^{1/p} ds$$

owing to the fact that $\sup_{0 \leq s \leq r} (\mathbb{E}|M(s)|^{2p})^{(p-1)/p}$ is nondecreasing w.r.t. r . This further implies that

$$\begin{aligned} \sup_{0 \leq s \leq r} (\mathbb{E}|M(s)|^{2p})^{1/p} &\leq p(2p-1) \int_0^r (\mathbb{E} \|\Gamma(t-s)u(s)\|_{HS}^{2p})^{1/p} ds \\ &\leq p(2p-1) \int_0^t (\mathbb{E} \|\Gamma(t-s)u(s)\|_{HS}^{2p})^{1/p} ds. \end{aligned}$$

Consequently, the desired assertion follows by taking $r \uparrow t$.

The lemma below gives some sufficient conditions to guarantee that (2.4) admits a unique invariant measure.

Lemma 2.4 *Let $v_\mu < 0$ and assume further that (H2) holds with $\lambda_3 > 0$ such that $c_\lambda^2 \lambda_3 (1 + e^{2\lambda\tau} \rho([- \tau, 0])) < \lambda$ for $\lambda \in (0, -v_\mu)$, where $c_\lambda > 0$ is introduced in (2.8). Then there exist $p > 1$ and $\lambda' > 0$ such that*

$$\sup_{t \geq 0} \mathbb{E} \|X_t(\xi)\|_\infty^{2p} \lesssim 1 + \|\xi\|_\infty^{2p}, \quad (2.9)$$

and

$$\mathbb{E} \|X_t(\xi) - X_t(\eta)\|_\infty^2 \lesssim e^{-\lambda' t} \|\xi - \eta\|_\infty^2. \quad (2.10)$$

Proof Owing to

$$c_\lambda^2 \lambda_3 (1 + e^{2\lambda\tau} \rho([- \tau, 0])) < \lambda \text{ for } \lambda \in (0, -v_\mu),$$

we can choose $\kappa, p > 1$ and $\epsilon > 0$ such that

$$\beta_{\kappa, \epsilon}(p) := p(2p - 1)(1 + \epsilon)\kappa^{1/p} c_\lambda^2 \lambda_3 (1 + e^{2\lambda\tau} \rho([- \tau, 0])) < \lambda. \quad (2.11)$$

In what follows, we fix $\kappa, p > 1$ and $\epsilon > 0$ such that (2.11) holds. Then there exists a constant $\gamma > 1$ such that $(a + b + c)^{2p} \leq \kappa a^{2p} + \gamma(b^{2p} + c^{2p})$ with $a, b, c > 0$. Thus, by the elementary inequality $(a + b)^\theta \leq a^\theta + b^\theta$ for $a, b \geq 0$ and $\theta \in (0, 1)$, it follows from (2.7) and (2.8) that

$$\begin{aligned} (\mathbb{E}|X(t)|^{2p})^{1/p} &\leq \kappa^{1/p} \Theta_p(t) + \gamma^{1/p} \left\{ |\Gamma(t)\xi(0)|^2 \right. \\ &\quad \left. + \left| \int_{[-\tau, 0]} \mu(d\theta) \int_\theta^0 \Gamma(t + \theta - s) \xi(s) ds \right|^2 \right\} \\ &\leq c e^{-2\lambda t} \|\xi\|_\infty^2 + \kappa^{1/p} \Theta_p(t), \end{aligned} \quad (2.12)$$

where

$$\Theta_p(t) := \left(\mathbb{E} \left| \int_0^t \Gamma(t - s) \sigma(X_s) dW(s) \right|^{2p} \right)^{1/p}.$$

Set $\alpha(p) := p(2p - 1)c_\lambda^2 \lambda_3$. Applying Lemma 2.3 with $u(s) = \sigma(X_s)$, followed by using (2.8), we derive from (H2) that

$$\begin{aligned} \Theta_p(t) &\leq \alpha(p) \int_0^t e^{-2\lambda(t-s)} \left\{ c + \left(\mathbb{E} \left(|X(s)|^2 + \int_{[-\tau, 0]} |X(s + \theta)|^2 \nu(d\theta) \right)^p \right)^{1/2p} \right\}^2 ds \\ &\leq \alpha(p) \int_0^t e^{-2\lambda(t-s)} \left\{ c + (1 + \epsilon) \left(\mathbb{E} \left(|X(s)|^2 + \int_{[-\tau, 0]} |X(s + \theta)|^2 \nu(d\theta) \right)^p \right)^{1/p} \right\} ds \\ &\leq \alpha(p) \int_0^t e^{-2\lambda(t-s)} \left\{ c + (1 + \epsilon) \left(\mathbb{E} |X(s)|^{2p} \right)^{1/p} \right\} ds \end{aligned}$$

$$\begin{aligned}
& + \int_{[-\tau, 0]} (\mathbb{E}|X(s + \theta)|^{2p})^{1/p} \nu(d\theta) \Big\} ds \\
& \leq c \left(\int_0^t e^{-2\lambda(t-s)} ds + e^{-2\lambda t} \|\xi\|_\infty^2 \right) \\
& \quad + 2\alpha(p)(1 + \varepsilon)(1 + e^{2\lambda\tau} \rho([-\tau, 0])) \int_0^t e^{-2\lambda(t-s)} (\mathbb{E}|X(s)|^{2p})^{1/p} ds
\end{aligned}$$

where in the first step we have used Minkovskii's inequality [99, p. 5], and in the second step utilized the elementary inequality: $(a + b)^2 \leq (1 + \theta)a^2 + (1 + \frac{1}{\theta})b^2$ with $\theta = \varepsilon$. Inserting the previous estimate on $\Theta_p(t)$ into (2.12) yields that

$$e^{2\lambda t} (\mathbb{E}|X(t)|^{2p})^{1/p} \leq c_\varepsilon \left(\int_0^t e^{2\lambda s} ds + \|\xi\|_\infty^2 \right) + 2\beta_{\kappa, \varepsilon}(p) \int_0^t e^{2\lambda s} (\mathbb{E}|X(s)|^{2p})^{1/p} ds,$$

where $\beta_{\kappa, \varepsilon}(p) > 0$ is defined in (2.11). Thus, by Gronwall's inequality, we have

$$\begin{aligned}
e^{2\lambda t} (\mathbb{E}|X(t)|^{2p})^{1/p} & \leq c_\varepsilon \left(\int_0^t e^{2\lambda s} ds + \|\xi\|_\infty^2 \right) \\
& \quad + 2c_\varepsilon \beta_{\kappa, \varepsilon}(p) \int_0^t \left(\int_0^s e^{2\lambda u} du + \|\xi\|_\infty^2 \right) e^{2\beta_{\kappa, \varepsilon}(p)(t-s)} ds.
\end{aligned}$$

Hence, by interchanging the order of integral, we obtain that

$$\begin{aligned}
(\mathbb{E}|X(t)|^{2p})^{1/p} & \lesssim 1 + \|\xi\|_\infty^2 + \int_0^t \int_0^s e^{-2\lambda(t-u)} e^{2\beta_{\kappa, \varepsilon}(p)(t-s)} du ds \\
& \quad + \|\xi\|_\infty^2 e^{-2\lambda t} \int_0^t e^{2\beta_{\kappa, \varepsilon}(p)(t-s)} ds \\
& \lesssim 1 + \|\xi\|_\infty^2 + \int_0^t e^{-2\lambda(t-u)} \int_u^t e^{2\beta_{\kappa, \varepsilon}(p)(t-s)} ds du \quad (2.13) \\
& \quad + \|\xi\|_\infty^2 e^{-2\lambda t} \int_0^t e^{2\beta_{\kappa, \varepsilon}(p)(t-s)} ds \\
& \lesssim 1 + \|\xi\|_\infty^2,
\end{aligned}$$

where in the last step we have used $\beta_{\kappa, \varepsilon}(p) < \lambda$ due to (2.11). Next, applying Hölder's inequality for integrals with respect to time and B-D-G's inequality for the term involving martingale, we deduce from (H2) and (2.13) that

$$\begin{aligned}
\mathbb{E}\|X_t\|_\infty^{2p} &\lesssim \mathbb{E}|X(t-\tau)|^{2p} + \int_{t-2\tau}^t \mathbb{E}|X(s)|^{2p} ds + \mathbb{E}\left(\int_{t-\tau}^t \|\sigma(X_u)\|_{HS}^2 du\right)^p \\
&\lesssim 1 + \mathbb{E}|X(t-\tau)|^{2p} + \int_{t-2\tau}^t \mathbb{E}|X(s)|^{2p} ds \\
&\lesssim 1 + \|\xi\|_\infty^{2p}, \quad t \geq \tau.
\end{aligned} \tag{2.14}$$

Similarly,

$$\mathbb{E}\left(\sup_{0 \leq t \leq \tau} |X(t)|^{2p}\right) \lesssim 1 + \|\xi\|_\infty^{2p}. \tag{2.15}$$

As a result, (2.9) follows from (2.14) and (2.15) immediately.

