

Preface and Introduction

This work focuses on dynamical systems that involve delays and random disturbances. The motivation of our study stems from a wide variety of systems in real life in which random noise has to be taken into consideration and the effect of delays cannot be avoided or ignored. In fact, there are numerous sources of delays that we encounter in every-day life. For example, everyone has the experience of delays in highway traffic, air traffic, and package delivery etc. Accompanying the rapid progress in technology, digital computers, internet, and various hand-held “smart” devices, delays become more and more prevalent in physical, cyber-physical, and biological systems. In the new era, delays are ubiquitous in wired or wireless communications, queueing, and networked control problems.

In this work, we concentrate on such systems that are described by functional stochastic differential equations. A functional stochastic differential equation is one whose state space depends on the past. Accompanying the extensive literature on functional differential equations for deterministic systems (see, e.g., the monograph [62]), the study of such stochastic systems was initiated in 1964 by Itô and Nisio [69]. For regularity and stochastic stability of solution processes, and Markov trajectories plus the infinitesimal generator for segment processes, we refer to the monographs [96] and [106], respectively. Because their importance, significant efforts have also been devoted to the study of delay differential systems and more generally functional differential systems, which are exemplified by chemical processes, biological systems, and communication systems. It is observed that delays may have detrimental impact on stability and performance of dynamical systems. Their presence adds substantial difficulties in analysis of long-term behavior of systems and stability. Facing the challenges, we need to have a thorough understanding on systems with delays and treat such systems with great care.

Motivational Examples

There are many examples and numerous applications involving functional differential equations with noise. Among the early works on controlled stochastic differential delay equations is [85], whereas early treatment of stability of stochastic delay equations can be found in [82]. In recent years, emerging applications in financial engineering have drawn much attention; see [2, 4, 7, 75] and references therein. Although Black-Scholes formula is one of the most important results in finance, the feasibility of the model has been questioned because of the assumption of constant appreciation and volatility rates. To treat more realistic situation, effort has also been directed to the development of dynamical models that take into consideration the influence of past history.

This brief is devoted to the study of large time behavior of stochastic functional differential equations. As a motivational example, we consider the following controlled functional stochastic differential equations in an infinite horizon, which is a modification of that considered in [84, Chapter 5]. Suppose that $X(t)$ is a diffusion process living in $\mathbb{R}^n$ and that X_t is the associated memory segment process $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$ with $\tau > 0$. Let U be the set of controls, and $b(\cdot)$ and $\sigma(\cdot)$ be appropriate functions (more precise notation will be given in later chapters), and $W(t)$ be a standard Brownian motion. Consider the following functional stochastic differential equation

$$dX(t) = b(X_t, u(X_t))dt + \sigma(X_t)dW(t).$$

We wish to minimize a long run average cost functional given by

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^u \left[\int_0^T k(X_t, u(X_t)) dt \right].$$

The idea of the ergodic cost problem is to replace the instantaneous measure by that of the ergodic measure so that the expectation becomes an average with respect to the ergodic measure leading to a substantial reduction of complexity. Such study is of great utility in many applications. However, to be able to carry out the desired task, one needs to make sure that under suitable conditions, there is an ergodic measure for such systems. This is one of the main objectives of the current brief.

As a second example, consider an adaptive control problem of controlled diffusion with past dependence treated in [46]. For a fixed $\tau > 0$, let $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^n)$ be the space of continuous functions. Consider a controlled process defined by

$$\begin{cases} dX(t) = \{f_1(X(t)) + f_2(X_t, u(X_t), \alpha^*)\}dt + g(X(t))dW(t) \\ X(0) = \xi \in \mathcal{C}, \end{cases}$$

where $X(t) \in \mathbb{R}^n$, α^* is an unknown parameter living in $\mathbb{R}^d$, and $W(\cdot)$ is a standard $\mathbb{R}^n$ -valued Brownian motion, $u(\cdot) \in U \subset \mathbb{R}^m$ with U being a compact set and $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$. The functions $f_1(\cdot)$ and $g(\cdot)$ satisfy global Lipschitz condition. The diffusion is assumed to be non-degenerate in that $g(x)g^*(x) \geq cI_{n \times n}$ for some $c > 0$ and all $x \in \mathbb{R}^n$ with g^* denoting the transpose of g . The function $f_2(\cdot)$ is assumed to be bounded and Borel measurable on $\mathcal{C}$. The objective is to minimize the long-run average cost

$$J(u, \xi, \alpha^*) = \limsup_{T \rightarrow \infty} \frac{1}{T} E_{\xi, u, \alpha^*} \int_0^T k(X_s, u(X_s)) ds,$$

where $k(\cdot) : \mathcal{C} \times U \mapsto \mathbb{R}$ is a bounded and continuous function. In [46], it was shown that the unknown parameter α^* can be estimated by a biased maximum likelihood estimator and a certainty equivalent control can be obtained. The control so designed yields an almost self-optimizing adaptive control. Again, in this process, the ergodicity plays an essential role.