Next, we show (2.10). Let $\Theta(t) = X(t, \xi) - X(t, \eta)$ for notational simplicity. Using the elementary inequality (1.21), followed by utilizing Hölder's inequality for the time integrals and Itô's isometry for the term involving martingale, we obtain that

$$\begin{aligned}
&\mathbb{E}|\Theta(t)|^2 \\
&\leq \frac{1+\alpha}{\alpha} \left| \Gamma(t)(\xi(0) - \eta(0)) + \int_{[-\tau, 0]} \mu(d\theta) \int_\theta^0 \Gamma(t+\theta-s)(\xi(s) - \eta(s)) ds \right|^2 \\
&\quad + (1+\alpha) \int_0^t \mathbb{E} \|\Gamma(t-s)(\sigma(X_s(\xi)) - \sigma(X_s(\eta)))\|_{HS}^2 ds \\
&\lesssim \frac{1+\alpha}{\alpha} e^{-2\lambda t} \|\xi - \eta\|_\infty^2 \\
&\quad + (1+\alpha) c_\lambda^2 \lambda_3 \int_0^t e^{-2\lambda(t-s)} \left(|\Theta(s)| + \int_{[-\tau, 0]} |\Theta(s+\theta)| \rho(d\theta) \right)^2 ds \\
&\lesssim \frac{1+\alpha}{\alpha} e^{-2\lambda t} \|\xi - \eta\|_\infty^2 + 2(1+\alpha) c_\lambda^2 \lambda_3 \int_0^t e^{-2\lambda(t-s)} \left(|\Theta(s)|^2 \right. \\
&\quad \left. + \rho([- \tau, 0]) \int_{[-\tau, 0]} |\Theta(s+\theta)|^2 \rho(d\theta) \right) ds \\
&\lesssim c_\alpha e^{-2\lambda t} \|\xi - \eta\|_\infty^2 + 2\beta(\alpha) \int_0^t e^{-2\lambda(t-s)} \mathbb{E}|\Theta(s)|^2 ds,
\end{aligned}$$

where $\beta(\alpha) := (1+\alpha) c_\lambda^2 \lambda_3 (1 + e^{2\lambda\tau} \rho([- \tau, 0]))$. So we arrive at

$$e^{2\lambda t} \mathbb{E}|\Theta(t)|^2 \lesssim c_\alpha \|\xi - \eta\|_\infty^2 + 2\beta(\alpha) \int_0^t e^{2\lambda s} \mathbb{E}|\Theta(s)|^2 ds.$$

Since $c_\lambda^2 \lambda_3 (1 + e^{2\lambda\tau} \rho([- \tau, 0])) < \lambda$, we can choose $\alpha > 0$ such that $\beta(\alpha) < \lambda$ holds. Thus, Gronwall's inequality leads to

$$\mathbb{E}|\Theta(t)|^2 \lesssim e^{-2(\lambda - \beta(\alpha))t} \|\xi - \eta\|_\infty^2.$$

Finally, (2.10) follows by carrying out the argument for deriving (2.14).

Theorem 2.5 *Assume that the assumptions of Lemma 2.4 hold. Then, (2.4) has a unique invariant measure π , which is exponentially mixing.*

Proof The proof on existence of an invariant measure is facilitated by use of the classical Arzelà–Ascoli theorem [20, Theorem 8.3, p. 56]. For any integer $q \geq 1$, set

$$\mu_q(\cdot) := \frac{1}{q} \int_0^q \mathbb{P}(t, \zeta, \cdot) dt,$$

where $\mathbb{P}(t, \zeta, \cdot)$ is the Markovian transition kernel of $X_t(\zeta)$. By virtue of the Krylov–Bogoliubov theorem (see, e.g., [37, Theorem 3.1.1, p. 21]), to establish the existence of an invariant measure, it is sufficient to verify that $(\mu_q(\cdot))_{q \geq 1}$ is relatively compact. Note that the phase space $\mathcal{C}$ for the segment process $(X_t(\zeta))_{t \geq 0}$ is a complete separable metric space under the uniform metric $\|\cdot\|_\infty$ (see, e.g., [20, p. 220]). Taking [20, Theorem 6.2, p. 37] into consideration, we need only prove that $(\mu_q(\cdot))_{q \geq 1}$ is tight. Moreover, by virtue of [20, Theorem 8.2, p. 55], it suffices to verify that

$$\lim_{\delta \downarrow 0} \sup_{q \geq 1} \mu_q(\varphi \in \mathcal{C} : w_{[-\tau, 0]}(\varphi, \delta) \geq \varepsilon) = 0 \quad (2.16)$$

for any $\varepsilon > 0$, where $w_{[-\tau, 0]}(\varphi, \delta)$, the modulus of continuity of φ , is defined by

$$w_{[-\tau, 0]}(\varphi, \delta) = \sup_{|s-t| \leq \delta, s, t \in [-\tau, 0]} |\varphi(s) - \varphi(t)|, \quad \delta > 0.$$

Noting that

$$\begin{aligned} I(t, \delta) &:= \sup_{t \leq v \leq u \leq t+\tau, 0 \leq u-v \leq \delta} |X(u) - X(v)| \\ &\leq \sup_{t \leq v \leq u \leq t+\tau, 0 \leq u-v \leq \delta} \int_v^u \left| \int_{[-\tau, 0]} \mu(d\theta) X(s+\theta) \right| ds \\ &\quad + \sup_{t \leq v \leq u \leq t+\tau, 0 \leq u-v \leq \delta} \left| \int_v^u \sigma(X_s) dW(s) \right| \\ &=: I_1(t, \delta) + I_2(t, \delta), \quad t \geq \tau, \end{aligned}$$

one has

$$\mathbb{P}(I(t, \delta) \geq \varepsilon) \leq \mathbb{P}(I_1(t, \delta) \geq \varepsilon/2) + \mathbb{P}(I_2(t, \delta) \geq \varepsilon/2).$$

Next, by virtue of Chebyshev's inequality and (2.9),

$$\mathbb{P}(I_1(t, \delta) \geq \varepsilon/2) \lesssim \varepsilon^{-2} \mathbb{E} I_1(t, \delta)^2 \lesssim (\mathbb{E} \|X_t\|_\infty^2 + \mathbb{E} \|X_{t+\tau}\|_\infty^2) \delta^2 \lesssim \delta^2. \quad (2.17)$$

On the other hand, for $p > 1$ satisfying (2.11) and arbitrary $0 \leq s \leq t$, by Lemma [99, Theorem 7.1, p. 39], together with (H2) and (2.9), it follows that

$$\mathbb{E} \left| \int_s^t \sigma(X_r) dW(r) \right|^{2p} \lesssim (t-s)^{p-1} \int_s^t \left\{ 1 + \sup_{u \geq 0} \mathbb{E} \|X_u\|_\infty^{2p} \right\} dr \lesssim (t-s)^p.$$

This, combined with the Kolmogorov tightness criterion (see, e.g., [77, Problem 4.11, p. 64]), implies that

$$\limsup_{\delta \downarrow 0} \sup_{t \geq \tau} \mathbb{P}(I_2(t, \delta) \geq \varepsilon/2) = 0. \quad (2.18)$$

Consequently, (2.16) follows from (2.17), (2.18), and

$$\mu_q(\varphi \in \mathcal{C} : w_{[-\tau, 0]}(\varphi, \delta) \geq \varepsilon) \leq \frac{2\tau}{q} + \frac{1}{q} \int_\tau^q \mathbb{P}(I(t, \delta) \geq \varepsilon) dt, \quad t \geq \tau.$$

With (2.10) in hand, the uniqueness of invariant measure and its mixing property can be proved by following the steps 2 and 3 of the proof for Theorem 1.5.

Remark 2.6 Under dissipative conditions, by the Arzelà–Ascoli tightness characterization, Es-Sarhir et al. [54] and Kinnally-Williams [79] exploited existence of invariant measures for FSDEs with super-linear drift terms and positivity constraints, respectively. Applying the Itô formula, they gave the uniform boundedness for higher moments of the segment processes, which plays a key role in analyzing the diffusion terms by the Kolmogorov tightness criterion. However, (2.4) need not satisfy any dissipative conditions, and therefore the techniques adopted in [54, 79] no longer work. In this section, adopting the variation-of-constants formula and utilizing the Arzelà–Ascoli tightness characterization, we provide verifiable criterion to capture a unique invariant measure for a class of semi-linear FSDEs.

Remark 2.7 Under dissipative conditions, Itô-Nisio [72] discussed existence of stationary solutions for FSDEs. According to [72, Theorem 3], we deduce from (2.9) that (2.4), without satisfying a dissipative condition, has a stationary solution. A solution $(X(t))_{t \geq -\tau}$ of (2.4) is strong stationary, or simply stationary, if the finite-dimensional distributions are invariant under time translation, i.e.,

$$\mathbb{P}\{X(t + t_k) \in \Gamma_k, k = 1, \dots, n\} = \mathbb{P}\{X(t_k) \in \Gamma_k, k = 1, \dots, n\}$$

for all $t \geq 0$, $t_k \geq -\tau$ and $\Gamma_k \in \mathcal{B}(\mathbb{R}^n)$. For stationary solutions of retarded O–U processes in Hilbert spaces, we refer to, for instance, [96]. It is worth pointing out that the stationary solutions discussed for example, in [72, 96] are related to the solution process $(X(t))_{t \geq -\tau}$. Whereas the invariant measure in our case involves the segment process $(X_t)_{t \geq 0}$. Before concluding this section, we give examples to show the validity of Theorem 2.5.

Example 2.8 Consider a semi-linear SDDE

$$dX(t) = -X(t-1)dt + \sigma(X(t-1))dW(t), \quad X_0 = \xi \in \mathcal{C}. \quad (2.19)$$

In this case, $\tau = -1$ and $\mu(\cdot) = -\delta_{-1}(\cdot)$. It is readily seen that the corresponding characteristic equation is

$$\lambda + e^{-\lambda} = 0. \quad (2.20)$$

A simple calculation using MATLAB yields that the unique root of (2.20) is

$$\lambda \approx -0.3181 + 1.3372i.$$

So, one has $v_\mu \approx -0.3181$. Thus, by Theorem 2.5, we deduce that (2.19) possesses a unique invariant measure $\pi \in \mathcal{P}(\mathcal{C})$ whenever the Lipschitz constant L of σ such that $c_\lambda^2 L^2 (1 + e^{2\lambda}) < \lambda$ for $\lambda \in (0, 0.3181)$. Note that (2.19) does not satisfy a dissipative condition even though the Lipschitz constant of σ is sufficiently small since it is impossible to choose constants $\lambda_1 > \lambda_2 > 0$ such that

$$-2(x_1 - x_2)(y_1 - y_2) + |\sigma(y_1) - \sigma(y_2)|^2 \leq -\lambda_1 |x_1 - x_2|^2 + \lambda_2 |y_1 - y_2|^2$$

holds for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Example 2.9 Consider a semi-linear SDDE

$$dX(t) = \{aX(t) + bX(t-1)\}dt + \sigma(X(t-1))dW(t), \quad X_0 = \xi \in \mathcal{C}, \quad (2.21)$$

where $a < 0, b \in \mathbb{R}$ and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz. For this case, $\mu(\cdot) = a\delta_0(\cdot) + b\delta_{-1}(\cdot)$. Note that the associated characteristic equation is

$$\lambda - a - be^{-\lambda} = 0. \quad (2.22)$$

By [90, Theorem 1], all the roots of (2.22) have negative real parts if and only if

$$a < b < -a. \quad (2.23)$$

Then, from Theorem 2.5, (2.21) admits a unique invariant measure $\pi \in \mathcal{P}(\mathcal{C})$ provided that the Lipschitz constant L of σ is sufficiently small.

2.3 Ergodicity for FSDEs of Neutral Type

In this section, we proceed to generalize Theorem 2.5 to an FSDE of neutral type in the form

$$d\left(X(t) - \int_{[-\tau, 0]} \rho(d\theta) X(t + \theta)\right) = \left(\int_{[-\tau, 0]} \mu(d\theta) X(t + \theta)\right) dt + \sigma dW(t) \quad (2.24)$$

with the initial value $X_0 = \xi \in \mathcal{C}$, where ρ, μ are $\mathbb{R}^n \otimes \mathbb{R}^n$ -valued finite signed measure on $[-\tau, 0]$, $\sigma \in \mathbb{R}^n \otimes \mathbb{R}^m$.

To begin, we give an overview of the variation-of-constants formula for linear functional differential equations of neutral type. By [65, Theorem 1.1, p. 256], the following linear functional differential equation of neutral type

$$d\left(Y(t) - \int_{[-\tau, 0]} \rho(d\theta) Y(t + \theta)\right) = \left(\int_{[-\tau, 0]} \mu(d\theta) Y(t + \theta)\right) dt \quad (2.25)$$

with the initial data $\xi \in \mathcal{C}$ has a unique solution $(Y(t; \xi))_{t \geq -\tau}$. Let

$$v_{\rho, \mu} = \sup\{\operatorname{Re}(\lambda) : \lambda \in \mathbb{C}, \Delta_{\rho, \mu}(\lambda) = 0\},$$

where

$$\Delta_{\rho, \mu}(\lambda) := \det\left(\lambda \left(I_{n \times n} - \int_{[-\tau, 0]} e^{\lambda\theta} \rho(d\theta)\right) - \int_{[-\tau, 0]} e^{\lambda\theta} \mu(d\theta)\right), \quad \lambda \in \mathbb{C}.$$

The fundamental solution or resolvent of (2.25) is the unique continuous function $\Gamma : [0, \infty) \mapsto \mathbb{R}^n \otimes \mathbb{R}^n$ satisfying

$$d\left(\Gamma(t) - \int_{[-\tau, 0]} \rho(d\theta) \Gamma(t + \theta)\right) = \left(\int_{[-\tau, 0]} \mu(d\theta) \Gamma(t + \theta)\right) dt$$

with the initial data $\Gamma(0) = I_{n \times n}$ and $\Gamma(\theta) = \mathbf{0}_{n \times n}$ for $\theta \in [-\tau, 0)$. It plays a role analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations. If $v_{\rho, \mu} < 0$, according to Theorem B.4, there exists $\lambda \in (0, -v_{\rho, \mu})$ such that

$$\|\Gamma(t)\| \leq c_\lambda e^{-\lambda t} \quad (2.26)$$

for some constant $c_\lambda > 0$. By (B.6) and the variation-of-constants formula [6, Theorem 1], the solution of (2.24) can be represented explicitly as

$$\begin{aligned}
X(t; \zeta) &= Y(t; \zeta) + \int_0^t \Gamma(t-s) \sigma dW(s) \\
&= \Gamma(t) \zeta(0) - \int_{[-\tau, 0]} \rho(d\theta) \Gamma(t+\theta) \zeta(0) \\
&\quad + \int_{[-\tau, 0]} \mu(d\theta) \int_\theta^0 \Gamma(t+\theta-s) \zeta(s) ds \\
&\quad + \int_{[-\tau, 0]} \rho(d\theta) \int_\theta^0 \Gamma(t-s+\theta) \zeta'(s) ds + \int_0^t \Gamma(t-s) \sigma dW(s)
\end{aligned} \tag{2.27}$$

for any $\zeta \in W^{1,2}([-\tau, 0]; \mathbb{R}^n)$ (the Sobolev space consisting of functions $f : [-\tau, 0] \mapsto \mathbb{R}^n$ such that $f(\cdot)$ and its distributional derivative $f'(\cdot)$ belong to $L^2([-\tau, 0]; \mathbb{R}^n)$). For more properties of Γ , refer to Theorem B.4.

Remark 2.10 For more details on the variation-of-constants formula of linear functional differential equations of neutral type, we refer the readers to, e.g., [6, 46, 65] for finite-dimensional case, and [95, 111, 145] for infinite-dimensional setting.

Let $\|\rho\| = \sup_{1 \leq k \leq n} \sqrt{\sum_{1 \leq j \leq n} \|\rho_{kj}\|_{\text{var}}^2}$, where $\|\rho_{kj}\|_{\text{var}}$ is the total variation of ρ_{kj} .

Theorem 2.11 *Let $v_{\rho, \mu} < 0$ and $\|\rho\| < 1/2$. Then (2.24) has a unique invariant measure $\pi \in \mathcal{P}(\mathcal{C})$, which is exponentially mixing.*

Proof In view of the argument in the proof of Theorem 2.5, to complete the proof of Theorem 2.1.1, it is sufficient to show that

$$\sup_{t \geq 0} \mathbb{E} \|X_t(\zeta)\|_\infty^p < \infty, \quad p > 2.$$

and that

$$\mathbb{E} \|X_t(\zeta) - X_t(\eta)\|_\infty^2 \lesssim e^{-\alpha t}$$

for some $\alpha > 0$.