Recently, there are much interests in treating the so-called consensus formation for networked systems and multi-agent systems. Related published works in physics, computer science, control engineering, ecology, biology, social sciences, as well as in the journal *Nature* among others, show the growing interests from a wide range of communities. The goal is to achieve a common objective such as position, speed, load distribution, etc. for a group of members often referred to as mobile agents. Such models have been successfully used since late 1980s in computer graphics, physics, control engineering, and social network modeling, to describe collective behavior such as flocking, schooling, autonomous vehicles, and other group behavior among others. A discrete-time model of autonomous agents, which can be viewed as points or particles, all moving in the same plane with the same speed but with different headings was proposed in [132], in which each agent updates its heading using a local rule based on the average headings of its own and its neighbors. This model turns out to be related to a formulation introduced earlier for simulating animation of flocking and schooling behaviors in [120]. A version of the scheme taking into consideration of time delay with communication latency and possibly random-switching topology may be written as a stochastic approximation type algorithm of the following form

$$x_{n+1} = x_n + \mu M(\alpha_n) x_{n-\lfloor d/\mu \rfloor} + \mu \hat{G}(x_{n-\lfloor d/\mu \rfloor}, \alpha_n, \zeta_{n-\lfloor d/\mu \rfloor}, \tilde{\zeta}_n),$$

with the initial segment x_k for $k = -\lfloor d/\mu \rfloor, \dots, 0$ being arbitrary. In the above, x_n is in a multi-dimensional Euclidean space, $\mu > 0$ is the stepsize of consensus control algorithm, $\hat{G}$ is an appropriate function, $d > 0$ is a constant and $\lfloor d/\mu \rfloor$ denotes the integer part of d/μ representing the time delays, α_n is a discrete-time Markov chain with state space $\mathcal{M} = \{1, \dots, m_0\}$ and transition matrix $P^\varepsilon = I + \varepsilon Q$ ($\varepsilon > 0$ is a small parameter, I is the identity matrix, and Q is a generator of a continuous-time Markov chain), $M(i)$ is the generator of a continuous-time Markov

chain for each $i \in \mathcal{M}$, $d > 0$ is a constant with $\lfloor d/\mu \rfloor$ representing the delay, and $\{\zeta_n\}$ and $\{\tilde{\zeta}_n\}$ are sequences representing measurement or observation noise. Assuming that $\mu = O(\varepsilon)$ and taking a continuous-time interpolation $x^\varepsilon(t) = x_n$ and $\alpha^\varepsilon = \alpha_n$ for $t \in [n\varepsilon, (n+1)\varepsilon)$. Then $x^\varepsilon(\cdot)$ belongs to a function space that consists of functions that are right continuous with left limits endowed with the Skorohod topology. Under appropriate conditions, it can be shown that the noises are averaged out, $x^\varepsilon(\cdot)$ has a weak limit characterized by the switched pure delay equation $\dot{x}(t) = M(\alpha(t))x(t-d)$, where $\alpha(\cdot)$ is a continuous-time Markov chain [the weak limit of $\alpha^\varepsilon(\cdot)$]. Furthermore, it can be demonstrated that $x^\varepsilon(\cdot + t_\varepsilon)$ (with $t_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$) converges weakly to θ , a vector with all components being the same (i.e., consensus is reached). With more effort, one can analyze the corresponding estimation error sequence $\{x_n - \theta\}$ and show that a suitable scaling leads to a stochastic differential delay equation with Markov switching of the form

$$dz(t) = M(\alpha(t))z(t-d)dt + \overline{G}(\alpha(t))dW(t),$$

for an appropriate function $\overline{G}(\cdot)$ and a standard multidimensional Brownian motion $W(\cdot)$; see [148] for more details together with many references therein.

Purposes of This Brief

This brief is mainly for research purposes. It is written for probabilists, applied mathematicians, engineers, and scientists who need to use delay systems in their work. Selected topics from the brief can also be used in a graduate level topics course in probability and stochastic processes.

The format of the SpringerBriefs in Mathematics gives us an excellent opportunity to focus on the long-term behavior of the functional stochastic differential equations. This very focused approach enables us to emphasize the central theme; our study encompasses ergodicity. Although there are many excellent treatises on stochastic differential delay equations and functional stochastic differential equations, short monographs devoted to ergodicity of functional stochastic differential equations seem to be scarce to date. It would certainly be welcomed to have a work collect a number of long-run properties about functional stochastic systems. Nevertheless, because of the format of the Briefs, we are not able to make this book a comprehensive treatment of functional stochastic differential equations. In fact, we are not even able to include the vast literature in a short book like this.

Outline of the Book

This book is organized as follows. Chapter 1 is devoted to ergodicity of functional stochastic differential equations and Chapter 2 focuses on ergodicity without dissipative conditions. Chapter 3 ascertains rates of convergence of Euler–Maruyama procedures. Chapter 4 obtains large deviations estimates for neutral functional stochastic differential equations with jumps. Chapter 5 gives an application to an interest model in an infinite horizon. Finally, to make the brief relatively self-contained, two appendices are given at the end of the book to collect a number of results on existence and uniqueness of solutions of functional stochastic differential equations, Markov properties, as well as certain technical results such as variation of constants formulas.

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