There exists a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ such that

- $\phi(x) \geq 0$ for all $x \in \mathbb{R}$;
- $\phi \in C_0^\infty(\mathbb{R})$;
- $\int_{\mathbb{R}} \phi(x) dx = 1$,

where $C_0^\infty(\mathbb{R})$ denotes the collection of C^∞ functions $f : \mathbb{R} \mapsto \mathbb{R}$ with compact support; see, for instance, [56]. For any $\delta > 0$, let $\phi_\delta(x) = \delta^{-k} \phi(\frac{x}{\delta})$ be the standard mollifier. Let ζ_δ be the convolution of ζ and ϕ_δ , i.e.,

$$\zeta_\delta(\theta) = (\zeta * \phi_\delta)(\theta) = \int_{\mathbb{R}} \zeta(\theta - s) \phi_\delta(s) ds = \int_{\mathbb{R}} \zeta(s) \phi_\delta(\theta - s) ds,$$

where we set $\xi(\theta) = \mathbf{0}_n$ for $\theta \in (-\infty, -\tau) \cup (0, \infty)$. It is easy to see that $\xi_\delta \in C_0^\infty(\mathbb{R})$. Moreover, $\xi_\delta \rightarrow \xi$ uniformly on compact subsets of $\mathbb{R}$. So ξ is mollified by convolution with ϕ_δ . Observe that

$$\mathbb{E}\|X_t(\xi)\|_\infty^2 \lesssim \mathbb{E}\|X_t(\xi) - X_t(\xi_\delta)\|_\infty^2 + \mathbb{E}\|X_t(\xi_\delta)\|_\infty^2,$$

and that

$$\begin{aligned} \mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 &\lesssim \mathbb{E}\|X_t(\xi) - X_t(\xi_\delta)\|_\infty^2 + \mathbb{E}\|X_t(\eta) - X_t(\eta_\delta)\|_\infty^2 \\ &\quad + \mathbb{E}\|X_t(\xi_\delta) - X_t(\eta_\delta)\|_\infty^2. \end{aligned}$$

By a straightforward calculation, one has

$$\lim_{\delta \rightarrow 0} \mathbb{E}\|X_t(\xi) - X_t(\xi_\delta)\|_\infty^2 = \lim_{\delta \rightarrow 0} \mathbb{E}\|X_t(\eta) - X_t(\eta_\delta)\|_\infty^2 = 0.$$

So, to finish the proof we need only show that

$$\sup_{t \geq 0} \mathbb{E}\|X_t(\xi_\delta)\|_\infty^p < \infty, \quad p > 2. \quad (2.28)$$

and

$$\mathbb{E}\|X_t(\xi_\delta) - X_t(\eta_\delta)\|_\infty^2 \lesssim e^{-\alpha t} \quad (2.29)$$

for some $\alpha > 0$. Observe that $\xi_\delta, \eta_\delta \in C_0^\infty(\mathbb{R})$. With (2.27) in hand, (2.29) can be done by carrying out the arguments to derive (2.10), while (2.28) can be finished by following the argument to deduce (2.9).

Remark 2.12 In fact, Theorem 2.11 can be generalized to the case of FSDEs of neutral type with multiplicative noises by following the main line of argument for Theorem 2.5. However, to avoid complicated calculations, in this section we focus only on the additive noise case.

Example 2.13 Consider an SDDE of neutral type

$$d\left(X(t) + \frac{1}{3}X(t-1)\right) = -X(t-1)dt + \sigma dW(t), \quad X_0 = \xi, \quad (2.30)$$

where $a \in \mathbb{R}$ and $(W(t))_{t \geq 0}$ is a real-valued Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The characteristic equation associated with the deterministic counterpart of (2.30) is

$$\lambda + \left(1 + \frac{\lambda}{3}\right)e^{-\lambda} = 0, \quad \lambda \in \mathbb{C}. \quad (2.31)$$

Using MATLAB, we can obtain that the unique root of (2.31) is $\lambda \approx -2.313474269$. Then, by Theorem 2.1.1 we deduce that (2.30) has a unique invariant measure, which is exponentially mixing.

2.4 Ergodicity for FSPDEs Driven by Cylindrical Wiener Processes

In this section, we are concerned with ergodicity for a range of FSPDEs. First, we introduce some notation. Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(K, \langle \cdot, \cdot \rangle_K, \|\cdot\|_K)$ be real separable Hilbert spaces. Let $Q \in \mathcal{L}(K)$ be symmetric and nonnegative so that there exists a nonnegative and symmetric element $Q^{\frac{1}{2}} \in \mathcal{L}(K)$ such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$ (see, e.g., [117, Theorem VI.9]). Let the trace of Q be finite, i.e., $\text{trace}(Q) < \infty$. Then, $Q^{\frac{1}{2}} \in \mathcal{L}_{HS}(K)$ due to $\|Q^{\frac{1}{2}}\|_{HS}^2 = \text{trace}(Q)$, and $S \circ Q^{\frac{1}{2}} \in \mathcal{L}_{HS}(K, H)$ for any $S \in \mathcal{L}(K, H)$. Let $K_0 = Q^{\frac{1}{2}}(K)$, which is a separable Hilbert space with the inner product given by $\langle u_0, v_0 \rangle_0 := \langle Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0 \rangle_K$ for $u_0, v_0 \in K$ (see, e.g., [113, Proposition C.0.3, p. 145]), where $Q^{-\frac{1}{2}}$ is the pseudoinverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Let $L_2^0 = \mathcal{L}_{HS}(K_0, H)$. According to [113, Proposition C.0.3 (ii), p. 145], $(Q^{\frac{1}{2}}e_k)_{k \geq 1}$ is an orthonormal basis of $(K_0, \langle \cdot, \cdot \rangle_0, \|\cdot\|_0)$ if $(e_k)_{k \geq 1}$ is an orthonormal basis of $(\text{Ker}(Q^{\frac{1}{2}}))^{\perp}$. Thus, $\|S\|_{L_2^0} = \|S \circ Q^{\frac{1}{2}}\|_{HS}$ for each $S \in L_2^0$.

In this section, we focus on an FSPDE on $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ in the form

$$dX(t) = \{AX(t) + \Phi(X_t)\}dt + \sigma dW(t), \quad t > 0, \quad X_0 = \zeta \in \mathcal{C}. \quad (2.32)$$

Herein, $(A, \mathcal{D}(A))$ is a self-adjoint operator on H generating a compact C_0 -semigroup $(e^{tA})_{t \geq 0}$ such that $\|e^{tA}\| \leq e^{-\alpha t}$ for some $\alpha > 0$, the linear operator $\Phi : \mathcal{C} \mapsto H$ is given by

$$\Phi(\phi) := \int_{[-\tau, 0]} (\mu(d\theta))\phi(\theta), \quad \phi \in \mathcal{C},$$

where $\mu : [-\tau, 0] \mapsto \mathcal{L}(H)$ is of bounded variation, $\sigma \in L_2^0$, and $(W(t))_{t \geq 0}$ is a K -valued Q -Wiener process under $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. By [11, Theorem A.1], (2.32) has a unique mild solution $(X(t))_{t \geq -\tau}$ satisfying

$$X(t) = e^{tA}\zeta(0) + \int_0^t e^{(t-s)A}\Phi(X_s)ds + \int_0^t e^{(t-s)A}\sigma dW(s), \quad t > 0. \quad (2.33)$$

Consider the following deterministic evolution equation

$$dY(t) = \{AY(t) + \Phi(Y_t)\}dt, \quad t > 0, \quad Y_0 = \zeta. \quad (2.34)$$

According to [145, Theorem 1.1, p. 37], equation (2.34) has a unique mild solution $(Y(t; \xi))_{t \geq -\tau}$ with the initial data $Y_0 = \xi$ such that

$$Y(t; \xi) = e^{tA}\xi(0) + \int_0^t e^{(t-s)A}\Phi(Y_s(\xi))ds, \quad t > 0 \quad (2.35)$$

holds. Let $(\Gamma(t))_{t \geq 0}$ solve the $\mathcal{L}(H)$ -valued evolution equation

$$d\Gamma(t) = \{A\Gamma(t) + \Phi(\Gamma_t)\}dt, \quad t > 0 \quad (2.36)$$

with the initial value $\Gamma(0) = I$, the identical operator on H , and $\Gamma(\theta) = O$ for $\theta \in [-\tau, 0)$, where O stands for the null operator on H . By the Laplace transform approach, the mild solution $Y(t; \xi)$ to (2.34) can be written explicitly as

$$Y(t; \xi) = \Gamma(t)\xi(0) + \int_0^t \Gamma(t-s)\Phi(\vec{\xi}_s)ds, \quad t \geq 0,$$

where $\vec{\xi} : [-\tau, \infty) \mapsto H$ is the right extension of ξ defined by

$$\vec{\xi}(\theta) = \xi(\theta), \quad \theta \in [-\tau, 0] \quad \text{and} \quad \vec{\xi}(\theta) = 0, \quad \theta \in (0, \infty);$$

see, for instance, [93, Theorem 3.2] & [111, Theorem 2.2]. Set

$$\alpha_0 := \sup\{\operatorname{Re}(\lambda) : \lambda \in \mathbb{C}, \text{ there exists an } x \in \mathcal{D}(A) \setminus \{0\} \text{ such that } (\lambda I - A - \Phi(e^{\lambda \cdot}))x = 0\},$$

where $(e^{\lambda \cdot}x)(\theta) = e^{\lambda\theta}x$, $\theta \in [-\tau, 0]$.

Theorem 2.14 *Assume that $(A, \mathcal{D}(A))$ is a self-adjoint operator on H generating a compact C_0 -semigroup $(e^{tA})_{t \geq 0}$ such that $\|e^{tA}\| \leq e^{-at}$ for some $a > 0$ and $\alpha_0 < 0$, and suppose further $\operatorname{trace}(Q) < \infty$ and $\sigma \in L_2^0$. Then, (2.32) admits a unique invariant measure π , which is exponentially mixing.*

Proof To show existence of an invariant measure, we adopt the stability-in-distribution approach (see, e.g., [14] & [155]). To complete the proof, it is sufficient to verify that

- (P1) $\sup_{t \geq 0} \mathbb{E}\|X_t(\xi)\|_\infty^2 < \infty$ for any $\xi \in U$;
- (P2) $\lim_{t \rightarrow \infty} \mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 = 0$ for any $\xi, \eta \in U$,

in which U is a bounded subset of $\mathcal{C}$. In what follows, we intend to show that (P1) and (P2) are fulfilled, respectively, for (2.32). By [93, Proposition 4.1], the unique mild solution $X(t; \xi)$ with the initial value $X_0 = \xi$ can be represented explicitly by

$$\begin{aligned}
X(t; \xi) &= Y(t; \xi) + \int_0^t \Gamma(t-s) \sigma dW(s) \\
&= \Gamma(t) \xi(0) + \int_0^t \Gamma(t-s) \Phi(\vec{\xi}_s) ds + \int_0^t \Gamma(t-s) \sigma dW(s). \quad (2.37)
\end{aligned}$$

Moreover, for any $\lambda \in (0, -\alpha_0)$, by virtue of [94, Lemma 2.1] there exists $c_\lambda > 0$ such that

$$\|\Gamma(t)\| \leq c_\lambda e^{-\lambda t}. \quad (2.38)$$

This, together with (2.37) and Itô's isometry, gives that

$$\begin{aligned}
\mathbb{E} \|X(t; \xi)\|_H^2 &\lesssim \|\xi\|_\infty^2 + \int_0^t \text{trace}(\Gamma(s) \sigma Q \sigma^* \Gamma(s)^*) ds \\
&\lesssim \|\xi\|_\infty^2 + \int_0^t \|\Gamma(s) \sigma Q^{\frac{1}{2}}\|_{HS}^2 ds \\
&\lesssim \|\xi\|_\infty^2 + \|\sigma\|_{L_2^0}^2 \int_0^t \|\Gamma(s)\| ds \\
&\lesssim 1 + \|\xi\|_\infty^2. \quad (2.39)
\end{aligned}$$

Recall that $\|e^{tA}\| \leq e^{-\alpha t}$ for some $\alpha > 0$. Next, for any $t \geq \tau$ observe from (2.33) and (2.39) that

$$\begin{aligned}
&\mathbb{E} \|X(t; \xi)\|_\infty^2 \\
&\lesssim \mathbb{E} \|X(t-\tau; \xi)\|_H^2 + \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left\| \int_{t-\tau}^s e^{(s-u)A} \Phi(X_u) du \right\|_H^2 \right) \\
&\quad + \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left\| \int_{t-\tau}^s e^{(s-u)A} \sigma dW(u) \right\|_H^2 \right) \quad (2.40) \\
&\lesssim 1 + \int_{t-\tau}^t \mathbb{E} \|\Phi(X_u)\|_H^2 du + \left(\mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left\| \int_{t-\tau}^s e^{(s-u)A} \sigma dW(u) \right\|_H^{2r} \right) \right)^{1/r} \\
&\lesssim 1 + \|\xi\|_\infty^2,
\end{aligned}$$

where in the second step we have utilized the Hölder inequality with $r > 1$ for the time integral and in the last step used [36, Proposition 7.3, p. 184] for the stochastic convolution. Hence, (P1) follows. Hereinafter, we intend to verify (P2). From (2.37), one has

$$X(t; \xi) - X(t; \eta) = \Gamma(t)(\xi(0) - \eta(0)) + \int_0^t \Gamma(t-s)(\Phi(\vec{\xi}_s) - \Phi(\vec{\eta}_s)) ds. \quad (2.41)$$

Moreover, for any $t \geq \tau$ by the notions of $\vec{\xi}$ and $\vec{\eta}$, we derive from Hölder's inequality that

$$\begin{aligned}
& \left\| \int_0^t \Gamma(t-s)(\Phi(\vec{\xi}_s) - \Phi(\vec{\eta}_s))ds \right\|_H^2 \\
& \lesssim \left\| \int_0^\tau \Gamma(t-s)(\Phi(\vec{\xi}_s) - \Phi(\vec{\eta}_s))ds \right\|_H^2 + \left\| \int_\tau^t \Gamma(t-s)(\Phi(\vec{\xi}_s) - \Phi(\vec{\eta}_s))ds \right\|_H^2 \\
& \lesssim \int_0^\tau \|\Gamma(t-s)\| \cdot \|\Phi(\vec{\xi}_s) - \Phi(\vec{\eta}_s)\|_H^2 ds \\
& \lesssim e^{-\lambda t} \|\xi - \eta\|_\infty^2,
\end{aligned}$$

which, combining (2.38) with (2.41), yields that

$$\|X(t; \xi) - X(t; \eta)\|_H^2 \lesssim e^{-\lambda t} \|\xi - \eta\|_\infty^2, \quad t \geq -\tau.$$

So one has

$$\|X(t+\theta; \xi) - X(t+\theta; \eta)\|_H^2 \lesssim e^{-\lambda(t+\theta)} \|\xi - \eta\|_\infty^2 \lesssim e^{-\lambda t} \|\xi - \eta\|_\infty^2, \quad \theta \in [-\tau, 0].$$

Thus we arrive at

$$\mathbb{E}\|X(t; \xi) - X(t; \eta)\|_\infty^2 \lesssim e^{-\lambda t} \|\xi - \eta\|_\infty^2. \quad (2.42)$$

Consequently, (P2) holds.

Example 2.15 Consider the following SPDE with memory

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) + bu(t - \tau, x) + c(x) \dot{W}(t), & t > 0, \quad x \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \geq 0, \\ u(t, x) = \phi(t, x), & t \in [-\tau, 0], \quad x \in [0, \pi], \end{cases} \quad (2.43)$$

where $b \in \mathbb{R}$, $c(\cdot)$, $\phi(t, \cdot) \in L^2([0, \pi])$, and $(W(t))_{t \geq 0}$ is a scalar Brownian motion. Let $A = \Delta$ be the Laplacian. Note that the characteristic equation associated with the deterministic counterpart of (2.43) is

$$(\lambda I - A - be^{-\lambda \tau} I)x = 0, \quad x \in \mathcal{D}(A) \setminus \{0\}. \quad (2.44)$$

Since the eigenvalues of A is $-n^2$, the eigenvalue of (2.44) satisfies

$$\lambda + n^2 - be^{-\lambda \tau} = 0, \quad n = 1, 2, \dots \quad (2.45)$$

By [90, Theorem 1], all the roots of (2.45) with each fixed $n \geq 1$ have negative real parts if and only if

$$-n^2 < b < n^2.$$

So (2.43) admits a unique invariant measure whenever $-1 < b < 1$.

In what follows, we consider the case $Q = I$ so that $\text{trace}(Q) = \infty$, i.e., $(W(t))_{t \geq 0}$ is a cylindrical Wiener process.

Theorem 2.16 *Assume that $(A, \mathcal{D}(A))$ is a self-adjoint operator on H generating a compact C_0 -semigroup $(e^{tA})_{t \geq 0}$ such that $\|e^{tA}\| \leq e^{-\alpha t}$ for some $\alpha > 0$ and $\alpha_0 < 0$. Suppose further that*

$$\sup_{t \geq 0} \left(\int_0^t s^{-2\alpha} \|e^{sA}\|_{HS}^2 ds \right) < \infty, \quad (2.46)$$

and

$$\sup_{t \geq 0} \left(\int_0^t \text{trace}(\Gamma(s) \sigma Q \sigma^* \Gamma(s)^*) ds \right) < \infty. \quad (2.47)$$

Then, (2.32) admits a unique invariant measure π , which is exponentially mixing.

Proof Theorem 2.16 can be proved by a slight modification of the argument for Theorem 2.14. To complete the proof, we still need to verify (P1) and (P2), respectively. Due to (2.47), we obtain from the first display of (2.39) that

$$\mathbb{E} \|X(t; \xi)\|_H^2 \lesssim 1 + \|\xi\|_\infty^2. \quad (2.48)$$

From the second inequality of (2.40), we deduce from (2.48) and [36, Proposition 7.9, p. 196] that

$$\begin{aligned} \mathbb{E} \|X(t; \xi)\|_\infty^2 &\lesssim 1 + \|\xi\|_\infty^2 + \int_{t-\tau}^t s^{-2\alpha} \|e^{sA}\|_{HS}^2 ds \\ &\lesssim 1 + \|\xi\|_\infty^2. \end{aligned}$$

As a result, (P1) follows. Also, (P2) holds as (2.42) shows.

2.5 Ergodicity for FSDEs Driven by Jump Processes

In the previous sections, we obtained existence and uniqueness of invariant measures for functional stochastic equations with continuous sample paths. In this section, we consider the case of FSDEs driven by jump processes. The discontinuity of the sample path makes the problem more difficult to deal with. Although the variation-of-constants approach can still be used, the verification of the tightness cannot be done as in the case of continuous sample paths. To overcome the difficulty, we use the Kurtz tightness criterion (see Lemma 2.17 below) to treat the underlying problem.

We start with some terminologies and notation. Let $(Z(t))_{t \geq 0}$ be an n -dimensional Lévy process defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. For each $t \geq 0$, $Z(t)$ is infinitely divisible by virtue of [3, Proposition 1.3.1, p. 40]. By the Lévy–Khintchine formula (see, e.g., [3, Theorem 1.2.14, p. 28]), $\Psi(\xi)$, the symbol or characteristic exponent of $Z(t)$, admits the form

$$\Psi(\xi) = i \langle b, \xi \rangle - \frac{1}{2} \langle \xi, A \xi \rangle + \int_{\mathbb{R}^n - \{0\}} \{e^{i \langle \xi, z \rangle} - 1 - i \langle \xi, z \rangle \mathbf{1}_{\{|z| < 1\}}\} \pi(dz),$$

in which $b \in \mathbb{R}^n$, $A \in \mathbb{R}^n \otimes \mathbb{R}^n$ is positive definite and symmetric, and $\pi(\cdot)$ is the Lévy measure, i.e., a σ -finite measure on $\mathbb{R}^n - \{0\}$ such that

$$\int_{\mathbb{R}^n - \{0\}} (1 \wedge |z|^2) \pi(dz) < \infty.$$

The lemma below is concerned with Kurtz's criterion on tightness of laws on $\mathcal{D}$.

Lemma 2.17 ([55, Theorem 8.6, p. 137–138]) *For each $t \geq 0$, let $Y^t(\cdot) \in D([0, \tau]; \mathbb{R}^n)$ and assume that*

$$\lim_{K_1 \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq s \leq T} |Y^t(s)| \geq K_1 \right) = 0 \text{ for each } T \leq \tau,$$

and, for $0 \leq u \leq \delta$ and $v \leq T$,

$$\begin{aligned} \mathbb{E}_v^t \{|Y^t(u+v) - Y^t(u)| \wedge 1\} &\leq \mathbb{E}_v^t \gamma(t, \delta) \\ \lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \mathbb{E} \gamma(t, \delta) &= 0, \end{aligned}$$

where $\mathbb{E}_v^t$ denotes the conditional expectation with respect to $\mathcal{F}_v^t$, the minimal σ -algebra that measures $(Y^t(s))_{0 \leq s \leq v}$. Then $(\mathcal{L}(Y^t(s)), s \in [0, \tau])_{t \geq 0}$ is tight in $D([0, \tau]; \mathbb{R}^n)$.

Consider a retarded O–U process driven by a Lévy process with the Lévy triple $(0, 0, \pi)$ in the form

$$dX(t) = \left(\int_{[-\tau, 0]} \mu(d\theta) X(t + \theta) \right) dt + \sigma dZ(t), \quad X_0 = \xi \in \mathcal{D}, \quad (2.49)$$

where $\mu(\cdot)$ is an $\mathbb{R}^n \otimes \mathbb{R}^n$ -valued finite signed measure on $[-\tau, 0]$, and $\sigma \in \mathbb{R}^n \otimes \mathbb{R}^n$. Hereinafter, $v_\mu \in \mathbb{R}$ is defined as in (2.5). The main result in this section is presented as follows:

Theorem 2.18 *Let $v_\mu < 0$ and assume further that*

$$\int_{|z| > 1} |z| \pi(dz) < \infty. \quad (2.50)$$

Then, (2.49) has a unique invariant measure $\mu \in \mathcal{P}(\mathcal{D})$.

Proof For each integer $q \geq 1$, set

$$\mu_q(\cdot) := \frac{1}{q} \int_0^q \mathbb{P}(t, \xi, \cdot) dt,$$

where $\mathbb{P}(t, \xi, \cdot)$ is the Markovian transition kernel of $X_t(\xi)$. If $(\mathcal{L}(X_t(\xi)))_{t \geq \tau}$ is tight under the Skorohod metric d_S , for any $\varepsilon > 0$ there exists a compact subset $U \in \mathcal{B}(\mathcal{D})$ such that

$$\mathbb{P}(X_t(\xi) \in U) \leq 1 - \varepsilon \text{ and hence } \mu_q(U) \leq 1 - \varepsilon.$$

Thus $(\mu_q(\cdot))_{q \geq 1}$ is tight. Recall that $(X_t(\xi))_{t \geq 0}$ is Markovian by Theorem B.2 (see also [119, Proposition 3.3]) and eventually Feller due to [119, Proposition 3.5], i.e., for all $t \geq \tau$, P_t maps $C_b(\mathcal{D})$ into itself. As a consequence, the Krylov–Bogoliubov theorem (see, e.g., [37, Theorem 3.1.1, p. 21]) implies that (2.49) has an invariant measure $\pi \in \mathcal{P}(\mathcal{D})$.

For $t > 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^n - \{0\})$, define the Poisson random measure generated by $Z(t)$ by

$$N(t, \Gamma) = \sum_{s \in (0, t]} \mathbf{1}_\Gamma(\Delta Z(s)),$$

where $\Delta Z(t) := Z(t) - Z(t-)$ for any $t \geq 0$, and the compensated Poisson random measure by

$$\tilde{N}(t, \Gamma) = N(t, \Gamma) - t\pi(\Gamma).$$

By the Lévy–Itô decomposition (see, e.g., [3, Theorem 2.4.16, p. 108]), one gets

$$Z(t) = \int_{|z| \leq 1} z \tilde{N}(t, dz) + \int_{|z| > 1} z N(t, dz). \quad (2.51)$$

By the variation-of-constants formula (see, e.g., Gushchin–Küchler [60]), the solution of (2.49) can be written explicitly as

$$\begin{aligned} X(t; \xi) &= \Gamma(t)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t + \theta - s) \xi(s) ds \\ &\quad + \int_0^t \Gamma(t - s) \sigma dZ(s), \end{aligned} \quad (2.52)$$

where $(\Gamma(t))_{t \geq -\tau}$ is the fundamental solution of (2.6). Substituting (2.51) into (2.52) leads to

$$\begin{aligned} X(t; \xi) &= \Gamma(t)\xi(0) + \int_{[-\tau, 0]} \mu(ds) \int_s^0 \Gamma(t + s - u) \xi(u) du \\ &\quad + \int_0^t \int_{|z| \leq 1} \Gamma(t - s) \sigma z \tilde{N}(ds, dz) + \int_0^t \int_{|z| > 1} \Gamma(t - s) \sigma z N(ds, dz) \\ &=: \sum_{j=1}^4 I_j(t). \end{aligned}$$

By (2.8), it is easy to see that

$$\sup_{t \geq 0} \mathbb{E}(|I_1(t)| + |I_2(t)|) \lesssim \|\xi\|_\infty.$$

By virtue of the Hölder inequality, the Itô isometry, and (2.8), we have

$$\sup_{t \geq 0} \mathbb{E}|I_3(t)| \leq \beta^{1/2} \|\sigma\| \sup_{t \geq 0} \left(\int_0^t \|\Gamma(t-s)\|^2 ds \right)^{1/2} < \infty,$$

where

$$\beta := \int_{|z| \leq 1} |z|^2 \pi(dz) < \infty \quad (2.53)$$

with $\pi(\cdot)$ being the Lévy measure. Also, by (2.8) it follows from (2.50) that

$$\sup_{t \geq 0} \mathbb{E}|I_4(t)| \leq \|\sigma\| \int_{|z| > 1} |z| \pi(dz) \sup_{t \geq 0} \left(\int_0^t \|\Gamma(t-s)\| ds \right) < \infty.$$

Hence we arrive at

$$\sup_{t \geq 0} \mathbb{E}|X(t; \xi)| < \infty. \quad (2.54)$$

For any $t \geq \tau$, we derive from (2.51) and (2.54) that

$$\begin{aligned} \mathbb{E}\|X_t(\xi)\|_\infty &\leq \mathbb{E}|X(t-\tau; \xi)| + \int_{t-\tau}^t \mathbb{E} \left| \int_{[-\tau, 0]} \mu(d\theta) X(u+\theta; \xi) \right| du \\ &\quad + \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left| \int_{t-\tau}^s \int_{|z| \leq 1} \sigma z \tilde{N}(dt, dz) \right| \right) \\ &\quad + \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left| \int_{t-\tau}^s \int_{|z| > 1} \sigma z N(du, dz) \right| \right) \\ &\leq c + \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left| \int_{t-\tau}^s \int_{|z| \leq 1} \sigma z \tilde{N}(du, dz) \right| \right) \\ &\quad + \tau \|\sigma\| \int_{|z| > 1} |z| \pi(dz), \end{aligned} \quad (2.55)$$

where in the last step we have used that, by [3, Theorem 2.3.8, p. 91],

$$\begin{aligned} \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left| \int_{t-\tau}^s \int_{|z| > 1} \sigma z N(du, dz) \right| \right) &\leq \|\sigma\| \mathbb{E} \int_{t-\tau}^t \int_{|z| > 1} |z| N(du, dz) \\ &= \|\sigma\| \tau \int_{|z| > 1} |z| \pi(dz). \end{aligned} \quad (2.56)$$

Next, by B–D–G’s inequality [116, Theorem 48, p. 193] and Jensen’s inequality, one finds that

$$\begin{aligned} \mathbb{E} \left(\sup_{t-\tau \leq s \leq t} \left| \int_{t-\tau}^s \int_{|z| \leq 1} \sigma z \tilde{N}(du, dz) \right| \right) &\lesssim \|\sigma\| \mathbb{E} \sqrt{\int_{t-\tau}^t \int_{|z| \leq 1} |z|^2 N(du, dz)} \\ &\lesssim \|\sigma\| \sqrt{\mathbb{E} \int_{t-\tau}^t \int_{|z| \leq 1} |z|^2 N(du, dz)} \quad (2.57) \\ &\lesssim \|\sigma\| \sqrt{\tau \beta}, \end{aligned}$$

where in the last step we have utilized [3, Theorem 2.3.8, p. 91], and $\beta > 0$ was given in (2.53). Consequently, (2.55) and (2.57) yield that

$$\sup_{t \geq 0} \mathbb{E} \|X_t(\xi)\|_\infty < \infty. \quad (2.58)$$

Set

$$\mathbb{E}_s(\cdot) := \mathbb{E}(\cdot | \mathcal{F}_s), \quad s \geq 0.$$

For $\theta \in [-\tau, 0)$ and $\tilde{\theta} \in [0, \delta]$, where $\delta > 0$ is an arbitrary constant such that $\theta + \delta \in [-\tau, 0]$, (2.49) and (2.51) lead to

$$\begin{aligned} \mathbb{E}_{t+\theta} |X_t(\theta + \tilde{\theta}) - X_t(\theta)| &= \mathbb{E}_{t+\theta} |X(t + \theta + \tilde{\theta}) - X(t + \theta)| \\ &\lesssim \int_{t+\theta}^{t+\theta+\delta} \mathbb{E}_{t+\theta} \left\{ \left| \int_{[-\tau, 0]} \mu(d\theta) X(s + \theta) \right. \right. \\ &\quad \left. \left. + \int_{|z| \leq 1} \sigma z \tilde{N}(ds, dz) + \int_{|z| > 1} \sigma z N(ds, dz) \right| \right\}. \end{aligned}$$

It follows that there is a $\gamma_0(t, \delta)$ satisfying

$$\mathbb{E}_{t+\theta} |X(t + \theta + \tilde{\theta}) - X(t + \theta)| \leq \mathbb{E}_{t+\theta} \gamma_0(t, \delta).$$

Taking expectation, following the arguments to derive (2.56) and (2.58), respectively, and taking $\limsup_{t \rightarrow \infty}$ followed by $\lim_{\delta \rightarrow 0}$, we obtain from (2.54) that

$$\lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \mathbb{E} \gamma_0(t, \delta) = 0. \quad (2.59)$$

In view of Lemma 2.17 and the time shift $t \mapsto t - \tau$, we conclude from (2.58) and (2.59) that $(X_t(\xi))_{t \geq \tau}$ is tight under the Skorohod metric d_S . Consequently, the solution of (2.49) enjoys an invariant measure.

Next, from (2.8) and (2.52), we get that

$$\begin{aligned}
& \mathbb{E} \|X_t(\zeta) - X_t(\eta)\|_\infty \\
&= \mathbb{E} \left(\sup_{-\tau \leq \theta \leq 0} \left| \Gamma(t + \theta)(\zeta(0) - \eta(0)) \right. \right. \\
&\quad \left. \left. + \int_{[-\tau, 0]} \mu(du) \int_u^0 \Gamma(t + \theta + u - s)(\zeta(s) - \eta(s)) ds \right| \right) \\
&\lesssim e^{-\lambda t} \|\zeta - \eta\|_\infty, \quad t \geq \tau.
\end{aligned}$$

This yields that

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \|\mathbb{P}(t, \zeta, \cdot) - \mathbb{P}(t, \eta, \cdot)\|_{\text{var}} \\
&= \lim_{t \rightarrow \infty} \sup_{\|\varphi\|_{\text{Lip}}=1} |\mathbb{E}\varphi(X_t(\zeta)) - \mathbb{E}\varphi(X_t(\eta))| = 0,
\end{aligned} \tag{2.60}$$

where $\|\varphi\|_{\text{Lip}}$ is the Lipschitz constant of φ w.r.t. the Skorohod metric d_S . If $\mu'(\cdot) \in \mathcal{P}(\mathcal{D})$ is also an invariant measure, then, by the invariance, one has

$$\|\mu - \mu'\|_{\text{var}} \leq \int_{\mathcal{D} \times \mathcal{D}} \|\mathbb{P}(t, \zeta, \cdot) - \mathbb{P}(t, \eta, \cdot)\|_{\text{var}} \mu(d\zeta) \mu'(d\eta). \tag{2.61}$$

Thus, the uniqueness of invariant measure follows by taking $t \rightarrow \infty$ in (2.61) and utilizing (2.60).

Remark 2.19 From (2.50), $\mathbb{E}|Z(t)| < \infty$ for all $t > 0$ because of [3, Theorem 2.5.2, p. 132]. (2.49) incorporates retarded O–U processes driven by symmetric α -stable processes with $1 < \alpha < 2$, which have finite p th moment for $p \in (0, \alpha)$, and subordinate Brownian motions $W_{S(t)}$, i.e., $W(t)$ is a standard Brownian motion and $S(t)$ is an $\alpha/2$ -stable subordination (i.e., a real-valued Lévy process with nondecreasing sample paths). For more details on stable distributions and subordinators, we refer to Applebaum [3, p. 33–62].

Remark 2.20 In many applications, one often encounters the so-called jump diffusion models, in which both Brownian type of noise and Lévy process appear. In view of our results in Section 2.2 and the current section, in lieu of (2.4) or (2.49), we can consider a process of the form

$$dX(t) = \left(\int_{[-\tau, 0]} \mu(d\theta) X(t + \theta) \right) dt + \sigma(X_t) dW(t) + \sigma_0 dZ(t), \quad X_0 = \zeta \in \mathcal{D}, \tag{2.62}$$

where $\sigma : \mathcal{D} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ such that (H2) in Chapter 1 holds. Continue to use the variation-of-constants formula. Comparing to the development in Theorem 2.18, we need to deal with an additional martingale term in the verification of tightness. This term can be easily handled using B–D–G’s inequality. The results of Theorem 2.18 continue to hold.

Remark 2.21 Examining the proof of Theorem 2.18, the technique employed therein applies to (2.49) with the Lévy triple $(0, A, \pi)$ and an FSDE with jumps

$$dX(t) = \left(\int_{[-\tau, 0]} \mu(d\theta) X(t + \theta) \right) dt + \sigma(X_{t-}) dZ(t), \quad X_0 = \xi \in \mathcal{D}, \quad (2.63)$$

where $\sigma : \mathcal{D} \mapsto \mathbb{R}^n \otimes \mathbb{R}^n$ is uniformly bounded.

Nevertheless, if the Lévy process $Z(t)$ has a finite second moment, the uniform boundedness of σ can indeed be removed as the following theorem shows. As we mention above, $v_\mu \in \mathbb{R}$ below is defined in (2.5). We assume that there exists an $L > 0$ such that

$$\|\sigma(\phi) - \sigma(\psi)\|_{HS}^2 \leq L \left(|\phi(0) - \psi(0)|^2 + \int_{[-\tau, 0]} |\phi(\theta) - \psi(\theta)|^2 \rho(d\theta) \right) \quad (2.64)$$

for any $\xi, \eta \in \mathcal{D}$.

Theorem 2.22 *Let $v_\mu < 0$ and (2.64) hold for a sufficiently small $L > 0$, and suppose further that*

$$\vartheta := \int_{|z| \geq 1} |z|^2 \pi(dz) < \infty. \quad (2.65)$$

Then (2.63) has a unique invariant measure $\pi \in \mathcal{P}(\mathcal{D})$.

Proof By the variation-of-constants formula (see, e.g., [119, Theorem 3.1]), one has

$$\begin{aligned} X(t; \xi) &= \Gamma(t)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t + \theta - s) \xi(s) ds \\ &\quad + \int_0^t \Gamma(t - s) \sigma(X_{s-}(\xi)) dZ(s), \end{aligned}$$

where $(\Gamma(t))_{t \geq -\tau}$ is the fundamental solution to (2.6). By (2.65), the Lévy-Itô decomposition (see, e.g., [3, Theorem 2.4.16, p. 126]) gives that

$$Z(t) = At + W(t) + \int_{z \neq 0} z \tilde{N}(t, dz), \quad A \in \mathbb{R}^n. \quad (2.66)$$

Thus, we have

$$\begin{aligned} X(t; \xi) &= \Gamma(t)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^0 \Gamma(t + \theta - s) \xi(s) ds \\ &\quad + \int_0^t \Gamma(t - s) \sigma(X_s(\xi)) A ds \\ &\quad + \int_0^t \Gamma(t - s) \sigma(X_s(\xi)) dW(s) + \int_0^t \int_{z \neq 0} \Gamma(t - s) \sigma(X_{s-}(\xi)) z \tilde{N}(ds, dz). \end{aligned}$$

According to (2.8), there exist $\varepsilon, \delta \in (0, -\lambda_0)$ with $\varepsilon + \delta \in (0, -\lambda_0)$ such that

$$\alpha := \int_0^\infty e^{2\varepsilon t} \|\Gamma(t)\|^2 dt < \infty \quad \beta := \int_0^\infty e^{2\varepsilon t} \|\dot{\Gamma}(t)\|^2 dt < \infty, \quad (2.67)$$

and

$$\kappa := \int_0^\infty e^{2\delta t} \|\dot{\Gamma}_1(t)\|^2 dt < \infty, \quad (2.68)$$

where $\dot{\Gamma}(t)$ denotes the derivative of $\Gamma(t)$, and $\Gamma_1(t) = e^{\varepsilon t} \Gamma(t)$. For any $\varepsilon \in (0, -\lambda_0)$ such that (2.67) holds, it is easily seen that

$$\begin{aligned} e^{2\varepsilon t} \mathbb{E}|X(t; \zeta)|^2 &\leq 5 \left\{ |e^{\varepsilon t} \Gamma(t) \zeta(0)|^2 + \left| \int_{[-\tau, 0]} \mu(d\theta) \int_\theta^0 e^{\varepsilon t} \Gamma(t + \theta - s) \zeta(s) ds \right|^2 \right. \\ &\quad + \mathbb{E} \left| \int_0^t e^{\varepsilon t} \Gamma(t - s) \sigma(X_s(\zeta)) \Lambda ds \right|^2 \\ &\quad + \mathbb{E} \left| \int_0^t e^{\varepsilon t} \Gamma(t - s) \sigma(X_s(\zeta)) dW(s) \right|^2 \\ &\quad \left. + \mathbb{E} \left| \int_0^t \int_{z \neq 0} e^{\varepsilon t} \Gamma(t - s) \sigma(X_{s-}(\zeta)) z \tilde{N}(ds, dz) \right|^2 \right\} \\ &=: 5\{\gamma_1(t) + \gamma_2(t) + \gamma_3(t) + \gamma_4(t) + \gamma_5(t)\}. \end{aligned}$$

By Hölder's inequality, we observe from (2.67) that

$$\begin{aligned} \gamma_3(t) &= \mathbb{E} \left| \int_0^t e^{\varepsilon(t-s)} \Gamma(t-s) e^{\varepsilon s} \sigma(X_s(\zeta)) \Lambda ds \right|^2 \\ &\leq \|\Lambda\|^2 \int_0^t e^{2\varepsilon s} \|\Gamma(s)\|^2 ds \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\zeta))\|_{HS}^2 ds \\ &\leq \alpha \|\Lambda\|^2 \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\zeta))\|_{HS}^2 ds. \end{aligned}$$

Next, by virtue of Fubini's theorem and (2.68), it follows that

$$\begin{aligned}
\Upsilon_4(t) &= \mathbb{E} \left| \int_0^t \left\{ I + \int_0^{t-s} \dot{I}_1(u) du \right\} e^{\varepsilon s} \sigma(X_s(\xi)) dW(s) \right|^2 \\
&\leq 2 \mathbb{E} \left| \int_0^t e^{\varepsilon s} \sigma(X_s(\xi)) dW(s) \right|^2 \\
&\quad + 2 \mathbb{E} \left| \int_0^t e^{\delta u} \dot{I}_1(u) e^{-\delta u} \int_0^{t-u} e^{\varepsilon s} \sigma(X_s(\xi)) dW(s) du \right|^2 \\
&\leq 2 \mathbb{E} \left| \int_0^t e^{\varepsilon s} \sigma(X_s(\xi)) dW(s) \right|^2 + 2\kappa \int_0^t e^{-2\delta u} \mathbb{E} \left| \int_0^{t-u} e^{\varepsilon s} \sigma(X_s(\xi)) dW(s) \right|^2 du \\
&\leq 2 \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds + 2\kappa \int_0^t e^{-2\delta u} \int_0^{t-u} e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds du \\
&\leq 2 \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds + 2\kappa \int_0^t \int_0^{t-s} e^{-2\delta u} du e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds \\
&\leq 2(1 + \kappa\delta^{-1}) \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds.
\end{aligned}$$

Similarly, we have

$$\Upsilon_5(t) \leq 2(\beta + \vartheta)(1 + \kappa\delta^{-1}) \int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds,$$

where $\beta, \vartheta > 0$ are introduced in (2.53) and (2.65), respectively. Furthermore, observe from (2.64) that

$$\begin{aligned}
&\int_0^t e^{2\varepsilon s} \mathbb{E} \|\sigma(X_s(\xi))\|_{HS}^2 ds \\
&\leq c(e^{2\varepsilon t} + \|\xi\|_\infty^2) + 2(1 + \varepsilon')L^2(1 + e^{2\varepsilon\tau} \|\rho\|_{\text{var}}^2) \int_0^t e^{2\varepsilon s} \mathbb{E} |X(s; \xi)|^2 ds
\end{aligned}$$

for any $\varepsilon' > 0$. Therefore, we arrive at

$$\begin{aligned}
e^{2\varepsilon t} \mathbb{E} |X(t; \xi)|^2 &\leq c(e^{2\varepsilon t} + \|\xi\|_\infty^2) + 10\{\alpha \|A\|^2 + 2(1 + \beta + \vartheta)(1 + \kappa\delta^{-1})\} \\
&\quad \times (1 + \varepsilon')L^2(1 + e^{2\varepsilon\tau} \|\rho\|_{\text{var}}^2) \int_0^t e^{2\varepsilon s} \mathbb{E} |X(s; \xi)|^2 ds.
\end{aligned}$$

So the Gronwall inequality leads to

$$\sup_{t \geq 0} \mathbb{E} |X(t, \xi)|^2 \lesssim 1 + \|\xi\|_\infty^2, \quad (2.69)$$

whenever $L > 0$ is sufficiently small. In the light of B–D–G’s inequality (see, e.g., [116, Theorem 48, p. 193]), besides (2.64), one derives from (2.66) and (2.69) that

$$\begin{aligned}
\mathbb{E}\|X_t(\xi)\|_\infty^2 &\lesssim 1 + \mathbb{E}\left(\sup_{t-\tau \leq s \leq t} \left|\int_{t-\tau}^s \sigma(X_s) dW(s)\right|^2\right) \\
&\quad + \mathbb{E}\left(\sup_{t-\tau \leq s \leq t} \left|\int_{t-\tau}^s \int_{z \neq 0} \sigma(X_{s-}) z \tilde{N}(ds, dz)\right|^2\right) \\
&\lesssim 1 + (1 + \beta + \vartheta) \int_{t-\tau}^t \mathbb{E}\|\sigma(X_s)\|_{HS}^2 ds \\
&\lesssim 1 + \|\xi\|_\infty^2, \quad t \geq \tau.
\end{aligned}$$

Carrying out the argument to deduce (2.69), we can derive that

$$\mathbb{E}|X(t; \xi) - X(t; \eta)|^2 \lesssim e^{-\lambda' t} \|\xi - \eta\|_\infty^2$$

for some $\lambda' > 0$. This, together with (2.64) and B–D–G’s inequality, yields that

$$\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \lesssim e^{-\lambda' t} \|\xi - \eta\|_\infty^2.$$

Finally, the desired assertion follows by imitating the corresponding argument of Theorem 2.18.

Remark 2.23 By a close scrutiny of the argument of Theorem 2.22, in fact, we can provide an upper bound of the Lipschitz constant $L > 0$.

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